NEW BEAUVILLE SURFACES, MODULI SPACES AND FINITE GROUPS

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Joint work with Matteo Penegini (Universität Bayreuth)
New Beauville surfaces, moduli spaces and finite groups

Outline

• Definitions and known results:
  ▪ Beauville surfaces.
  ▪ Moduli spaces.

• Hurwitz groups and braid group actions.

• New results:
  1. Alternating and symmetric groups.
  2. The groups PSL(2,q).
  3. Finite simple groups of low Lie rank.
  4. Abelian groups.
New Beauville surfaces, moduli spaces and finite groups

Beauville surfaces – definitions

Group theoretical property: 
[Bauer, Catanese, Grunewald]

G admits a Beauville structure of type $(\tau_1, \tau_2)$, where $\tau_i = (n_i, m_i, l_i)$ (i=1,2), if there exist two triples $T_i = (x_i, y_i, z_i)$ (i=1,2) s.t.

- $(x_i, y_i, z_i) = G,$
- $x_i, y_i, z_i = 1,$
- $\{ x_i \cap z_i \} = \{ 1 \},$ where

$$\sum_{i=1}^{l_i} \Sigma_i \cap \Sigma_j = \{ 1 \},$$

Algebro-geometric definition:
[Beauville '78; Catanese '00]

A Beauville surface $S$ (over $C$) is a quotient $S = (C_1 \times C_2)/G$, where $C_1$ and $C_2$ are curves of genus at least 2, $G$ is a finite group acting freely on their product, and $S$ is rigid.
Beauville structures – known results

**Theorem.** The following groups admit a Beauville structure:

- The alternating groups $A_n$ for $n \geq 6$ and the symmetric groups $S_n$ for $n \geq 5$. [Bauer, Catanese, Grunewald '05; Fuertes, Gonzáles-Diez '09].

- The groups $SL(2,p)$ and $PSL(2,p)$ for primes $p \neq 2,3,5$. [Bauer, Catanese, Grunewald '05].

- The Suzuki groups $Sz(2^p)$ for odd primes $p$. [Bauer, Catanese, Grunewald '05].

- A finite abelian group admits a Beauville structure if and only if $G=(\mathbb{Z}/n\mathbb{Z})^2$ for $(n,6)=1$. [Bauer, Catanese, Grunewald '05].

- For every prime $p$, there exists a $p$-group which admits a Beauville structure. [Bauer, Catanese, Grunewald '05; Fuertes, Gonzáles-Diez, Jaikin-Zapirain '09].
In algebraic geometry...
If we fix $G$ and type $(\tau_1, \tau_2)$, then we can look at all Beauville surfaces $S=S(G; \tau_1, \tau_2)$.
Let $\chi = \chi(S)$ and $K^2 = K^2(S)$ be the standard surface invariants, and let:

$M_{\chi, k^2}$ be the moduli space of surfaces of general type with those invariants.

$M(G; \tau_1, \tau_2) \subset M_{\chi, k^2}$ is the moduli space of Beauville surfaces $S(G; \tau_1, \tau_2)$.

$M(G; \tau_1, \tau_2)$ consists of a finite number of points.

Group theoretical method to count the number of points:
[Bauer, Catanese '02]
Let $U(G; \tau_1, \tau_2)$ be the set of all Beauville structures $(T_1; T_2)$ of $G$ of (unordered) type $(\tau_1, \tau_2)$, then:

The number $h(G; \tau_1, \tau_2)$ of Hurwitz components (i.e. points) is the number of orbits of $U(G; \tau_1, \tau_2)$ under the action of $B_3 \times B_3 \times \text{Aut}(G)$, given by:

$\left( \gamma_1, \gamma_2, \phi \right): (T_1, T_2) \mapsto (\gamma_1(\phi(T_2)), \gamma_2(\phi(T_2)))$

for $\gamma_1, \gamma_2 \in B_3$, $\phi \in \text{Aut}(G)$. 

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for $\gamma_1, \gamma_2 \in B_3$, $\phi \in \text{Aut}(G)$.
The space $\mathcal{M}_{\chi,k^2}$ consists of finitely many different connected components (of different dimensions).

