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Master's thesis

Forking independence in the free group

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Contents

1 Introduction

This thesis deals with C. Perin's and R. Sklinos's purely group theoretic characterization of forking independence in non-abelian finitely generated free groups which they gave in [PS20]. To understand this characterization one has to have knowledge of some topics that are not treated in detail in [PS20]. To give the reader a more thorough understanding of these topics, this thesis introduces some aspects of model theory and Bass-Serre theory, as well as the notion of a tree of cylinders with a focus on the properties it has, if we consider a non-abelian finitely generated free group. Furthermore, it gives a short introduction to modular groups and explains why the main results of [PS16] are in fact special cases of the main theorem of [PS20]. Apart from that, this thesis closely follows [PS20], giving more detailed proofs and explanations and specifying where it is needed.

In [Sel13] Z. Sela proved that the first order theory of torsion-free hyperbolic groups, and thus in particular the first order theory of free groups, is stable. This implies that there exists a notion of independence, which we call forking independence, in all models of the first order theory of free groups. This independence is akin to linear independence in vector spaces or algebraic independence in algebraically closed fields. To characterize forking independence in every model of the first order theory of free groups would give an alternative proof of the stability of the theory, see for example [TZ12, Theorem 8.5.10]. However, this is not easy because we do not know what most of the models of the first order theory of free groups look like. In particular, a group elementary equivalent to a non-abelian free group does not need to be a free group. The difficulty arising from this can be seen in the definition of forking independence:

Definition ([PS20, Definition 1.1])**.** Let M be a stable structure and let A be a subset of M. Tuples $\bar{b}$ and $\bar{c}$ of elements of M *fork* over A, i.e. are not independent over A, if and only if there exists a set X definable over $A\bar{c}$ which contains $\bar{b}$, and a sequence of automorphisms $\theta_n \in \mathrm{Aut}_A(\hat{M})$ for some elementary extension $\hat{M}$ of M, such that the translates $\theta_n(X)$ are k-wise disjoint for some $k \in \mathbb{N}$.

(For more details about forking see Section 2.1 and [Pil96].) Thus, even to understand forking independence in models of the first order theory of free groups one has to move to a *saturated model* of the theory. But we do not know what a saturated model of the theory of non-abelian free groups looks like. Nevertheless, Perin and Sklinos succeeded to characterize the model theoretic notion of forking independence for all non-abelian finitely generated free groups in purely group theoretic terms. This characterization is the main theorem treated in this thesis, see Theorem 4.15.

Theorem ([PS20, Theorem 3.15]). *Let $\mathbb{F}$ be a non-abelian finitely generated free group, let $A \subseteq \mathbb{F}$ be a set of parameters, and let $\bar{b}, \bar{c}$ be tuples from $\mathbb{F}$.*

Then $\bar{b}$ is independent from $\bar{c}$ over A if and only if there exists a normalized pointed cyclic JSJ decomposition Λ for $\mathbb{F}$ relative to A in which the intersection of any two blocks of the minimal subgraphs $\Lambda^{min}_{A\bar{b}}$ and $\Lambda^{min}_{A\bar{c}}$ of $\langle A, \bar{b}\rangle$ and $\langle A, \bar{c}\rangle$ respectively is contained in a disjoint union of envelopes of rigid vertices.

(For the definitions of a normalized pointed cyclic JSJ decomposition, blocks, envelopes, and rigid vertices see Section 4.) In other words, the tuples are independent over A if and only if there exists a certain JSJ decomposition in which they live in 'essentially disjoint' parts. Unlike in a special case that Perin and Sklinos proved in [PS16] before their main characterization, which is Theorem 4.18, it is important to consider all JSJ decompositions relative to A of a certain form. The idea behind this is that one can create 'artificial intersection' between the minimal subgraphs of $\langle A, \bar{b}\rangle$ and $\langle A, \bar{c}\rangle$ with trivially stabilized edges. Thus, it is important to add the trivially stabilized edges 'at the right places'.

Contents

This thesis is structured as follows: Section 2 gives a short overview of the basic concepts and notions we work with. In Section 2.1 we give a short introduction to model theory, including *types* and *stability theory*, in particular *forking independence*. A crucial method we use in this thesis is Bass-Serre theory which we introduce in Section 2.2. Here, the correspondence between the action of a group on a tree and a *graph of groups* plays an important role.

Section 3 first introduces the *tree of cylinders*, which will be important for defining a normalized pointed cyclic JSJ decomposition, in general. Afterwards it focusses on finitely generated free groups and trees with infinite cyclic edge stabilizers. It turns out that in this case we know a lot about the structure of the tree of cylinders.

Section 4 deals with *cyclic JSJ decompositions* in general and a *normalized* pointed cyclic JSJ decomposition in particular. A cyclic JSJ decomposition of a group G relative to a set of parameters A is a way to encode all possible splittings of G as amalgamated product or HNN extension over a cyclic subgroup in a graph of groups such that A is contained in one of the factors. The JSJ trees, which correspond to JSJ decompositions, form a *deformation space*. Under certain conditions this deformation space has an (up to isomorphism) unique tree of cylinders that is in fact also a JSJ tree, see [GL11]. We slightly modify this tree of cylinders to define normalized pointed cyclic JSJ trees. Because we studied trees of cylinders before, we know what a normalized pointed cyclic JSJ tree looks like. It turns out that tuples can only be independent if their minimal subgraphs intersect solely in certain edges and vertices, thus we define an *envelope* of a vertex. Furthermore, we consider the two special cases with which Perin and Sklinos dealt in [PS16] before their complete characterization, where the set of parameters A is either a free factor of the non-abelian finitely generated free group $\mathbb{F}$, or $\mathbb{F}$ is freely indecomposable with respect to A. We show that they are in fact special cases of the main theorem. Afterwards, we introduce *Dehn twists*, automorphisms forming the so called *modular group*, and *vertex automorphisms* on trees in Section 4.5. We need these automorphisms to prove independence in the case where an adequate decomposition exists.

In Section 5 we prove the main result, which is Theorem 4.15, following [PS20]. To prove the right to left direction of the main result, namely the independence of $\bar{b}$ and $\bar{c}$ over A if their minimal subgraphs fulfil the disjointness condition, we first prove a special case where $\bar{b}$ is contained in the smallest free factor of $\mathbb{F}$ containing A and $\mathbb{F}$ is freely indecomposable with respect to $A\bar{c}$. In fact we prove that the orbit of $\bar{b}$ under modular automorphisms fixing A is the same as the orbit of $\bar{b}$ under automorphisms fixing both A and $\bar{c}$, which is enough to prove independence. For the other direction we construct an adequate decomposition, using an idea that was already used in the proof of the second direction of [PS16, Theorem 1].

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2 Preliminaries

In this section we give a short overview of the model- and group-theoretic notions we need for the remainder of the thesis.

2.1 Model theory

The exposition in this subsection closely follows [PS16, Sections 2.1 & 2.2]. We fix a language $\mathcal{L}$, an $\mathcal{L}$-structure $\mathcal{M}$ with universe M, and its complete first-order theory $\mathrm{Th}(\mathcal{M})$.

Definition 2.1 ([PS16, p. 1985])**.**

i) An *n-type* $p(v_1, \dots, v_n)$ of $\mathrm{Th}(\mathcal{M})$ is a set of $\mathcal{L}$-formulas with at most n free variables $v_1, \dots, v_n$ which is consistent with $\mathrm{Th}(\mathcal{M})$, i.e. there exists a model of $\mathrm{Th}(\mathcal{M}) \cup p(v_1, \dots, v_n)$.

ii) Let $A \subseteq M$ be a set of parameters, and let $\mathrm{Th}_A(\mathcal{M})$ be the set of all $\mathcal{L}(A)$ sentences which are true in $\mathcal{M}$. An *n-type* $p(v_1, \dots, v_n)$ *over* A is a set of $\mathcal{L}(A)$-formulas in at most n free variables $v_1, \dots, v_n$ that is consistent with $\mathrm{Th}_A(\mathcal{M})$. Here, $\mathcal{L}(A)$ denotes the language obtained from $\mathcal{L}$ by adding a constant symbol for every element of A. The set of n-types over A is denoted by $S_n^{\mathcal{M}}(A)$.

A type $p(\bar{v})$ is called *complete* if for every $\mathcal{L}$-formula $\phi(\bar{v})$ either $\phi(\bar{v})$ or $\neg\phi(\bar{v})$ is in $p(\bar{v})$. Similarly, a type $p(\bar{v})$ over $A \subseteq M$ is called complete if for every $\mathcal{L}(A)$-formula $\phi(\bar{v})$ either $\phi(\bar{v})$ or $\neg\phi(\bar{v})$ is in $p(\bar{v})$.

Every element $\bar{a} = (a_1, \dots, a_n) \in M^n$ determines a complete n-type

$$\mathrm{tp}^{\mathcal{M}}(\bar{a}\,/A) = \{\phi(v_1, \dots, v_n) \mid \mathcal{M} \models \phi(\bar{a}) \text{ and } \phi(v_1, \dots, v_n) \text{ is an } \mathcal{L}(A)\text{-formula}\},$$

the *type of* $\bar{a}$ *over* A. Equivalently, the type of $\bar{a}$ over A can be thought of as the collection of all sets which are definable over A and contain $\bar{a}$, see [LPS13, p. 523]. We write $\mathrm{tp}^{\mathcal{M}}(\bar{a})$ for $\mathrm{tp}^{\mathcal{M}}(\bar{a}/\varnothing)$. An element $\bar{a} \in M^n$ *realizes* a type $p(\bar{v}) \in S_n^{\mathcal{M}}(A)$ if $\bar{a}$ satisfies all formulas from $p(\bar{v})$, i.e. if $p(\bar{v}) = \mathrm{tp}(\bar{a}/A)$.

There is a topology on the set of complete n-types $S_n^{\mathcal{M}}(A)$, defined by the basis of open sets

$$[\phi] = \{p \in S_n^{\mathcal{M}}(A) : \phi \in p\}$$

for an $\mathcal{L}(A)$-formula ϕ in at most n variables. We call $S_n^{\mathcal{M}}(A)$ equipped with this topology a *Stone space*. A type $p \in S_n^{\mathcal{M}}(A)$ is *isolated* if there is a formula $\phi \in p$ such that $[\phi] = \{p\}$.

Definition 2.2 ([PS16, Definition 2.1])**.** Let A be a subset of M. Then $\mathcal{M}$ is called *atomic over A* if every type in $S_n^{\mathcal{M}}(A)$ which is realized in $\mathcal{M}$ is isolated.

If $\mathcal{M}$ is a countable structure, i.e. M is countable, then $\mathcal{M}$ is atomic if the orbit of every finite tuple $\bar{a}$ of $\mathcal{M}$ under $\mathrm{Aut}(\mathcal{M})$ is definable over $\varnothing$.

In the case where $\mathcal{L}$ is the language of groups and $\mathcal{M}$ is the model of a finitely generated free group, Perin and Sklinos proved in [PS16, Theorem 5.3] the following useful result which is also true for torsion-free hyperbolic groups, although we only state it for free groups.

Theorem 2.3. *Let $\mathbb{F}$ be a finitely generated free group and let A be a subset of $\mathbb{F}$ such that A is not contained in any proper free factor of $\mathbb{F}$. Then for any tuple $\bar{b}$ of $\mathbb{F}$ the orbit of $\bar{b}$ under $\mathrm{Aut}_A(\mathbb{F})$ is definable over A.*

We will also need the notion of homogeneity.

Definition 2.4 ([PS16, Definition 2.2])**.** Let $\mathcal{M}$ be a countable structure. Then $\mathcal{M}$ is called *homogenous* if for every two tuples $\bar{a}, \bar{b}$ of $\mathcal{M}$ such that $\mathrm{tp}^{\mathcal{M}}(\bar{a}) = \mathrm{tp}^{\mathcal{M}}(\bar{b})$ there is an automorphism of $\mathcal{M}$ sending $\bar{a}$ to $\bar{b}$.

Lemma 2.5. *Let $\mathcal{M}$ be a homogenous structure, $A \subseteq \mathcal{M}$ be a finite subset, and $\bar{b}, \bar{c}$ be tuples of $\mathcal{M}$. Then $\bar{b}$ and $\bar{c}$ have the same type over A if and only if there exists an automorphism of $\mathcal{M}$ fixing A that sends $\bar{b}$ to $\bar{c}$.*

Proof. Let $\sigma \in \mathrm{Aut}_A(\mathcal{M})$ such that $\sigma(\bar{b}) = \bar{c}$. Then for any formula $\phi(\bar{x}, \bar{a})$ over A we have

$$\mathcal{M} \models \phi(\bar{b}, \bar{a}) \Leftrightarrow \mathcal{M} \models \phi(\sigma(\bar{b}), \sigma(\bar{a})) \Leftrightarrow \mathcal{M} \models \phi(\bar{c}, \bar{a}).$$

In particular, it holds that $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{c}/A)$. Note that we do not need homogeneity for this.

Now assume that $\bar{b}$ and $\bar{c}$ have the same type over A. Because A is finite this implies $\mathrm{tp}(\bar{b}A/\varnothing) = \mathrm{tp}(\bar{c}A/\varnothing)$. By homogeneity there exists an automorphism σ of $\mathcal{M}$ sending $\bar{b}A$ to $\bar{c}A$. In particular, σ is an automorphism of $\mathcal{M}$ fixing A that sends $\bar{b}$ to $\bar{c}$. $\square$

Perin and Sklinos as well as A. Ould Houcine proved in [PS12, Theorem 1.1] respectively [OH11, Theorem 1.1] the following:

Theorem 2.6. *Let $\mathbb{F}$ be a non-abelian finitely generated free group. Then $\mathbb{F}$ is homogenous.*

Remark 2.7. Notice that if $\mathbb{F}$ is a non-abelian finitely generated group, and A a finitely generated subgroup of $\mathbb{F}$, then Lemma 2.5 implies that tuples $\bar{b}$ and $\bar{c}$ have the same type over A if and only if there exists an automorphism of $\mathbb{F}$ fixing A that sends $\bar{b}$ to $\bar{c}$: Let X be a finite generating set of A. Then by Lemma 2.5 it holds that $\mathrm{tp}(\bar{b}/X) = \mathrm{tp}(\bar{c}/X)$ if and only if there exists $\sigma \in \mathrm{Aut}_X(\mathbb{F})$ such that $\sigma(\bar{b}) = \bar{c}$. Since X is a generating set of A, the automorphism σ also fixes A pointwise.

A further crucial notion is the one of algebraic elements over a subset of M.

Definition 2.8 ([PS16, p. 1986])**.** Let $A \subseteq M$, and let $\bar{a}$ be a tuple of $\mathcal{M}$. Then $\bar{a}$ is called *algebraic over* A if there is an $\mathcal{L}(A)$-formula $\phi(\bar{v})$ such that $\bar{a} \in \phi(\mathcal{M})$ and $\phi(\mathcal{M})$ is finite. The set of algebraic tuples over A is called the *algebraic closure of* A and denoted by $\mathrm{acl}_{\mathcal{M}}(A)$.

We will sometimes use the algebraic closure in the extended theory T^{eq}, where we added a sort for every definable equivalence relation such that equivalence classes can be thought of as elements of a T^{eq}-structure. We denote the algebraic closure in T^{eq} by acl^{eq}. For more details on the T^{eq} construction see [TZ12, p. 140].

The following Lemma is very useful for proving that a given tuple is algebraic.

Lemma 2.9 ([PS16, Lemma 2.3])**.** *Let $\mathcal{M}$ be a countable structure which is atomic over a subset A of M, and let $\bar{a}$ be a tuple of $\mathcal{M}$. Then $\bar{a}$ is algebraic over A if and only if $|\{f(\bar{a}) : f \in \mathrm{Aut}_A(\mathcal{M})\}|$ is finite.*

Stability theory

The beginnings of stability were developed by M. Morley in his dissertation [Mor65], where he introduced several fundamental concepts. But most of the results concerning stability theory were developed by S. Shelah in [She90]. Roughly speaking, stability draws a line between ‘well behaved’ theories, which are stable, and theories whose models are too complicated to classify, which are not stable.

A first-order formula $\phi(\bar{x}, \bar{y})$ in a structure $\mathcal{M}$ has the *order property* if there are sequences $(\bar{a}_n)_{n<\omega}, (\bar{b}_n)_{n<\omega}$ such that

$$\mathcal{M} \models \phi(\bar{a}_n, \bar{b}_m) \text{ if and only if } m < n.$$

Definition 2.10 ([PS16, Definition 2.4])**.** A first-order theory T is *stable* if no formula has the order property in a model of T.

Fix a stable first-order theory T and a 'big' saturated model $\mathbb{M}$ of T, which is usually called the *monster model*, with universe M. For more explanations on the monster model see [Mar02, p.218]. Capital letters $A, B, \ldots$ denote parameter sets, i.e. subsets of M, and lower letters $\bar{a}, \bar{b}, \ldots$ denote tuples of elements of $\mathbb{M}$. To make expressions more readable, we write $\mathrm{tp}(\bar{a}/A)$ for $\mathrm{tp}^{\mathbb{M}}(\bar{a}/A)$ and $S_n(A)$ for $S_n^{\mathbb{M}}(A)$.

Now we can define what it means for a formula to fork.

