Crowded Particles - From Ions to Humans

Bärbel Schlake

Westfälische Wilhelms-Universität Münster
Institut für Numerische und Angewandte Mathematik

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1. Ions
   - Motivation
   - One-Dimensional Model
   - Entropy
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   - Motivation
   - One-Dimensional Model
   - Entropy

2 Pedestrians
   - Observations
   - Social Force Model and Pedestrian Motion
   - 1D Movement
     - Investigations for the 1D Model
     - Limiting Behaviour
   - Further Work
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Biological Background

- located in membrane cells
- proteins with a hole in their middle
- regulate movement of inorganic ions through impermeable cell membrane
- exact structure not known
- topic of interest in biophysics, medicine
One-Dimensional Hopping Model

Probabilities

\[ n(x, t) = P(\text{negatively charged ion at position } x \text{ at time } t) \]
\[ p(x, t) = P(\text{positively charged ion at position } x \text{ at time } t) \]
transition rate:

\[ \Pi_{n/p}^{n/p}(x, t) = \]

\[ P(\text{jump of } n/p \text{ from position } x \text{ to } x + h \text{ in } (t, t + \Delta t)) \cdot \frac{1}{\Delta t} P(x + h \text{ is empty}) \]
transition rate:
\[ \Pi_n^p(x, t) = \]
\[ P(\text{jump of } n/p \text{ from position } x \text{ to } x + h \text{ in } (t, t + \Delta t)) \cdot \]
\[ \frac{1}{\Delta t} P(x + h \text{ is empty}) \]

potential \( V(x, t) \) given
transition rate:
$$\Pi_{n/p}^{+}(x, t) = P(\text{jump of } n/p \text{ from position } x \text{ to } x + h \text{ in } (t, t + \Delta t)) \cdot \frac{1}{\Delta t} P(x + h \text{ is empty})$$

- potential $V(x, t)$ given
- probability that a negative particle is located at position $x$ at time $t + \Delta t$:
  $$n(x, t + \Delta t) = n(x, t)(1 - \Pi_{+}(x, t) - \Pi_{-}(x, t)) + n(x + h, t)\Pi_{-}(x + h, t) + n(x - h, t)\Pi_{+}(x - h, t)$$
transition rate:
\[ \Pi_{\pm}^{n/p}(x, t) = P(\text{jump of } n/p \text{ from position } x \text{ to } x + h \text{ in } (t, t + \Delta t)) \cdot \frac{1}{\Delta t} P(x + h \text{ is empty}) \]

potential \( V(x, t) \) given

probability that a negative particle is located at position \( x \) at time \( t + \Delta t \):
\[
n(x, t + \Delta t) = n(x, t)(1 - \Pi_{+}(x, t) - \Pi_{-}(x, t)) + n(x + h, t)\Pi_{-}(x + h, t) + n(x - h, t)\Pi_{+}(x - h, t)
\]

---

**Resulting Model**

\[
\begin{align*}
\partial_t n &= \nabla \cdot (D(1 - m)\nabla n - Dn\nabla m - n(1 - m)\nabla V) \\
\partial_t p &= \nabla \cdot (D(1 - m)\nabla p - Dp\nabla m + p(1 - m)\nabla V)
\end{align*}
\]

- \( D \) diffusion coefficient, mass \( m(x, t) = n(x, t) + p(x, t) \)
\( V \) satisfies the Poisson equation:
\[
\lambda^2 V_{xx}(x, t) = n(x, t) - p(x, t) + f(x)
\]
\( f(x) \): Permanent charge on the membrane
$\partial_t n = \nabla \cdot (D(1 - m)\nabla n - Dn\nabla m - n(1 - m)\nabla V)$

$\partial_t p = \nabla \cdot (D(1 - m)\nabla p - Dp\nabla m + p(1 - m)\nabla V)$

Solutions for Equilibrium

$n(x, t) = \frac{\exp(V(x,t)/D)}{\exp(V(x,t)/D) + \beta \cdot \exp(-V(x,t)/D) + \alpha}$

$p(x, t) = \frac{\beta \cdot \exp(-V(x,t)/D)}{\exp(V(x,t)/D) + \beta \cdot \exp(-V(x,t)/D) + \alpha}$

- $\alpha$ and $\beta$ are constants, $\alpha, \beta > 0$
Numerical Results
Entropy

\[ E = \int (n \cdot \log (n) + p \cdot \log (p) + (1 - m) \cdot \log (1 - m) - nV + pV) \, dx \]

- decreasing during the process
- minimal in equilibrium state
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Under normal Condition

