Oberseminar: Percolation and L^2-Betti numbers

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Introduction

The aim of this seminar is to study interactions between stochastic processes on graphs (often Cayley-graphs), geometry, and L^2-invariants. The most prominent example of such an interaction is a theorem by Kesten that provides equivalent characterizations of amenability of a group in terms of the return probability of a simple random walk on the Cayley graph, Følner sets, and the spectral radius of the Laplace operator on the Cayley graph.

An important notion is that of *percolation*. Percolation theory originated from the following example: One wants to model the spread of a fluid through a random medium. The medium is modelled as a system of channels, some of which are wide and others narrow. The fluid passes only through the wide ones. Let us say the system of channels is represented by (edges of) the Cayley graph of $\mathbb{Z}^d$. Each edge is labeled as *wide (or open, or passable)* with probability $p \in (0,1)$ or as *narrow (or closed)* with probability $1-p$. Note that the randomness of this model is completely in the medium and not in the motion of the fluid. The underlying probability space is $\Omega = \{0,1\}^E$, where E are the edges in the Cayley graph and the probability measure is the infinite product of the measure that gives 0 probability p and 1 probability $(1-p)$. A point $\omega \in \Omega$ can be thought of as a random subgraph of the Cayley graph. If $0 \in \mathbb{Z}^d$ is the source of the fluid, the set of points that are wetted is exactly the connected component W_ω of 0 in $\omega \in \Omega$.

What is the probability that points (infinitely) far out are wetted? More mathematically: One studies the *percolation probability* in dependence of p:

$$\theta(p) = \mathrm{P}\big(W_\omega \text{ is infinite for a.e. } \omega\big)$$

We will study *phase transitions*, i.e. the critical probability p, where $\theta(p)$ jumps from zero to positive, on many more groups than just $\mathbb{Z}^d$ using L^2-Betti numbers in lectures 7-9.

Percolation models are also considered by statistical physicists because they demonstrate many aspects of phase transitions in the *Ising model* in statistical mechanics.

The surveys [2, 8] are a good starting point for learning about percolation.

Percolation theory gives a new perspective on the calculation of the spectrum of certain Laplace operators on groups. In the lectures 5 and 6 we will recover a well-known computation of Grigorchuk and Zuk (that led to a counterexample to the *strong Atiyah conjecture*) using percolation clusters.

Another topic that appears throughout lectures 10-13 is that of *random forests*. A random forest is a G-invariant probability measure on the set of sub-forests of the Cayley graph of G. We will learn in lecture 11 how random forests are related to a problem about unitary representations (*Dixmier's problem*).

Program

1. *Introduction to L^2-Betti numbers I*
(Wolfgang Lück; 15.4)

2. *Introduction to L^2-Betti numbers II*
(Wolfgang Lück; 22.4)

3. *Introduction to percolation*
(Nina Gantert; 29.4)

4. *Introduction to random walks*
(Nina Gantert; 6.5)

5. *Percolation clusters and the spectrum of lamplighter groups I* [6]
(Thomas Kochler; 20.5)

6. *Percolation clusters and the spectrum of lamplighter groups II* [6]
(Andrea Winkler; 10.6)

7. *L^2-Betti numbers of measured equivalence relations* [4]
(Roman Sauer; 17.6)

8. *Invariant percolation and harmonic Dirichlet functions I* [1, 4]
(Raphael Zentner; 24.6)

9. *Invariant percolation and harmonic Dirichlet functions II* [1, 4]
(David Rosenthal; 1.7)

10. *A measurable solution to von Neumann's problem* [5]
(Clara Löh; 8.7)

11. *Spanning trees and random forests* [3, 7]
(Martin Langer; 15.7)

12. *Non-unitarisable representations and random forests* [3]
(Pascal Fabig; 22.7)

References

[1] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm, *Group-invariant percolation on graphs*, Geom. Funct. Anal. **9** (1999), no. 1, 29–66.

[2] Itai Benjamini and Oded Schramm, *Percolation beyond $\mathbf{Z}^d$, many questions and a few answers*, Electron. Comm. Probab. **1** (1996), no. 8, 71–82 (electronic).

[3] Inessa Epstein and Nicolas Monod, *Non-unitarisable representations and random forests*, Int. Math. Res. Not. **22** (2009), 4336–4353.

[4] D. Gaboriau, *Invariant percolation and harmonic Dirichlet functions*, Geom. Funct. Anal. **15** (2005), no. 5, 1004–1051.

[5] Damien Gaboriau and Russell Lyons, *A measurable-group-theoretic solution to von Neumann's problem*, Invent. Math. **177** (2009), no. 3, 533–540.

[6] Franz Lehner, Markus Neuhauser, and Wolfgang Woess, *On the spectrum of lamplighter groups and percolation clusters*, Math. Ann. **342** (2008), no. 1, 69–89.

[7] Russell Lyons and Yuval Peres, *Probability on Trees and Networks.* In preparation. Current version available at `http://mypage.iu.edu/ rdlyons`.

[8] Russell Lyons, *Phase transitions on nonamenable graphs*, J. Math. Phys. **41** (2000), no. 3, 1099–1126. Probabilistic techniques in equilibrium and nonequilibrium statistical physics.

[9] ———, *Determinantal probability measures*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 167–212.