

μ -Measure equivalence rigidity of hyperbolic lattices

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l^p -Measure equivalence

Measure equivalence with l^p -condition

A **ME-coupling** (Ω, μ) of Γ and Λ is a measure space with a μ -preserving action of $\Gamma \times \Lambda$ such that Γ, Λ both have a μ -finite fundamental domain. If Γ and Λ admit a ME-coupling with l^p -integrable cocycles w.r.t. some fundamental domains then we call them **l^p -measure equivalent**.

l^p -cocycles

A measurable cocycle $\alpha : \Gamma \times (X, \mu) \rightarrow \Lambda$ is **l^p -integrable** if for every $\gamma \in \Gamma$

$$\int_X l(\alpha(\gamma, x))^p d\mu(x) < \infty,$$

where $l : \Lambda \rightarrow \mathbb{N}$ is the length function for some word metric on Λ .

- l^p -ME interpolates between $p = \infty$ (\Rightarrow QI) and $p = 0$ (=ME).
- l^p -ME is an equivalence relation on groups.

Rigidity result for hyperbolic lattices

Theorem (informal)

Let Γ be a lattice in $G = \text{Isom}(\mathbb{H}^n)$, $n \geq 3$. Then any l^1 -ME-coupling of Γ with another group basically comes from the standard example of lattices in G or atomic couplings of commensurable groups.

- The standard coupling of hyperbolic lattices is l^1 -integrable.
- A corresponding rigidity result for **orbit equivalence (OE)** can be formulated.
- Analogous rigidity results (without any l^1 -integrability condition) for lattices in higher rank Lie groups hold true **[Furman, 2000]**.
- Lack of rigidity for $n = 2$: $\mathbb{Z}^2 * \mathbb{Z}^2$ OE to $\mathbb{Z} * \mathbb{Z}$.

Precise rigidity result – ME-version

Theorem (Bader-Furman-S.)

Let Γ be a lattice in $G = \text{Isom}(\mathbb{H}^n)$, $n \geq 3$. Let (Ω, μ) be an ergodic, l^1 -integrable ME-coupling with another group Λ .

Then the following holds:

- a) There exists a homomorphism $\rho : \Lambda \rightarrow G$ with finite kernel and image being a lattice in G .
- b) There exists a $\Gamma \times \Lambda$ -equivariant measurable map $\phi : \Omega \rightarrow G$; the push-forward measure $\phi_*\mu$ is the Haar measure corresponding either
 - i) to G ,
 - ii) or to its index two subgroup $G^0 = \text{Isom}_+(\mathbb{H}^n)$,
 - iii) or to a lattice $\Gamma' < G$.

In the latter case, Γ' contains Γ and $\rho(\Lambda)$ as subgroups of finite index.

Classical Mostow rigidity

Theorem (Mostow rigidity – Lie-theoretic version)

Any isomorphism $\Gamma \rightarrow \Lambda$ of lattices in $G = \text{Isom}(\mathbb{H}^n)$, $n \geq 3$, extends to an automorphism of G .

Theorem (Mostow rigidity – topological version)

Let M and N be closed hyperbolic n -dimensional manifolds. Then any homotopy equivalence $M \rightarrow N$ is homotopic to an isometry.

topological version \Rightarrow Lie-theoretic version:

Induction over skeleta

$$\begin{array}{ccc} \coprod_{S_i} \Gamma \times \partial\Delta^i & \longrightarrow & \tilde{M}^{(i-1)} \\ \downarrow \text{hook} & & \downarrow \\ \coprod_{S_i} \Gamma \times \Delta^i & \longrightarrow & \tilde{M}^{(i)} \dashrightarrow \tilde{N} \end{array}$$

Thurston's proof of (topological) Mostow rigidity

Proof for closed manifolds

Step 1) $f : M \xrightarrow{\cong} N \Rightarrow \|M\| = \|N\| \Rightarrow \text{vol}(M) = \text{vol}(N)$

[Gromov-Thurston].

Step 2) $\tilde{f} : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is a quasi-isometry, thus induces a homeomorphism

$$\partial_\infty \tilde{f} : \partial_\infty \mathbb{H}^n \xrightarrow{\cong} \partial_\infty \mathbb{H}^n.$$

Step 3) Regular, ideal n -simplices are exactly the geodesic n -simplices with maximal volume [Haagerup-Munkholm].

$\partial_\infty \tilde{f}$ preserves regular, ideal simplices.

Step 4) Hyperbolic geometry: $\partial_\infty \tilde{f}$ induced by an isometry.

Modification for finite volume manifolds

Only from volume considerations, Thurston constructs a measurable $\partial_\infty \tilde{f}$ that preserves regular, ideal n -simplices almost everywhere.

