

ON UNIFORM K -HOMOLOGY

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BACKGROUND

Coarse geometric ideas were first introduced into index theory by John Roe (1988). Subsequently this was put into the context of K -theory of C^* -algebras.

The coarse assembly map:

$$\begin{array}{ccc} K\text{-homology} & \xrightarrow{\mu} & K\text{-theory of Roe } C^*\text{-algebras} \\ [\text{elliptic op } D] & \longmapsto & [\text{index}(D)] \end{array}$$

Two points of view:

- given D , detect its index in the right-hand side
- the left-hand side is “computable” \implies if μ is an isomorphism, it computes the right-hand side

UNIFORM CASE: MOTIVATION

Why uniform Roe algebras instead of Roe algebras?

- smaller, more manageable C^* -algebra; can provide insight for Roe algebras
- C^* -algebraic properties relate to coarse geometric properties: e.g. if $X = \Gamma$ (a discrete group) then nuclearity (and also exactness) of uniform Roe alg. $\iff \Gamma$ has property A ($\iff C_r^*\Gamma$ is exact) [Guentner–Kaminker–Ozawa]
- typically much bigger K -theory (if nonzero): makes it “easier” to detect indices

UNIFORM K -HOMOLOGY: PLAN

Construct “uniform K -homology”: groups $K_*^u(X)$, with an index map $\mu_u : K_*^u(X) \rightarrow K_*(C_*^*X)$. Further properties:

- homological properties (e.g. Mayer–Vietoris sequence)
- show that for a group Γ , $\lim_{d \rightarrow \infty} K_*^u(P_d(\Gamma))$ is isomorphic to $K_{*,\text{top}}(\Gamma; \ell^\infty \Gamma) = \lim_{Y \subseteq \underline{E}\Gamma, \Gamma\text{-inv.}, Y/\Gamma \text{ cpct}} KK_\Gamma^*(C_0(Y), \ell^\infty \Gamma)$

uniform K -homology $\xrightarrow{\mu_u}$ K -theory of uniform Roe C^* -algebras
 [“uniform” elliptic op D] \longmapsto [index(D)]

Application:

- a characterization of amenability of X in terms of $K_0^u(X)$

COARSE GEOMETRY

In the category of (e.g.) metric spaces X . A function $f : X \rightarrow Z$ is *coarse*, if

- for every $r \geq 0$ there is $R \geq 0$, such that $\forall x_1, x_2 \in X$

$$d(x_1, x_2) \leq r \implies d(f(x_1), f(x_2)) \leq R.$$

- $f^{-1}(\{z\})$ is bounded $\forall z \in Z$.

We say f is *uniformly cobounded*, if

- for any $r \geq 0$: $\sup_{z \in Z} \text{diam}(f^{-1}(B(z, r))) < \infty$.

Note f might not be continuous. Any local information is irrelevant.

Two functions $f, g : X \rightarrow Z$ are *close*, if there exists $C > 0$, so that $d(f(x), g(x)) \leq C$.

$f : X \rightarrow Z$ is a coarse equivalence, if there is a coarse $g : Z \rightarrow X$, such that $f \circ g$ and $g \circ f$ are close to the identity maps.

Example: \mathbb{R}^n is coarsely equivalent with \mathbb{Z}^n .

RIPS COMPLEX

- Switching from “discrete” to “continuous” and back in the coarse category:

Let Y be a discrete metric space with bounded geometry.

Rips complex: For $d > 0$, let $P_d(Y)$ be a (locally finite, finite dimensional) simplicial polyhedron as follows:

- the vertex set is Y ,
- (y_0, \dots, y_q) span a simplex iff $\forall i, j: d(y_i, y_j) \leq d$.