- *Fastest* route, not the *shortest* one
- Individual speed: most comfortable one (dependent of age, sex, purpose of the trip...)
  Speeds are Gaussian distributed (mean value: 1.34 m/s, St-d 0.26 m/s)
- Certain distance from other pedestrians and boundaries
- Resting pedestrians are uniformly distributed about the available space
In Case of Competitive Evacuation (Panic)

- Nervousness
- Pedestrians try to move faster than normal
- Interactions become physical in nature
- Uncoordinated passing of bottlenecks
- Jams build up. Arching and clogging at exits
- Physical interactions add up ⇒ dangerous pressures up to 4.5 tons
- Escape is slowed down by injured or fallen people
- Herding behaviour
Equation of Motion

\[
\frac{d\mathbf{r}_i(t)}{dt} = \mathbf{v}_i(t)
\]

\[
m_i \frac{d\mathbf{v}_i(t)}{dt} = m_i \frac{\mathbf{v}_i^0 \mathbf{e}_i^0(t) - \mathbf{v}_i(t)}{\tau_i} + \sum_{j(\neq i)} f_{ij} + \sum_{W} f_{iW}
\]

- \( \mathbf{r}_i \) position of pedestrian \( i \)
- \( \mathbf{v}_i \) velocity
- \( \frac{d\mathbf{v}_i(t)}{dt} \) acceleration
- \( m_i \) mass
- \( \tau_i \) acceleration time
- \( \mathbf{v}_i^0 \) desired velocity, \( \mathbf{e}_i^0 \) desired direction
**Forces**

\[ \mathbf{f}_{ij} = \mathbf{f}_{ij}^{\text{interact}} + \mathbf{f}_{ij}^{\text{body}} + \mathbf{f}_{ij}^{\text{slid}} \]

\[ \mathbf{f}_{ij}^{\text{interact}} = A_i \cdot e\left[\frac{(R_{ij} - d_{ij})}{B_{ij}}\right] \mathbf{n}_{ij} \]

\[ \mathbf{f}_{ij}^{\text{body}} = k(R_{ij} - d_{ij}) \mathbf{n}_{ij} \]

\[ \mathbf{f}_{ij}^{\text{slid}} = \kappa(R_{ij} - d_{ij}) \Delta \mathbf{v}_{ji}^t \cdot \mathbf{t}_{ij} \]

\( \mathbf{f}_{ij}^{\text{body}} \) and \( \mathbf{f}_{ij}^{\text{slid}} \) only if pedestrians \( i \) and \( j \) touch each other (panic)

\( A_i, B_i, k, \kappa \) constants

\( d_{ij} \) distance

\( \mathbf{n}_{ij} = (n_{ij}^1, n_{ij}^2) = (\mathbf{r}_i - \mathbf{r}_j)/d_{ij} \) normalized vector from \( j \) to \( i \)

\( \mathbf{t}_{ij} = (-n_{ij}^2, n_{ij}^1) \) tangential direction

\( \Delta \mathbf{v}_{ji}^t = (\mathbf{v}_j - \mathbf{v}_i) \cdot \mathbf{t}_{ij} \) tangential velocity difference

\( R_{ij} = (R_i + R_j) \) sum of radii
Walls are treated analogously

Anisotropy:

\[ f_{ij} = A_i \exp[\{(R_{ij} - d_{ij})/B_i\}] n_{ij} \cdot \left( \lambda_i + (1 - \lambda_i) \frac{1 + \cos \varphi_{ij}}{2} \right) \]  \hspace{1cm} (1)

\( \lambda < 1 \): happenings in front more weighted than events behind

\( \varphi_{ij} \): angle between \( e_i \) and \( -n_{ij} \)

Sights: forces of type (1) (\( B \) larger, \( A \) smaller and negative)

Groups: \( f_{ij}^{att} = C_{ij} n_{ij} \)

Fluctuation term \( \xi_i \)
Helbing’s Model describes the following phenomena quite realistically:

- Segregation
- Lane Formation

Oscillations

Bottlenecks: Passing of direction

Intersections

Unstable traffic
Helbing’s Model describes the following phenomena quite realistically:

- **Segregation**
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- **Oscillations**
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Helbing’s Model describes the following phenomena quite realistically:

- **Segregation**
  Lane Formation

- **Oscillations**
  Bottlenecks: Passing of direction

- **Intersections**
  Unstable traffic
Behaviour in Competitive Evacuation (Panic)

- People are getting **nervous** ⇒ Higher level of fluctuations
- Higher desired velocity
- Herding behaviour:

\[
\mathbf{e}_i^0 = \frac{(1 - n_i)\mathbf{e}_i + n_i \langle \mathbf{e}_j^0 \rangle_i}{\| (1 - n_i)\mathbf{e}_i + n_i \langle \mathbf{e}_j^0 \rangle_i \|}
\]