Definition 2.11 ([PS16, Definition 2.5])**.**

i) Let $\phi(\bar{x}, \bar{b})$ be a formula in $\mathbb{M}$. Then $\phi(\bar{x}, \bar{b})$ *forks over* A if there is an $n < \omega$ and an infinite sequence $(\bar{b}_i)_{i<\omega}$ such that $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{b}_i/A)$ for $i < \omega$, and the set $\{\phi(\bar{x}, \bar{b}_i) : i < \omega\}$ is n-inconsistent, i.e. every subset of n formulas is inconsistent.

ii) A tuple $\bar{a}$ is *independent from* B *over* A (denoted $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_A B$) if there is no formula in $\mathrm{tp}(\bar{a}/A \cup B)$ which forks over A. Otherwise we say that $\bar{a}$ *forks with* B *over* A.

We usually write AB for $A \cup B$ and $\bar{a}A$ for the union of sets $\bar{a} \cup A$.

One can think of $\bar{a}$ forking with B over A as the fact that the type of $\bar{a}$ over AB contains much more information than the type of $\bar{a}$ over A alone, see [LPS13, p. 523].

A consequence of stability is the existence of non-forking extensions.

Definition 2.12 ([PS16, Definitions 2.7 & 2.8])**.**

i) Let $p \in S_n(A)$ and $A \subseteq B$. Then $q := \mathrm{tp}(\bar{a}/B)$ is called a *non-forking extension of* p if $p \subseteq q$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_A B$.

ii) A type $p \in S_n(A)$ is called *stationary* if for every $B \supseteq A$ the type p admits a unique non-forking extension over B.

Forking independence satisfies certain axioms. We record the ones important for this thesis. Normality is an easy observation, the other facts can be found in [Mar02, pp. 234–235] and [PS16, Lemma 2.12].

Fact 2.13.

i) *(Symmetry)* $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\;|}_A \bar{b}$ *if and only if* $\bar{b} \mathop{\smile\!\!\!\!\!\!\;\;|}_A \bar{a}$.

ii) *(Transitivity) Let* $A \subseteq B \subseteq C$. *Then* $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\;|}_A C$ *if and only if* $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\;|}_A B$ *and* $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\;|}_B C$.

iii) *(Monotonicity) Let* $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\;|}_A B$ *and* $C \subseteq B$. *Then* $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\;|}_A C$.

iv) *(Finite Basis)* $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\;|}_A B$ *if and only if* $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\;|}_A B_0$ *for all finite* $B_0 \subseteq B$.

v) *(Normality)* $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\;|}_A \bar{b}$ *if and only if* $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\;|}_A A\bar{b}$.

vi) *Let* $A \subseteq B$. *Then* $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\;|}_A B$ *if and only if* $\mathrm{acl}(\bar{a}A) \mathop{\smile\!\!\!\!\!\!\;\;|}_{\mathrm{acl}(A)} \mathrm{acl}(B)$.

vii) $\bar{a}\bar{b} \mathop{\smile\!\!\!\!\!\!\;\;|}_A C$ *if and only if* $\bar{a} \mathop{\smile\!\!\!\!\!\!\;\;|}_A C$ *and* $\bar{b} \mathop{\smile\!\!\!\!\!\!\;\;|}_{A\bar{a}} C$.

Furthermore, by [Pil96, Remark 2.25 (ii)] we get the following:

Fact 2.14. *Every type over a set* $A = \mathrm{acl}^{eq}(A)$ *is stationary.*

The following lemma is [PS16, Lemma 2.10].

Lemma 2.15. *Let* $\mathcal{M}$ *be a model of* T. *Let* $\bar{b}, A \subseteq \mathcal{M}$ *and let* $\mathcal{M}$ *be countable and atomic over* A. *Suppose the set* X *defined by a formula* $\phi(\bar{v}, \bar{b})$ *contains a non-empty almost* A*-invariant subset, i.e. a subset that has finitely many images under* $\mathrm{Aut}_A(\mathcal{M})$. *Then* $\phi(\bar{v}, \bar{b})$ *does not fork over* A.

Moreover, if we consider a non-abelian finitely generated free group $\mathbb{F}$, then on one hand Sela as well as O. Kharlampovich and A. Myasnikov proved in [Sel06, Theorem 4] respectively [KM06, Theorem 1] that all non-abelian free factors of $\mathbb{F}$ are elementary substructures of $\mathbb{F}$. On the other hand, every elementary substructure of $\mathbb{F}$ is a non-abelian free factor of $\mathbb{F}$ by [Per11, Theorem 1.3]. Thus we get:

Theorem 2.16. *Let* $\mathbb{F}$ *be a non-abelian finitely generated free group and let* H *be a subgroup of* $\mathbb{F}$. *Then* H *is an elementary substructure of* $\mathbb{F}$ *if and only if* H *is a non-abelian free factor of* $\mathbb{F}$.

2.2 Bass-Serre theory

From now on we work with the language of groups $\mathcal{L}_{Group} = \{\cdot, ^{-1}, 1\}$. Every group gives rise to an $\mathcal{L}_{Group}$ structure in a natural way. Let G be a group. To make expressions more readable, we omit the multiplication dot, i.e. for $g, h \in G$ we write gh instead of $g \cdot h$. For a subgroup H of G and an element $g \in G$ we define $H^g = gHg^{-1}$. Every element $g \in G$ defines a map $\mathrm{Conj}(g) : G \longrightarrow G$ by $h \mapsto ghg^{-1}$.

We consider simplicial trees T with an action of G on T, they are called *G-trees*, and we identify two trees if there is a G-equivariant isomorphism between them. The action of G on T is denoted by $\cdot$, and we always assume that the action of G on T is without inversion, i.e. that for every edge $e = (v, w)$ in T there exists no element $g \in G$ such that g maps e to its inverse edge $\bar{e} = (w, v)$. The stabilizer of an edge e of T and the stabilizer of a vertex v of T are denoted by $\mathrm{Stab}(e)$ and $\mathrm{Stab}(v)$ respectively. We denote by $V(T)$ and $E(T)$ the set of vertices and edges of T respectively. Let $\alpha : E(T) \longrightarrow V(T)$ and $\omega : E(T) \longrightarrow V(T)$ be the maps which map every edge to its beginning and ending respectively. For vertices v, w of T the *segment* $[v, w]$ is defined as the unique path in T from v to w. If we consider sets $X \subseteq Y$, we denote their difference by $X - Y$. For basic definitions and facts about trees see [Ser80].

J.-P. Serre developed in [Ser80] in collaboration with H. Bass a theory which helps to analyse the structure of a group which acts on a tree by looking at this action. Graphs of groups play an important role. The exposition in this section closely follows [Bog08, Chapter 2, Sections 16 and 18]. All citations in this section refer to the contents of Chapter 2 of [Bog08].

Definition 2.17 ([Bog08, Definition 16.1])**.** A *graph of groups* $(\mathbb{G}, Y)$ consists of a connected graph Y, a group G_v for each $v \in V(Y)$, called *vertex group*, a group G_e for each $e \in E(Y)$, called *edge group*, and monomorphisms $\alpha_e : G_e \longrightarrow G_{\alpha(e)}$ for every edge $e \in E(Y)$. We furthermore require that $G_{\bar{e}} = G_e$, where $\bar{e}$ is the inverse edge of e.

Define the group $F(\mathbb{G}, Y)$ for a graph of groups $(\mathbb{G}, Y)$ as the free product

$$\left(*_{v \in V(Y)} G_v\right) * F$$

where F is the free group with basis $\{t_e \mid e \in E(Y)\}$, subject to the following relations:

- $t_e^{-1}\alpha_e(g)t_e = \alpha_{\bar{e}}(g)$ for every $e \in E(Y)$ and every $g \in G_e$,
- $t_e t_{\bar{e}} = 1$ for every $e \in E(Y)$.

There are two definitions of the fundamental group of a graph of groups, one with respect to a vertex p and one with respect to a subtree T. But the two constructed fundamental groups are isomorphic for every choice of the vertex p and every maximal subtree T of Y by [Bog08, Corollary 16.7].

Definition 2.18 ([Bog08, Definitions 16.2 & 16.3])**.**

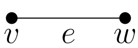
i) Let $(\mathbb{G}, Y)$ be a graph of groups, and let p be a vertex of the graph Y. The *fundamental group* $\pi_1(\mathbb{G}, Y, p)$ *of the graph of groups* $(\mathbb{G}, Y)$ *with respect to the vertex* p is the subgroup of $F(\mathbb{G}, Y)$ consisting of all elements of the form $g_0 t_{e_1} g_1 t_{e_2} \dots t_{e_n} g_n$, where $e_1 e_2 \dots e_n$ is a closed path in Y with initial vertex p and $g_0 \in G_p$, $g_i \in G_{\omega(e_i)}$ for all $i = 1, \dots, n$.

ii) Let $(\mathbb{G}, Y)$ be a graph of groups, and let T be a maximal subtree of the graph Y. The *fundamental group* $\pi_1(\mathbb{G}, Y, T)$ *of the graph of groups* $(\mathbb{G}, Y)$ *with respect to the subtree* T is the group $F(\mathbb{G}, Y)$ subject to the relation $t_e = 1$ for all $e \in E(T)$.

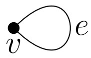
Since the fundamental groups $\pi_1(\mathbb{G}, Y, p)$ and $\pi_1(\mathbb{G}, Y, T)$ are isomorphic for a graph of groups $(\mathbb{G}, Y)$, every vertex p of Y, and every maximal subtree T, we will henceforth just write $\pi_1(\mathbb{G}, Y)$.

We give some basic examples to better grasp this concept.

Example 2.19 ([PS16, Example 16.4])**.**

i) Let Y be a segment 

. Then the group $\pi_1(\mathbb{G}, Y)$ is isomorphic to the amalgamated product $G_v *_{G_e} G_w$.

ii) Let Y be a loop 

. Then the group $\pi_1(\mathbb{G}, Y)$ is isomorphic to the HNN extension $G_v *_{G_e} = \langle G_v, t \mid t^{-1}\alpha_e(g)t = \alpha_{\bar{e}}(g),\, g \in G_e \rangle$.

iii) For an arbitrary graph of groups $(\mathbb{G}, Y)$ the fundamental group $\pi_1(\mathbb{G}, Y, T)$ can be obtained from $\pi_1(\mathbb{G}, T, T)$ by successively applying HNN extensions. The number of applications equals the number of mutually inverse pairs of edges of Y not lying in T. The fundamental group $\pi_1(\mathbb{G}, T, T)$ can be obtained (for $|V(T)| \geq 2$) from the fundamental group of a segment consisting of only one edge by successive applications of amalgamated products.

Every vertex group of a graph of groups $(\mathbb{G}, Y)$ can be canonically embedded in the fundamental group $\pi_1(\mathbb{G}, Y)$, see [Bog08, Theorem 16.10].

To give the main theorems in this section, it remains to define factor graphs. The definition is taken from [Bog08, p. 48]. Let G be a group acting on a graph Y without inversion of edges. For $x \in V(Y) \cup E(Y)$ denote by $\mathcal{O}(x)$ the orbit of x with respect to the action of G, that is $\mathcal{O}(x) = \{g \cdot x \mid g \in G\}$. The *factor graph* $G \backslash Y$ is defined as the graph with vertices $\mathcal{O}(v), v \in V(Y)$, and edges $\mathcal{O}(e), e \in E(Y)$, such that

i) $\mathcal{O}(v)$ is the beginning of $\mathcal{O}(e)$ if there exists an element $g \in G$ such that $g \cdot v$ is the beginning of e;

ii) the inverse of the edge $\mathcal{O}(e)$ is the edge $\mathcal{O}(\bar{e})$.

The edges $\mathcal{O}(e)$ and $\mathcal{O}(\bar{e})$ do not coincide because G acts without inversion of edges on Y. The map $p : Y \longrightarrow G \backslash Y$ given by $x \mapsto \mathcal{O}(x)$ for $x \in V(Y) \cup E(Y)$ is a morphism of graphs, called *projection* or *quotient map*. A preimage of a vertex or edge y of $G \backslash Y$ with respect to the map p is called a *lift* of y in Y. If G is finitely generated, the factor graph $G \backslash Y$ is finite.

Every subtree T' of the factor graph $G \backslash Y$ is in fact isomorphic to a subtree T of Y by [Bog08, Proposition 1.12]. We call T a *lift* of the tree T' in Y.

Definition 2.20 ([Bog08, Definition 18.1])**.** Let $p : X \longrightarrow Y$ be a morphism from a tree X to a connected graph Y, and let T be a maximal subtree in Y. A pair $(\widetilde{T}, \widetilde{Y})$ of subtrees in X is called a *lift of the pair* of graphs (T, Y) if $\widetilde{T} \subseteq \widetilde{Y}$ and

i) for every edge $e \in \widetilde{Y} - \widetilde{T}$ either $\alpha(e) \in \widetilde{T}$ or $\omega(e) \in \widetilde{T}$;

ii) p maps $\widetilde{T}$ isomorphically onto T, and p maps $E(\widetilde{Y}) - E(\widetilde{T})$ bijectively onto $E(Y) - E(T)$.

For every vertex $v \in V(Y) = V(T)$ let $\tilde{v}$ denote its preimage in $V(\widetilde{T})$, and for every edge $e \in E(Y)$ let $\tilde{e}$ denote its preimage in $E(\widetilde{Y})$.

Now we can state the two main theorems of this section. They give a duality between a group acting without inversion on a tree and the fundamental group of a graph of groups. The first theorem asserts that for every fundamental group $G = \pi_1(\mathbb{G}, Y, T)$ of a graph of groups there exists a tree X such that

G acts on X without inversion of edges and the graph Y is isomorphic to the factor graph $G \setminus X$. The second one states that every group acting without inversion on a tree is isomorphic to the fundamental group of a graph of groups.

Theorem 2.21 ([Bog08, Theorem 18.2]). *Let $G = \pi_1(\mathbb{G}, Y, T)$ be the fundamental group of a graph of groups $(\mathbb{G}, Y)$ with respect to a maximal subtree T. Then the group G acts without inversion of edges on a tree X such that the factor graph $G \backslash X$ is isomorphic to the graph Y and the stabilizers of the vertices and edges of the tree X are conjugate to the canonical images in G of the groups $G_v, v \in V(X)$, and $\alpha_e(G_e), e \in E(X)$, respectively.*

Moreover, for the projection $p : X \longrightarrow Y$ corresponding to this action, there exists a lift $(\widetilde{T}, \widetilde{Y})$ of the pair (T, Y) such that

i) the stabilizer of any vertex $\tilde{v} \in V(\widetilde{T})$ (any edge $\tilde{e} \in E(\widetilde{Y})$ with initial vertex in $V(\widetilde{T})$) in the group G is equal to the group G_v (respectively to the group $\alpha_e(G_e)$);

ii) if the terminal vertex of an edge $\tilde{e} \in E(\widetilde{Y})$ does not lie in $V(\widetilde{T})$, then the element t_e^{-1} carries this vertex into $V(\widetilde{T})$.

For the other direction we need to construct a graph of groups from an action of a group G on a tree X. Let $Y = G \backslash X$ be a factor graph and $p : X \longrightarrow Y$ the canonical projection. Let T be a maximal subtree of Y and $(\widetilde{T}, \widetilde{Y})$ a lift of the pair (T, Y). We define a graph of groups $(\mathbb{G}, Y)$ as follows: For every vertex and every edge y of Y let G_y be equal to the stabilizer of $\tilde{y}$ under the action of G. For every edge $e \in E(Y) - E(T)$ with $\omega(\tilde{e}) \notin V(\widetilde{T})$ choose an arbitrary element $t_e \in G$ such that $\omega(\tilde{e}) = t_e \cdot \widetilde{\omega(e)}$ (recall that $\widetilde{\omega(e)} \in V(\widetilde{T})$). Set $t_{\bar{e}} = t_e^{-1}$.

For each edge $e \in E(Y)$ define an embedding $\omega_e : G_e \longrightarrow G_{\omega(e)}$ by:

$$\omega_e(g) = \begin{cases} g & \text{if } \omega(\tilde{e}) \in V(\widetilde{T}), \\ t_e^{-1} g t_e & \text{if } \omega(\tilde{e}) \in V(\widetilde{Y}) - V(\widetilde{T}). \end{cases}$$

Then Y together with the vertex and edge groups G_y for every edge or vertex y of Y and the embeddings ω_e for every $e \in E(Y)$ defines a graph of groups.

Theorem 2.22 ([Bog08, Theorem 18.5])**.** *Let G be a group acting without inversion of edges on a tree X. Then there exists a canonical isomorphism from G onto the group $\pi_1(\mathbb{G}, Y, T)$, where the graph of groups $(\mathbb{G}, Y)$ is defined as above. This isomorphism extends the identity isomorphisms* $\mathrm{Stab}(\tilde{v}) \longrightarrow G_v$ *for $v \in V(Y)$ and carries t_e to t_e for $e \in E(Y) - E(T)$.*

Thus, we can 'encode' a group by its action on a tree. To make notation easier, we denote graphs of groups $(\mathbb{G}, Y)$ henceforth by Λ, meaning that Λ is the underlying graph with associated vertex and edge groups and monomorphisms.

For a group G acting without inversion on a tree T we get a corresponding graph of groups Λ by Theorem 2.22 such that G is the fundamental group of Λ. Now we can construct a so called Bass-Serre presentation of this graph of groups using Theorem 2.21.