Reduction of main theorem to cocycle Mostow rigidity I

Theorem (adapted from Furman's earlier work)

Let Γ be a lattice in $G = \text{Isom}(\mathbb{H}^n)$, and Λ be an arbitrary group ME to Γ via the coupling (Ω, m) . Let $(\Sigma, n) = (\Omega, m) \times_{\Lambda} (\Omega^{\text{op}}, m)$ be the corresponding self-coupling of Γ . Assume that there exists a measurable $\Gamma \times \Gamma$ -equivariant map $\Phi : \Sigma \rightarrow G$ ("*untwisting map*"), i.e. n -a.e.

$$\Phi([\gamma x, \gamma' y]) = \gamma \Phi([x, y]) \gamma'^{-1} \quad (\gamma, \gamma' \in \Gamma).$$

Then there exist measurable maps $f : \Omega \rightarrow G$ and a homomorphism $\rho : \Lambda \rightarrow G$ so that

$$f((\gamma, \lambda)x) = \gamma f(x) \rho(\lambda)^{-1}.$$

Then elementary observations (for lattice image) and an application of Ratner's theorems (for identifying $\Phi_* n$) eventually yield the main theorem.

... How do we get the untwisting map?

Reduction of main theorem to cocycle Mostow rigidity II

Setting

- Let $X \subset \Sigma$ be a common fundamental domain of both copies of Γ , and $\alpha : \Gamma \times X \rightarrow X$ be the corresponding OE-cocycle.
- We may assume that $\Gamma \subset \text{Isom}(\mathbb{H}^n)$ is co-compact. Let $M = \Gamma \backslash \mathbb{H}^n$.

Proof of main theorem – outline

- Step 1) Extend $\alpha : X \rightarrow \text{map}(\Gamma, \Gamma)$ to a α -equivariant, measurable map $\psi : X \rightarrow \text{map}(\tilde{M}, \tilde{M})$.
- Step 2) Show that ψ induces a measurable, α -equivariant map $\partial_\infty \psi : X \rightarrow \mathcal{M}(\partial \mathbb{H}^n, \partial \mathbb{H}^n)$ that **preserves regular, ideal n -simplices**.
- Step 3) Hyperbolic geometry $\Rightarrow \partial_\infty \psi$ comes from of a α -equivariant map $\phi : X \rightarrow \text{Isom}(\tilde{M}) = G$ (**cocycle Mostow rigidity**).
- Step 4) ϕ is a coboundary for α ; thus we can also untwist Σ .

A crucial step in the proof – controlling volume

Lemma

For any geodesic simplex σ with $\text{vol}(\sigma) \approx v_{\max}$ we have

$$\int_X \int_{\Gamma \backslash G} \text{vol}(\psi_x(g\sigma)) d\mu_X(x) d\mu_{\Gamma \backslash G}(g) \approx \text{vol}(\sigma).$$

Volume and degree 1 maps

Let $f : M \rightarrow M$ be a degree 1 map. Let $c = \sum a_i \sigma_i$ be an n -cycle. Then:

$$\sum a_i \text{vol}^{\text{or}}(\sigma_i) = \sum a_i \text{vol}^{\text{or}}(f(\sigma_i)).$$

- Find **suitable homology theories** for our situation.
- Show that $\psi : X \rightarrow \text{map}(\tilde{M}, \tilde{M})$ is of degree 1.
- View left side of lemma as the **evaluation of a homology class at the volume form**.

L^1 -homology and induced maps

Maps induced by $\alpha : \Gamma \times X \rightarrow \Gamma$ and $\psi : X \rightarrow \text{map}(\tilde{M}, \tilde{M})$

$$\begin{array}{ccc} \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_*(\Gamma) \hookrightarrow & \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_*^{\text{geo}}(\tilde{M}) \\ C_*(\alpha) \downarrow & & \downarrow C_*^{\text{geo}}(\psi) \\ \overline{L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\Gamma)}^1 \hookrightarrow & \longrightarrow & \overline{L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*^{\text{geo}}(\tilde{M})}^1 \end{array}$$

Remarks

- $L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*^{\text{geo}}(\tilde{M}) = \bigoplus_F L^1(X; \mathbb{Z})$
- Vertical maps are inclusions of orbits.
- $C_0(\alpha)(1 \otimes \gamma) = \sum \chi_{X_i} \otimes \gamma_i$ where $\alpha(x, \gamma) = \gamma_i$ constant on $x \in X_i$.

A new deformation-rigidity phenomenon

Integrality, Poincare duality, simplicial volume

We have by Poincare duality and ergodicity

$$H_n(L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*^{\text{geo}}(\tilde{M})) \cong H^0(\tilde{M}; L^1(X; \mathbb{Z})) = L^1(X; \mathbb{Z})^\Gamma \cong \mathbb{Z}.$$

Since the simplicial volume of M is > 0 every Cauchy sequence of cycles in $L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n^{\text{geo}}(\tilde{M})$ is eventually constant!

Sobolev homology and l^1 -condition

Under l^1 -integrability, we show that $H_n(\phi)$ already lands in

$$H_n(L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*^{\text{geo}}(\tilde{M})) \subset \overline{H_n(L^1(X; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*^{\text{geo}}(\tilde{M}))}^1$$

For this we use a new tool (**Sobolev homology**) and the ability to subdivide geodesic simplices in negative curvature very efficiently.

... THANK YOU!