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Some remarks on Grothendieck's paper *Sur la classification des fibres holomorphes sur la sphere de Riemann.*

Grothendieck wrote this paper [1] during his stay as a Visiting Associate Professor at the University of Kansas in Lawrence where he spent the first half of the year 1955. It is certainly not one of his most important work; he himself does not mention it specifically in his *Esquisse thematique des principaux travaux mathematiques de A. Grothendieck* from 1972 [2]. Nevertheless, the paper contains an important theorem with an interesting history.

The main result of the paper is the following theorem.

Theorem *Let X be the Riemann sphere and E an n-dimensional holomorphic vector bundle over X. Then*

$$E \approx L(d_1) \oplus \ldots \oplus L(d_n)$$

where $L(d)$ denotes the 1-dimensional bundle of degree d. The direct summands $L(d_i)$ are uniquely determined up to a permutation.

Actually, Grothendieck proves a somewhat more general result, where the structure group of E can be an arbitrary reductive complex Lie group. Then the structure group can be reduced to a Cartan subgroup H. In case of $G = GL(n, \mathbb{C})$ this is the subgroup of diagonal matrices and this leads to the "diagonalization" of the bundle E, stated in the theorem.

Grothendieck mentions that by Serre's GAGA theorems it is sufficient to prove the theorem for *algebraic* vector bundles over projective 1-space. But he is not aware of the fact, that this algebraic result has been known for a long time. It goes back to R. Dedekind and H. Weber and their fundamental paper *Theorie der algebraischen Funktionen einer Veränderlichen* published in 1882 in Crelle's Journal [3] The following historical remarks are based on a thorough investigation of this subject due to W.-D. Geyer: *Die Theorie der algebraischen Funktionen einer Veränderlichen nach Dedekind und Weber* [4].

The crucial point of the classification of algebraic vector bundles is the following lemma.

Lemma *Let K be a field, T an indeterminante and $K[T, T^{-1}]$ the ring of Laurent polynomials. Let $A \in GL(n, K[T, T^{-1}])$. Then there exist matrices $B \in GL(n, K[T])$, $C \in GL(n, K[T^{-1})$ such that*

$$BAC = \begin{pmatrix} T^{d_1} & & 0 \\ & \ddots & \\ 0 & & T^{d_n} \end{pmatrix}$$

with $d_1 \geq d_2 \geq \ldots \geq d_n$. The sequence of the d_i is uniquely determined.

It is obvious that this lemma implies the classificaton theorem for vector bundles: Let $X = \mathbb{P}_1$ and

$$X_1 = \mathbb{P}_1 \setminus \text{ "south pole" } = \operatorname{Spec}(K[T]),$$
$$X_2 = \mathbb{P}_1 \setminus \text{ "north pole" } = \operatorname{Spec}(K[T^{-1}]).$$

If E is the bundle, then $E|X_1$ and $E|X_2$ are free bundles given by the free modules $K[T]^n$, $K[T^{-1}]^n$, respectively. These free modules are glued together over the intersection

$$X_1 \cap X_2 = \operatorname{Spec}(K[T, T^{-1}])$$

by a matrix $A \in GL(n, K[T, T^{-1}])$. One may perform the base change B in $K[T]^n$ and C^{-1} in $K[T^{-1}]$ and gets immediately the unique decomposition of E into line bundles.

One may ask: How did Dedekind and Weber find this result? Certainly they were not interested in "vector bundles". The answer is easy: Dedekind and Weber wanted to prove the Riemann-Roch theorem for an arbitrary smooth projective curve X. Their idea and procedure was very simple and natural and can be described in modern terms as follows: A divisor over X corresponds to a line bundle L. One is interested in the dimension of the space of global sections

$$\dim \Gamma(X, L) = \dim H^0(X, L).$$

One may consider X as an n-fold covering of $\mathbb{P}_1$. The direct image of L with respect to $X \to \mathbb{P}_1$ is an n-dimensional vector bundle E over $\mathbb{P}_1$. Using the diagonalization of E, the computation of $\dim \Gamma(\mathbb{P}_1, E)$ can be reduced to line bundles, i.e. the Riemann-Roch theorem for $\mathbb{P}_1$. (Of course, Dedekind and Weber used the language of algebraic funtion fields.)

