GENERA, THE CHERN CHARACTER AND THE THOM ISOMORPHISM

MANUEL AMANN

In this talk we shall provide the necessary terminology which will enable us to formulate a cohomological version of the Atiyah–Singer Index Theorem.[1]

Throughout the talk cohomology is taken with rational coefficients unless indicated differently.

Structure of the talk. In section 1 we shall recall the splitting principle and we shall apply it to characteristic classes. In section 2 we shall introduce multiplicative sequences and genera. This section is central to the talk. Section 3 is devoted to the Chern character and in section 4 the Thom isomorphism is depicted.

1. Splitting Principles

Let us first briefly recall the main tool for various constructions, namely the *splitting principle*. We shall need

Lemma 1.1 (Hirsch–Leray). *Let E be a complex vector bundle of rank k. Then $H^*(\mathbf{P}_{\mathbb{C}}(E), \mathbb{Z})$ is a free $H^*(X, \mathbb{Z})$-module with basis $1, u, u^2, \ldots, u^{k-1}$ for some $u \in H^2(\mathbf{P}_{\mathbb{C}}(E), \mathbb{Z})$.*

This may be generalised replacing $1, u, u^2, \ldots, u^{k-1}$ by a basis of the fibre of a proper map.

Proposition 1.2 (Splitting Principle). *Let $E \to M$ be a complex vector bundle of rank n over a manifold X. Then there is a manifold $F(E)$ and a smooth fibration $\pi : F(E) \to M$ such that*

- *The homomorphism $\pi^* : H^*(M, \mathbb{Z}) \to H^*(F(E), \mathbb{Z})$ is injective.*
- *The bundle $\pi^* E$ splits into a direct sum of complex line bundles, i.e.*

$$\pi^* E = l_1 \oplus \cdots \oplus l_n$$

Proof. Consider the projectivisation $p : \mathbf{P}_{\mathbb{C}}(E) \to M$ and the complex bundle $p^* E \to \mathbf{P}_{\mathbb{C}}(E)$. We have $p^* E = \mathbf{P}_{\mathbb{C}}(E) \times_M E$ and this bundle admits the section $l \mapsto (l, 1_l)$ over $\mathbf{P}_{\mathbb{C}}(E)$, i.e. there is a line subbundle l_1 of $p^*(E)$. Choosing a metric we may decompose $p^*(E) = l_1 \oplus l_1^{\perp}$. Proceeding inductively for $l_1^{\perp}$, we decompose $p^*(E)$ as described. □

The space $F(E)$ can be chosen to be the *flag bundle* of E over M.

On the level of classifying spaces we make the following observation: Complex bundles have transition functions in $\mathbf{U}(n)$. Hence they are classified by $\mathbf{BU}(n)$. The inclusion of the maximal torus $T(n)$ into $\mathbf{U}(n)$ induces a map

Date: January 19th, 2010.

[1]The talk is based upon the chapters III.11 and III.12, p. 225-243, in [1].

$\mathbf{B}(T(n)) \to \mathbf{B}(\mathbf{U}(n))$. Pulling back the bundle along this map splits it into line bundles, as $\mathbb{S}^1 = \mathbf{U}(1)$ is the classifying space for line bundles.

Analogous statements hold for real bundles.

Proposition 1.3. *Let E be an oriented real vector bundle of rank $2n$ over a manifold M. Then there is a smooth fibration $\pi : \tilde{F}(E) \to M$ such that*

$$\pi^* E = E_1 \oplus \cdots \oplus E_n$$

with oriented real 2-plane bundles E_i. The morphism $\pi^* : H^*(M, \mathbb{Z}) \to H^*(\tilde{F}(E), \mathbb{Z})$ is injective and there is the decomposition

$$\pi^*(E \otimes \mathbb{C}) \cong l_1 \oplus \bar{l}_1 \oplus \cdots \oplus l_n \oplus \bar{l}_n$$

into complex line bundles and their conjugates given by $E_i = l_i \oplus \bar{l}_i$.

Proof. The first splitting can be obtained as in the the proof of proposition 1.2: We just replace the complex Grassmanian $\mathbf{P}_{\mathbb{C}}(E)$ by the Grassmanian $\widetilde{\mathbf{Gr}}_2(E)$ of real oriented 2-planes, i.e. E_1 is the tautological bundle over a point in $\widetilde{\mathbf{Gr}}_2(E)$. The injectivity of the morphism follows from the generalised version of Hirsch–Leray.

It remains to see that for a two-dimensional bundle $E \to M$ the bundle $E \otimes \mathbb{C}$ splits as $l \oplus \bar{l}$ with a complex line bundle l.

(Recall that the scalar multiplication on $\bar{l}$ is given by $\lambda(v) = \bar{\lambda} v$.) An orientation on a 2-dimensional real bundle is the same as a complex structure; this is reflected by the equality of structure groups $\mathbf{SO}(2) = \mathbf{U}(1)$. Thus we may consider E as a complex bundle with complex structure j.