Their number can grow exponentially as a function of $\chi$ (or $K^2$). [Catanese '92; Manetti '97].

**Proposition** [GP]. The number of points in the space $\mathcal{M}_{(G;\tau_1,\tau_2)}$ grows polynomially as a function of $\chi$ (or $K^2$).
Triangle groups and Hurwitz groups

$\Delta = \Delta(m_1,m_2,m_3) = \langle x_1,x_2,x_3 : x_1^{m_1} = x_2^{m_2} = x_3^{m_3} = x_1x_2x_3 = 1 \rangle$.

The type $\tau = (m_1,m_2,m_3)$ is hyperbolic $\iff 1/m_1 + 1/m_2 + 1/m_3 < 1$.

Example: Hurwitz groups are finite quotients of $\Delta(2,3,7)$, (automorphism group of a Riemann surface of genus $g>1$).

Question: Which interesting finite groups $G$ are quotients of triangle groups?

Answers: Higman, Conder ('80), Macbeath ('69), Everitt ('00), …

$G$ is a finite quotient of the triangle group $\Delta(m_1,m_2,m_3)$
$\iff G$ is generated by $(x_1,x_2,x_3)$ with $x_1x_2x_3 = 1$ of type $\tau = (m_1,m_2,m_3)$.

$G$ admits a Beauville structure $(T_1,T_2)$ of type $(\tau_1,\tau_2)$, where $\tau_i = (n_i,m_i,l_i)$ ($i = 1,2$)
$\iff$ There are surjections $\Delta(n_i,m_i,l_i) \to G$, and the generators satisfy $\Sigma_1 \cap \Sigma_2 = \{1\}$. 
Braid group actions

$S(G,\tau) := \{(x,y,z): <x,y,z>=G, xyz=1 \text{ or } yxz=1, (x,y,z) \text{ has (unordered) type } \tau\}.$

$B_3 = \langle \sigma_1, \sigma_2 \rangle$ acts on $S(G,\tau)$ by :

$\sigma_1: (x,y,z) \rightarrow (xyx^{-1},x,z),$

$\sigma_2: (x,y,z) \rightarrow (x,zy^{-1},y).$

Lemma. $d(G,\tau):= \{B_3 \times \text{Inn}(G)\text{-orbits of } S(G,\tau)\}$ =

$= \{\text{Inn}(G)\text{-orbits of } S(G,\tau)\} = \{B_3\text{-orbits of } S(G,\tau)\}.$

Proof. $\sigma_1(x,y,y^{-1}x^{-1})=(xyx^{-1},x,y^{-1}x^{-1})=x(y,x,x^{-1}y^{-1})x^{-1}, \sigma_2(x,y,y^{-1}x^{-1})=(x,x^{-1}y^{-1},y).$

[Biggers,Fried; Völklein]: $g(x,y,z)g^{-1}$ is obtained from $(x,y,z)$ by a applying $\sigma_1,\sigma_2.$

$U(G;\tau_1,\tau_2) := \{\text{Beauville structures } (T_1,T_2) \text{ of } G \text{ of (unordered) type } (\tau_1,\tau_2)\}.$

$h(G;\tau_1,\tau_2):= \{B_3 \times B_3 \times \text{Aut}(G)\text{-orbits of } U(G;\tau_1,\tau_2)\}$

Corollary. $|d(G,\tau_1)| \cdot |d(G,\tau_2)| / |\text{Out}(G)| \leq h(G;\tau_1,\tau_2) \leq |d(G,\tau_1)| \cdot |d(G,\tau_2)|.$
Result 1 – Alternating and symmetric groups

The following theorem was conjectured by Bauer, Catanese, Grunewald:

**Theorem [GP].** Let $\tau_i=(n_i,m_i,l_i)$ ($i=1,2$) be two hyperbolic types. Then almost all alternating groups $A_n$ admit a Beauville structure of type $(\tau_1,\tau_2)$.

