RADEMACHER'S THEOREM IN $\mathbb{R}^n$

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1. Introduction

The purpose of these notes is to prove the classical Rademacher's Theorem. There are versions that are far more general, notably Cheeger obtained recently a Rademacher type theorem only assuming the space has a Vitali Covering Theorem, which incidentally gives a new proof of the classical version, see [3]. We will not take his approach, for it is too technical, and we wish to keep our exposition elementary. We refer also to the survey article [7] on nonsmooth calculus for a discussion on generalizations of the theorem.

Besides Cheeger's proof, we know of two ways of proving the theorem, one using Sobolev spaces, and the other Fubini and absolute continuity on lines (ACL) (see [4], [6]). One could argue that they are in essence equivalent, as the Sobolev spaces are the same as the space of ACL-functions. Nonetheless, we adopt the first method due to Calderón, as it provides an important generalization of the theorem to Sobolev spaces.

Throughout, U will denote an open subset of $\mathbb{R}^n$. A function $f : U \to \mathbb{R}^m$ is Lipschitz if $|f(x) - f(y)| \leq L|x - y|$ (in the usual distance) for some L. A function is locally Lipschitz on U if for every point $x \in U$, there exists an open neighbourhood V of x on which the restriction of f is Lipschitz.

Recall that a function f is differentiable at the point $x \in \mathbb{R}^n$ if there exists a linear function $Df_x : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{y \to x} \frac{f(y) - f(x) - Df_x(y - x)}{|y - x|} = 0.$$

Theorem 1.1 (Rademacher, 1919)**.** *If $f : U \to \mathbb{R}^m$ is locally Lipschitz, then it is differentiable at almost every $x \in U$ (with respect to the Lebesgue measure).*

2. proof

By considering the coordinate functions of f instead, we can assume $m = 1$, i.e. f is real-valued. Also, since the statement is purely local,

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we restrict our attention to a small ball, and therefore we assume f to be Lipschitz.

Recall that a function $g : U \to \mathbb{R}^n$ is a weak-derivative for f if for any test function ϕ (smooth and compactly supported), $\int g\phi = -\int f\nabla\phi$. We denote the weak (or strong) derivative of f at x by the usual notation ∇f_x. Also, we denote $W^{1,p}(U)$, $1 \leq p \leq \infty$, the Sobolev space of functions $f \in L^p(U)$ admitting a weak-derivative $g \in L^p(U)$ (note f, g are defined a.e.), and by $W^{1,p}_{loc}$ the functions that are locally in $W^{1,p}$. We will prove Theorem 1.1 through the 2 theorems of this section, but first we need the following proposition related to Sobolev embedding theorems (see [1] or [4] for a proof).

Proposition 1 (Morrey Inequality)**.** *For $n < p < \infty$, $f \in W^{1,p}$, there exists a constant $C = C(n, p)$ depending on n and p such that*

$$|f(y) - f(x)| \leq C\,|y - x| \left(⨍_{B(x,r)} |\nabla f|^p \right)^{1/p},$$

where $⨍_{B(x,r)}$ denotes the average over the ball of radius r centered at x.

Theorem 2.1 (Calderón, 1951)**.** *For $p > n$, any function $f \in W^{1,p}$ is almost everywhere differentiable.*

Proof. If $p = \infty$, then f is locally in $W^{1,p}$ for any $p \geq 1$, so we can consider $p < \infty$. ∇f being in L^p has Lebesgue points almost everywhere. Let x be such a point, i.e.

$$\lim_{r\to 0} ⨍_{B(x,r)} |\nabla f_y - \nabla f_x|^p \, dy = 0.$$

Define g to be the difference between f and its linear approximation:

$$g(y) := f(y) - f(x) - \nabla f_x (y - x).$$

Then $g \in W^{1,p}$, $g(x) = 0$ and $\nabla g_y = \nabla f_y - \nabla f_x$. To finish the proof, apply Morrey's inequality to g:

$$\begin{aligned} |f(y) - f(x) - \nabla f_x(y - x)| &= |g(y) - g(x)| \leq C\,|y - x| \left(⨍_{B(x,r)} |\nabla g_y|^p \, dy \right)^{1/p} = \\ &= C\,|y - x| \left(⨍_{B(x,r)} |\nabla f_y - \nabla f_x|^p \, dy \right)^{1/p} \in o(|y - x|). \end{aligned}$$

□

The following theorem implies that Rademacher's theorem is a special case of Calderón's theorem.

Theorem 2.2. *$W^{1,\infty}_{loc}(U)$ identifies with the space of locally Lipschitz functions on U.*

Remark 1. The theorem here states that one can find a representative for $f \in W^{1,\infty}_{loc}(U)$ that is defined almost everywhere and locally Lipschitz.

Remark 2. The theorem does not follow from Morrey's inequality, as the latter is not valid for $p = \infty$.

Remark 3. There is a global version of this, but it is not true that $W^{1,\infty}(U)$ identifies with Lipschitz functions if U is unbounded. Instead, we get locally uniformly Lipschitz functions.

Proof. Since the statement is local, we restrict to a small ball, and prove that then $W^{1,\infty}$ corresponds to Lipschitz functions.