Definition 2.23 ([PS20, Definition 3.2])**.** Let G be a finitely generated group, and let T be a G-tree. Denote by Λ the corresponding graph of groups and by $p : T \longrightarrow \Lambda$ the quotient map.

A *Bass-Serre presentation* for Λ is a triple $(T^1, T^0, \{t_e\}_{e \in E_1})$ consisting of

i) a subtree T^1 of T which contains exactly one edge of $p^{-1}(e)$ for each edge e of Λ;

ii) a subtree T^0 of T^1 which contains exactly one vertex of $p^{-1}(v)$ for each vertex v of Λ;

iii) for each edge $e \in E_1 := \{e = (v, w) \mid v \in T^0, w \in T^1 - T^0\}$ an element t_e of G such that $t_e^{-1} \cdot v$ lies in T^0.

We call t_e the *Bass-Serre element* or the *stable letter* associated to e.

By the theory we now developed, we can view the action of a group G on a tree T as a *splitting* of G as a graph of groups since there exists an isomorphism between G and the fundamental group of a graph of groups Λ. We call Λ a *one-edge splitting* if Λ has one edge, i.e. if we can write G as an amalgam $G = G_v *_{G_e} G_w$ or as an HNN-extension $G = G_v *_{G_e}$ for $v, w \in V(\Lambda), e \in E(\Lambda)$. If all edge groups of Λ are cyclic, then we call Λ a *cyclic splitting*.

A corollary of Theorems 2.21 and 2.22 is Kurosh's Theorem.

Theorem 2.24 ([Kur34, Untergruppensatz]). *Let G be a group that can be written as $G = *_{i\in I}G_i$ for subgroups G_i of G, and let $H \leq G$ be a subgroup of G. Then H can be written as a free product*

$$H = *_{j\in J}H_j$$

where each H_j is either an infinite cyclic group or conjugated to a subgroup of a G_i.

3 The tree of cylinders

In Section 4 we want to construct a special kind of JSJ tree which we need to formulate the main theorem. This JSJ tree is in fact constructed from a special tree of cylinders, so in this section we introduce cylinders and the tree of cylinders. First we establish the notions for an arbitrary finitely generated group and an arbitrary *admissible* equivalence relation. In the next subsection we restrict to *commensurability* as equivalence relation and to a non-abelian finitely generated free group.

3.1 The basic construction

Let G be a finitely generated group, and let $\mathcal{E}$ be a family of subgroups of G which is closed under conjugation but not closed under taking subgroups (if it is, all trees of cylinders will be trivial). However, $\mathcal{E}$ is mostly invariant under $\mathrm{Aut}(G)$, the group of automorphisms of G, and sandwich closed, which means that if $A, B \in \mathcal{E}$ and H is a subgroup of G such that $A \subseteq H \subseteq B$, then $H \in \mathcal{E}$. An *$\mathcal{E}$-tree* is a G-tree T such that the edge stabilizers of T belong to $\mathcal{E}$. We fix an $\mathcal{E}$-tree T. The exposition in this subsection follows [GL11, Beginning of Section 3 & Section 4].

Definition 3.1 ([GL11, Definition 3.1]). An equivalence relation $\sim$ on $\mathcal{E}$ is called *admissible* if the following axioms hold for $A, B \in \mathcal{E}$:

i) If $A \sim B$ and $g \in G$, then $A^g \sim B^g$.

ii) If $A \subseteq B$, then $A \sim B$.

iii) Let T be an $\mathcal{E}$-tree. If $A \sim B$ and A, B fix vertices $a, b \in V(T)$ respectively, then for every edge e in $[a, b]$ we have $\mathrm{Stab}(e) \sim A \sim B$.

Fix an arbitrary admissible equivalence relation $\sim$ on $\mathcal{E}$. To define a tree of cylinders T_C for every $\mathcal{E}$-tree T, we first define cylinders in T.

Definition 3.2 ([GL11, Definition 4.1])**.** Let T be an $\mathcal{E}$-tree. Define an equivalence relation $\sim_T$ on the set of edges of T as follows for all edges e, e' of T:

$$e \sim_T e' \Leftrightarrow \mathrm{Stab}(e) \sim \mathrm{Stab}(e').$$

A *cylinder* of T is an equivalence class C of edges of T.

Rather than viewing C as an equivalence class, we identify it with the union of its edges and view it as a sub-forest of T. A very important property of cylinders is their connectedness.

Lemma 3.3 ([GL11, Lemma 4.2])**.** *Every cylinder C of T is a subtree.*

Proof. Clearly there are no cycles in C since it is a sub-forest of T and T is a tree. Furthermore, let e, f be edges in C, i.e. $\mathrm{Stab}(e) \sim \mathrm{Stab}(f)$. By axiom iii) the stabilizer of every edge e' in the path from e to f is equivalent to $\mathrm{Stab}(e)$ and $\mathrm{Stab}(f)$, which means that e' lies in C. Thus, C is connected and a subtree of T. $\square$

An easy but important observation is that the intersection of two distinct cylinders consists at most of one vertex: Suppose on the contrary that the intersection of distinct cylinders C_1, C_2 contains vertices $x \neq y$. Since cylinders are connected, the path from x to y has to lie in C_1 as well as in C_2. It contains at least one edge e, which means that e is in the intersection of C_1 and C_2. Thus, for all edges $e_1 \in C_1$ and $e_2 \in C_2$ we get

$$\mathrm{Stab}(e_1) \sim \mathrm{Stab}(e) \sim \mathrm{Stab}(e_2).$$

This contradicts our assumption of C_1 and C_2 being distinct cylinders.

Remark 3.4 ([GL11, p. 990])**.** Let $\mathcal{E}$ be sandwich closed and let T be an $\mathcal{E}$-tree. Then the stabilizer of a vertex v of T which belongs to at least two cylinders is not contained in a group of $\mathcal{E}$. Suppose on the contrary that there exists a group $A \in \mathcal{E}$ such that $\mathrm{Stab}(v) \subseteq A$. Then for every edge e incident to v we have $\mathrm{Stab}(e) \subseteq \mathrm{Stab}(v) \subseteq A$. Since $\mathcal{E}$ is sandwich closed this implies that $\mathrm{Stab}(v) \in \mathcal{E}$, so by axiom ii) we get $\mathrm{Stab}(v) \sim \mathrm{Stab}(e)$ for every edge e incident to v. Thus, all edges incident to v have equivalent stabilizers which contradicts our assumption that v belongs to at least two cylinders.

Now we can define the tree of cylinders for a given $\mathcal{E}$-tree T.

Definition 3.5 ([GL11, Definition 4.3])**.** Let T be an $\mathcal{E}$-tree. The *tree of cylinders* T_C of T is the bipartite tree with vertex set $V(T_C) = V_0(T_C) \sqcup V_1(T_C)$ defined as follows:

i) $V_0(T_C)$ is the set of vertices v of T which either belong to at least two distinct cylinders or are of valence one in T.

ii) $V_1(T_C)$ is the set of cylinders C of T.

iii) There exists an edge $\varepsilon = (v, C)$ between vertices $v \in V_0(T_C)$ and $C \in V_1(T_C)$ if and only if v (viewed as a vertex of T) belongs to C (viewed as a subtree of T).

Note that this definition is slightly different from Definition 4.3 of [GL11] because V. Guirardel and G. Levitt only consider *minimal* G-trees, i.e. G-trees with no proper G-invariant subtree. A minimal G-tree T does not contain vertices of valence one: Suppose on the contrary that T contains a vertex v of valence one. Then removing the orbit of v and the orbit of the unique edge incident to v gives a connected G-invariant tree, a contradiction to T being a minimal G-tree. Thus, if we consider minimal G-trees, the definitions are the same.

Remark 3.6. If $\mathcal{E}$ is sandwich closed, the stabilizer of a vertex v of $V_0(T_c)$, viewed as a vertex of T, belongs to $\mathcal{E}$ if and only if v has valence one in T: Let $\mathrm{Stab}(v)$ be in $\mathcal{E}$. Then by Remark 3.4 it cannot stabilize a vertex which lies in two distinct cylinders of T, thus v must be of valence one. Now let v be a vertex of valence one in T and let e be the unique edge incident to v. Then the stabilizer of v fixes e, thus we have $\mathrm{Stab}(v) = \mathrm{Stab}(e) \in \mathcal{E}$.

Depending on the context it is useful to think of a cylinder of T as a subtree of T, a vertex of T_c, or an equivalence class. Similarly, we sometimes think about a vertex of $V_0(T_c)$ as a vertex of T or a vertex of T_c.

An important property of the tree of cylinders is, as the name suggests, that it is a tree.

Lemma 3.7. *Let T be an $\mathcal{E}$-tree and T_c be its tree of cylinders. Then T_c is a tree.*

Proof. (uses [Gui04, p. 1447]) Clearly, T_c is connected because the cylinders cover T and T is connected. To see that there are no circles in T_c, consider the path $p = v_0, C_0, v_1, \dots, v_{n-1}, C_{n-1}, v_n$ in T_c where $v_i \in V_0(T_c)$ and $C_i \in V_1(T_c)$ and let $\tilde{p} = [v_0, v_1][v_1, v_2] \dots [v_{n-1}, v_n]$ be the corresponding path in T. Without loss of generality we may assume that p is a reduced path, i.e. that it does not backtrack. Then $C_i \cap C_{i+1} = v_{i+1}$ because cylinders have at most one vertex in common, and $\tilde{p}$ does not backtrack. Since T is a tree this means $v_0 \neq v_n$ and p is not a closed path. □

There is a natural action of G on T_c. The action of G on $V_0(T_c)$ is the same as the action of G on the vertices of T. This is well-defined because if v is contained in at least two cylinders of T, i.e. $v \in V_0(T_c)$, then there exist edges e and e' in T incident to v such that $e \nsim_T e'$. Since $\mathrm{Stab}(g \cdot e) = \mathrm{Stab}(e)^g$ for every $g \in G$, we get by axiom i) that $g \cdot e \nsim_T g \cdot e'$, and therefore $g \cdot v$ lies in at least two different cylinders.

To define the action of G on $V_1(T_c)$ we first notice that the action of G on T sends cylinders to cylinders: If e and e' are edges of a cylinder C of T, then again by axiom i) the edges $g \cdot e$ and $g \cdot e'$ are in the same cylinder for every $g \in G$. Now define the action of G on elements $C \in V_1(T_c)$ as $g \cdot C = C'$ for $g \in G$, where C' is the cylinder in T such that in T we have $g \cdot w \in C'$ for all $w \in C$. We define the action of G on the edges of T_c accordingly. Then G acts without inversion of edges on T_c.

Note that because G sends cylinders to cylinders we can now also talk about cylinders of the graph of groups Λ corresponding to a G-tree T, meaning that they are images of cylinders of T.

Lemma 3.8. *If T is a minimal G-tree, then T_c is a minimal G-tree.*

Proof. Let T_c' be a G-invariant subtree of T_c and let T' be the subtree of T consisting of the vertices and cylinders in $V_0(T_c') \cup V_1(T_c')$ viewed as vertices and subtrees of T. Then T' is a G-invariant subtree of T, hence minimality of T implies $T' = T$ and therefore $T_c' = T_c$. □

In particular, the tree of cylinders of a minimal G-tree does not contain vertices of valence one.

The definition of the action of G on T_c gives us a few easy observations:

Remark 3.9 ([GL11, p. 989])**.**

i) The stabilizer of an arbitrary vertex v of T fixes a point in T_c, the vertex $v \in V_0(T_c)$ if v belongs to at least two cylinders, and the vertex $C \in V_1(T_c)$ if C is the only cylinder containing v.

ii) A vertex $v \in V_0(T_c)$ may either be viewed as a vertex of T or a vertex of T_c. Its stabilizer in T_c is the same as in T.

iii) The stabilizer of a vertex $C \in V_1(T_c)$ is the setwise stabilizer of the corresponding cylinder C of T. It does not have to fix a vertex in T, for example if T_c is a point and T is not.

iv) The edge stabilizers of T_c fix a point in T and contain a group in $\mathcal{E}$: If $\varepsilon = (v, C)$ is an edge in T_c and e is an edge of C incident to v, then $\mathrm{Stab}(\varepsilon) \supseteq \mathrm{Stab}(e)$.

Let C be a cylinder of T. Then all edges of C have equivalent stabilizers and we can therefore associate C with an equivalence class $E_C \in \mathcal{E}/\!\sim$. If all edge stabilizers of T_c are in $\mathcal{E}$, we can build the tree of cylinders of T_c and it is exactly T_c itself.

Remark 3.10 ([GL11, Remark 4.6])**.** Let $\varepsilon = (v, C)$ be an edge of T_c, and let e be an edge of C incident to v in T. Then the stabilizer of e is a representative of the equivalence class E_C and is contained in $\mathrm{Stab}(\varepsilon) = \mathrm{Stab}(v) \cap \mathrm{Stab}(C)$. If $\mathrm{Stab}(\varepsilon)$ is in $\mathcal{E}$, then by axiom ii) of admissible relations the stabilizers of e and ε are equivalent, which means that $\mathrm{Stab}(\varepsilon)$ is in E_C.

In particular, if every edge stabilizer of T_c is in $\mathcal{E}$, then for every edge $\varepsilon_i = (v_i, C)$ of T_c incident to C, the stabilizer $\mathrm{Stab}(\varepsilon_i)$ is in E_C and no edge of T_c incident to another vertex of $V_1(T_c)$ is in E_C. Thus, cylinders in T_c are subtrees of diameter 2, we call them *star graphs*, with central vertex belonging to $V_1(T_c)$ and not contained in another cylinder of T_c. This implies that $(T_c)_c = T_c$.

3.2 Commensurability

Let G be a finitely generated group. In the last subsection we developed the theory for certain sets $\mathcal{E}$ of subgroups of G and admissible equivalence relations

$\sim$ on $\mathcal{E}$. Now we restrict to the special case with which we will be working for the rest of this thesis.

Definition 3.11. Let $\mathcal{E}$ be the set of infinite cyclic subgroups of G, and let $\sim$ be the equivalence relation defined for $A, B \in \mathcal{E}$ by

$$A \sim B \Leftrightarrow A \cap B \text{ has finite index in both } A \text{ and } B.$$

We say that A and B are *commensurable*.

Clearly, the set $\mathcal{E}$ is stable under conjugation and sandwich closed. It is not stable under taking subgroups because the trivial group is not in $\mathcal{E}$. By [GL11, Section 3.1], commensurability is an admissible equivalence relation.

Lemma 3.12. *Commensurability is an admissible equivalence relation.*

Proof. Let $A, B \in \mathcal{E}$ be infinite cyclic subgroups of G generated by elements $a, b \in G$ respectively, and let $g \in G$. For axiom i) of admissible relations notice that if $A \cap B = \langle a^i \rangle = \langle b^j \rangle$ for $i, j \in \mathbb{N}$, then $A^g \cap B^g = \langle g a^i g^{-1} \rangle = \langle g b^j g^{-1} \rangle$. Therefore, if the index of $A \cap B$ in A and in B is finite, then it is i and j respectively, and the index of $A^g \cap B^g$ in A^g and in B^g is also i and j, respectively. Thus, if $A \sim B$, then $A^g \sim B^g$.

If $A \subseteq B$, then $A = \langle b^k \rangle$ for $k \in \mathbb{N} - \{0\}$. This means that the index of A in B is finite and axiom ii) holds.

To prove axiom iii) let T be an $\mathcal{E}$-tree, let $A \sim B$, and let A and B fix the vertices v, w of T respectively. Then $A \cap B$ fixes the path between v an w, and for every edge e in the path we have $A \cap B \subseteq \operatorname{Stab}(e)$. The group $A \cap B$ is cyclic as an intersection of two infinite cyclic subgroups, and it is not trivial because A and B are commensurable, thus $A \cap B \in \mathcal{E}$. Since T is an $\mathcal{E}$ tree, the stabilizer of e is in $\mathcal{E}$ as well. Hence, by axiom ii) the groups $A \cap B$ and $\operatorname{Stab}(e)$ are commensurable. Finally, since $A \cap B$ is also commensurable with A and B by axiom ii), we get $\operatorname{Stab}(e) \sim A \cap B \sim A \sim B$. $\square$

For the rest of this thesis we denote by $\sim$ commensurability.

Now we leave the theory developed by Guirardel and Levitt and restrict to non-abelian finitely generated free groups. Fix a non-abelian finitely generated free group $\mathbb{F}$, the set $\mathcal{E}$ of infinite cyclic subgroups of $\mathbb{F}$, commensurability as equivalence relation $\sim$, an $\mathcal{E}$-tree T, and a cylinder C of T. We want to see what cylinders of T look like. First we notice the following:

Lemma 3.13. *Let e be an edge of C, and let $g \in \mathbb{F}$ such that $g \cdot e \in C$. Then g commutes with the generator of* Stab(e)*.*

Proof. The tree T is an $\mathcal{E}$-tree and therefore there exists a non-trivial element $h \in \mathbb{F}$ which generates the stabilizer of e. The stabilizer $\mathrm{Stab}(g \cdot e) = \mathrm{Stab}(e)^g$ is generated by ghg^{-1}. Since e and $g \cdot e$ lie in the same cylinder, their stabilizers are commensurable, which means that the index of $\langle ghg^{-1} \rangle \cap \langle h \rangle$ in $\langle ghg^{-1} \rangle$ and in $\langle h \rangle$ is finite. Thus, the intersection $\langle ghg^{-1} \rangle \cap \langle h \rangle$ is not trivial and there exist $i, j \in \mathbb{N} - \{0\}$ such that $gh^ig^{-1} = h^j$. Since $\mathbb{F}$ is a free group, this implies $i = j$ and that g and h commute. □

All edge stabilizers of C lie in the same maximal cyclic subgroup of $\mathbb{F}$.