**Theorem [GP].** $h(A_n;\tau_1,\tau_2) = \Omega(n^6)$, hence $h(A_n;\tau_1,\tau_2) = \Omega((\log \chi)^5)$.

**Theorem [GP].** Let $\tau_i=(n_i,m_i,l_i)$ ($i=1,2$) be two hyperbolic types, and assume that for $i=1,2$ at least two of $n_i,m_i,l_i$ are even. Then almost all symmetric groups $S_n$ admit a Beauville structure of type $(\tau_1,\tau_2)$.

**Theorem [GP].** $h(S_n;\tau_1,\tau_2) = \Omega(n^6)$, hence $h(S_n;\tau_1,\tau_2) = \Omega((\log \chi)^5)$.
New Beauville surfaces, moduli spaces and finite groups

Background – alternating groups as Hurwitz groups

**Theorem** [Higman; Conder '80].

\[ A_n \text{ is a Hurwitz group} \text{ (i.e. quotient of } \Delta(2,3,7) \text{) if } n \text{ is large enough.} \]

Higman's Conjecture (60's).
Every triangle group, and more generally – every Fuchsian group, surjects to all but finitely many alternating groups.

[Everitt '00]. Proved Higman's conjecture.

[Liebeck,Shalev '04]. An alternative (more general) proof:
- Based on probabilistic group theory.
- Generalized to other finite simple groups (in [Liebeck,Shalev '05]).
Theorem of Liebeck and Shalev

Definitions. A conjugacy class in $S_n$ of cycle-shape $(m^a)$, where $m | n \ (n=ma)$, is called homogeneous:

$$\underbrace{(*,*,\ldots,*)}_{m \text{ times}} \underbrace{(*,*,\ldots,*)}_{m \text{ times}} \cdots \underbrace{(*,*,\ldots,*)}_{m \text{ times}} \underbrace{\overbrace{\cdots}^a \overbrace{\cdots}^a \overbrace{\cdots}^a}_{\underbrace{\cdots}_a \underbrace{\cdots}_a \underbrace{\cdots}_a}$$

A conjugacy class in $S_n$ of cycle-shape $(m^a,1^f)$, $(n=ma+f)$, with $f$ bounded, is called almost homogeneous:

$$\underbrace{(*,*,\ldots,*)}_{m \text{ times}} \underbrace{(*,*,\ldots,*)}_{m \text{ times}} \cdots \underbrace{(*,*,\ldots,*)}_{m \text{ times}} \underbrace{\overbrace{\cdots}^a \underbrace{\cdots}^a \underbrace{\cdots}^a}_{\underbrace{\cdots}_a \underbrace{\cdots}_a \underbrace{\cdots}_a} \underbrace{\overbrace{\cdots}^f \overbrace{\cdots}^f \overbrace{\cdots}^f}_{\underbrace{\cdots}_f \underbrace{\cdots}_f \underbrace{\cdots}_f}$$

Theorem [Liebeck, Shalev '04]. Assume that $(m_1,m_2,m_3)$ is hyperbolic.

Then the probability that three random elements $x_1,x_2,x_3 \in A_n$, with product 1, from almost homogeneous classes $C_1,C_2,C_3$, of orders $m_1,m_2,m_3$, will generate $A_n$, tends to 1 as $n \to \infty$. 
Example: alternating groups as Hurwitz groups

Example: assume that \( \tau=(2,3,7) \). Conjugacy classes in \( A_n \) of orders...

\[
\begin{array}{ccc}
2 & 3 & 7 \\
(*,*)(*,*) & (*,*) & (*,*,*,*,*,*,*) \\
(*,*)(*,*)(*,*)(*,*) & (*,*,*)(*,*,*) & (*,*,*,*,*,*,*)(*,*,*,*,*,*,*) \\
\vdots & \vdots & \vdots \\
\end{array}
\]

[Liebeck, Shalev '04]. Random Hurwitz generation of \( A_n \):

\[
Pr \{<x,y,z>=A_n: \text{xyz}=1, x \in C_2, y \in C_3, z \in C_7, \text{Fix}(C_i)<f\} \rightarrow 1, \text{ as } n \rightarrow \infty.
\]

- Since \( f \) can be arbitrarily large:
  \[
  \# \{(C_2,C_3,C_7), \text{Fix}(C_i)<f\} \rightarrow \infty \text{ as } n \rightarrow \infty.
  \]

- Moreover, as the number of almost homogeneous conjugacy classes of a certain order grows linearly in \( n \),
  \[
  \# \{(C_2,C_3,C_7), \text{Fix}(C_i)<f\} = \Theta(n^3).
  \]
On the Proof of our Theorems for $A_n$

Proof of the Beauville structure:
Let $\tau_1=(n_1, m_1, l_1)$ and $\tau_2=(n_2, m_2, l_2)$ be two hyperbolic types.