\( n_i \): nervousness

- Freezing by heating:
‘Faster is slower Effect’:

\[ v_i^0(t) = [1 - n_i(t)] v_i^0(0) + n_i(t) v_i^{max} \]

\[ n_i(t) = 1 - \frac{\bar{v}_i(t)}{v_i^0(0)} \]
Optimization

- Series of columns in the middle of a corridor
- Funnelshaped geometrie at bottlenecks
- Two small doors better than a large one
- Intersections: Guidance arrangements which lead to a roundabout, an obstacle in the centre
- Slim queues
- Zigzag shape
- Avoidance of staircases
- Escape routes should not have a constant width
- Column placed asymmetrically in front of exits
Model for hard Bodies with Remote Action

General Model (Helbing)

\[ m_i \frac{dv_i}{dt} = f_i \]

\[ f_i = m_i \frac{v_i^0 - v_i}{\tau_i} - \sum_{j \neq i} A'_i(||r_j - r_i|| - d_i) \]

- \( A' \) potential
- \( d_i = a_i + b_i v_i \) safety margin; \( a_i \) and \( b_i \) are constants
Seyfried’s Model

The Model

\[ f_i(t) = \begin{cases} G_i(t) & ; \text{if } v_i(t) > 0 \\ \max(0, G_i(t)) & ; \text{if } v_i(t) \leq 0 \end{cases} \]

\[ G_i(t) = \frac{v_i^0 - v_i(t)}{\tau_i} - e_i \left( \frac{1}{r_{i+1}(t) - r_i(t) - d_i(t)} \right)^{B_i} \]

- \( e_i \) and \( B_i \): strength and the range of the force
Investigations for the Model

Old Model

\[
\frac{dv_i(t)}{dt} = \begin{cases} 
G_i(t) , & \text{if } v_i(t) > 0 \\
\max(0, G_i(t)) , & \text{if } v_i(t) \leq 0 
\end{cases}
\]

\[
G_i(t) = \frac{v_i^0 - v_i(t)}{\tau_i} - e_i \left( \frac{1}{r_{i+1}(t) - r_i(t) - d_i(t)} \right)^{g_i}
\]

Problem:
Do pedestrians go backwards?
Seyfried’s term:

\[ G_i(t) = \frac{v_i^0 - v_i(t)}{\tau_i} - e_i \left( \frac{1}{r_{i+1}(t) - r_i(t) - d_i(t)} \right) g_i \]

New Acceleration Term

\[ \tilde{G}_i(t) = \begin{cases} 
\frac{v_i^0 - v_i(t)}{\tau_i} - \frac{e_i}{m_i} \left( \frac{h}{r_{i+1}(t) - r_i(t) - d_i(t)} \right) g_i & , \quad r_{i+1} - r_i - d_i > 0 \\
-\infty & , \quad \text{else}
\end{cases} \]
Improved Model

New Model

\[
\frac{dv_i}{dt} = \tilde{G}_i(t) \cdot H(v_i(t), \tilde{G}_i(t))
\]

\[
H(v_i(t), \tilde{G}_i(t)) = \begin{cases} 
0 & \text{, if } v_i(t) \leq 0 \text{ and } \tilde{G}_i \leq 0 \\
1 & \text{, if } \tilde{G}_i(t) \geq \delta \text{ or } v_i(t) \geq \epsilon 
\end{cases}
\]

Old model:

\[
\frac{dv_i(t)}{dt} = \begin{cases} 
G_i(t) & \text{, if } v_i(t) > 0 \\
\max(0, G_i(t)) & \text{, if } v_i(t) \leq 0 
\end{cases}
\]

Main difference: \textbf{Interpolation}

Interpolation decreases the deceleration in case the velocity comes close to zero.
Pedestrians do not go backwards, if their initial safety margin is large enough, because

- The interpolation we used lessened the deceleration in case the velocity came close to zero
- Pedestrians start deceleration sufficiently long before they approach an obstacle
- The more a person decelerates, the less becomes the velocity and the required safety margin

⇒ Model works well!
Existence and Uniqueness

**Lemma 1:**

Let $r_{i+1}(0) - r_i(0) > d_i(0) \; \forall i$. Then the ODE-System

$$
\begin{pmatrix}
\dot{r}_1 \\
\dot{v}_1 \\
\dot{r}_2 \\
\dot{v}_2 \\
\vdots \\
\dot{r}_N \\
\dot{v}_N
\end{pmatrix}
= \begin{pmatrix}
v_1 \\
\tilde{G}_1(t, r_1, r_2, v_1) \cdot H(v_1(t), \tilde{G}_1(t)) \\
v_2 \\
G_2(t, r_2, r_3, v_2) \cdot H(v_2(t), \tilde{G}_2(t)) \\
\vdots \\
v_N \\
\tilde{G}_N(t, r_N, r_1, v_N) \cdot H(v_N(t), \tilde{G}_N(t))
\end{pmatrix}
\Leftrightarrow \dot{y} = f(t, y)
$$

has a local solution. The solution is unique.
Velocity Density Relation

![Graph showing the velocity density relation with data points forming a decreasing trendlin]
Limiting Behaviour

What happens if the number of pedestrians $N$ tends to $\infty$?