Notation from now on: “ $Y \subset X$ is a discrete model of X ”:

- Y : a uniformly discrete metric space with bounded geometry
- X : a space (with bounded geometry)
- $Y \subset X$: a coarse equivalence

ROE C^* -ALGEBRAS

An operator $T \in \mathcal{B}(\ell^2 Y \otimes H)$ (a matrix $(t_{yx})_{x,y \in Y}$, $t_{yx} \in \mathcal{B}(H)$) has *propagation at most R* , if $d(x,y) \geq R$ implies $t_{yx} = 0$.

Furthermore T is said to be *locally compact*, if $t_{yx} \in \mathcal{K}(H)$.

If $\dim(H) = \infty$, define the *Roe algebra*:

$$C^*Y = \overline{\{T \in \mathcal{B}(\ell^2 Y \otimes H) \mid T \text{ is loc.comp. and has fin.prop.}\}}^{\|\cdot\|}$$

If $H = \mathbb{C}$, i.e. $T = (t_{yx})_{x,y \in Y} \in \mathcal{B}(\ell^2 Y)$, we obtain the *uniform Roe algebra*

$$C_u^*Y = \overline{\{T \in \mathcal{B}(\ell^2 Y) \mid T \text{ has fin.prop.}\}}^{\|\cdot\|}$$

Roe algebras are functorial under coarse maps;

Uniform Roe algebras are functorial under unif. cobounded coarse maps on the K -theory level.

EXAMPLE: $K_*(C_u^*\mathbb{Z})$

Since $C_u^*\mathbb{Z} \cong \ell^\infty\mathbb{Z} \rtimes \mathbb{Z}$, we can use the Pimsner–Voiculescu exact sequence.

$$\begin{array}{ccccc}
 K_0(\ell^\infty\mathbb{Z}) & \xrightarrow{1-\alpha_*} & K_0(\ell^\infty\mathbb{Z}) & \xrightarrow{\text{incl}_*} & K_0(C_u^*\mathbb{Z}) \\
 \uparrow & & & & \downarrow \\
 K_1(C_u^*\mathbb{Z}) & \xleftarrow{\text{incl}_*} & K_1(\ell^\infty\mathbb{Z}) & \xleftarrow{1-\alpha_*} & K_1(\ell^\infty\mathbb{Z})
 \end{array}$$

Denote $B(\mathbb{Z}) = \{\text{all bounded sequences } \mathbb{Z} \rightarrow \mathbb{Z}\}$
 and $I(\mathbb{Z}) = \{(a_n) \in B(\mathbb{Z}) \mid -\infty < \sup_{n \in \mathbb{Z}} (\sum_{i=0}^n a_i) < \infty\}$.

Compute $(1-\alpha)_*(a_n)_n = (a_n - a_{n-1})_n$.

Known: $K_0(\ell^\infty\mathbb{Z}) \cong B(\mathbb{Z})$ and $K_1(\ell^\infty\mathbb{Z}) = 0$.

Obtain: $K_1(C_u^*\mathbb{Z}) \cong \mathbb{Z}$, $K_0(C_u^*\mathbb{Z}) = B(\mathbb{Z})/I(\mathbb{Z})$.

ANALYTIC K-HOMOLOGY

“A space” is always assumed to be 2^{nd} countable, locally compact, Hausdorff topological space. Let X be a space.

K_0 -cycles for X : *Fredholm modules* (H, ϕ, F) :

- H — Hilbert space
- $\phi : C_0(X) \rightarrow \mathcal{B}(H)$ — *-homomorphism
- $F \in \mathcal{B}(H)$, such that $\forall f \in C_0(X)$:
 - (I) $(1 - FF^*)\phi(f) \sim 0$ and $(1 - F^*F)\phi(f) \sim 0$ (Fredholmness)
 - (II) $[F, \phi(f)] \sim 0$ (pseudolocality)

Notation: $T \sim S$ iff $T - S$ is compact.

Operation: direct sum of Fredholm modules.

To form $K_0(X)$, we mod out by an equivalence relation on Fredholm modules generated by unitary equivalence, operator homotopy and addition of degenerate modules (those where \sim is actually =).