Then, one can construct 6 different almost homogeneous conjugacy classes $C_{n_1}^1, C_{m_1}^1, C_{l_1}^1, C_{n_2}^2, C_{m_2}^2, C_{l_2}^2$, of orders $n_1, m_1, l_1, n_2, m_2, l_2$.

By [Liebeck, Shalev '04], the probability that three random elements $x_i, y_i, z_i$, with product 1, from $C_{n_i}^i, C_{m_i}^i, C_{l_i}^i$, will generate $A_n$ tends to 1 as $n \to \infty$.

Moreover, since all 6 conjugacy classes are different, then $\Sigma_1 \cap \Sigma_2 = \{1\}$.

Therefore, almost all groups $A_n$ admit a Beauville structure of type $(\tau_1, \tau_2)$.

Counting points in the moduli space:
$$h(A_n; \tau_1, \tau_2) \geq \# \{(C_{n_1}^1, C_{m_1}^1, C_{l_1}^1, C_{n_2}^2, C_{m_2}^2, C_{l_2}^2) : \text{Fix}(C_j^i) < f\} = \Omega(n^6).$$
Result 2 – The groups PSL(2,q)

Let $p$ be a prime number, and let $q=p^e$.

**Theorem** [GP]. If $q \geq 7$, then PSL(2,q) admits a Beauville structure.

**Remark.** An alternative proof was given by [Fuertes,Jones].

**Theorem** [GP]. Let $\tau_1$ and $\tau_2$ be two hyperbolic types, then there exists a constant $c=c(\tau_1, \tau_2)$, independent of $p$, such that:

$$h(\text{PSL}(2,p);\tau_1, \tau_2) \leq c.$$ 

**Corollary** [GP]. Let $\tau_1=(n,n,n)$ and $\tau_2=(m,m,m)$ where $n>m>5$ are primes. Then there exist infinitely many $p$ for which PSL(2,p) admits a Beauville structure of type $(\tau_1, \tau_2)$ and moreover $h(\text{PSL}(2,p);\tau_1, \tau_2) \leq c$.

This provides an infinite family of surfaces for which $h$ is bounded while $\chi \rightarrow \infty$. 
Elements in PSL(2,q)

$$\text{PSL}(2,q) = \text{SL}(2,q)/\langle -I \rangle,$$

of order: $$q(q+1)(q-1)/d$$ (d=1 if q is even, d=2 if q is odd)

$$A \in \text{PSL}(2,q), \alpha = \text{tr}(A) \Rightarrow \text{characteristic polynomial: } P_A(\lambda) = \lambda^2 - \alpha \lambda + 1.$$

<table>
<thead>
<tr>
<th>Element type</th>
<th>Canonical form</th>
<th>Conjugacy classes</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Split</td>
<td>$\begin{pmatrix} a &amp; 0 \ 0 &amp; a^{-1} \end{pmatrix}, a \in F_q^*$, $a + a^{-1} = \alpha$</td>
<td>Each $\alpha$: one conjugacy class in PSL(2,q)</td>
<td>$\frac{q-1}{d}$</td>
</tr>
<tr>
<td>Non-split</td>
<td>$\begin{pmatrix} a &amp; 0 \ 0 &amp; a^q \end{pmatrix}, a \in F_q^* \setminus F_q$, $a^{q+1} = 1$</td>
<td>Each $\alpha$: one conjugacy class in PSL(2,q)</td>
<td>$\frac{q+1}{d}$</td>
</tr>
<tr>
<td>Unipotent</td>
<td>$\begin{pmatrix} 1 &amp; 1 \ 0 &amp; 1 \end{pmatrix}, \alpha = \pm 2$</td>
<td>two conjugacy classes in PSL(2,q), and one class in PGL(2,q)</td>
<td>$p$</td>
</tr>
</tbody>
</table>
Subgroups of PSL(2,q)