Geyer points out that the analytic version of the Lemma (where $K[T]$ is replaced by the ring of convergent power series over $\mathbb{C}$) can be found in work of Birkhoff (1913) on Fredholm's integral equation. Seshari used this to give independently a proof of Grothendiecks theorem (1957) [5].

Taking Grothendieck's (or Dedekind-Weber's) result as starting point it is very natural to ask for a classification of vector bundles over an arbitrary projective variety. Using the Krull-Schmidt theorem in this situation - proved by Atiyah [6] at about the same time - this classificaton is reduced to indecomposable bundles. It seems that Grothendieck, when he wrote his paper, had only a few vague ideas about this question. In *Remarques finales* he writes:

Il semble plausible que la seule variété projective X sur laquelle tout fibré vectoriel holomorphe soit décomposable en somme de fibrés holomorphes de fibre $\mathbb{C}$, soit la sphére de Riemann.

Then he remarks that over curves of genus $g \neq 0$ always 2-dimensional indecomposable exist, namely a non-trivial extension of a line bundle by a line bundle. He also makes a few remarks concerning higher dimensional varieties.

Following Grothendieck, Atiyah classified vector bundles over elliptic curves [7]. It seems that this classification is much more difficult than for the Riemann sphere. It turns out that this classification problem is "tame". This means that the indecomposable vector bundles of the same dimension and degree form 1-dimensional "moduli spaces". The description of these can be made explicit.

It seems nowadays "well known" that over all other projective varieties X the classification problem is "wild". To prove this one can proceed as follows: Construct two vector bundles E, F with the following properties

$$\begin{aligned} &\mathrm{End}(E) = \mathrm{End}(F) = K \\ &\mathrm{Hom}(E, F) = 0, \mathrm{Hom}(F, E) = 0 \\ &\dim \mathrm{Ext}^1(E, F) = r \geq 3. \end{aligned}$$

(Here K denotes the ground field which can be any field.) Having such bundles it is easy to see that the classification of vector bundes V which can be obtained as an extension

$$0 \to E^n \to V \to F^n \to 0$$

can be reduced to the simultanous classification of $(r-1)$-tuples of $n \times n$-matrices over K - and this is the standard "wild" classification problem. (Details of this have been worked out in a thesis written in Münster; see [8].) Partial results in this direction have been published by Greuel and Oort [9].

Let us come back to Grothendieck's paper. He proves his main result essentially in the following very natural way: First it is shown that E contains an 1-dimensional subbundle E_1. This, obviously, leads to a "flag" of subbundles $0 \subset E_1 \subset E_2 \subset \ldots \subset E_n = E$. Then it is shown that the degrees of all such E_1 are bounded from above. Then one can show that an E_1 of maximal degree is a direct summand so that one can proceed by induction. The last step follows for a 2-dimensional E from the fact $\mathrm{Ext}^1(E_1, E/E_1) = 0$ so that the extension

$$0 \to E_1 \to E \to E/E_1 \to 0$$

is trivial. The Ext-group is 0 by Serre duality.

Though Grothendieck deals with a quite specific result he always uses methods as general as possible: GAGA theorems, the formalism of cohomology, Riemann-Roch and Serre duality, the Krull-Schmidt theorem for vector bundles (proved by Atiyah and as yet unpublished) are used. In retrospect one can clearly recognize what later became characteristic for his work in Algebraic Geometry: utmost generality.

Having proved the main result Grothendieck discusses the reduction of the structure group $GL(n, \mathbb{C})$ to the complex orthogonal group $O(n, \mathbb{C})$. Clearly, this is possible if and only if E is isomorphic to the dual bundle E^*. One finds here a first trace of the theory of additive categories with duality and the corresponding theory of symmetric bilinear forms. For example, we find the "hyperbolic form" on $E \oplus E^*$.

References

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