Lemma 3.14. *Every stabilizer of an edge of C is contained in the same maximal cyclic subgroup Z_C of $\mathbb{F}$. Furthermore, Z_C is the centralizer of all edge stablizers of C.*

Proof. Let e be an edge of C, let $h \in \mathbb{F}$ be the generator of Stab(e), and let $g \in \mathbb{F}$ be an element of the setwise stabilizer of C. Then g and h commute by Lemma 3.13, hence there exists an element $z \in \mathbb{F}$ as well as $m, n \in \mathbb{N}$ such that

$$z^m = g \text{ and } z^n = h.$$

We may choose z to have no roots in $\mathbb{F}$, i.e. there does not exist an $x \in \mathbb{F}$ such that $x^k = z$ for $k \in \mathbb{N}$. Then z generates a maximal cyclic subgroup Z_C of $\mathbb{F}$ which contains Stab(e).

To see that every edge stabilizer of C is contained in Z_C, let e' be another edge of C whose stabilizer is generated by $h' \in \mathbb{F}$. By Lemma 3.13 the elements g and h' commute and since $g = z^m$ and z has no roots in $\mathbb{F}$, there exists an $n' \in \mathbb{N}$ such that $h' = z^{n'}$. Thus, the stabilizer of e' is also contained in Z_C.

Clearly, Z_C is by definition the centralizer of every edge stabilizer of C. □

The maximal cylic subgroup we constructed in the last lemma is exactly the setwise stabilizer of C.

Lemma 3.15. *The maximal cyclic subgroup Z_C is the setwise stabilizer of C.*

Proof. Let Z_C be generated by $z \in \mathbb{F}$. First let $g \in \mathbb{F}$ be an element of the stabilizer of C. Then analogous to the proof of Lemma 3.14 the element g may be written as $g = z^m$ for $m \in \mathbb{N}$, and thus $g \in Z_C$.

To see that every element of Z_C is an element of the setwise stabilizer of C, consider $z^n \in Z_C$ for $n \in \mathbb{N}$, let e be an arbitrary edge of C, and let $\mathrm{Stab}(e)$ be generated by $h \in \mathbb{F}$. Then $\langle z^n h z^{-n} \rangle$ is the stabilizer of the edge $z^n \cdot e$. Since Z_C is the centralizer of $\langle h \rangle$ by Lemma 3.14, we have $\langle z^n h z^{-n} \rangle = \langle h \rangle$. Clearly, $\langle h \rangle$ is commensurable to itself, hence e and $z^n \cdot e$ lie in the same cylinder and z^n is in the setwise stabilizer of C. □

This implies that the stabilizers of all vertices $C \in V_1(T_c)$ are infinite cyclic. We can also express the stabilizer of C in terms of commensurability, see [PS20, Behind Definition 3.3].

Lemma 3.16. *The setwise stabilizer of C is the commensurator of the stabilizer of any one of its edges e, that is, the set of elements $g \in \mathbb{F}$ such that* $\mathrm{Stab}(e)^g \sim \mathrm{Stab}(e)$.

Proof. First let $g \in \mathbb{F}$ be an element of the setwise stabilizer of C, and let e be an arbitrary edge of C. Then the edge $g \cdot e$ lies in the cylinder C and we have $\mathrm{Stab}(e) \sim \mathrm{Stab}(g \cdot e) = \mathrm{Stab}(e)^g$.

Now let e be an arbitrary edge of C, and let $g \in \mathbb{F}$ be an element of the commensurator of $\mathrm{Stab}(e)$, i.e. an element of $\mathbb{F}$ such that $\mathrm{Stab}(e)^g \sim \mathrm{Stab}(e)$. Then the edges e and $g \cdot e$ lie in the same cylinder and g commutes with all elements of $\mathrm{Stab}(e)$ by Lemma 3.13. Thus, by Lemma 3.14 it lies in Z_C which is the stabilizer of C by Lemma 3.15. □

Let T be an $\mathcal{E}$-tree, i.e. a tree with infinite cyclic edge stabilizers. In general, the corresponding tree of cylinders T_c does not need to have infinite cyclic edge stabilizers. However, for example in the case of torsion-free hyperbolic groups, and thus in particular in the case of free groups, the commensurator of an infinite cyclic group is itself infinite cyclic, see [PS20, p. 6]. Thus, the stabilizer of a cylinder C of T, and therefore the stabilizer of a vertex C of $V_1(T_c)$, is infinite cyclic by Lemma 3.16. The stabilizer $\mathrm{Stab}(\varepsilon) = \mathrm{Stab}(v) \cap \mathrm{Stab}(C)$ of an edge $\varepsilon = (v, C)$ in T_c incident to C is cyclic as a subgroup of the stabilizer of C. It cannot be the trivial group because it contains a group in $\mathcal{E}$ by the last item of Remark 3.9. Therefore, all edge stabilizers of T_c are infinite cyclic, i.e. elements of $\mathcal{E}$, and we can construct the tree of cylinders of T_c. By Remark 3.10 we have $(T_c)_c = T_c$.

Before we describe the cylinders of T, remember that the stabilizer of a vertex v in $V_0(T_c)$ is infinite cyclic if and only if v is of valence one in T by

Remark 3.6. Now we connect this fact and Lemma 3.15 with Remark 3.9 and include Remark 3.10 to describe the tree of cylinders T_c of an $\mathcal{E}$-tree T.

Remark 3.17. The tree of cylinders T_c is a bipartite graph where

- vertices $C \in V_1(T_c)$ are cyclically stabilized, and they are each the centre of a cylinder of T_c;
- vertices $v \in V_0(T_c)$ are cyclically stabilized if and only if v is of valence one in T_c;
- edges $\varepsilon = (v, C)$ are cyclically stabilized, join vertices $v \in V_0(T_c)$ to vertices $C \in V_1(T_c)$, and are contained in the cylinder of T_c whose centre is C.

In particular, cylinders in T_c are star graphs with a central infinite cyclically stabilized vertex.

Remember that if T is minimal, and thus T_c is minimal by Lemma 3.8, vertices of valence one do not exist in T_c. Thus, the tree of cylinders T_c is a bipartite graph with infinite cyclically stabilized vertices on one side and not infinite cyclically stabilized vertices on the other side.

4 JSJ decompositions and the modular group

JSJ decompositions play an important role in Perin's and Sklinos's characterization of forking independence in non-abelian finitely generated free groups. Cyclic JSJ decompositions are a generalization of Grushko decompositions. A Grushko decomposition of a finitely generated group G relative to a set of parameters A is a way to see all splittings of G as a free product where A is contained in one of the free factors. A cyclic JSJ decomposition of G relative to A is a way to see all the splittings of G as an amalgamated product or an HNN extension over a cyclic subgroup where A is contained in one of the factors. To prove independence, we need a special kind of cyclic JSJ decomposition relative to a set of parameters. Indeed, this JSJ decomposition corresponds to a tree which is a slightly modified tree of cylinders.

Furthermore, we consider the modular group which consists of some automorphisms that can be read locally from a splitting of a group. These automorphisms are also useful for proving independence.

4.1 JSJ trees

First we give some definitions needed to define cyclic JSJ trees, taken from [PS20, Section 3]. Let G be a finitely generated group. A G-tree whose edge stabilizers are cyclic (possibly finite or trivial) is called a *cyclic* G-tree.

Remember that a G-tree is called *minimal* if there exists no proper G-invariant subtree. In this thesis, G-trees are usually **not** minimal, unless otherwise mentioned. However, if H is a finitely generated subgroup of G, then there exists a unique subtree of T preserved by H on which the action of H is minimal. It is denoted by T_H^{min}. The image of T_H^{min} under the quotient map $T \longrightarrow \Lambda$ is called the *minimal subgraph of H in* Λ and denoted by Λ_H^{min}. For a finitely generated subgroup H and a tuple $\bar{b}$ of G we often consider the minimal subgraph of $\langle H, \bar{b} \rangle$, which we denote by $T_{H\bar{b}}^{min}$ for better readability. Similarly, we denote its minimal subgraph in Λ by $\Lambda_{H\bar{b}}^{min}$.

Let A be a subset of G. Then a *(G, A)-tree* is a G-tree in which A fixes a point. The group G is *freely indecomposable relative to* A if G does not act non-trivially on a (G, A)-tree with trivial edge stabilizers. Equivalently (unless $G = \mathbb{Z}$ and A is trivial), one cannot write $G = G_1 * G_2$ with G_1, G_2 non-trivial and A is contained in a conjugate of G_1 or G_2.

An element g or a subgroup H of G is called *elliptic* if it fixes a point in T. Considering the corresponding graph of groups Λ, this is equivalent to g or H being contained in a conjugate of a vertex group of Λ. A minimal cyclic (G, A)-tree is *universally elliptic* if its edge stabilizers are elliptic in every minimal cyclic (G, A)-tree.

A map between trees is called *simplicial*, if it sends vertices to vertices and edges to edges or vertices. A surjective simplicial map $p : T_1 \longrightarrow T_2$ between trees T_1 and T_2 is called a *collapse map* if it is obtained by collapsing some orbits of edges to points. Then we say that T_1 *refines* T_2.

We always assume that maps between G-trees are G-equivariant. A G-tree T_1 *dominates* a G-tree T_2 if there is a (not necessarily simplicial) surjective map $d : T_1 \longrightarrow T_2$. The map d is called *domination map*. It sends edges to edge paths in T_2. Equivalently, T_1 dominates T_2 if every subgroup of G which is elliptic in T_1 is also elliptic in T_2. In particular, every refinement of T_1 dominates T_1.

The *deformation space* of a minimal cyclic (G, A)-tree T is the set of all minimal cyclic (G, A)-trees T' such that T dominates T' and T' dominates T. By [GL17, Lemma 7.3(3)] the tree of cylinders of T only depends on the deformation space of T, i.e. for two non-trivial minimal cyclic (G, A)-trees which belong to the same deformation space, there exists a canonical G-equivariant isomorphism between their trees of cylinders, see [GL11, Corollary 4.10]. Hence, we often talk about 'the' tree of cylinders of a deformation space.

Definition 4.1 ([PS20, p. 7])**.** A *cyclic JSJ tree for G relative to A* is a minimal cyclic (G, A)-tree T such that

- T is universally elliptic and
- T dominates any other minimal universally elliptic cyclic (G, A)-tree.

We call the associated graph of groups of a JSJ-tree a *JSJ decomposition.*

The second condition is a maximality condition: the vertex stabilizers are as small as necessary to fit into any decomposition.

All these JSJ-trees belong to a common deformation space by [GL17, Definition 2.12], which we denote by $JSJ_A(G)$. If G is finitely presented and A is finitely generated, then the JSJ deformation space relative to A always exists by [GL17, Corollary 2.21]. It is unique by [GL17, Definition 2.12]. Note that this implies for a finitely generated free group $\mathbb{F}$ and a finitely generated subgroup A of $\mathbb{F}$ that the JSJ deformation space $JSJ_A(\mathbb{F})$ relative to A always exists.

Definition 4.2 ([PS20, p. 7])**.** Let T be a JSJ tree for G relative to A, and let H be the stabilizer of a vertex v in T (or a vertex group in the associated graph of groups). Then H is *rigid* if it is elliptic in any cyclic (G, A)-tree and *flexible* if not. We also say that v is rigid or flexible.

If G is a torsion-free hyperbolic group and A is a finitely generated subgroup of G such that G is freely indecomposable with respect to A, the flexible vertices of a cyclic JSJ tree for G relative to A are *surface-type* vertices by [GL17, Theorem 6.6]. This means that their stabilizers are fundamental groups of hyperbolic surfaces with boundary, any adjacent edge group is contained in a maximal boundary subgroup, and any maximal boundary subgroup contains either exactly one adjacent edge group or exactly one conjugate of A. For more details see [PS16, p. 1993].

4.2 Normalized JSJ decompositions

We want to define a special kind of JSJ decomposition which we need to formulate the main result. The exposition in this subsection closely follows [PS20, Sections 3.2].

Guirardel and Levitt proved in [GL11, Theorem 2] that if G is a torsion free hyperbolic group freely indecomposable with respect to a finitely generated subgroup A, the tree of cylinders T_c of the cyclic JSJ deformation $JSJ_A(G)$ is itself a JSJ tree. Note that T_c is a minimal G-tree by Lemma 3.8 because all trees in $JSJ_A(G)$ are minimal G-trees. Moreover, T_c is *2-acylindrical* by [GL11, Theorem 2], i.e. every non-trivial element of G fixes at most a segment of length 2. If we consider a non-abelian finitely generated free group $\mathbb{F}$, which we do in this subsection, we even know that T_c is *strongly* 2-acylindrical.

Lemma 4.3. *Let $\mathbb{F}$ be a non-abelian finitely generated free group which is freely indecomposable with respect to a finitely generated subgroup A. Then the tree of cylinder T_c of the cyclic JSJ deformation space $JSJ_A(\mathbb{F})$ is even strongly 2-acylindrical, i.e. if a non-trivial elements stabilizes two distinct edges, then they are incident to a common infinite cyclically stabilized vertex.*

Proof. Let $1 \neq g \in \mathbb{F}$ be an element which stabilizes two distinct edges e_1, e_2 of T_c. Since T_c is 2-acylindrical, they are incident to a common vertex v of T_c and g stabilizes only the path u, e_1, v, e_2, w in T_c where u, w are the other vertices incident to e_1, e_2 respectively. By Remark 3.17 the tree T_c is a bipartite graph with infinite cyclically stabilized vertices on one side and not infinite cyclically stabilized vertices on the other side, since T_c is a minimal $\mathbb{F}$-tree. Thus, either v is infinite cyclically stabilized or both u and w are infinite cyclically stabilized.

Suppose that u and w have infinite cyclic stabilizers, and let $\mathrm{Stab}(e_1)$ and $\mathrm{Stab}(e_2)$ be generated by elements h_1 and h_2 of $\mathbb{F}$ respectively. The element g is contained in $\mathrm{Stab}(e_1) \cap \mathrm{Stab}(e_2)$, thus there exist $i, j \in \mathbb{N} - \{0\}$ such that $h_1^i = g = h_2^j$. In particular, the group $\langle g \rangle$ is a subgroup of $\mathrm{Stab}(e_1)$ and $\mathrm{Stab}(e_2)$ of index i, j respectively. Thus, $\langle g \rangle \cap \mathrm{Stab}(e_k) = \langle g \rangle$ has finite index in both $\langle g \rangle$ and $\mathrm{Stab}(e_k)$ for $k = 1, 2$, and we have

$$\mathrm{Stab}(e_1) \sim \langle g \rangle \sim \mathrm{Stab}(e_2). \tag{1}$$

The vertices u and w are centres of distinct cylinders of T_c by Remark 3.17, and the edges e_1 and e_2 lie in distinct cylinders of T_c, a contradiction to Equation 1. Hence, the vertex v has an infinite cyclic stabilizer. $\square$

Note that this means an element of $\mathbb{F}$ can at most stabilize a cylinder of T_c.

Lemma 4.4. *Let $\mathbb{F}$ be a non-abelian finitely generated free group which is freely indecomposable with respect to a non-trivial finitely generated subgroup A. Let T_c be the tree of cylinders of $JSJ_A(\mathbb{F})$. If A is cyclic, then it either fixes exactly one vertex or a cylinder of T_c. If A is not cyclic, then it fixes exactly one vertex of T_c.*

Proof. The tree T_c is an $(\mathbb{F}, A)$-tree, so A fixes at leat one vertex of T_c. Let A be infinite cyclic and assume that A fixes two distinct vertices v and w of T_c. Then it also fixes the path between v and w, which can consist of at most two edges by Lemma 4.3 since A is cyclic. If it consists of only one edge, then clearly this edge is contained in a cylinder C of T_c. If the path consists of two edges, then the middle vertex is cyclically stabilized because T_c is strongly 2-acylindrical, and v and w lie in the same cylinder C of T_c by Remark 3.17. Let e be a edge of $[v, w]$. Then A is a subgroup of $\mathrm{Stab}(e)$ and contained in the centralizer of $\mathrm{Stab}(e)$ by Lemma 3.14, which is the stabilizer of C by Lemma 3.15. Thus, A fixes the whole cylinder C of T_c setwise.

Now assume that A is not cyclic. Suppose that it fixes two distinct vertices v, w of T_c, which means it also fixes the path $[v, w]$ in T_c. If e is an edge in $[v, w]$, then A is a subgroup of $\mathrm{Stab}(e)$. All edge stabilizers of T_c are infinite cyclic, thus A is cyclic as a subgroup of an infinite cyclic group, a contradiction. Hence, A fixes a unique vertex of T_c. □

Now we slightly modify the tree of cylinders of the cyclic JSJ deformation space $JSJ_A(\mathbb{F})$ to define the pointed cyclic JSJ tree for $\mathbb{F}$ relative to A.