The complex structure on the bundle $E \otimes \mathbb{C} = E \oplus iE$ is given by multiplication with i. The map $f : E \to E \otimes \mathbb{C}$ given by $e \mapsto (e, -je) = e - ije$ is complex linear, i.e. $f(je) = if(e)$; the map $g : E \to E \otimes \mathbb{C}$ given by $e \mapsto (e, je) = e + ije$ is conjugate linear, i.e. $g(je) = -ig(e)$. Evidently, $f(e) = \bar{g}(e)$. Thus, f is the inclusion of the complex line bundle $l := E$ and g includes E as the conjugate line bundle $\bar{E} = \bar{l}$. Hence, by dimension and after choosing a metric, we obtain the decomposition $E \otimes \mathbb{C} = l \oplus \bar{l}$. □

Remark 1.4. Recall that we may identify the conjugate bundle $\bar{E}$ of a complex vector bundle $E \to M$ with the dual bundle of the latter. Given a hermitian metric $\langle \cdot, \cdot \rangle$ in E—linear in the first component, anti-linear in the second one—the correspondence $v \mapsto \langle \cdot, v \rangle$ identifies $\bar{E}$ with $E^* = \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$.

In particular, we see that for line bundles

$$l \otimes \bar{l} \cong l^* \otimes l = \mathrm{Hom}_{\mathbb{C}}(l, \mathbb{C}) \otimes l = \mathrm{Hom}(l, l)$$

is trivial, as it is a line bundle which admits the section corresponding to the identity morphism from l to l.

This makes line bundles a group under tensor products. This group is isomorphic to $H^2(M, \mathbb{Z})$. Indeed, line bundles are classified by $\mathbf{BU}(1) = \mathbb{C}\mathbf{P}^\infty$, which is $K(2)$, where K is the Eilenberg–MacLane spectrum for cohomology. ⊡

In the following we shall use the splitting principle to characterise characteristic classes. We shall start with the Euler class: Split the oriented real bundle E of rank $2n$ as $E = E_1 \oplus \cdots \oplus E_n$ a direct sum of 2-dimensional bundles. As the Euler class $\chi(E)$ satisfies $\chi(E \oplus E') = \chi(E)\chi(E')$, we obtain that $\chi(E) = x_1 \cdot \dots \cdot x_n$ with $\chi(E_i) =: x_i$. Complexifying $E \otimes \mathbb{C}$ we obtain that $E \otimes \mathbb{C} = l_1 \oplus \bar{l}_1 \oplus \dots \oplus l_n \oplus \bar{l}_n$ with $E_i \cong l_i$ as complex bundles, i.e. $x_i = \chi(E_i) = c_1(E_i) = c_1(l_i)$.

Let now E be a complex vector bundle of rank n. The total Chern class $c(E) = 1 + c_1(E) + \dots + c_n(E)$ satisfies $c(E \oplus E') = c(E)c(E')$. Again using the splitting principle we obtain $E = l_1 \oplus \dots l_n$ and

$$c(E) = \prod_{i=1}^{n} c(l_i) = \prod_{i=1}^{n} (1 + c_1(l_i)) = \prod_{i=1}^{n} (1 + x_i)$$

where again $c_1(l_i) = x_i$. This yields

$$\begin{aligned} c_1(E) &= x_1 + \dots + x_n \\ c_2(E) &= \sum_{i \neq j} x_i x_j \\ c_k(E) &= \sigma_k(x_1, \dots, x_n) \end{aligned}$$

where $\sigma_k(x_1, \dots, x_n)$ is the k-th elementary symmetric polynomial in the x_i. (More precisely, we pull back E to $F(E)$ where it splits as line bundles and we may describe its Chern classes in the cohomology of $F(E)$. As the latter contains the one of M, we may identify the Chern classes of E with the σ_k by the functoriality of the Chern classes.)

We may do similar with the Pontryagin classes $p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C})$ for $1 \leq i \leq n$ of a real oriented bundle E of rank $2n$.

Recall that the odd-degree Chern classes of $E \otimes \mathbb{C}$ vanish modulo 2-torsion, since

$$E \otimes \mathbb{C} \cong l_1 \oplus \bar{l}_1 \oplus \dots \oplus l_n \oplus \bar{l}_n \cong \overline{E \otimes \mathbb{C}}$$

and since $c_1(\bar{l}) = -c_1(l)$—this can be seen from the identification with the second cohomology group from remark 1.4—which yields $c_i(\bar{E}) = (-1)^j c_j(E)$.