K-HOMOLOGY – MAIN EXAMPLE/MOTIVATION

Reformulate the defn (“grading”): $T = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix} \in \mathcal{B}(H \oplus H)$.

Requirements now: T is odd and selfadjoint;

(i) $(T^2 - 1)\phi(f) \sim 0$; (ii) $[T, \phi(f)] \sim 0$.

Example:

- M a connected Riemannian manifold
- D a essentially self-adjoint, elliptic differential operator acting on sections of a graded vector bundle $S = S^- \oplus S^+$ over M
- $\chi : \mathbb{R} \rightarrow \mathbb{R}$ a chopping function (odd, $\chi(\pm\infty) \rightarrow \pm 1$)
- $H = L^2(M, S)$, $\phi : C_0(M) \rightarrow \mathcal{B}(H)$ by multiplication
- $T = \chi(D) \in \mathcal{B}(H)$
- Then (H, ϕ, T) is a Fredholm module.

K-HOMOLOGY – PROPERTIES

Analytic K -homology is a generalized homology theory on the category of “spaces” (i.e. it satisfies homotopy axiom and has a long exact sequence).

There are only two distinct K -homology groups $K_0(X)$ and $K_1(X)$ (periodicity).

Interpretation of K -homology as K -theory:

One can fix a suitable $*$ -representation $\phi : C_0(X) \rightarrow \mathcal{B}(H)$, and represent any $K_0(X)$ -element as (H, ϕ, F) for some F (Voiculescu’s theorem).

Define

$$\mathcal{D}_\phi = \{T \in \mathcal{B}(H) \mid [T, \phi(f)] \sim 0 \ \forall f\}.$$

One can show that $K_i(X) \cong K_{1-i}(\mathcal{D}_\phi)$.

ROE C^* -ALGEBRA — CONTINUOUS VERSION

Let $\phi : C_0(X) \rightarrow \mathcal{B}(H)$ be a $*$ -homomorphism. An operator $T \in \mathcal{B}(H)$ is *locally compact*, if $\phi(f)T\phi(g) \in \mathcal{K}(H)$ for all $f, g \in C_0(X)$. T has *finite propagation*, if for some $R \geq 0$, $\phi(f)T\phi(g) = 0$ for all $f, g \in \mathcal{B}(H)$ satisfying $d(\text{supp}(f), \text{supp}(g)) \geq R$.

We can define the Roe C^* -algebra of X also as

$$C^*X = \overline{\{T \in \mathcal{B}(H) \mid T \text{ is loc.comp. and has fin.prop.}\}}^{\|\cdot\|}$$

COARSE ASSEMBLY MAP $\mu : K_0(X) \rightarrow K_0(C^*X)$

If (H, ϕ, S) is an even Fredholm module for X , ie.

$$[\phi(f), S] \sim 0, \quad (1 - S^*S)\phi(f) \sim 0, \quad \phi(f)(1 - SS^*) \sim 0 \quad \text{for } f \in C_0(X).$$

Construct an invertible element

$$W = \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -S^* & 1 \end{pmatrix} \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathcal{B}(H)).$$

Then put $\text{index}(S) = W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^{-1} \in \mathcal{M}_2(\mathcal{B}(H))$. Concretely,

$$\text{index}(S) = \begin{pmatrix} SS^* + (1 - SS^*)SS^* & S(1 - S^*S) + (1 - SS^*)S(1 - S^*S) \\ S^*(1 - SS^*) & (1 - S^*S)^2 \end{pmatrix}.$$

If S had finite propagation, then we can check that

$$[\text{index}(S)] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \in K_0(C^*X).$$

In the odd case: $[(H, \phi, Q)] \mapsto [\exp(-2\pi i \frac{Q+1}{2})] \in K_1(C^*X)$.