Definition 4.5 ([PS20, Definition 3.6])**.** Let $\mathbb{F}$ be a finitely generated free group which is freely indecomposable relative to a non-trivial finitely generated subgroup A, and let T_c be the tree of cylinders of $JSJ_A(\mathbb{F})$. The *pointed cyclic JSJ tree of cylinders* T for $\mathbb{F}$ relative to A with base point v_A is defined as follows:

i) If A is not infinite cyclic, set $T = T_c$ and let v_A be the unique vertex of T_c fixed by A.

ii) If A is infinite cyclic, let w be either the unique vertex fixed by A, or the centre of the cylinder fixed by A. We define T as the tree T_c where we added one orbit of vertices $\mathbb{F} \cdot v_A$ and one orbit of edges $\mathbb{F} \cdot e$ with $e = (v_A, w)$. Furthermore, we set $\mathrm{Stab}(v_A) = \mathrm{Stab}(e) = A$.

Mostly we just call T the pointed cyclic JSJ tree. We can define T in this way because of Lemma 4.4. Clearly, all edges of a pointed cyclic JSJ tree have infinite cyclic stabilizers.

Note that cylinders in the pointed cyclic JSJ tree of cylinders T are still star graphs: Clearly this is the case, if A is not infinite cyclic. If A is infinite cyclic and fixes a cylinder C of T_c, then A is contained in every edge stabilizer of C and therefore commensurable to every edge stabilizer of C. Hence, the edge e defined above, which is fixed by A, is also an edge of the cylinder C in T. If A is infinite cyclic and fixes a unique vertex of T_c, then the edge e defined above forms its own cylinder. Otherwise A, which is the stabilizer of e, would be commensurable to the edge stabilizers of a cylinder C of T_c and thus stabilize C.

Remember that the tree of cylinders of $JSJ_A(\mathbb{F})$ is a minimal $\mathbb{F}$-tree, thus it does not have vertices of valence one, and in the pointed cyclic JSJ tree there is at most one orbit of valence one vertices, namely that of the base point v_A.

When we consider the minimal subtree of a subgroup H containing A in a pointed cyclic JSJ tree relative to A, we require that v_A belongs to T_H^{min}.

Definition 4.6 ([PS20, Definition 3.7])**.** Let T be a pointed cyclic JSJ tree for $\mathbb{F}$ relative to A, and let T' be the subtree of T which lies in $JSJ_A(\mathbb{F})$. A vertex of T is called *rigid* if it is a translate of the base vertex or if it belongs to T' and is rigid in T'. It is called *flexible* or of *surface-type* if it belongs to T' and is flexible or of surface-type in T'. It is called a *Z-type vertex* if it is distinct from translates of the base vertex and its stabilizer is infinite cyclic.

We also call vertices of the associated graph of groups rigid, flexible, surface-type and Z-type, meaning that they are images of rigid, flexible, surface-type or Z-type vertices respectively.

Note that this definition is slightly different from Definition 3.7 of [PS20]. Here we also call translates of the base vertex rigid, not only the base vertex itself. Furthermore, translates of the base vertex are not of Z-type in this definition, which makes a difference if A is infinite cyclic.

Also note that every flexible vertex of the pointed cyclic JSJ tree T for $\mathbb{F}$ relative to A is of surface-type. As we saw before, this holds for the flexible vertices of the subtree T' of T lying in $JSJ_A(\mathbb{F})$ because $\mathbb{F}$ is a finitely generated free group which is freely indecomposable relative to the finitely generated subgroup A of $\mathbb{F}$. The only vertices of T not lying in T' are translates of the base vertex in the case where A is infinite cyclic, and these vertices are all rigid.

Moreover, every vertex of T is either rigid or flexible, not both. All Z-type vertices as well as the base vertex are rigid.

Remark 4.7. In the pointed cyclic JSJ tree T for $\mathbb{F}$ relative to A all Z-type vertices are only adjacent to non-Z-type vertices because this is the case in the subtree T' which is the tree of cylinders of $JSJ_A(\mathbb{F})$ and we added at most non-Z-type vertices, i.e. translates of the base vertex v_A, to T'. Furthermore, all non-Z-type vertices are only adjacent to Z-type vertices and translates of v_A because they are only adjacent to Z-type vertices in T' and we added at most translates of v_A to T'.

Finally we can define the normalized JSJ decomposition needed for the main result.

Definition 4.8 ([PS20, Definition 3.9])**.** Let $\mathbb{F}$ be a non-abelian finitely generated free group, let A be a non-trivial finitely generated subgroup of $\mathbb{F}$, and let $\mathbb{F}_A$ be the smallest free factor of $\mathbb{F}$ containing A. A cyclic JSJ tree T for $\mathbb{F}$ relative to A with base point v_A is called a *normalized pointed cyclic JSJ tree* for $\mathbb{F}$ relative to A if

i) the minimal subtree $T^{min}_{\mathbb{F}_A}$ of $\mathbb{F}_A$ in T is the pointed cyclic JSJ tree of cylinders for $\mathbb{F}_A$ relative to A,

ii) trivially stabilized edges join a translate of the base vertex v_A to a vertex which lies in a translate of $T^{min}_{\mathbb{F}_A}$.

The associated graph of groups Λ of T is called a *normalized pointed cyclic JSJ decomposition* for $\mathbb{F}$ relative to A.

We can build the pointed cyclic JSJ tree for $\mathbb{F}_A$ relative to A because $\mathbb{F}_A$ is finitely generated by [Per19, Lemma 1.2] as it is a free factor of $\mathbb{F}$.

If we consider a normalized pointed cyclic JSJ tree T for $\mathbb{F}$ relative to A and its associated graph of groups Λ, we will always refer to the base point of T and its image in Λ as v_A.

4.3 Statement of the main result

We still need some definitions to state the main theorem. This subsection follows Section 3.3 of [PS20].

Definition 4.9 ([PS20, Definition 3.10])**.** Let $\mathbb{F}$ be a non-abelian finitely generated free group which is freely indecomposable with respect to a finitely generated subgroup A of $\mathbb{F}$. Let T be the pointed cyclic JSJ tree for $\mathbb{F}$ relative to A, and let T' be the subtree of T which lies in $JSJ_A(\mathbb{F})$, i.e. the minimal subtree of $\mathbb{F}$ in T.

Let v be a rigid vertex of T. If v is contained in T', then an *envelope of v in T* is a union of edges incident to v which meets at most one orbit of edges of each cylinder of T', such that none of these edges joins v to a surface type vertex. If $v \notin T'$, i.e. v is a translate of the base vertex v_A and has valence one in T, then an envelope of v consists at most of the unique edge incident to v. If this edge joins v to a surface type vertex, then an envelope of v in T only consists of v itself.

Let $p : T \longrightarrow \Lambda$ be the quotient map to the associated graph of goups Λ. The image of an envelope of v in Λ is called an envelope of $p(v)$ in Λ. It is a union of edges incident to $p(v)$.

Remark 4.10. This definition is slightly different from Definition 3.10 of [PS20], which means that the statement of the main theorem is slightly changed. In the case where v is a rigid vertex which is also of Z-type or it is a translate of v_A, we excluded edges incident to a surface type vertex. If we do not exclude them, there occurs a problem in the proof of Proposition 5.1. If v is a rigid non-Z-type vertex, all vertices of T' adjacent to v are of Z-type by Remark 3.17. Thus, this definition implies that an envelope of a rigid vertex cannot contain a surface type vertex.

Note that the definition implies that for a rigid vertex v of T an envelope of $p(v)$ is a star graph, i.e. no two edges in it have both endpoints in common: If $v \notin T'$, this is trivial because $p(v)$ has valence one in Λ. Now let v be a rigid vertex of T'. If v is of Z-type, then it is the centre of a cylinder of T' by Remark 3.17, thus all edges incident to v in T' belong to the same cylinder. An envelope of v in T therefore consists of edges incident to v which lie in the same orbit. Hence, an envelope of $p(v)$ in Λ only consists of one edge, which is a star graph. If v is not of Z-type, we have to show that for two distinct edges

$e = (p(v), w)$ and $f = (p(v), w)$ of Λ incident to $p(v)$ and the vertex w of Λ, there exist lifts in T' which lie in the same cylinder. First remember that T' is a bipartite graph connecting Z-type and non-Z-type vertices by Remark 3.17. Since v is not of Z-type, lifts of the vertex w are of Z-type. There exist lifts e' and f' of the edges e and f respectively in T' which are incident to a common lift w' of w. The stabilizers of e' and f' are infinite cyclic and contained in the infinite cyclic stabilizer $\mathrm{Stab}(w')$. Hence, they are commensurable which means that e' and f' lie in the same cylinder.

Figure 1 is an example of an envelope of a rigid vertex v of T and its image in Λ.

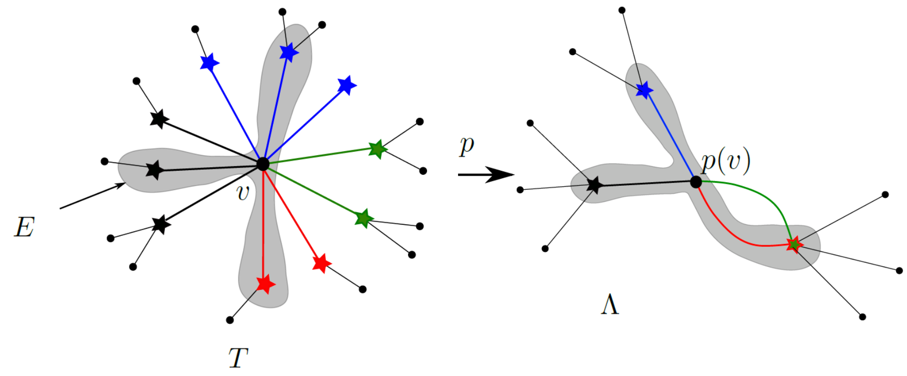


Figure 1: E is an envelope of v. Each colour of edge incident to v denotes a different orbit of edges. Note that E is not an envelope anymore if we add a green edge to E. (Dot vertices represent rigid non-Z-type vertices, stars represent Z-type vertices and the edges have an associated infinite cyclic edge group.) (SOURCE: [PS20, p. 10])

Since $\mathbb{F}$ will not always be freely indecomposable with respect to A, we generalize this definition.

Definition 4.11 ([PS20, Definition 3.11])**.** Let $\mathbb{F}$ be a finitely generated free group, let A be a finitely generated subgroup of $\mathbb{F}$, and let $\mathbb{F}_A$ be the smallest free factor of $\mathbb{F}$ containing A. Let T be a normalized pointed cyclic JSJ tree for $\mathbb{F}$ relative to A.

Let v be a rigid vertex of T. A subtree E of T is an envelope of v in T if $v \in E$ and there exists an element $g \in \mathbb{F}$ such that $g \cdot E$ is an envelope of $g \cdot v$ in $T_{\mathbb{F}_A}^{min}$.

Note that an envelope of a rigid vertex does not contain trivially stabilized edges because the tree $T^{min}_{\mathbb{F}_A}$ is a pointed cyclic JSJ tree and thus has infinite cyclic edge stabilizers.

Finally, we define sandwich terms and blocks of the minimal subgraph of a subgroup. These are the last definitions we need to formulate the main result.

Definition 4.12 ([PS20, Definition 3.12])**.** Let $\mathbb{F}$ be a finitely generated free group, let A be a finitely generated subgroup of $\mathbb{F}$, and let $\mathbb{F}_A$ be the smallest free factor of $\mathbb{F}$ containing A. Let T be a normalized pointed cyclic JSJ tree for $\mathbb{F}$ relative to A with base vertex v_A.

An element β of $\mathbb{F}$ is called a *sandwich term* if the path $[v_A, \beta \cdot v_A]$ does not contain any translates of v_A other than its endpoint and can be subdivided into three subpaths $[v_A, u] \cup [u, v] \cup [v, \beta \cdot v_A]$ where $[v_A, u]$ and $[v, \beta \cdot v_A]$ are either empty or consist of a single trivially stabilized edge and $[u, v]$ lies entirely in a translate of $T^{min}_{\mathbb{F}_A}$.

The image of the path $[u, v]$ in $\Lambda^{min}_{\mathbb{F}_A}$ is called the *imprint* of the sandwich term β in $\Lambda^{min}_{\mathbb{F}_A}$ and denoted by $\mathrm{imp}(\beta)$. The trivially stabilized edges in $[v_A, \beta \cdot v_A]$ and their images in Λ are called the trivially stabilized edges of the sandwich term β.

Definition 4.13 ([PS20, Definition 3.13])**.** Let $\mathbb{F}$ be a finitely generated free group, and let A be a finitely generated subgroup of $\mathbb{F}$. Let T be a normalized pointed cyclic JSJ tree for $\mathbb{F}$ relative to A and let Λ be the associated graph of groups.

For every subgroup H of $\mathbb{F}$ we define $\mathcal{S}(H)$ to be the set of minimal subgraphs of sandwich terms β such that some translate of $[v_A, \beta \cdot v_A]$ is contained in T^{min}_H. It is a collection of subgraphs of Λ^{min}_H.

We declare two subgraphs of Λ to be equivalent if their intersection contains either a vertex which is neither of Z-type nor the base vertex v_A, or an edge, and we consider the equivalence relation generated by this. A *block* of Λ^{min}_H is the union of all the subgraphs of $\mathcal{S}(H)$ in a given equivalence class.

Note that the intersection of two distinct blocks of Λ consists at most of the base vertex v_A and a disjoint union of Z-type vertices.

Before we formulate the main result, we give an example to visualize the definition of blocks.

Example 4.14 ([PS20, Example 3.14])**.** Let $\mathbb{F}$ be a finitely generated free group, let A be a finitely generated subgroup of $\mathbb{F}$ and let Λ be a normalized pointed cyclic JSJ decomposition for $\mathbb{F}$ relative to A as in Figure 2. The vertex

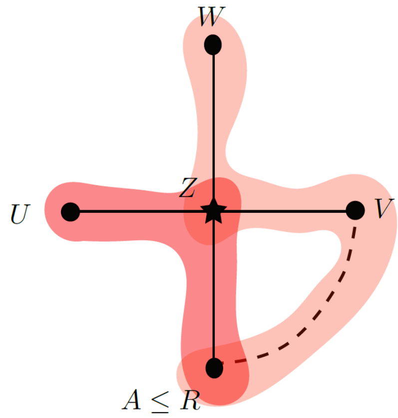


Figure 2: The minimal subgraph of $(u, tvwt^{-1})$ has two blocks, the dark red one is the minimal subgraph of the sandwich term u, and the light red one is the minimal subgraph of the sandwich term $tvwt^{-1}$. (Dot vertices represent rigid non-Z-type vertices, stars represent Z-type vertices, dashed edges have an associated trivial edge group and full ones have an associated infinite cyclic edge group.) (SOURCE: [PS20, p. 10])

groups R, U, V, and W are rigid with $A \leq R$, and the central vertex is of Z-type, i.e. the vertex group Z is infinite cyclic. Let t be a Bass-Serre element associated to the trivially stabilized edge. Let $u \in U, v \in V$ and $w \in W$ be elements of the rigid vertex groups which do not lie in the adjacent edge groups. They exist because the edge group is infinite cyclic and the vertex groups are not infinite cyclic. Let $\bar{c}$ be the tuple $(u, tvwt^{-1})$. We want to see what the blocks of the minimal subgraph of groups $\Lambda^{min}_{A\bar{c}}$ of $\langle A, \bar{c}\rangle$ in Λ looks like.

Let T be the tree corresponding to Λ with base vertex v_A. Consider the element u and the path $[v_A, u \cdot v_A]$ in T. It consists of four edges, the first and the last being lifts of the edge between the vertices with vertex groups R and Z and the other two being lifts of the edge between the vertices with vertex groups Z and U. This path does not contain any translate of v_A other than its ending, and lies entirely in $T^{min}_{\mathbb{F}_A}$. Thus, the element u is a sandwich term with minimal subgraph consisting of the vertices with vertex groups R, Z and U and the edges between them.

Consider the element $tvwt^{-1} = (tvt^{-1})(twt^{-1})$ and the path $[v_A, tvwt^{-1} \cdot v_A]$ in T. It consists of six edges, the first and the last one are trivially stabilized and the middle four edges lie in the translate of $T^{min}_{\mathbb{F}_A}$ by t. The second and fifth are lifts of the edge between vertices with vertex groups V and Z and the other two are lifts of the edge between vertices with vertex groups W and Z. Thus, the path can be subdivided into three subpaths of the desired form, and $tvwt^{-1}$ is a sandwich term. Its minimal subgraph in Λ consists of the vertices with vertex groups R, V, Z and W as well as the trivially stabilized edge together with the edge between vertices with vertex groups V and Z and the edge between vertices with vertex groups Z and W.

Therefore, $\Lambda^{min}_{A\bar{c}}$ is the whole graph Λ but consists of two blocks.

Finally we can state the main result:

Theorem 4.15 ([PS20, Theorem 3.15])**.** *Let $\mathbb{F}$ be a non-abelian finitely generated free group, let $A \subseteq \mathbb{F}$ be a set of parameters, and let $\bar{b}, \bar{c}$ be tuples from $\mathbb{F}$.*

Then $\bar{b}$ is independent from $\bar{c}$ over A if and only if there exists a normalized pointed cyclic JSJ decomposition Λ for $\mathbb{F}$ relative to A in which the intersection of any two blocks of the minimal subgraphs $\Lambda^{min}_{A\bar{b}}$ and $\Lambda^{min}_{A\bar{c}}$ of $\langle A, \bar{b}\rangle$ and $\langle A, \bar{c}\rangle$ respectively is contained in a disjoint union of envelopes of rigid vertices.