The total Pontryagin class

$$1 + p_1(E) + \dots + p_n(E)$$

clearly has the property $p(E \oplus E') = p(E)p(E')$. Using the splitting principle we compute $E \otimes \mathbb{C} = l_1 \oplus \bar{l}_1 \oplus \dots \oplus l_n \oplus \bar{l}_n$ and

$$c(E \otimes \mathbb{C}) = \prod_{1 \leq i \leq n} c(l_i)c(\bar{l}_i) = \prod_{1 \leq i \leq n} (1 + x_i)(1 - x_i) = \prod_{1 \leq i \leq n} (1 - x_i^2)$$

and

$$p(E) = \sum_{0 \leq i \leq n} (-1)^i c_i(E \otimes \mathbb{C}) = \prod_{1 \leq i \leq n} (1 + x_i^2)$$

where $c_1(l_i) = x_i$. Thus, $p_i(E) = \sigma_i(x_1^2, \dots, x_n^2)$ for $1 \leq i \leq n$.

2. Multiplicative Sequences and Genera

By $\mathbb{Q}[\![x]\!]^\wedge$ we denote the set of formal rational power series with constant term equal to one. This set forms a multiplicative group. Given a power series $f \in \mathbb{Q}[\![x]\!]$ we observe that

$$f(x_1) \cdot \ldots \cdot f(x_n) = 1 + F_1(\sigma_1) + F_2(\sigma_1, \sigma_2) + F_3(\sigma_1, \sigma_2, \sigma_3) + \ldots$$

where F_k is a polynomial of degree k and σ_k is the k-th elementary symmetric function in the $x_1, \ldots, x_n$. Indeed, every symmetric polynomial may be expressed as a polynomial in the elementary symmetric functions.

We observe that the polynomials $F_k(\sigma_1, \ldots, \sigma_k)$ are independent of the numbers of variables x_i, i.e. in order to compute F_k we need to form $f(x_1) \cdot \ldots \cdot f(x_k)$—the number of variables x_i corresponds to the number of functions σ_i we need to express the symmetric polynomial in the x_i—and F_k remains the same polynomial when augmenting the number of variables. Indeed, the latter property is due to the fact that $\sigma_i(x_1, \ldots, x_n) = \sigma_i(x_1, \ldots, x_n, 0, \ldots)$ for $i \leq n$ and $\sigma_i(x_1, \ldots, x_n, 0, \ldots) = 0$ for $i > n$. That is, whenever we have additional variables, we can set them to zero and the polynomial F_k (in more variables) becomes exactly the polynomial F_k we had before. Thus these two polynomials have to coincide.

The sequence of polynomials $(F_k(\sigma_1, \ldots, \sigma_k))_{1 \leq k}$ is called the *multiplicative sequence* of f. Let B be a graded algebra. Denote by $B^\wedge$ the set of all formal (infinite) sums $1 + b_1 + b_2 + \ldots$ where $b_i \in B_i$. Multiplication in $B^\wedge$ is defined in analogy to multiplication in B. Again, every element in $B^\wedge$ has a multiplicative inverse. This makes $B^\wedge$ an abelian group under multiplication. As an example we obtain $B^\wedge = \mathbb{Q}[\![x]\!]^\wedge$ if $B = \mathbb{Q}[x]$.

Given a multiplicative sequence $(F_k(\sigma_1, \ldots, \sigma_k))_{1 \leq k}$, we associate a map $F : B^\wedge \to B^\wedge$ to it via

$$b = 1 + b_1 + b_2 + \ldots \mapsto F(b) = 1 + F_1(b_1) + F_2(b_1, b_2) + \ldots$$

That is, the morphism F is some sort of substitution morphism. Let us now describe the multiplicativity property of multiplicative sequences:

Lemma 2.1. *The map $F : B^\wedge \to B^\wedge$ is a group homomorphism, i.e. $F(bc) = F(b)F(c)$ for all $b, c, \in B^\wedge$.*

Proof. Consider the ring of symmetric polynomials $\mathbb{Q}[\sigma_1, \ldots, \sigma_{n+m}]$ in the variables $x_1, \ldots, x_{n+m}$. That is,

$$\begin{aligned}\sigma &= 1 + \sigma_1 + \cdots + \sigma_n \\ &= (1 + x_1) \cdot \ldots \cdot (1 + x_{n+m}) \\ &= (1 + x_1) \cdot \ldots \cdot (1 + x_n) \cdot (1 + x_{n+1}) \cdot \ldots \cdot (1 + x_{n+m}) \\ &= \sigma' \sigma''\end{aligned}$$

where σ' and σ'' are the total symmetric polynomials in the $x_1, \dots, x_n$ and $x_{n+1}, \dots, x_{n+m}$ respectively. By definition we obtain that

$$\begin{aligned} F(\sigma) &= 1 + F_1(\sigma_1) + F_2(\sigma_1, \sigma_2) + \dots \\ &= f(x_1) \cdot \dots \cdot f(x_{n+m}) \\ &= f(x_1) \cdot \dots \cdot f(x_n) \cdot f(x_{n+1}) \cdot \dots \cdot f(x_{n+m}) \\ &= F(\sigma')F(\sigma'') \end{aligned}$$

