

Oberseminar: Percolation and L^2 -Betti numbers
 L^2 -Betti numbers: Basic Exercises

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Exercise 1 (Group rings). Which of the following rings are isomorphic?

- The group ring $\mathbb{Z}[\mathbb{Z}/5]$
- the quotient $\mathbb{Z}[T]/(T^5)$ of the ring of polynomials
- the ring $\mathbb{Z}[\zeta_5] \subset \mathbb{C}$, where $\zeta_5 \in \mathbb{C}$ is a primitive fifth root of unity.

Exercise 2 (Cayley graphs).

1. Let $a, b \in \mathbb{F}_2$ be two elements freely generating the free group \mathbb{F}_2 of rank 2. Sketch the Cayley graph of \mathbb{F}_2 with respect to the following generating systems:
 - $\{a, b\}$
 - $\{a, a \cdot b\}$
 - $\{a, b, a \cdot b\}$.
2. Are there finite generating sets of \mathbb{Z} and of \mathbb{F}_2 respectively such that the corresponding Cayley graphs are isomorphic?

Exercise 3 (Group actions).

1. Let $\mathbb{Z}/2$ act on the unit circle $S^1 \subset \mathbb{C}$ by reflection at the x -axis. Describe the orbits of this action. Is this action free? Is it transitive?
2. Let \mathbb{Z} act on the unit circle $S^1 \subset \mathbb{C}$ by rotation around an irrational angle $\alpha \in [0, 2\pi)$, i.e., the quotient $\alpha/2\pi$ is irrational. Is this action free? Is it transitive? Is it a topologically nice action?

Exercise 4 (Free equivariant CW-complexes).

1. Consider the translation action of \mathbb{Z} on the Cayley graph of \mathbb{Z} with respect to the generating system $\{1, 2\}$ of \mathbb{Z} . Describe a free \mathbb{Z} -CW-structure on this Cayley graph compatible with this action and compute the corresponding cellular chain complex.
2. Construct a CW-complex X with fundamental group \mathbb{Z} in such a way that the differential in degree 3 of the corresponding cellular chain complex of the universal covering of X is given by the matrix

$$(3 \cdot t - 4 \cdot t^7) \in M_{1 \times 1}(\mathbb{Z}[\mathbb{Z}]),$$

where $t \in \mathbb{Z}$ is a chosen generator of the group \mathbb{Z} . [Some knowledge in algebraic topology is necessary to solve this exercise.]

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Exercise 5 (Homology). Let C be the chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2 \longrightarrow 0,$$

where $\pi: \mathbb{Z} \longrightarrow \mathbb{Z}/2$ is the canonical projection. Compute the homology of C , of $C \otimes_{\mathbb{Z}} \mathbb{Z}/2$, and of $C \otimes_{\mathbb{Z}} \mathbb{Q}$.

Exercise 6 (L^2 -Homology). Find an L^2 -cycle that represents a non-zero class in $H_1^{(2)}(\mathbb{F}_2)$.

Exercise 7 (Von Neumann dimension). Let $F: (\ell^2(\mathbb{Z}))^3 \longrightarrow (\ell^2(\mathbb{Z}))^3$ be the map given by the matrix

$$\sqrt{42} \cdot \begin{pmatrix} f & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $f := \pi + \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{e^{i \cdot k} - 1}{\pi \cdot i \cdot k} \cdot t^k \in \ell^2(\mathbb{Z})$ (here, $t \in \mathbb{Z}$ stands for the standard generator). Compute the von Neumann dimension (over \mathbb{Z}) of the image of F .

Exercise 8 (L^2 -Betti numbers).

1. Is there a group G with $b_0^{(2)}(G) = 1/42$ and $b_1^{(2)}(G) = 9 \cdot 6$?
2. Is there a group G with $b_0^{(2)}(G) = 0 = b_1^{(2)}(G)$ and $b_2^{(2)}(G) \neq 0$?