By Remark 8.3 of [PS16] we may assume that A is a finitely generated subgroup of $\mathbb{F}$.

We give an example to see why the blocks are important.

Example 4.16 ([PS20, Example 3.16])**.** Let Λ and $\bar{c}$ be as in Example 4.14, and let b be an element of W which is not contained in the incident edge group. Then b is a sandwich term and the minimal subgraph Λ^{min}_{Ab} of $\langle A, b\rangle$ consists of the edges between the vertices with vertex groups R, Z and W. Clearly, Λ^{min}_{Ab} consists of only one block.

Note that the minimal subgraphs of $\Lambda^{min}_{A\bar{c}}$ and Λ^{min}_{Ab} intersect in more than a disjoint union of envelopes of rigid vertices. But the intersection of Λ^{min}_{Ab} with the dark red block, which is the minimal subgraph of u, consists of the edge between the vertices with vertex groups R and Z, and the intersection of Λ^{min}_{Ab} with the light red block, which is the minimal subgraph of $tvwt^{-1}$, consists of the edge between the vertices with vertex groups Z and W. Both of these edges join a rigid vertex to a Z-type vertex and are thus envelopes of rigid vertices. By Theorem 4.15, the element b is independent from $\bar{c}$ over A. We will see a proof of this in Example 5.2.

4.4 Two special cases of the main result

Perin and Sklinos first described forking independence for two families of parameter sets in [PS16]. The first one contains all parameter sets which are free factors of $\mathbb{F}$ and the second one contains all parameter sets which are not contained in any proper free factor of $\mathbb{F}$. In this section we show that the two main results of [PS16] are indeed special cases of the main theorem of this thesis. The ideas for this can be found in the introduction of [PS20].

The first main result deals with parameter sets which are free factors of $\mathbb{F}$.

Theorem 4.17 ([PS16, Theorem 1])**.** *Let $\bar{b}, \bar{c}$ be tuples of elements in a non-abelian finitely generated free group $\mathbb{F}$, and let A be a free factor of $\mathbb{F}$. Then $\bar{b}$ and $\bar{c}$ are independent over A if and only if $\mathbb{F}$ admits a free decomposition $\mathbb{F} = \mathbb{F}_b * A * \mathbb{F}_c$ with $\bar{b} \in \mathbb{F}_b * A$ and $\bar{c} \in A * \mathbb{F}_c$.*

In other words, the tuples $\bar{b}$ and $\bar{c}$ are independent over A if and only if there is a Grushko decomposition of $\mathbb{F}$ relative to A in which $\bar{b}$ and $\bar{c}$ live in 'essentially disjoint' parts.

To see that this is really a special case of the main result, Theorem 4.15, let A be a free factor of a non-abelian finitely generated free group $\mathbb{F}$, and let $\bar{b}$ and $\bar{c}$ be tuples of elements of $\mathbb{F}$. Let Λ be a normalized pointed cyclic JSJ decomposition for $\mathbb{F}$ relative to A. We want to show that the intersection of the minimal subgraphs $\Lambda^{min}_{A\bar{b}}$ and $\Lambda^{min}_{A\bar{c}}$ of $\langle A, \bar{b}\rangle$ and $\langle A, \bar{c}\rangle$ respectively is contained in a disjoint union of envelopes of rigid vertices if and only if $\mathbb{F}$ admits a free decomposition $\mathbb{F} = \mathbb{F}_b * A * \mathbb{F}_c$ with $\bar{b} \in \mathbb{F}_b * A$ and $\bar{c} \in A * \mathbb{F}_c$.

Let T be the tree corresponding to Λ. Then T is a normalized pointed cyclic JSJ tree for $\mathbb{F}$ relative to A. Remember that this means that the minimal subtree T^{min}_A of A (as the smallest free factor of $\mathbb{F}$ containing A) in T is the pointed cyclic JSJ tree of cylinders of A relative to A, and that trivially stabilized edges join a translate of the base vertex v_A to a vertex which lies in a translate of T^{min}_A. If A is not infinite cyclic, the tree T^{min}_A is the tree of cylinders T' of $JSJ_A(A)$ and otherwise it is T' with an attached orbit of vertices and an attached orbit of edges. Since the tree T' is in $JSJ_A(A)$, it is in particular a minimal (A, A)-tree, i.e. an A-tree with no proper A-invariant subtree and in which A fixes a point. Thus, T' is itself a point.

We first consider the case where A is not infinite cyclic. Then T consists of an orbit of vertices stabilized by conjugates of A (corresponding to translates of T') and trivially stabilized edges joining them. Thus, the graph of groups Λ is a rose with a single central vertex with vertex group A and some loops attached to it, all with trivial edge group, see Figure 3.

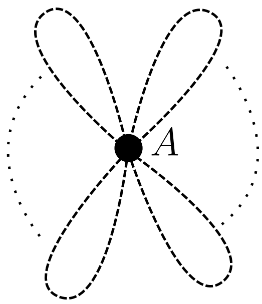


Figure 3: The normalized pointed cyclic JSJ decomposition Λ for $\mathbb{F}$ relative to A in the case where A is not infinite cyclic. (Dot vertices represent rigid non-Z-type vertices, dashed edges have an associated trivial edge group and the dotted lines indicate an unknown number of edges of this kind.)

Let D be the set of Bass-Serre elements associated to the loops of Λ. Then Λ corresponds to a free product decomposition $\mathbb{F} = A * \mathbb{F}'$ of $\mathbb{F}$ where $\mathbb{F}'$ is the free group with basis D.

Since Λ has only one vertex, which is rigid as it is the image of the base vertex of T, the minimal subgraphs $\Lambda^{min}_{A\bar{b}}$ and $\Lambda^{min}_{A\bar{c}}$ satisfy the disjointness condition of the main theorem if and only if their intersection is at most this central vertex. Thus, they satisfy the disjointness condition if and only if there exist disjoint subsets D_b, D_c of D such that $\langle A, \bar{b}\rangle \subseteq \langle A \cup D_b\rangle$ and $\langle A, \bar{c}\rangle \subseteq \langle A \cup D_c\rangle$. Without loss of generality we may assume that $D = D_b \cup D_c$. Thus, if we set $\mathbb{F}_b = \langle D_b\rangle$ and $\mathbb{F}_c = \langle D_c\rangle$, then we can write $\mathbb{F} = \mathbb{F}_b * A * \mathbb{F}_c$ where $\bar{b} \in \mathbb{F}_b * A$ and $\bar{c} \in A * \mathbb{F}_c$ if and only if the disjointness condition is satisfied for $\Lambda^{min}_{A\bar{b}}$ and $\Lambda^{min}_{A\bar{c}}$, which we wanted to show.

Now let A be infinite cyclic. Remember that T^{min}_A consists of T', which is a single vertex w, together with an attached orbit of vertices $A \cdot v_A$ and an orbit of edges $A \cdot e$ where $e = (v_A, w)$. All these vertices and edges are stabilized by A. Now T consists of translates of T^{min}_A that are joint by trivially stabilized edges, where each trivially stabilized edge joins a translate of v_A to a vertex in a translate of T^{min}_A. Then Λ has only two vertices, v'_A and w', where the inverse image of v'_A in T contains v_A and the inverse image of w' in T contains w. The vertex group of both vertices is A. The graph of groups Λ furthermore contains an edge e' between v'_A and w' with edge group A and a number of edges with trivial edge group which either join v'_A and w' or are loops attached to v'_A, see Figure 4.

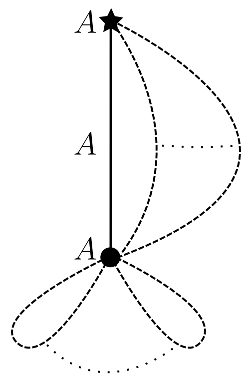


Figure 4: The normalized pointed cyclic JSJ decomposition Λ for $\mathbb{F}$ relative to A in the case where A is infinite cyclic. (Dot vertices represent rigid non-Z-type vertices, full edges have an associated infinite cyclic edge group, dashed ones have an associated trivial edge group and the dotted lines indicate an unknown number of edges of this kind.)

Let D_1 be the set of Bass-Serre elements associated to the loops of Λ, and let D_2 be the set of Bass-Serre elements associated to the other edges of Λ with trivial edge group. Then Λ corresponds to a free product decomposition $\mathbb{F} = F_1 * (F_2 * (A *_A A)) \cong \mathbb{F}' * A$ where F_1 and F_2 are the free groups with basis D_1 and D_2 respectively and $\mathbb{F}' = F_1 * F_2$.

The vertex v'_A is rigid as it is the image of v_A in Λ, and the vertex w' is of Z-type because A is infinite cyclic. Thus, the edge e' is an envelope of v'_A in Λ and no other edge can be part of this envelope because no other edge lies in the pointed cyclic JSJ decomposition Λ_A^{min}. Hence, the minimal subgraphs $\Lambda_{A\bar{b}}^{min}$ and $\Lambda_{A\bar{c}}^{min}$ satisfy the disjointness condition of the main theorem if and only if their intersection is at most the edge e', i.e. if and only if there exist disjoint subsets D_b, D_c of $D_1 \cup D_2$ such that $\langle A, \bar{b}\rangle \subseteq \langle A \cup D_b\rangle$ and $\langle A, \bar{c}\rangle \subseteq \langle A \cup D_c\rangle$. We now may proceed as in the case where A is not infinite cyclic and obtain the desired result. Therefore, Theorem 4.17 is indeed a special case of Theorem 4.15.

The second main result of [PS16] deals with parameter sets which are not contained in any proper free factor of $\mathbb{F}$.

Theorem 4.18 ([PS16, Theorem 2]). *Let $\mathbb{F}$ be a non-abelian finitely generated free group which is freely indecomposable with respect to a set $A \subseteq \mathbb{F}$ of parameters, and let Λ be the pointed cyclic JSJ decomposition for $\mathbb{F}$ relative to A. Let $\bar{b}$ and $\bar{c}$ be tuples in $\mathbb{F}$ and denote by $\Lambda^{min}_{A\bar{b}}$ (respectively $\Lambda^{min}_{A\bar{c}}$) the minimal subgraphs of groups of Λ whose fundamental group contains the subgroup $\langle A, \bar{b}\,\rangle$ (respectively $\langle A, \bar{c}\,\rangle$) of $\mathbb{F}$.*

Then $\bar{b}$ and $\bar{c}$ are independent over A if and only if each connected component of $\Lambda^{min}_{A\bar{b}} \cap \Lambda^{min}_{A\bar{c}}$ contains at most one non-Z-type vertex and such a vertex is of non-surface-type.

In other words, the tuples $\bar{b}$ and $\bar{c}$ are independent over A if and only if they live in 'essentially disjoint' parts of the pointed cyclic JSJ decomposition for $\mathbb{F}$ relative to A.

To see that this is indeed a special case of the main result, Theorem 4.15, let A be a subset of a non-abelian finitely generated free group $\mathbb{F}$, which is not contained in any proper free factor of $\mathbb{F}$, and let $\bar{b}$ and $\bar{c}$ be tuples in $\mathbb{F}$. Since $\mathbb{F}$ itself is the smallest free factor of $\mathbb{F}$ containing A, Definition 4.8 implies that there exists a unique normalized pointed cyclic JSJ decomposition Λ for $\mathbb{F}$ relative to A which is exactly the pointed cyclic JSJ decomposition for $\mathbb{F}$ relative to A. Let T be the tree corresponding to Λ.

Any two blocks of the minimal subgraph $\Lambda^{min}_{A\bar{b}}$ of $\langle A, \bar{b}\rangle$ as well as any two blocks of the minimal subgraph $\Lambda^{min}_{A\bar{c}}$ of $\langle A, \bar{c}\rangle$ only intersect in the base vertex v_A: Consider the blocks of $\Lambda^{min}_{A\bar{b}}$, the proof is exactly the same for the blocks of $\Lambda^{min}_{A\bar{c}}$. By definition any two blocks of $\Lambda^{min}_{A\bar{b}}$ intersect at most in the base vertex v_A and in some Z-type vertices. Suppose there exist distinct blocks B, C of $\Lambda^{min}_{A\bar{b}}$ whose intersection contains a Z-type vertex w. Take a non-backtracking path in B from v_A to w and a non-backtracking path in C from w back to v_A, this gives a loop α. Now choose a lift of α in $T^{min}_{A\bar{b}}$ starting at v_A. Then this lift is a path in $T^{min}_{A\bar{b}}$ from v_A to a translate $g \cdot v_A$ of v_A where $g \in \langle A, \bar{b}\rangle$. Since T is a pointed cyclic JSJ tree, it does not contain any trivially stabilized edges. Hence, the path $[v_A, g \cdot v_A]$ does not contain any trivially stabilized edges, and because it is a lift of the loop α it does not contain any translates of v_A other than its endpoints. Therefore, the element g is a sandwich term with minimal subgraph α. The loop α meets both blocks B and C in at least one edge, thus it is equivalent to both B and C, a contradiction to B and C being distinct blocks. This proves that any two blocks of $\Lambda_{A\bar{b}}^{min}$ as well as any two blocks of $\Lambda_{A\bar{c}}^{min}$ intersect only in v_A.

Now we prove that Theorem 4.18 is really a special case of Theorem 4.15, i.e. we show that each connected component of $\Lambda_{A\bar{b}}^{min} \cap \Lambda_{A\bar{c}}^{min}$ contains at most one non-Z-type vertex and this vertex is of non-surface-type if and only if the intersection of any two blocks of $\Lambda_{A\bar{b}}^{min}$ and $\Lambda_{A\bar{c}}^{min}$ is contained in a disjoint union of envelopes of rigid vertices. Remember that an envelope of a rigid non-Z-type vertex v consists of the central vertex v itself with adjacent Z-type vertices, and an envelope of a Z-type vertex z consists at most of an edge joining z and a rigid vertex. Thus, using Remark 4.7 it obviously holds that each connected component of $\Lambda_{A\bar{b}}^{min} \cap \Lambda_{A\bar{c}}^{min}$ contains at most one non-Z-type vertex and this vertex is of non-surface-type if and only if the intersection of $\Lambda_{A\bar{b}}^{min}$ and $\Lambda_{A\bar{c}}^{min}$ is contained in a disjoint union of envelopes of rigid vertices, which implies that any two blocks of $\Lambda_{A\bar{b}}^{min}$ and $\Lambda_{A\bar{c}}^{min}$ also fulfil this disjointness property. We prove the other direction by contraposition.

Assume there exists a connected component X of $\Lambda_{A\bar{b}}^{min} \cap \Lambda_{A\bar{c}}^{min}$ which contains more than one non-Z-type vertex or its only non-Z-type vertex is of surface-type. Then X contains in particular more than an envelope of a rigid vertex. We show that there exist blocks of $\Lambda_{A\bar{b}}^{min}$ and $\Lambda_{A\bar{c}}^{min}$ whose intersection is more than a disjoint union of envelopes of rigid vertices. We may assume that at least one of the minimal subgraphs $\Lambda_{A\bar{b}}^{min}, \Lambda_{A\bar{c}}^{min}$, say $\Lambda_{A\bar{b}}^{min}$, consists of more than one block, otherwise the result is immediate.

Let X be the connected component of $\Lambda_{A\bar{b}}^{min} \cap \Lambda_{A\bar{c}}^{min}$ which is not contained in an envelope of a rigid vertex. If X meets all blocks of $\Lambda_{A\bar{b}}^{min}$ in an envelope of a rigid vertex, it meets several blocks and thus contains v_A because the blocks of $\Lambda_{A\bar{b}}^{min}$ only intersect in v_A. In particular, its intersection with each block of $\Lambda_{A\bar{b}}^{min}$ consists of the union of some edges incident to v_A. Hence, X is contained in an envelope of the rigid vertex v_A, a contradiction. Therefore, there exists a block B of $\Lambda_{A\bar{b}}^{min}$ which intersects X in more than an envelope of a rigid vertex.

If $\Lambda_{A\bar{c}}^{min}$ consists of only one block, then the intersection of B with $\Lambda_{A\bar{c}}^{min}$, which is the only block of itself, consists of more than a disjoint union of envelopes of rigid vertices, which we wanted to show. Otherwise suppose that $X \cap B$ intersects all blocks of $\Lambda_{A\bar{c}}^{min}$ in an envelope of a rigid vertex. Then $X \cap B$ meets several blocks, thus contains v_A, and its intersection with each block of $\Lambda_{A\bar{c}}^{min}$ consists of a union of some edges incident to v_A. Hence, $X \cap B$ is contained in an envelope of the rigid vertex v_A, a contradiction. This implies that there exists a block C of $\Lambda_{A\bar{c}}^{min}$ that intersects $X \cap B$ in more than a disjoint union of envelopes of rigid vertices. In particular, the intersection of B and C is more than a disjoint union of envelopes of rigid vertices, which we wanted to show.

Therefore, Theorem 4.18 is indeed a special case of Theorem 4.15.

4.5 The modular group

Let G be a torsion-free hyperbolic group which is freely indecomposable with respect to a subgroup H. The relative modular group is a subgroup of the group of automorphisms of G fixing H, which we need for proving independence. The exposition in this subsection follows Sections 4.2 till 4.5 of [PS16].