Now let B be arbitrary and let $b, c \in B^\wedge$. Consider the subalgebra $\langle b_1, \dots b_n, c_1, \dots c_m \rangle \subseteq B$, which is a quotient of the abstract polynomial algebra

$$\mathbb{Q}[b_1, \dots, b_n] \otimes \mathbb{Q}[c_1, \dots, c_m] \cong \mathbb{Q}[\sigma'_1, \dots, \sigma'_n] \otimes \mathbb{Q}[\sigma''_1, \dots, \sigma''_m]$$

under the projection homomorphism p. By the definition $F(1 + b_1 + \dots + b_n) = 1 + F_1(b_1) + \dots$ we see that F commutes with p.

Thus the elements $b, c \in B$ are represented by the elements σ' and σ'' in the polynomial algebra, i.e. $p(\sigma') = b$, $p(\sigma'') = c$. We compute

$$\begin{aligned} F((1 + b_1 + \dots b_n)(1 + c_1 + \dots c_m)) &= F(p(\sigma'\sigma'')) \\ &= p(F(\sigma'\sigma'')) \\ &= p(F(\sigma)) \\ &= p(F(\sigma') \cdot F(\sigma'')) \\ &= p(F(\sigma')) \cdot p(F(\sigma'')) \\ &= F(1 + b_1 + \dots b_n)F(1 + c_1 + \dots c_m) \end{aligned}$$

We derive the general result for $B^\wedge$ from this result for B, since a confirmation of this identity in degree k requires the terms b_r, c_r for $r \leq k$ only. □

Note that we may gain the power series f by $F(1 + x) = 1 + F_1(x) + F_2(x, 0) + \dots = f(x)$ in $\mathbb{Q}[x]$.

Let us now define the notion of a *genus* and relate it to multiplicative sequences in characteristic classes.

Definition 2.2. A genus φ is a ring homomorphism $\varphi : \Omega \otimes \mathbb{Q} \to R$ from the rationalised oriented cobordism ring $\Omega \otimes \mathbb{Q}$ into an integral domain R over $\mathbb{Q}$.

Prescribing the values on the complex projective spaces, the generators of the rationalised cobordism ring, one sees that every genus can be expressed via multiplicative sequences. Let us construct the genus φ_f belonging to a multiplicative sequence (F_k).

Let (F_k) as always be the multiplicative sequence associated to the formal power series f. Let E be a real vector bundle over a space X. We associate to E the *total F-class* $F(E) = F(p(E)) \in H^{4*}(X, \mathbb{Q})^\wedge$ where $p \in H^{4*}(X, \mathbb{Q})$ is the total (rational) Pontryagin class of E. Given two such bundles E and E' over X, we have $p(E \oplus E') = p(E)p(E')$ and $F(E \oplus E') = F(p(E \oplus E')) = F(p(E)p(E')) = F(E)F(E')$.

The (rational) Pontryagin classes of E (with $E \otimes \mathbb{C} = l_1 \oplus \bar{l}_1 \oplus \dots \oplus l_n \oplus \bar{l}_n$) can be described as the elementary symmetric polynomials in the x_i^2 where $x_i = c_1(l_i)$. This yields that $F(E) = f(x_1^2) \cdot \dots \cdot f(x_n^2)$.

Definition 2.3. The F-genus of a compact oriented manifold of dimension n is

$$F(X)[X] = \begin{cases} F_k(p_1(X), \ldots, p_k(X))[X] & \text{for } n = 4k \\ 0 & \text{otherwise} \end{cases}$$

By $[X] \in H_n(M)$ we denote the fundamental class of the manifold X over which we integrate.

Let us now see that the F-genus is a well-defined genus. Essentially, we need to prove multiplicativity and the vanishing on boundaries. Multiplicativity is clear, since by Künneth

$$\begin{aligned} F(X \times X')[X \times X'] &= F(p(X)p(X'))[X][X'] \\ &= (F(p(X))[X])(F(p(X'))[X']) \\ &= F(X)[X] \cdot F(X')[X'] \end{aligned}$$

