A. Eskin: Coarse differentiation

$\varphi: X \to Y$ quasi-isometric embedding of metric spaces

Asymptotic cone construction:

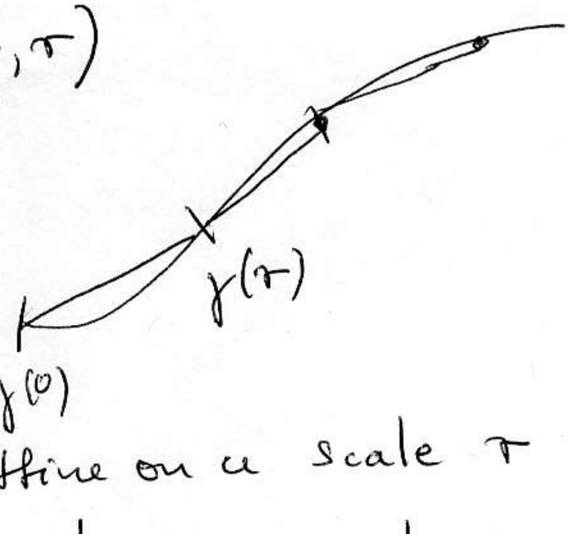
$$\hat{\phi}: AC(X) \to AC(Y)$$

bilipschitz

Def: $\Phi: X \to Y$ is affine if $\varphi(\text{geodesic}) = \text{geodesic}$.

Def: A path $\gamma: [0, L] \to Y$ is called ε-efficient on a scale r if

$$\underbrace{\sum_{i=0}^{L/r} d\big(\gamma(ir), \gamma((i+1)r)\big)}_{= \Delta(\gamma, r)} < (1+\varepsilon)\, d(\gamma(0), \gamma(L)) \qquad (*)$$



Def: A map $\varphi: X \to Y$ is ε-affine on a scale r if $\varphi(\text{geodesic})$ is ε-efficient on a scale r.

Lemma (Coarse differentiation)

Suppose $\varphi : X \to Y$ is a (K,C) qi-embedding.

Suppose $\mathcal{F}$ is a locally finite family of geodesics on X

set

$$\mathcal{F}_{B,L} = \left\{ \gamma \cap B \mid \gamma \in \mathcal{F},\ \frac{1}{10} \cancel{\text{radius}(B)} L \le |\gamma \cap B| \le 10 \cancel{\text{radius}(B)} L,\ \gamma \cap B \text{ connected} \right\}$$

for a ~~ball~~ set B. ~~of rad~~

$\forall \varepsilon > 0\ \forall \vartheta > 0$

$\exists L_0$ s.th. if $L > L_0$ $\exists$ r, R: $C \ll r < R \ll L$ s.th.

the following holds:

Let $\mathcal{F}'$ = collection of segments obtained by subdividing segments in $\mathcal{F}_{B,L}$ on the scale R.

Then at least $(1-\vartheta)$-fraction of the segments in $\mathcal{F}'$ have images which are ε-efficient on the scale r.

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Proof:

Pick large scales $C \leq r_0 \ll r_1 \ll r_2 \ll \dots$

~~Fix~~ $[0,L] \xrightarrow{\gamma} Y$ sequent

Let $I \subset [0,L]$ subsequent of length r_m

If $\gamma|_I$ is not ε-efficient on scale r_m then

$$\Delta(\gamma|_I, r_{m-1}) \geq (1+\varepsilon)\,\Delta(\gamma|_I, r_m)$$

Subdivide γ on scale r_m; let $\delta_m(\gamma)$ = fraction of these which are not ε-efficient on scale r_{m-1}.

Then: $\Delta(\gamma, r_{m-1}) \geq \Delta(\gamma, r_m) + \varepsilon\,\delta_m(\gamma)\,\Delta(\gamma, r_m)$

Now we use $(*)$ in the form:

$$\Delta(\gamma, \cancel{L}\, r) \leq \Delta(\gamma, L) + \varepsilon\,\frac{L}{2K}.$$

then we get: ~~$\Delta(\gamma, r)$~~ $\Delta(\gamma, r_{m-1}) \geq \Delta(\gamma, r_m) + \varepsilon\,\delta_m(\gamma)\,|\gamma|$