Definition 4.19 ([PS16, Definition 4.5])**.** Let G be a finitely generated group, and let T be G-tree. Let $e = (v, w)$ be an edge in T, and let a be an element in the centralizer of $\mathrm{Stab}(e)$. By collapsing all edges of T which do not lie in the same orbit as e we obtain a G-tree T'. This tree corresponds to a one-edge splitting of G as an amalgamated product $G = V *_{\mathrm{Stab}(e)} W$ or as an HNN extention $V*_{\mathrm{Stab}(e)}$ with stable letter t, where V and W are the stabilizers of the images of v and w in T', respectively.

The *Dehn twist* by a about e is the automorphism of G which restricts to the identity on V and to a conjugation by a on W (respectively restricts to the identity on V and sends t to at in the HNN case).

For a vertex x of T we denote by G_x the stabilizer of its image in T'.

Lemma 4.20 ([PS16, Lemma 4.7])**.** *Let G be a finitely generated group, and let T be a minimal cyclic G-tree with Bass-Serre presentation $(T^1, T^0, (t_f)_{f\in E_1})$. Let τ_e be a Dehn twist by an element a about an edge $e = (v, w)$ of T^1. Then*

- *for every vertex x of T^0 the restriction of τ_e to G_x is a conjugation by an element g_x of G, which is 1 if x and v lie in the same connected component of $T^1 - \{e\}$, and a otherwise;*
- *for every edge $f = (x, y')$ of T^1 with $x \in T^0$ and $t_f^{-1} \cdot y' = y \in T^0$ we have*

$$\tau_e(t_f) = \begin{cases} g_x t_f g_y^{-1} & if f \neq e \\ at_f & if f = e \\ t_f a^{-1} & if f = \bar{e}. \end{cases}$$

Now we can define the relative modular group which consists of Dehn twists about certain edges.

Definition 4.21 ([PS16, pp. 2000])**.** Let G be a torsion-free hyperbolic group which is freely indecomposable with respect to a subgroup H. The *relative modular group* $\mathrm{Mod}_H(G)$ is the subgroup of $\mathrm{Aut}_H(G)$ generated by Dehn twists of one-edge cyclic splittings of G in which H is elliptic.

The following result of E. Rips and Sela, proven in [RS94, Corollary 4.4], will be useful later.

Theorem 4.22. *Let G be a torsion-free hyperbolic group which is freely indecomposable with respect to a (possibly trivial) subgroup H. The modular group $\mathrm{Mod}_H(G)$ has finite index in $\mathrm{Aut}_H(G)$.*

The second kind of automorphisms we consider are vertex automorphisms. They extend automorphisms of stabilizers of vertices in a G-tree to automorphisms of G.

Definition 4.23 ([PS16, Defnition 4.10])**.** Let G be a finitely generated group, and let T be a G-tree. Let v be a vertex of T, and let T' be the G-tree obtained by collapsing all orbits of edges of T except those of edges incident to v.

A *vertex automorphism* σ associated to v is an automorphism of G such that $\sigma(G_v) = G_v$, where G_v is the stabilizer of the image of v in T', and for every edge $e = (v, w)$ of T' incident to v, the automorphism σ restricts to a conjugation on the stabilizer of e as well as on the stabilizer of w if w does not lie in the orbit of v.

Vertex automorphisms as well as Dehn twists are automorphisms of G that can be read from a splitting of G. We call them elementary automorphisms.

Definition 4.24 ([PS16, Definition 4.16])**.** Let G be a finitely generated group, and let T be a G-tree. An *elementary automorphism* ρ associated to T is a Dehn twist about an edge e of T or a vertex automorphism associated to a vertex v of T. We call the edge e or the vertex v the *support* of ρ and write $\mathrm{Supp}(\rho)$.

Furthermore, under certain conditions an elementary automorphism can be written as a product of a modular automorphism and a conjugation.

Lemma 4.25 ([PS16, Lemma 4.26]). *Let $A \leq H$ be subgroups of a torsion-free hyperbolic group G which is freely indecomposable with respect to A. Let T be a minimal cyclic (G, A)-tree with a distinguished set of orbits of vertices which are of surface-type. Let ρ be an elementary automorphism associated to T whose support does not lie in any translate of the minimal subtree of H in T. Then there exist $g \in G$ and $\sigma \in \mathrm{Mod}_H(G)$ such that*

$$\rho = \mathrm{Conj}(g) \circ \sigma.$$

In a pointed cyclic JSJ tree every element of the relative modular group can be written as a product of certain elementary automorphisms. This is a merge of Lemma 4.21 and Proposition 4.22 of [PS16].

Lemma 4.26. *Let G be a torsion-free hyperbolic group which is freely indecomposable with respect to a subgroup H. Let T be the pointed cyclic JSJ tree for G relative to H, and let $(T^1, T^0, (t_e)_{e \in E_1})$ be a Bass-Serre presentation for G with respect to T.*

Every element θ of $\mathrm{Mod}_H(G)$ can be written as a product of the form

$$\mathrm{Conj}(z) \circ \rho_1 \circ \ldots \circ \rho_r$$

where $z \in G$ and the ρ_i are Dehn twists about distinct edges of T^1 or vertex automorphisms associated to distinct surface-type vertices of T^0. Furthermore, up to changing z, we can permute the list of supports of the ρ_i.

Finally, if H fixes a non-surface-type vertex x of T^0, we can in fact choose the ρ_i to fix $\mathrm{Stab}_G(x)$ pointwise, and thus z to lie in the centralizer of H.

5 Proof of the main theorem

In this section we prove the main theorem, which is Theorem 4.15, namely:

Theorem ([PS20, Theorem 3.15]). *Let $\mathbb{F}$ be a non-abelian finitely generated free group, let $A \subseteq \mathbb{F}$ be a set of parameters, and let $\bar{b}, \bar{c}$ be tuples from $\mathbb{F}$.*

Then $\bar{b}$ is independent from $\bar{c}$ over A if and only if there exists a normalized pointed cyclic JSJ decomposition Λ for $\mathbb{F}$ relative to A in which the intersection of any two blocks of the minimal subgraphs $\Lambda^{min}_{A\bar{b}}$ and $\Lambda^{min}_{A\bar{c}}$ of $\langle A, \bar{b} \rangle$ and $\langle A, \bar{c} \rangle$ respectively is contained in a disjoint union of envelopes of rigid vertices.

In Subsections 5.1 and 5.2 we assume that an adequate decomposition exists and prove independence. First we deal with a special case of this in Subsection 5.1, and then prove the general case in Subsection 5.2. In Subsection 5.3 we prove the other direction, namely the existence of an adequate decomposition if the tuples are independent over the parameter set.

5.1 A special case

The exposition in this section closely follows Section 4.1 of [PS20].

Let $\mathbb{F}$ be a non-abelian finitely generated free group, let $A \subseteq \mathbb{F}$ be a set of parameters, and let $\bar{b}, \bar{c}$ be tuples coming from $\mathbb{F}$. The aim of this section is to prove one direction of Theorem 4.15, namely the independence of $\bar{b}$ and $\bar{c}$ if a normalized pointed cyclic JSJ decomposition with the desired property exists, in the case where one of the tuples is contained in the smallest free factor $\mathbb{F}_A$ of $\mathbb{F}$ containing A.

Assume that $\bar{b}$ belongs to $\mathbb{F}_A$. Then we may assume that $\mathbb{F}$ is freely indecomposable with respect to $A\bar{c}$. Otherwise let $\mathbb{F}_{A\bar{c}}$ be the smallest free factor of $\mathbb{F}$ containing $A\bar{c}$. Then $\mathbb{F}_A$, and thus $\bar{b}$, is contained in $\mathbb{F}_{A\bar{c}}$. Since $\mathbb{F}_{A\bar{c}}$ is an elementary substructure of $\mathbb{F}$ by Theorem 2.16, proving independence of $\bar{b}$ and $\bar{c}$ over A in $\mathbb{F}_{A\bar{c}}$ proves independence in $\mathbb{F}$.

The idea of the proof is to show that the orbit of $\bar{b}$ under $\mathrm{Mod}_A(\mathbb{F}_A)$ is contained in the orbit of $\bar{b}$ under $\mathrm{Aut}_{A\bar{c}}(\mathbb{F})$, i.e. any tuple that is the image of $\bar{b}$ by a modular automorphism fixing A is in fact the image of b by an automorphism fixing both A and $\bar{c}$. This implies that the set $X := \mathrm{Aut}_{A\bar{c}}(\mathbb{F})$ contains a non-trivial almost A-invariant subset, which is enough to prove independence.

First we prove the following:

Proposition 5.1 ([PS20, Proposition 4.1])**.** *Let $\bar{b}, \bar{c}$ be tuples in a non-abelian finitely generated free group $\mathbb{F}$, let $A \subseteq \mathbb{F}$ be a set of parameters, and denote by $\mathbb{F}_A$ the smallest free factor of $\mathbb{F}$ containing A. Assume that $\mathbb{F}$ is freely indecomposable with respect to $A\bar{c}$ and that $\bar{b} \in \mathbb{F}_A$.*

Suppose there exists a normalized pointed cyclic JSJ decomposition Λ for $\mathbb{F}$ relative to A in which the intersection of any two blocks of the minimal sugraphs $\Lambda^{min}_{A\bar{b}}$ and $\Lambda^{min}_{A\bar{c}}$ is contained in a disjoint union of envelopes of rigid vertices.

Then the orbit of $\bar{b}$ under $\mathrm{Aut}_{A\bar{c}}(\mathbb{F})$ contains the orbit of $\bar{b}$ under $\mathrm{Mod}_A(\mathbb{F}_A)$.

Note that we consider the intersection of any two blocks of $\Lambda^{min}_{A\bar{b}}$ and $\Lambda^{min}_{A\bar{c}}$, whereas Proposition 4.1 of [PS20] considers the intersection of $\Lambda^{min}_{A\bar{b}}$ with any block of $\Lambda^{min}_{A\bar{c}}$. We apply the proposition above in the proof of Proposition 5.3 where we assume that the intersection of any two blocks of $\Lambda^{min}_{A\bar{b}}$ and $\Lambda^{min}_{A\bar{c}}$ is contained in a disjoint union of envelopes of rigid vertices. This does not imply that the intersection of $\Lambda^{min}_{A\bar{b}}$ with any block of $\Lambda^{min}_{A\bar{c}}$ also fulfils this property, see for example Example 4.16. Thus, to apply Proposition 5.1 in the proof of Proposition 5.3 we have to formulate the proposition as we did in this thesis.

First we give the proof on an example.

Example 5.2 ([PS16, Example 4.2])**.** Consider the example given in 4.14 and 4.16, where Λ is as in Figure 2, $\bar{c} = (u, tvwt^{-1})$, and b lies in W but not in the incident edge group. Denote by T be the tree corresponding to Λ, and let e_1, e_2, e_3, e_4 be edges in T incident to a common lift of the Z-type vertex of Λ which are lifts of the edges joining the vertices of R and Z, V and Z, W and Z and U and Z respectively.

Let $\theta \in \mathrm{Mod}_A(\mathbb{F}_A)$. We want to find an element α of $\mathrm{Aut}_{A\bar{c}}(\mathbb{F})$ such that $\theta(b) = \alpha(b)$. Let z be the generator of Z. Then z is in the centralizer of the stabilizer of every e_i. Thus, for every $i = 1, \ldots, 4$ there exists a Dehn twist τ_i by z about e_i. By Lemma 4.20 the Dehn twist τ_i restricts to a conjugation by z on R, V, W, U if $i = 1, 2, 3, 4$ respectively, and fixes all the other vertex groups pointwise.

Note that the minimal subgraph of $\mathbb{F}_A$ in Λ consists of all edges of Λ except the trivially stabilized one because Λ is a normalized pointed cyclic JSJ decomposition, and remember that $T^{min}_{\mathbb{F}_A}$ is the pointed cyclic JSJ tree for $\mathbb{F}_A$ relative to A. Thus, we may choose a Bass-Serre representation $(T_1, T_0, (t_f)_{f \in E_1})$ for $\Lambda^{min}_{\mathbb{F}_A}$, where $T^0 = T^1$ consists of the edges e_i with incident vertices. Now Lemma 4.26 implies that we may write θ as a product of the form

$$\theta = \mathrm{Conj}(g) \circ \rho_1 \circ \cdots \circ \rho_r$$

where $g \in \mathbb{F}_A$ and the ρ_j are Dehn twists about the edges e_i or vertex automorphisms associated to distinct surface-type of T^0. There are no surface-type vertices in Λ, and thus there are none in $T^{min}_{\mathbb{F}_A}$, so the ρ_j are Dehn twists about the edges e_i. Furthermore, again by Lemma 4.26 we may permute the ρ_j such that they are ordered by their support. Remember that only τ_1 restricts to a conjugation by z on R, the other Dehn twists fix R pointwise, thus we may write

$$\theta = \mathrm{Conj}(z^{-k_1}) \circ \tau_1^{k_1} \circ \tau_2^{k_2} \circ \tau_3^{k_3} \circ \tau_4^{k_4}$$

for $k_i \in \mathbb{N}$. We get $\theta(b) = z^{k_3-k_1} b z^{k_1-k_3}$ because $b \in W$ and only τ_3 restricts to a conjugation by z on W.

Now we define an automorphism α of $\mathbb{F}$ by

$$\alpha\,|_{\mathbb{F}_A} = \mathrm{Conj}(z^{-k_1}) \circ \tau_1^{k_1} \circ \tau_2^{k_3} \circ \tau_3^{k_3} \circ \tau_4^{k_1} \text{ and } \alpha(t) = tz^{k_1-k_3}.$$

Note that α fixes A because $A \subseteq R$. Since b is contained in W, if follows $\alpha(b) = z^{k_3-k_1} b z^{k_1-k_3} = \theta(b)$. Moreover, $\alpha(u) = \mathrm{Conj}(z^{-k_1})(\tau_4^{k_1}(u)) = u$, and $\alpha(vw) = \mathrm{Conj}(z^{-k_1})(\tau_2^{k_3}(v)) \cdot \mathrm{Conj}(z^{-k_1})(\tau_3^{k_3}(w)) = z^{k_3-k_1} vw z^{k_1-k_3}$. Therefore, we get

$$\alpha(tvwt^{-1}) = \alpha(t)\alpha(vw)\alpha(t)^{-1} = tz^{k_1-k_3}(z^{k_3-k_1} vw z^{k_1-k_3}) z^{k_3-k_1} t^{-1} = tvwt^{-1}.$$

Hence, $\alpha(c) = c$ and $\alpha \in \mathrm{Aut}_{A\bar{c}}(\mathbb{F})$ such that $\alpha(b) = \theta(b)$.

Proof of Proposition 5.1. The tree T is normalized, so the minimal subtree $T^{min}_{\mathbb{F}_A}$ of $\mathbb{F}_A$ in T is the pointed cyclic JSJ tree of cylinders for $\mathbb{F}_A$ relative to A. Remember that the tuple $\bar{b}$ is contained in $\mathbb{F}_A$. We may choose a Bass-Serre presentation $(T^1, T^0, (t_f)_{f \in E_1})$ such that the vertices and edges of T^1 whose orbits meet T^{min}_{Ab} actually lie in T^{min}_{Ab}. Then by Lemma 4.26 every $\theta \in \mathrm{Mod}_A(\mathbb{F}_A)$ can be written as

$$\theta = \mathrm{Conj}(z) \circ \rho_1 \circ \cdots \circ \rho_r$$

where $z \in \mathbb{F}_A$ and the ρ_i are Dehn twists about distinct edges of T^1 or vertex automorphisms associated to distinct surface-type vertices of T^0. This lemma also tells us that, up to changing z, we may permute the ρ_i. Hence, we may assume that there exists $k \in \mathbb{N}$ such that $\rho_1, \ldots, \rho_k$ all have support in T^{min}_{Ab} and $\rho_{k+1}, \ldots, \rho_r$ all have support outside any translate of T^{min}_{Ab}. Furthermore, again by Lemma 4.26 we may choose the ρ_i to fix $\mathrm{Stab}(v_A)$ pointwise, and thus z to lie in the stabilizer of v_A under $\mathbb{F}_A$, which is the stabilizer of v_A under $\mathbb{F}$. We want to show that there exists an $\alpha \in \mathrm{Aut}_{A\bar{c}}(\mathbb{F})$ such that $\theta(\bar{b}) = \alpha(\bar{b})$.

By Lemma 4.25 the composition $\rho_{k+1} \circ \cdots \circ \rho_r$ is the composition of an element of $\mathrm{Mod}_{A\bar{b}}(\mathbb{F}_A)$ with a conjugation. Hence, the image of $\bar{b}$ by θ is, up to conjugation, the same as the image of $\bar{b}$ by $\rho_1 \circ \cdots \circ \rho_k$, and we may assume that $k = r$, i.e. that all the ρ_i have support in $T^{min}_{A\bar{b}}$.