The fact that $F(X)[X]$ vanishes for $X = \partial Y$ follows from Stoke's theorem. Indeed, the Pontryagin numbers of X are all zero in this case. This is due to the fact that the normal bundle of X in Y is trivial, since X and Y are orientable and thus X has a normal orientation in Y, i.e. a trivial normal bundle. Thus we have $p(TY|_X) = p(X)p(\mathbb{R}) = p(X)$ for rational Pontryagin classes (where $\mathbb{R}$ denotes the trivial normal bundle). Consider the exact sequence of the pair

$$H^n(Y) \xrightarrow{i^*} H^n(X) \xrightarrow{\delta} H^{n+1}(Y, X)$$

where $i : X \hookrightarrow Y$ is the inclusion. The morphism i^* thus maps the Pontryagin classes of $TY|_X$ exactly to the ones of TX and δ maps a product of Pontryagin classes to zero by exactness. We compute:

$$\langle p_1^{r_1} \cdot \ldots \cdot p_n^{r_n}, [\partial Y] \rangle = \langle \delta p_1^{r_1} \cdot \ldots \cdot p_n^{r_n}, [Y] \rangle = 0$$

Since $F(X)[X]$ is a linear combination of Pontryagin numbers, it also vanishes.

Let us now give the crucial examples. The genus belonging to the formal power series

$$\hat{a}(x) = \frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)} = 1 - \frac{1}{24}x + \frac{7}{2^7 \cdot 3^2 \cdot 5}x^2 + \ldots$$

is called the *$\hat{A}$-genus*. That is, the corresponding multiplicative sequence starts with

$$\begin{aligned} \hat{A}_1(p_1) &= -\frac{1}{24}x \\ \hat{A}_2(p_1) &= \frac{1}{2^7 \cdot 3^2 \cdot 5}(-4p_2 + 7p_1^2) \end{aligned}$$

As the (rational) Pontryagin classes are the elementary symmetric polynomials in the x_i^2, from a splitting as above we see that for a vector bundle E we obtain

$$\hat{A}(E) = \prod_{i=1}^{n} \frac{x_i/2}{\sinh(x_i/2)}$$

The L-genus corresponds to the formal power series

$$l(x) = \frac{\sqrt{x}}{\tanh \sqrt{x}}$$

It results from the Index Theorem or it can be deduced by direct verification that the *L-genus* of the tangent bundle coincides with the signature of the manifold.

One may replace the oriented cobordism ring by the complex one, which results in complex genera. These can be obtained by evaluation of multiplicative sequences on Chern classes instead of Pontryagin classes. An example of this is the *Todd-genus* belonging to the power series

$$\mathrm{td}(x) = \frac{x}{1 - e^{-x}}$$

The total Todd class of a vector bundle is given by $\mathrm{Td}(E) = \mathrm{Td}(c(E)) \in H^2(M, \mathbb{Q})^\wedge$. The genus of a complex manifold with respect to a power series is defined analogously. As above we see that this construction is well-defined, i.e. that it is multiplicative and vanishes on boundaries.

Proposition 2.4. *For any oriented real vector bundle E we compute*

$$\mathrm{Td}(E \otimes \mathbb{C}) = \hat{A}(E)^2$$

PROOF. Using the splitting principle and the notation from above we obtain that

$$\mathrm{Td}(E \otimes \mathbb{C}) = \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}} \cdot \frac{-x_i}{1 - e^{x_i}}$$

Multiplication by $e^{x_i/2} e^{-x_i/2}$ yields

$$\begin{aligned} \mathrm{Td}(E \otimes \mathbb{C}) &= \prod_{i=1}^{n} \left[\frac{x_i}{e^{x_i/2} - e^{-x_i/2}} \right]^2 \\ &= \prod_{i=1}^{n} \left[\frac{x_i/2}{\sinh(x_i/2)} \right]^2 \\ &= [\hat{A}(E)]^2 \end{aligned}$$

□

3. The Chern Character

Let E be a complex vector bundle of rank n over a manifold X. As above we use the splitting principle to write

$$c(E) = 1 + c_1 + \cdots + c_n = \prod_{i=1}^{n} (1 + x_i)$$

for the total Chern class of E so that $c_k = \sigma_i(x_1, \ldots, x_n)$.

Definition 3.1. The element

$$\begin{aligned}\operatorname{ch}(E) &= e^{x_1} + \cdots + e^{x_n}\\ &= n + \sum_{i=1}^{n} x_i + \frac{1}{2}\sum_{i=1}^{n} x_i^2 + \dots\\ &= n + c_1 + \frac{c_1^2 - 2c_2}{2} + \frac{c_1^3 - 3c_1c_2 + 3c_3}{6} + \cdots \in H^*(M, \mathbb{Q})\end{aligned}$$

is called the *Chern character* of E. The Chern character of M is defined to be the one of TM.

Proposition 3.2. *The Chern character is a ring homomorphism* $ch : K(X) \to H^{2*}(X, \mathbb{Q})$.