**Theorem** [Dickson, 1901]. The subgroups of PSL(2,q) are:

- **Structural subgroups**, conjugate (in PSL(2, \( \overline{F_p} \))) to a subgroup of:

  \[
  B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \ a, b \in F_q \right\} \quad \text{or} \quad T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^q \end{pmatrix}, \ a \in F_{q^2}^* / F_q \right\}.
  \]

- **Small triangle subgroups**, isomorphic to:
  \[
  \Delta = \Delta(m_1,m_2,m_3), \ \text{where} \ 1/m_1 + 1/m_2 + 1/m_3 > 1.
  \]
  - \( \Delta(2,2,m) = D_{2m} \), (dihedral group).
  - \( \Delta(2,3,3) = A_4 \).
  - \( \Delta(2,3,4) = S_4 \), (if \( p \equiv \pm 1 \bmod 8 \)).
  - \( \Delta(2,3,5) = A_5 \), (if \( p=5 \) or \( p \equiv \pm 1 \bmod 10 \)).

- **Subfield subgroups**, isomorphic to: PSL(2,\( p^r \)) if \( r \| e \), or PGL(2,\( p^r \)) if \( 2r \| e \).
Generation theorems of $\text{PSL}(2,q)$

$$E(\alpha, \beta, \gamma) := \{A, B, C \colon ABC = I, \, \text{tr}(A) = \alpha, \, \text{tr}(B) = \beta, \, \text{tr}(C) = \gamma\}.$$  

**Definition.** $(\alpha, \beta, \gamma)$ is singular if $\alpha^2 + \beta^2 + \gamma^2 - \alpha \beta \gamma = 4$.

$(\alpha, \beta, \gamma)$ is small if the orders $(k, m, n)$ of $(A, B, C) \in E(\alpha, \beta, \gamma)$ satisfy $(k, m, n) = (2, 2, n)$ or $2 \leq k, m, n \leq 5$.

**Theorem** [Macbeath '69]. $E(\alpha, \beta, \gamma) \neq \emptyset$ for all $(\alpha, \beta, \gamma) \in F_q^3$.

**Theorem** [Macbeath '69].

$(\alpha, \beta, \gamma)$ is singular $\iff$ Any $(A, B, C) \in E(\alpha, \beta, \gamma)$ generates a structural subgroup.

**Corollary** [Macbeath '69]. If $(\alpha, \beta, \gamma)$ is not singular nor small, then any $(A, B, C) \in E(\alpha, \beta, \gamma)$ generates a subfield subgroup of $\text{PSL}(2,q)$.

**Theorem** [Macbeath '69]. Let $(\alpha, \beta, \gamma)$ be non-singular. If $q$ is odd, $E(\alpha, \beta, \gamma)$ contains two $\text{PSL}(2,q)$-conjugacy classes and one $\text{PGL}(2,q)$-conjugacy class. If $q$ is even, $E(\alpha, \beta, \gamma)$ contains one $\text{PSL}(2,q)$-conjugacy class.
Proof – Beauville structures of PSL(2,q)

**Case q odd.** If $q \geq 13$, we construct a Beauville structure for $\text{PSL}(2,q)$ of type

$$\tau_1 = \left( \frac{q-1}{2}, \frac{q-1}{2}, \frac{q-1}{2} \right), \quad \tau_2 = \left( \frac{q+1}{2}, \frac{q+1}{2}, \frac{q+1}{2} \right).$$

Let $r = \frac{q-1}{2}$ (resp. $r = \frac{q+1}{2}$), and let $(\alpha, \beta, \gamma)$ be three traces of elements of order $r$.

By [Macbeath '69], $E(\alpha, \beta, \gamma) \neq \emptyset$.