We investigate the general formulation:

$$\frac{dv_i}{dt} = \frac{v_i^0 - v_i}{\tau_i} + \frac{1}{m} A'(N(x_{i+1} - x_i) - d_i)$$
Equation for particle distribution function

\[ f^N = f^N(x_1, x_2, ..., x_N, t) : \]

\[ \partial_t f^N + \sum_i \partial x_i \left( \frac{dx_i}{dt} f^N \right) = 0 \]
General Distribution of Particles

- Equation for particle distribution function
  \[ f^N = f^N(x_1, x_2, ..., x_N, t) : \]
  \[
  \partial_t f^N + \sum_i \partial_{x_i} \left( \frac{dx_i}{dt} f^N \right) = 0
  \]

- In our case
  \[
  \partial_t f^N + \sum_i \partial_{x_i} \left( \left[ v_i^0 - \frac{1}{m} A' (N(x_{i+1} - x_i) - d_i) \right] f^N \right) = 0
  \]
General Distribution of Particles

- Equation for particle distribution function
  \[ f^N = f^N(x_1, x_2, ..., x_N, t) : \]
  \[ \partial_t f^N + \sum_i \partial x_i \left( \frac{dx_i}{dt} f^N \right) = 0 \]

- In our case
  \[ \partial_t f^N + \sum_i \partial x_i \left( \left[ v^0_i - \frac{1}{m} A'(N(x_{i+1} - x_i) - d_i) \right] f^N \right) = 0 \]

- Analogon of BBGKY hierarchy (classical kinetic theory):
  \[ m^N_k(x_k, x_{k-1}, ..., x_1, t) = \int_{\mathbb{R}^{N-k}} f^N(x_1, ..., x_N, t) dx_{k+1} dx_{k+2} ... dx_N \]
General Distribution of Particles

- Equation for $k$-th marginal:

\[
\partial_t m_k^N + \sum_{i \leq k} \int_\mathbb{R} \partial_{x_i} \left( \left[ v_i^0 - \frac{1}{m} \nabla A(N(x_{i+1} - x_i) - d_i)^{-B_i} \right] m_{k+1}^N \right) dx_{k+1} = 0
\]
General Distribution of Particles

- Equation for $k$-th marginal:

$$
\partial_t m_k^N + \sum_{i \leq k} \int_{\mathbb{R}} \partial_{x_i} \left( \left[ v_i^0 - \frac{1}{m} \nabla A(N(x_{i+1} - x_i) - d_i)^{-B_i} \right] m_{k+1}^N \right) dx_{k+1} = 0
$$

- We assume $m_k^N$ to be dependent of $m_1^N$ and the distances $x_i - x_{i-1} \ \forall i \leq k$, furthermore

$$
m_1^N(x_1) \xrightarrow{N \to \infty} \rho(x)
$$

$\rho$ denotes density function.
Equation from $k$-th particle marginal in the limit $N \to \infty$

\[
\frac{\partial_t \rho(x, t)}{\partial_t} + \frac{\partial_x}{\partial_x} \left( \left[ v^0 - \frac{1}{m} A' \left( \frac{1}{\rho(x, t)} - d \right) \right] \rho(x, t) \right) = 0
\]
Results

Equation from \(k\)-th particle marginal in the limit \(N \to \infty\)

\[
\partial_t \rho(x, t) + \partial_x \left( \left[ v^0 - \frac{1}{m} A' \left( \frac{1}{\rho(x, t)} - d \right) \right] \rho(x, t) \right) = 0
\]

- Continuity equation:

\[
\partial_t \rho + \partial_x (v \rho) = 0
\]

\[
v = v^0 - \frac{1}{m} A' \left( \frac{1}{\rho(x, t)} - d \right)
\]
In the limits $N \to \infty$ and $\tau \to 0$ all stochastic fluctuations disappear

Motion can be described by a deterministic equation!

Motion does not depend on initial distribution
Further Work

- Analyze existing data
- New experiments (train evacuation)
- Group dynamics
- Influence of audible signals (message, voice)
- Influence of fitness
Thank you for your Attention!