PROOF. Due to the nature of the Grothendieck construction it suffices to show that $\operatorname{ch}(E \oplus E') = \operatorname{ch}(E) \oplus \operatorname{ch}(E')$ and $\operatorname{ch}(E \otimes E') = \operatorname{ch}(E)\operatorname{ch}(E')$ for complex vector bundles E, E'. These two properties follow from an application of the splitting principle:

$$\begin{aligned}c(E \oplus E') &= c(E)c(E') = \prod_{i=1}^{n}(1 + x_i)\prod_{i=1}^{m}(1 + x_i')\\ c(E \otimes E') &= c((l_1 \oplus \cdots \oplus l_n) \otimes (l_1' \oplus \cdots \oplus l_m))\\ &= c\left(\sum_{i,j} l_i \otimes l_j'\right)\\ &= \prod_{i,j} c(l_i l_j')\\ &= \prod_{i,j}(1 + c_1(l_1) + c_1(l_j'))\\ &= \prod_{i,j}(1 + x_i + x_j')\end{aligned}$$

(For this we make use of the fact that $c_1(l \otimes l') = c_1(l) + c_1(l')$—cf. remark 1.4.)

Thus we obtain

$$\begin{aligned}\operatorname{ch}(E \oplus E') &= \sum_{i=1}^{n} e^{x_i} + \sum_{i=1}^{m} e^{x_j'} = \operatorname{ch}(E) + \operatorname{ch}(E')\\ \operatorname{ch}(E \otimes E') &= \sum_{i=1}^{n}\sum_{j=1}^{m} e^{x_i + x_j'} = \left(\sum_{i=1}^{n} e^{x_i}\right) \cdot \left(\sum_{j=1}^{m} e^{x_j'}\right) = \operatorname{ch}(E)\operatorname{ch}(E')\end{aligned}$$

□

In the following let us compute the Chern characters of several bundles. Consider the bundle $\bigwedge^k E$ of k-th exterior powers $(1 \leq k \leq n)$ of the complex bundle E of rank n. Now consider the ring $K(X)[\![x]\!]$ of formal power series in $K(X)$. We define a group homomorphism $\lambda_t : K(X) \to K(X)[\![x]\!]$ on vector bundles by

$$\lambda_t(E) = \sum_k \left[\bigwedge^k E\right] t^k$$

This map then satisfies $\lambda_t(E \oplus F) = \lambda_t(E)\lambda_t(F)$ due to

$$\bigwedge^k (E \oplus F) = \sum_{i+j=k} \left(\bigwedge^i E\right) \otimes \left(\bigwedge^j F\right)$$

which we derive from the corresponding equality of vector spaces.

We remark that for vector spaces $\lambda_t(E)$ is a polynomial whilst for general elements in $K(X)$ this is false. For example,

$$\lambda_t(-[l]) = (1 + t[l])^{-1} = \sum (-t)^m [l]^m$$

as $-[l]$ is the inverse of $[l]$, which is mapped to $1+t[l]$, and due to the formula for the geometric series.

Using the splitting principle we see from the homomorphism property that

$$\lambda_t(E) = \prod_{i=1}^n \lambda_t(l_i) = \prod_{i=1}^n (1 + t[l_i])$$

from which we deduce the formula

$$\mathrm{ch}(\lambda_t E) = \prod_{i=1}^n (1 + te^{x_i})$$

In particular, this yields

$$\mathrm{ch}(\lambda_{-1} E) = \prod_{i=1}^n (1 - te^{x_i}) = \mathrm{ch}\left(\bigwedge^{\mathrm{even}} E - \bigwedge^{\mathrm{odd}} E\right)$$

Now let E be a real oriented Riemannian vector bundle of dimension $2n$ and let $\omega_E = i^n e_1 \cdot \ldots \cdot e_{2n}$ be the oriented unit volume element in the complex Clifford bundle $\mathbb{C}\mathrm{l}(E) = \mathrm{Cl}(E) \otimes \mathbb{C}$. There is the splitting

$$\mathbb{C}\mathrm{l}(E) = \mathbb{C}\mathrm{l}^+(E) \oplus \mathbb{C}\mathrm{l}^-(E)$$

with $\mathbb{C}\mathrm{l}^{\pm}(E) = (1 \pm w_E)\mathbb{C}\mathrm{l}(E)$. The *Clifford difference element* is defined to be

$$\delta(E) := [\mathbb{C}\mathrm{l}^+(E)] - [\mathbb{C}\mathrm{l}^-(E)] \subseteq K(X)$$