$W^{1,\infty} \Rightarrow$ *Lipschitz*:
The argument is similar to the case when f has a continuous derivative, but we need to mollify to make it work. So let η_ϵ be the standard approximation to the identity, $f^\epsilon = \eta_\epsilon * f$. The standard properties (see [4]) of mollification that we need are the following: f^ϵ is smooth, for any Lebesgue point x of f and ∇f, $f^\epsilon(x) \to f(x)$ and $\eta_\epsilon * \nabla f = \nabla f^\epsilon_x \to \nabla f_x$ as $\epsilon \to 0$. In particular, $\|\nabla f^\epsilon\|_\infty \to \|\nabla f\|_\infty$. Now by the fundamental theorem of calculus, $\gamma(t) = (1-t)x + ty$,

$$|f^\epsilon(y) - f^\epsilon(x)| = \left| \int_0^1 \nabla f^\epsilon_{\gamma(t)}(y-x)\, dt \right| \le \|\nabla f^\epsilon\|_\infty\, |y-x|.$$

Taking the limit as $\epsilon \to 0$, for Lebesgue points of f

$$|f(y) - f(x)| \le \|\nabla f\|_\infty\, |y-x|.$$

f is therefore Lipschitz almost everywhere, and we extend it to be Lipschitz everywhere.

Lipschitz $\Rightarrow W^{1,\infty}$:
Let $\{e_i\}$ be the standard basis for $\mathbb{R}^n$, and define for small enough h

$$g_i^h := \frac{f(x+he_i) - f(x)}{h}.$$

By the Lipschitz condition, g_i^h is bounded uniformly for any h. In particular, g_i^h is weak*-precompact in $L^\infty \cong (L^1)'$ (the dual of L^1) by the Banach-Alaoglu theorem, and therefore there exists a sequence $h \to 0$ such that $g_i^h \to g_i$ weakly, $g_i \in L^\infty$. We apply a discrete version of integration by parts: let ϕ be a test function,

$$\int f(x)\frac{\phi(x-he_i) - \phi(x)}{h}dx = -\int \frac{f(x+he_i) - f(x)}{h}\phi(x)dx = -\int g_i^h \phi.$$

Taking the limit $h \to 0$ and using dominated convergence, we obtain

$$\int f \partial_i \phi = -\int g_i \phi,$$

where ∂_i is the i^{th} directional derivative. We therefore have showed that $\nabla f = (g_i)$ is the weak derivative of f, and is essentially bounded, as required.

□

3. Further Developments

We'd like to conclude by mentioning some developments after Rademacher's theorem. There is another important generalization due to Stepanov, needed for example in the study of quasiconformal maps. It is different then the Calderón generalization, and we refer the interested reader to [5] for the classical version (stated below), and to [2] for a version in metric spaces.

Theorem 3.1 (Stepanov)**.** *$f : U \to \mathbb{R}$ is differentiable at almost every $x \in S(f)$, where*

$$S(f) := \left\{ x \in \mathbb{R}^n \ : \ \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} < \infty \right\}.$$

Finally, in vague terms, in the context of Cheeger's work, we have the following interpretation of Rademacher's theorem. Let (X, d, μ, x) be a pointed complete doubling measure metric space, and define $X_{x,r} := (X, r^{-1}d, \mu_r, x)$, where μ_r has the same measure on the unit ball in the rescaled metric as μ does in the inital metric. Then these converge in the Gromov-Hausdorff limit (after choosing an ultrafilter) to a space (X_∞, x_∞) that we term a weak tangent space. If $f : X \to \mathbb{R}$ is L-Lipschitz, then the rescaled functions $f_{x,r} := (f(y) - f(x))/r$ converge in some sense to a L-Lipschitz function on a weak tangent space (using the same scales).

In our situation, $X = U$ is an open subset of Euclidean space, and at any scale $X_{x,r}$ converges to a unique tangent space, the usual $\mathbb{R}^n$. Rademacher's theorem states that at all scales, $f_{x,r}$ converges to a unique linear function on the tangent space, ∇f_x, and this at almost every point $x \in U$.

References

[1] R.A. Adams and J. Fournier *Sobolev Spaces, second edition*, Academic Press, Oxford, 2003. Pure and Applied Mathematics, Vol. 140.

[2] Z.M. Balogh, K. Rogovin and T. Zürcher *The Stepanov Differentiability Theorem in Metric Measure Spaces*, the Journal of Geometric Analysis, **14**(3), 405-422, (2004).

[3] J. Cheeger *Differentiability of Lipschitz functions on metric measure spaces*, Geometric and Functional Analysis, **9** (1999), 428-517.

[4] L.C. Evans and R.F. Gariepy *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, Florida, 1992.

[5] H. Federer *Geometric Measure Theory*, Die Grundlehren deer mathematischen Wisseschaften in Einzeldarstellungen, no.153, Springer-Verlag, Berlin, Heidelberg, (1969).

[6] J. Heinonen *Lectures on analysis on metric spaces*, Springer-Verlag, New York, 2001.

[7] J. Heinonen *Nonsmooth Calculus*, Bulletin of the American Mathematical Society (N.S.) **44** (2007), 163-232.