COARSE BAUM–CONNES PROBLEM

CONJECTURE (COARSE BAUM–CONNES CONJECTURE)

If Y is a space with bounded geometry, then

$$\mu : \lim_{d \rightarrow \infty} K_*(P_d(Y)) \rightarrow K_*(C^*Y)$$

is an isomorphism.

- In principle, this conjecture provides a way to compute $K_*(C^*Y)$ and investigate higher indices of some elliptic operators.
- If true for a discrete group Γ , it implies rational injectivity of the Baum–Connes map, and hence e.g. the Novikov conjecture on invariance of higher signatures.
- Known to be true for spaces which coarsely embed into a Hilbert space [Yu].
Known to be false for certain expander graphs [Higson].

UNIFORM K -HOMOLOGY: PREPARATION

Let X be a metric space. Fix a Hilbert space H with $\phi : C_0(X) \rightarrow \mathcal{B}(H)$.

Notation: for $R, L > 0$, let

$$C_{R,L}(X) = \{f \in C_0(X) \mid \text{diam}(\text{supp}(f)) \leq R, \|f\|_\infty \leq 1, f \text{ is } L\text{-Lipschitz}\}.$$

An operator $T \in \mathcal{B}(H)$ is (ε, N) -approximable, if there is $k \in \mathcal{B}(H)$ with $\text{rank}(k) \leq N$ and $\|T - k\| < \varepsilon$.

An expression $E(f)$ with operators from $\mathcal{B}(H)$ and $\phi(f)$ (for instance $E(f) = [T, \phi(f)]$, or $E(f) = T\phi(f)$) is l -uniformly approximable, if for any $\varepsilon, R, L > 0$, there exists $N \geq 0$, such that for every $f \in C_{R,L}(X)$, $E(f)$ is (ε, N) -approximable. We write $E(f) \sim_{lua} 0$.

We say that $T \in \mathcal{B}(H)$ is *uniform*, if $T\phi(f) \sim_{lua} 0 \sim_{lua} \phi(f)T$. [stronger than locally compact]

UNIFORM K -HOMOLOGY: DEFINITION

(An even) *uniform Fredholm module* for X is a Fredholm module (H, ϕ, F) , such that

- $(1 - F^*F)\phi(f) \sim_{luc} 0$ and $(1 - FF^*)\phi(f) \sim_{luc} 0$.
- $[\phi(f), F] \sim_{luc} 0$ (F is uniformly pseudolocal).

Again, to form $K_0^u(X)$, we mod out by the equivalence relation generated by unitary equivalence, (certain) homotopies and adding degenerate modules.

The groups are functorial under Lipschitz, uniformly cobounded maps.

EXAMPLES OF UNIFORM K -HOMOLOGY CLASSES

THEOREM

Let M be a complete Riemannian manifold with bounded geometry, D be a Dirac-type operator over M (on a Clifford bundle S over M). Then the Fredholm module associated to it is uniform.

“The fundamental class”: X is an infinite graph (for instance a Cayley graph of a finitely generated group) with vertex set Y . Let $S \in \mathcal{B}(\ell^2\mathbb{N})$ be the unilateral shift, and let $\tilde{S} \in \mathcal{B}(\ell^2Y \otimes \ell^2\mathbb{N})$ be a “ Y -diagonal” operator, with S on the diagonal. Let $C_0(X)$ act on ℓ^2Y by multiplication. Then

$$\mathbf{S} = [(\ell^2Y \otimes \ell^2\mathbb{N}, \phi, \tilde{S})] \in K_0^u(X).$$

PROPERTIES OF UNIFORM K -HOMOLOGY

THEOREM

K_*^u has a Mayer–Vietoris sequence.

THEOREM

If $Y \subset X$ is discrete model of X , then there is a homomorphism $\mu_u : K_*^u(X) \rightarrow K_*(C_u^*Y)$, the assembly map.