Lemma 3.3. *The Clifford difference element satisfies*

$$\delta(E \oplus E') = \delta(E)\delta(E')$$

PROOF. Using $w_{E \oplus E'} = w_E w_{E'}$ we compute

$$\begin{aligned}
\delta(E)\delta(E') =& ([\mathbb{C}\mathrm{l}^+(E)] - [\mathbb{C}\mathrm{l}^-(E)]) \cdot ([\mathbb{C}\mathrm{l}^+(E')] - [\mathbb{C}\mathrm{l}^-(E')]) \\
=& \left(\frac{1}{2}(1 + w_E)[\mathbb{C}\mathrm{l}(E)] - \frac{1}{2}(1 - w_E)[\mathbb{C}\mathrm{l}(E)]\right) \\
& \cdot \left(\frac{1}{2}(1 + w_{E'})[\mathbb{C}\mathrm{l}(E')] - \frac{1}{2}(1 - w_{E'})[\mathbb{C}\mathrm{l}(E')]\right) \\
=& w_E w_{E'} [\mathbb{C}\mathrm{l}(E)][\mathbb{C}\mathrm{l}(E')]
\end{aligned}$$

and

$$\begin{aligned}\delta(E \oplus E') =&\delta([\mathbb{C}\mathrm{l}(E) \otimes \mathbb{C}\mathrm{l}(E')])^+ - \delta([\mathbb{C}\mathrm{l}(E) \otimes \mathbb{C}\mathrm{l}(E')])^- \\ =&\frac{1}{2}(1 + w_E w E')[\mathbb{C}\mathrm{l}(E) \otimes \mathbb{C}\mathrm{l}(E')] \\ &- \frac{1}{2}(1 - w_E w_{E'})[\mathbb{C}\mathrm{l}(E) \otimes \mathbb{C}\mathrm{l}(E')] \\ =&w_E w'_E[\mathbb{C}\mathrm{l}(E)][\mathbb{C}\mathrm{l}(E')]\end{aligned}$$

□

Let us now compute the Chern character of the Clifford difference element. Using the splitting principle we may write $E = E_1 \oplus \cdots \oplus E_n$ as a sum of 2-plane bundles E_i and $E \otimes \mathbb{C} = l_1 \oplus \bar{l}_1 \oplus \cdots \oplus l_n \oplus \bar{l}_n$. where $E_i \otimes \mathbb{C} = l_i \oplus \bar{l}_i$. Thus we only have to deal with the case $\dim E = 2$ with $E \otimes \mathbb{C} = l \oplus \bar{l}$. As $\mathbb{C}\mathrm{l}(E)$ is isomorphic to the complexified exterior bundle as vector spaces, we obtain

$$\mathbb{C}\mathrm{l}(E) = \mathbb{C} \cdot 1 \oplus l \oplus \bar{l} \oplus \mathbb{C} \cdot w_E$$

Choose an oriented orthonormal basis of E_x at some point x as (e_1, e_2). Then we obtain $l_x = \mathbb{C} \cdot (e_1 - ie_2)$ and $\bar{l}_x = \mathbb{C} \cdot (e_1 + ie_2)$. With $w_E = ie_1e_2$ we compute

$$\begin{aligned}&(1 + w_E)\mathbb{C}(e_1 - ie_2) = (1 + ie_1e_2)(e_1 - ie_2)\mathbb{C} = (e_1 - ie_2 + ie_2 - e_1)\mathbb{C} = 0 \\ &(1 + w_E)\mathbb{C}(e_1 + ie_2) = \mathbb{C}(e_1 + ie_2 + ie_2 + e_1) = \mathbb{C}(e_1 + ie_2) \\ &(1 - w_E)\mathbb{C}(e_1 - ie_2) = \mathbb{C}(e_1 - ie_2) \\ &(1 - w_E)\mathbb{C}(e_1 + ie_2) = 0\end{aligned}$$

so that

$$\begin{aligned}\mathbb{C}\mathrm{l}^+(E) &= (1 + w_E)(\mathbb{C} \cdot 1 \oplus l \oplus \bar{l} \oplus \mathbb{C} \cdot w_E) = (2 + 2w_E)\mathbb{C} + \bar{l} \\ \mathbb{C}\mathrm{l}^-(E) &= (1 - w_E)(\mathbb{C} \cdot 1 \oplus l \oplus \bar{l} \oplus \mathbb{C} \cdot w_E) = (1 - 1)\mathbb{C} + l\end{aligned}$$

(By $(1 - 1)\mathbb{C}$ we actually denote the trivial line bundle $(1 - w_E)(\mathbb{C} + \mathbb{C}w_E)$ of all elements $(1 - w_E)(a + bw_E) = (a - b) - w_E(a - b)$ for $(a, b) \in \mathbb{C}^2$.)