By replacing $(\alpha, \beta, \gamma)$ with $(-\alpha, -\beta, -\gamma)$, if necessary, we may assume that $(\alpha, \beta, \gamma)$ is not singular. Since it is also not small, then any $(A, B, C) \in E(\alpha, \beta, \gamma)$ generates $\text{PSL}(2,q)$, using [Macbeath '69].

Since $\frac{q-1}{2}$ and $\frac{q+1}{2}$ are relatively prime, then $\Sigma_1 \cap \Sigma_2 = \{1\}$.

For $7 \leq q \leq 11$ odd, one finds Beauville structures using computer calculations.

**Case q even.** Similarly, we construct a Beauville structure for $\text{PSL}(2,q)$ of type

$$\tau_1 = (q-1, q-1, q-1), \quad \tau_2 = (q+1, q+1, q+1).$$
Proof – counting points in the $\text{PSL}(2,p)$ moduli space

$G = \text{PSL}(2,p) \Rightarrow \text{Aut}(G) = \text{PGL}(2,p).$

**Lemma 1.** $\# \{\text{PSL}(2,p)\text{-orbits of } S(G,\tau)\} = 2 \cdot \# \{\text{PGL}(2,p)\text{-orbits of } S(G,\tau)\}.$

**Lemma 2.** $\{\text{PGL}(2,p)\text{-orbits of } S(G,\tau)\} = \{(\alpha,\beta,\gamma): \text{traces of three elements of type } \tau \text{ which are not singular nor small}\}.$

$\Upsilon_k := \{\pm \alpha \in \mathbb{F}_p: \exists A \in \text{PSL}(2,p) \text{ of order } k \text{ with } \text{tr}(A) = \alpha\}.$

**Lemma 3.** Let $2 \leq k \leq m \leq n$ and assume that $m>2$ and $n>5,$ then

$\# \{\text{PGL}(2,p)\text{-orbits of } S(G,(k,m,n))\} = \# \{(\pm \alpha, \pm \beta, \pm \gamma): \alpha \in \Upsilon_k, \beta \in \Upsilon_m, \gamma \in \Upsilon_n, \alpha^2 + \beta^2 + \gamma^2 - \alpha \beta \gamma \neq 4\}.$

**Example.** $\# \{\text{PGL}(2,p)\text{-orbits of } S(G,(2,3,7))\} = \# \{(0, \pm 1, \pm \gamma): \gamma \in \Upsilon_7\} = 0, 1 \text{ or } 3.$

**Proposition.** There exists $c = c(\tau)$ s.t. $\# \{\text{PGL}(2,p)\text{-orbits of } S(G,\tau)\} < c.$

**Corollary.** $h(\text{PSL}(2,p); \tau_1, \tau_2) < (2c)^2.$
Result 3 – Finite simple groups of low Lie rank

**Theorem** [GP]. The following finite simple groups of Lie type $G = G(q)$ admit a Beauville structure, provided that $q$ is large enough:

1. Suzuki groups, $G = Sz(q) = ^2B_2(q)$, where $q = 2^{2e+1}$;
2. Ree groups, $G = ^2G_2(q)$, where $q = 3^{2e+1}$;
3. $G = G_2(q)$, where $q = p^e$ for a prime $p > 3$;
4. $G = ^3D_4(q)$, where $q = p^e$ for a prime $p > 3$;
5. $G = PSL(3,q)$, where $q = p^e$, $p$ prime;
6. $G = PSU(3,q)$, where $q = p^e$, $p$ prime.

**Remark.** It was proved by [Fuertes, Jones] that all Suzuki groups $Sz(q) = ^2B_2(q)$ and Ree groups $^2G_2(q)$ admit a Beauville structure.
Theorem of Marion

**Theorem** [Marion]. Let $G$ be one of the following groups and let $p_1 \leq p_2 \leq p_3$ be a hyperbolic triple of primes, such that $\text{lcm}(p_1, p_2, p_3) \mid |G|:

1. Suzuki groups, $G = 2B_2(q)$, where $q = 2^{2e+1};$
2. Ree groups, $G = 2G_2(q)$, where $q = 3^{2e+1};$
3. $G = G_2(q)$, where $q = p^e$, $p > 3$, and $(p_1, p_2, p_3) \not\in \{(2,5,5), (3,3,5), (3,5,5), (5,5,5)\};$
4. $G = 3D_4(q)$, where $q = p^e$, $p > 3$, and $(p_1, p_2, p_3)$ are distinct, s.t. $\{p_1, p_2\} \neq \{2,3\};$
5. $G = PSL(3,q)$, where $q = p^e$, and $(p_1, p_2, p_3)$ are odd primes;
6. $G = PSU(3,q)$, where $q = p^e$, and $(p_1, p_2, p_3)$ are odd primes.

Then, if $\phi \in \text{Hom}(\Delta(p_1, p_2, p_3), G)$, then $\text{Prob}\{\phi \text{ is surjective}\} \to 1$ as $q \to \infty$.

**Proof of the Beauville structures:**

For any of the groups $G$ above we choose two hyperbolic triples of primes $\tau_1 = (n_1, m_1, l_1)$ and $\tau_2 = (n_2, m_2, l_2)$, s.t. $n_i, m_i, l_i \mid |G|$ and $(n_1 m_1 l_1, n_2 m_2 l_2) = 1$. Thus, by [Marion], $G$ admits a **Beauville structure** of type $(\tau_1, \tau_2)$, if $q$ is large enough.
Conjectures on finite simple groups of Lie type

**Conjecture** [Bauer,Catanese,Grunewald '05].
All finite simple non-abelian groups (except $A_5$) admit a Beauville structure.

**Remark.** There are infinitely many finite simple groups which are not Hurwitz!

**Theorem** [Macbeath '69]. $\text{PSL}(2,q)$ is a Hurwitz group if and only if
\[ q=p=7; \text{ or } q=p \equiv \pm 1 \text{ mod } 7; \text{ or } q=p^3 \text{ where } p \equiv \pm 2, \pm 3 \text{ mod } 7. \]

**Conjecture** [Liebeck,Shalev '05]. For any Fuchsian group $\Gamma$ there is an integer $r(\Gamma)$, such that if $G$ is a finite simple classical group of Lie rank at least $r(\Gamma)$, then the probability that a randomly chosen homomorphism from $\Gamma$ to $G$ is an epimorphism tends to 1 as $|G|\to\infty$.

**Conjecture** [GP]. Let $\tau_i=(n_i,m_i,l_i)$ ($i=1,2$) be two hyperbolic types. If $G$ is a finite simple classical group of Lie rank sufficiently large, then it admits a Beauville structure of type $(\tau_1,\tau_2)$. 
Result 4 – Abelian groups

**Theorem** [GP]. Let $p$ be a prime number and let $G=\mathbb{Z}/p\mathbb{Z}^2$.

Then $G$ admits a Beauville structure of type $(\tau,\tau)$, where $\tau=(p,p,p)$, and

$$h(G;\tau,\tau) = \Theta(p^4), \text{ hence } h(G;\tau,\tau) = \Theta(\chi^2).$$

Moreover, $N_p/36 < h(G;\tau,\tau) < N_p/6$, where $N_p = (p-1)(p-2)(p-3)(p-4)$.

**Proof.** The $\text{Aut}(G)$-representatives in $U(G;\tau,\tau)$ are exactly the vectors

$$\left( (1,0),(0,1),(-1,-1) ; (a,b),(c,d),(-a-c,-b-d) \right),$$

where $a,b,c,d \in \mathbb{Z}/p\mathbb{Z}^*$.  

Since $\Sigma_1 \cap \Sigma_2 = \{0\}$, all pairs of vectors are linearly independent, and so

$$a-b, a+c, c-d, b+d, a+c-b-d, ac-bd \in \mathbb{Z}/p\mathbb{Z}^*.$$  

Thus, the number of $\text{Aut}(G)$-representatives equals $N_p = (p-1)(p-2)(p-3)(p-4)$.

Now, the $(B_3 \times B_3)$-action on these representatives, which is equivalent to the $(S_3 \times S_3)$-action, since $G$ is abelian, gives orbits of sizes between 6 and 36.