Getting around Voiculescu’s theorem:

For fixed H and $\phi : C_0(X) \rightarrow \mathcal{B}(H)$, define $K_*^u(X, \phi)$ (then $K_*^u(X) = \lim_{\phi} K_*^u(X, \phi)$). Similarly, define $\mathcal{D}_{\phi}^u \subset \mathcal{B}(H)$ as the C^* -algebra of uniformly pseudolocal operators.

PROPOSITION

$K_i^u(X, \phi) \cong K_{1-i}(\mathcal{D}_{\phi}^u)$.

PROPERTIES OF UNIFORM K -HOMOLOGY

THEOREM

If Γ is a torsion–free countable discrete group, then there is a natural isomorphism

$$K_*^{\text{top}}(\Gamma, \ell^{\infty}\Gamma) \cong \lim_{d \rightarrow \infty} K_*^u(P_d\Gamma),$$

Recall that

$$K_*^{\text{top}}(\Gamma, A) = \lim_{\substack{\Delta \subset B\Gamma, \\ \Delta \text{ compact}}} KK_*^{\Gamma}(C_0(\rho^{-1}(\Delta)), A)$$

is the left-hand side of the Baum–Connes conjecture with coeffs in $\ell^{\infty}\Gamma$.

COROLLARY

The statement that

$$\mu_u : \lim_{d \rightarrow \infty} K_*^u(P_d\Gamma) \rightarrow K_*(C_u^*\Gamma)$$

is an isomorphism for a discrete group Γ is equivalent to the Baum–Connes conjecture for Γ with coefficients in $\ell^{\infty}\Gamma$.

CONSTRUCTING THE UNIFORM INDEX/ASSEMBLY MAP

Starting with an even uniform Fredholm module (H, ϕ, F) for X , applying the usual formula, we get an idempotent $T \in \mathcal{B}(H \oplus H)$. The task is to prove that we can actually arrange $[T] - [(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})] \in K_0(C_u^*Y)$.

For simplicity, take just $T \in \mathcal{B}(H)$, which is “uniform” (i.e. $T\phi(f) \sim_{ua} 0 \sim_{ua} \phi(f)T$), and also assume that ϕ is “sufficiently big” (the range of ϕ misses $\mathcal{K}(H) \setminus \{0\}$).

Using a suitable decomposition of the space X into the pieces corresponding to Y , we obtain a unitary isomorphism $U : H \cong \ell^2 Y \otimes \ell^2 \mathbb{N}$. The point is to use uniformity to choose U such that for any $\varepsilon > 0$ there is $n \in \mathbb{N}$, so that T is ε -far from an operator in $\mathcal{M}_n(C_u^*Y) \subset \mathcal{M}_n(\mathcal{B}(\ell^2 Y)) \subset \mathcal{B}(\ell^2 Y \otimes \ell^2 \mathbb{N})$.

This places $T \in C_u^*Y \otimes \mathcal{K}(\ell^2 \mathbb{N})$.

AMENABILITY: THEOREM AND RELATED RESULTS

Let X be a graph (bounded vertex degree) with vertex set Y . Y is said to be *amenable*, if for any $r, \delta > 0$, there exists a finite $U \subset Y$ with $\frac{\#\{o_r U\}}{\#\{U\}} < \delta$. (aka Følner condition. Works well for groups, not so much for general metric spaces.)

The element $\mathbf{S} \in K_0^u(X)$ defined before satisfies $\mu_u(\mathbf{S}) = \mathbf{1} \in K_0(C_u^*Y)$.

THEOREM

Y is amenable iff $\mathbf{S} \neq \mathbf{0} \in K_0^u(X)$.

THEOREM (G. ELEK, '97)

Y is amenable iff $\mathbf{1} \neq \mathbf{0} \in K_0(C_u^*Y)$.

THEOREM (J. BLOCK AND S. WEINBERGER, '92)

Y is amenable iff $H_0^{uf}(Y) \neq \{0\}$.

That's it!

Thank you!