In particular, we obtain that $[\mathbb{C}\mathrm{l}^+(E)] = [\bar{l}]$ and $[\mathbb{C}\mathrm{l}^-(E)] = [l]$ (since the additional summands are trivial bundles). Thus

$$\delta(E) = [\bar{l}] - [l]$$

Using the splitting principle for bundles E of rank $2n$ and lemma 3.3 we compute

$$\delta(E) = \prod_{i=1}^{n}([\bar{l}_i] - [l_i])$$

and due to the homomorphism properties of the Chern character

$$\mathrm{ch}[\delta(E)] = \prod_{i=1}^{n}(e^{-x_i} - e^{x_i})$$

By the multiplicativity of the Euler class we computed

$$\chi(E) = x_1 \cdot \ldots \cdot x_n$$

which now yields

$$\begin{aligned}
\mathrm{ch}[\delta(E)] &= \chi(E) \prod_{i=1}^{n} \frac{e^{-x_i} - e^{x_i}}{x_i} \\
&= (-1)^n \chi(E) \prod_{i=1}^{n} \frac{e^{x_i/2} - e^{-x_i/2}}{x_i} (e^{x_i/2} + e^{-x_i/2}) \\
&= (-2)^n \chi(E) \prod_{i=1}^{n} \frac{\sinh(x_i/2)}{x_i/2} \cdot \frac{x_i/2}{\tanh(x_i/2)} \\
&= (-2)^n \chi(E) \hat{L}(E) \hat{A}(E)^{-2}
\end{aligned}$$

where the $\hat{L}$-genus belongs to the multiplicative sequence satisfying $L_m = 4^m \hat{L}_m$ (for $m \geq 2$). One may do similar for a spinor difference element on spin bundles.

4. The Thom Isomorphism

On a not necessarily compact manifold X^n we have Poincaré Duality by

$$D_X : H^p_{\mathrm{cpt}}(X) \xrightarrow{\cong} H_{n-p}(X)$$

Let Y^m be another manifold. Define for each $p \geq m - n$ *integration over the fibre*

$$f_! : H^p_{\mathrm{cpt}}(Y) \to H^{p-(m-n)}_{\mathrm{cpt}}(X)$$

by $f_! = D_X^{-1} f_* D_Y$. Let $\pi : E \to X$ be an oriented vector bundle of rank k. Denote by $i : X \to E$ the zero section. As π and i are homotopy equivalences, both $\pi_! : H^{p+k}_{\mathrm{cpt}}(E) \to H^p_{\mathrm{cpt}}(X)$ and $i_! : H^p_{\mathrm{cpt}}(X) \to H^{p+k}_{\mathrm{cpt}}(E)$ are isomorphisms for all p. (Using deRham forms we remark that $\pi_!$ really is integration over the fibre.) We have $\pi_* i_* = i_* \pi_* = \mathrm{id}$ and

$$\pi_! i_! = D_X^{-1} \pi_* D_E D_E^{-1} i_* D_X = \mathrm{id} = i_! \pi_!$$

Definition 4.1. The isomorphism $i_! : H^p_{\mathrm{cpt}}(X) \to H^{p+k}_{\mathrm{cpt}}(E)$ is called the *Thom isomorphism* of E for compactly supported cohomology.

By Lefschetz Duality we obtain a more classical version:

$$H^{k+p}_{\mathrm{cpt}}(E) \cong H_{n-p}(E) \cong H_{n-p}(D_E) \cong H^{k+p}(D_E, \partial D_E)$$

In K-theory there is also a Thom isomorphism $i_! : K_{\mathrm{cpt}}(X) \to K_{\mathrm{cpt}}(E)$ for a complex vector bundle E.

Given X and Y smooth manifolds with a smooth proper embedding $f : X \hookrightarrow Y$ we assume the normal bundle of $f(X)$ in Y to admit a complex structure. In this case we then can define the map

$$f_! : K_{\mathrm{cpt}}(X) \to K_{\mathrm{cpt}}(Y)$$

as the composition of the Thom isomorphism $i_! : K_{\mathrm{cpt}}(X) \to K_{\mathrm{cpt}}(N)$ and the extension map $K_{\mathrm{cpt}}(N) \to K_{\mathrm{cpt}}(Y)$ where N is identified with a tubular neighbourhood of $f(X)$.

Given a smooth proper embedding of $f : X \hookrightarrow Y$ we obtain the smooth proper embedding $f_* : TX \hookrightarrow TY$. The normal bundle of TX in TY is just the pullback to TX of $N \oplus N$ where N is the normal bundle of X in Y—i.e. one summand N corresponds to the normals in the manifolds, the other one to the normals in the fibres. On this bundle we may define a complex structure J by

$$J = \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

Thus we have a well-defined Thom isomorphism

$$f_! : K_{\mathrm{cpt}}(TX) \to K_{\mathrm{cpt}}(TY)$$

Using the Chern character $\mathrm{ch} : K_{\mathrm{cpt}}(X) \to H^{\mathrm{even}}_{\mathrm{cpt}}(X)$ we may form the obvious square, which, however, needs not commute.

References

[1] H. B. Lawson, Jr. and M.-L. Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

Manuel Amann
Fachbereich Mathematik und Informatik
Westfälische Wilhelms-Universität Münster
Einsteinstr. 62
48149 Münster, Germany

mamann@uni-muenster.de
http://wwwmath.uni-muenster.de/u/mamann