For each i we define an automorphism ρ'_i as follows: If $\mathrm{Supp}(\rho_i)$ does not lie in any translate of $T^{min}_{A\bar{c}}$, we set $\rho'_i = \rho_i$. If $\mathrm{Supp}(\rho_i)$ lies in a translate of $T^{min}_{A\bar{c}}$, then consider the blocks of $\Lambda^{min}_{A\bar{b}}$ and $\Lambda^{min}_{A\bar{c}}$ containing the image of $\mathrm{Supp}(\rho_i)$

in Λ. The hypothesis on the minimal subgraphs of $\langle A, \bar{b}\rangle$ and $\langle A, \bar{c}\rangle$ ensures that the support is contained in an envelope of a rigid vertex. Remember that an envelope of a rigid vertex never contains a surface type vertex by Remark 4.10. Thus, $\mathrm{Supp}(\rho_i)$ is not a surface-type vertex but an edge e_0, and ρ_i is a Dehn twist by some element γ_i about e_0. Denote by e the image of e_0 in Λ and by Γ_e the block of $\Lambda^{min}_{A\bar{c}}$ containing it. Again by our hypothesis on the minimal subgraphs Γ_e intersects each block of $\Lambda^{min}_{A\bar{b}}$ at most in a disjoint union of envelopes of rigid vertices. Keep in mind that an envelope of a rigid vertex v of Λ is a star graph which consists of only one edge if v is of Z-type or the base vertex. The edges of a cylinder are all incident to a common Z-type vertex (or incident to v_A if A is infinite cyclic and does not stabilize a cylinder). So the cylinders in Λ contain at most one edge of every envelope of a rigid vertex. Hence, the edge e is the only edge of its cylinder belonging to both Γ_e and $\Lambda^{min}_{A\bar{b}}$. Note that by Lemma 3.14 the element γ_i is in the centralizer of $\mathrm{Stab}(f)$ for every edge f in the cylinder of e_0, thus there exist Dehn twists by γ_i about every edge in the cylinder of e_0. Now let ρ'_i be the product of Dehn twists $\tau_0, \ldots, \tau_m$ by γ_i about edges $e_0, e_1, \ldots, e_m$ respectively where $\tau_o = \rho_i$ and every e_j is a representative of an orbit of edges of the cylinder of e_0 whose image lies in Γ_e. We choose the Dehn twists $\tau_1, \ldots, \tau_m$ to restrict to a conjugation by γ_i (and not the identity) on the stabilizer of the connected component not containing e_0 in the graph where we removed the orbit of $e_1, \ldots, e_m$ respectively. Since e_j lies outside any translate of $T^{min}_{A\bar{b}}$ for every $j = 1, \ldots, m$ and e_0 lies in $T^{min}_{A\bar{b}}$, this implies that $T^{min}_{A\bar{b}}$ is in the connected component whose stabilizer is fixed by τ_j. Thus, the Dehn twists $\tau_1, \ldots, \tau_m$ fix $\langle A, \bar{b}\rangle$, and in particular they restrict to the identity on $\mathrm{Stab}(v_A)$. Remember that $\tau_o = \rho_i$ also restricts to the identity on $\mathrm{Stab}(v_A)$, so ρ'_i restricts to the identity on $\mathrm{Stab}(v_A)$.

Set $\theta' = \mathrm{Conj}(z) \circ \rho'_1 \circ \cdots \circ \rho'_r$. Then the image of $\bar{b}$ by $\theta = \mathrm{Conj}(z) \circ \rho_1 \circ \ldots \circ \rho_r$ is the same as the image of $\bar{b}$ by θ' because all Dehn twists we added fix $\bar{b}$. We want to extend θ' to an automorphism α of $\mathbb{F}$ fixing $A\bar{c}$.

First we modify the tree T to get an adequate presentation of $\mathbb{F}$. Denote by $p : T \longrightarrow T'$ the map defined by folding together in each cylinder all the edges whose images in Λ belong to a common block of $\Lambda^{min}_{A\bar{c}}$. In other words, if β is a sandwich term whose minimal subgraph lies in $\Lambda^{min}_{A\bar{c}}$ and the path $[v_A, \beta \cdot v_A]$ contains edges from the same cylinder, then we fold them together. Two examples of this can be seen in Figures 5 and 6. Denote by Λ' the graph of groups corresponding to T'.

Note that every ρ_i' corresponds to an elementary automorphism of T'. If $\mathrm{Supp}(\rho_i)$ lies outside any translate of $T_{A\bar{c}}^{min}$, then ρ' itself is an elementary automorphism of T'. Otherwise it corresponds to the Dehn twist by γ_i about the edge $p(e_0) = p(e_1) = \ldots = p(e_m)$ of T'.

Pick a maximal subtree of Λ' which does not include any trivially stabilized edges, lift it to a subtree S^0 of T', and extend this to a Bass-Serre presentation $(S^0, S^1, (t_f)_{f\in E_1})$ for Λ'. Note that $\mathbb{F} = \mathbb{F}_A * \langle \{t_f\}_{f\in E_{triv^+}} \rangle$ where E_{triv^+} consists of all the trivially stabilized edges of S^1 of the form $(p(v_A), w)$ (to fix an orientation).

Let $v^1, \ldots, v^n$ be the vertices of S^0 which come from rigid vertices of T or whose inverse image by p is not a single vertex, and let $H^1, \ldots, H^n$ be the stabilizers of $v^1, \ldots, v^n$ respectively. Each block of $(\Lambda')_{A\bar{c}}^{min}$ now consists of an envelope of the image of one of the vertices v^i together with at most two trivially stabilized edges, and each trivially stabilized edge is associated in this way with at most one vertex v^i: A lift of a block of $\Lambda_{A\bar{c}}^{min}$ in T is a path $[v_A, \beta \cdot v_A]$ where β is a sandwich term and only the first and the last edge may be trivially stabilized. Remember that cylinders in T are star graphs, that the cylinders cover T, and that if $[v_A, \beta \cdot v_A]$ contains edges from the same cylinder, then they are folded together in T'. Thus, if $[v_A, \beta \cdot v_A]$ does not contain any trivially stabilized edge, then the image of this path in T', and therefore its image in Λ', is a star graph with a central rigid vertex. For an example of this case see Figure 5.

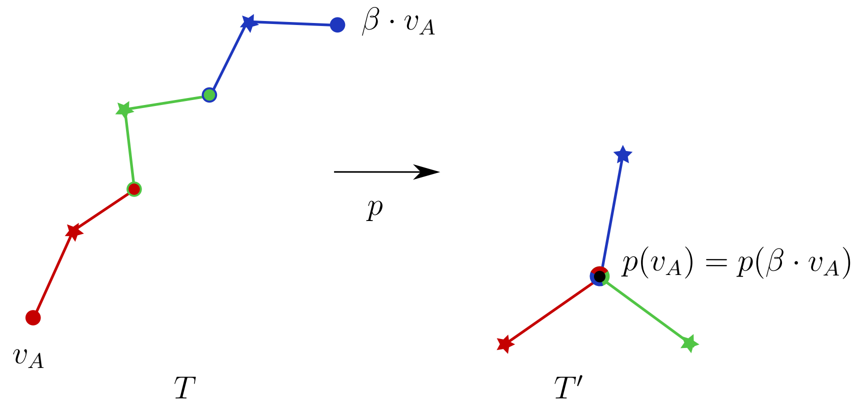


Figure 5: An example of the path $[v_A, \beta \cdot v_A]$ of a sandwich term β whose minimal subgraph lies in $\Lambda_{A\bar{c}}^{min}$ and does not contain any trivially stabilized edge. On the left hand side there is the path in T and on the right hand side its image in T'. Different colours denote different cylinders. (Dot vertices represent rigid non-Z-type vertices, stars represent Z-type vertices, and full edges have an associated infinite cyclic edge group.)

If $[v_A, \beta \cdot v_A]$ contains at least one trivially stabilized edge, then its image in T' consists of a star graph with a central rigid vertex and at least one trivially stabilized edge attached to a vertex of this star graph. An example of this case can be seen in Figure 6.

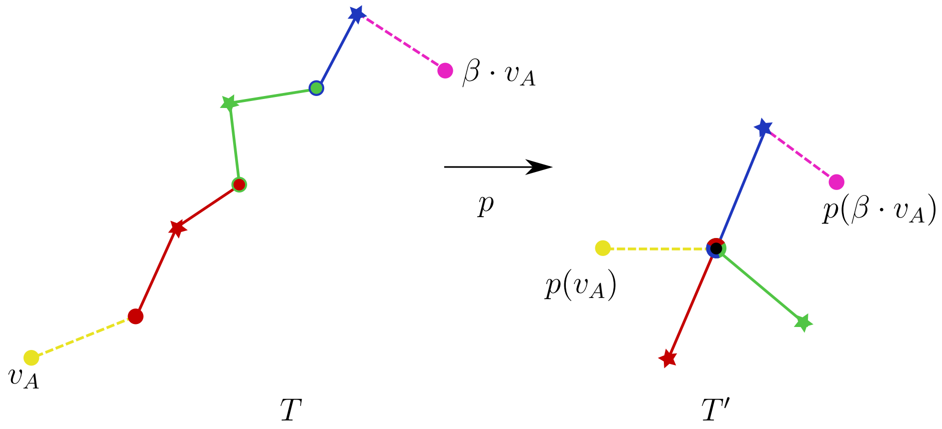


Figure 6: An example of the path $[v_A, \beta \cdot v_A]$ of a sandwich term β whose minimal subgraph lies in $\Lambda^{min}_{A\bar{c}}$ and contains two trivially stabilized edges. On the left hand side there is the path in T and on the right hand side its image in T'. Different colours denote different cylinders. (Dot vertices represent rigid non-Z-type vertices, stars represent Z-type vertices, full edges have an associated infinite cyclic edge group, and dashed ones have an associated trivial edge group.)

Each trivially stabilized edge is associated in this way with at most one v^i because if two sandwich terms have the same trivially stabilized edges at the beginning or the end, their minimal subgraphs are equivalent.

All elementary automorphisms ρ'_j of θ' restrict to the identity on $\mathrm{Stab}(v_A)$ in T, thus they restrict to the identity on the stabilizer of $p(v_A)$ in T', and they restrict to a conjugation on each H^i. Since θ' is the composition of $\mathrm{Conj}(z)$ with the ρ'_j and z lies in the centralizer of $\mathrm{Stab}(v_A)$, the automorphism θ' restricts to the identity on the stabilizer of $p(v_A)$ and to a conjugation by some element δ_i on each H^i.

Now define $\hat{H}^i$ to be the subgroup generated by H^i and all the Z-type vertex groups adjacent to H^i in Λ' by a non-trivially stabilized edge, i.e. $\hat{H}^i$ is generated by H^i together with some roots of elements of H^i. Let g be an element of $\hat{H}^i - H^i$. Then there exists $h \in H^i$ and $k \in \mathbb{N}$ such that $h^k = g$.

Now it holds that $\theta'(g) = \theta'(h^k) = \theta'(h)^k = \delta_i h^k \delta_i^{-1} = \delta_i g \delta_i^{-1}$, thus θ' also restricts to a conjugation by δ_i on $\hat{H}^i$.

Remember that the blocks of $(\Lambda')^{min}_{A\bar{c}}$ consist of an envelope of the image of one of the vertices v^i together with at most two trivially stabilized edges, and that we defined $\hat{H}^i$ to be the stabilizer of this envelope of v^i. Thus, every element in the tuple $\bar{c}$ can be written as a product of the form

$$g_1(t_{e_1}h_1t_{f_1}^{-1})g_2(t_{e_2}h_2t_{f_2}^{-1})\dots g_s(t_{e_s}h_st_{f_s}^{-1})g_{s+1} \tag{2}$$

where for each j

i) $g_j \in \mathrm{Stab}(v_A)$ in T';

ii) there exists an index l_j such that e_j, f_j are trivially stabilized edges lying in the block of $(\Lambda')^{min}_{A\bar{c}}$ containing the image of v^{l_j};

iii) $h_j \in \hat{H}^{l_j}$.

Finally define $\alpha \in \mathrm{Aut}(\mathbb{F})$ by setting $\alpha\,|_{\mathbb{F}_A} = \theta'$, and for each i and every edge $e \in E_{triv^+}$ whose image lies in a block of $(\Lambda')^{min}_{A\bar{c}}$ containing the image of v^i we set $\alpha(t_e) = t_e\delta_i^{-1}$, and $\alpha(t_f) = t_f$ for every other edge $f \in E_{triv^+}$. As we saw above, $\mathbb{F}$ is the free product of $\mathbb{F}_A$ together with the free group generated by the elements $\{t_e\}_{e\in E_{triv^+}}$ so this indeed defines an automorphism of $\mathbb{F}$.

We know that $\alpha\,|_{\mathbb{F}_A} = \theta'$ and θ' restricts to the identity on the stabilizer of $p(v_A)$ in T', which contains A. Furthermore, using the representation 2 of every element in $\bar{c}$ we get $\alpha(\bar{c}) = \bar{c}$, thus $\alpha \in \mathrm{Aut}_{A\bar{c}}(\mathbb{F})$. Since $\bar{b} \in \mathbb{F}_A$ we get $\alpha(\bar{b}) = \theta'(\bar{b})$, and as shown above $\theta'(\bar{b}) = \theta(\bar{b})$. Hence, it also holds that $\alpha(\bar{b}) = \theta(\bar{b})$. ☐

Now we can prove that under these conditions the tuples $\bar{b}$ and $\bar{c}$ are independent over A.

Proposition 5.3 ([PS20, Proposition 4.3])**.** *Let $\bar{b}, \bar{c}$ be tuples in a non-abelian finitely generated free group $\mathbb{F}$, let $A \subseteq \mathbb{F}$ be a set of parameters, and denote by $\mathbb{F}_A$ the smallest free factor of $\mathbb{F}$ containing A. Assume that $\mathbb{F}$ is freely indecomposable with respect to $A\bar{c}$ and that $\bar{b} \in \mathbb{F}_A$.*

Suppose there exists a normalized pointed cyclic JSJ decomposition Λ for $\mathbb{F}$ relative to A in which the intersection of any two blocks of the minimal subgraphs $\Lambda^{min}_{A\bar{b}}$ and $\Lambda^{min}_{A\bar{c}}$ is contained in a disjoint union of envelopes of rigid vertices.

Then $\bar{b}$ is independent from $\bar{c}$ over A.

Proof. Assume on the contrary that $\bar{b}$ forks with $\bar{c}$ over A. Let $X := \mathrm{Aut}_{A\bar{c}}(\mathbb{F})\cdot \bar{b}$ be the orbit of $\bar{b}$ under automorphisms of $\mathbb{F}$ fixing $A\bar{c}$. For every set Y containing $\bar{b}$ we have $X = \mathrm{Aut}_{A\bar{c}}(\mathbb{F})\cdot\bar{b} \subseteq \mathrm{Aut}_{A\bar{c}}(\mathbb{F})\cdot Y$. If Y is furthermore $A\bar{c}$-definable, then Proposition 1.3.5 of [Mar02] implies that Y already contains X.

The group $\mathbb{F}$ is freely indecomposable with respect to $A\bar{c}$, so X is definable over $A\bar{c}$ by Theorem 2.3. Let ϕ be the formula defining X, i.e. $\phi(\mathbb{F}) = X$. We want to show that ϕ forks over A. Since $\bar{b}$ forks with $\bar{c}$ over A, there exists a formula $\psi \in \mathrm{tp}(\bar{b}/A\bar{c})$ which forks over A. Denote by Y the set defined by ψ, i.e. $\psi(\mathbb{F}) = Y$. Then Y is definable over $A\bar{c}$ and contains $\bar{b}$, hence it contains X. The definition of forking and the fact that $X = \phi(\mathbb{F}) \subseteq \psi(\mathbb{F}) = Y$ imply that ϕ forks over A because ψ forks over A

By Proposition 5.1 the set X contains $\mathrm{Mod}_A(\mathbb{F}_A) \cdot \bar{b}$. Moreover, $\mathrm{Mod}_A(\mathbb{F}_A)$ has finite index in $\mathrm{Aut}_A(\mathbb{F}_A)$ by Theorem 4.22, so $\mathrm{Mod}_A(\mathbb{F}_A)\cdot \bar{b}$ is a non-trivial almost A-invariant subset of X. Since $\bar{b}$ is contained in $\mathbb{F}_A$, the orbit X is a subset of $\mathbb{F}_A$ and $\mathbb{F}_A$ is atomic over A by Theorem 2.3. By Lemma 2.15 we see that ϕ cannot fork over A, which is a contradiction. □

5.2 Proving independence

The exposition in this subsection closely follows Section 4.2 of [PS20]. We want to prove the first direction of Theorem 4.15, namely:

Theorem 5.4 ([PS20, Theorem 4.4]). *Let $\bar{b}, \bar{c}$ be tuples in a non-abelian finitely generated free group $\mathbb{F}$, let $A \subseteq \mathbb{F}$ be a set of parameters, and denote by $\mathbb{F}_A$ the smallest free factor of $\mathbb{F}$ containing A.*

Suppose there exists a normalized pointed cyclic JSJ decomposition Λ for $\mathbb{F}$ relative to A in which the intersection of any two blocks of the minimal subgraphs $\Lambda^{min}_{A\bar{b}}$ and $\Lambda^{min}_{A\bar{c}}$ is contained in a disjoint union of envelopes of rigid vertices.

Then $\bar{b}$ is independent from $\bar{c}$ over A.

Note that for a sandwich term β the minimal subgraph $\Lambda^{min}_{A\beta}$ consists only of one block which is exactly $\Lambda^{min}_{A\beta}$.

We first prove Theorem 5.4 for sandwich terms.

Proposition 5.5 ([PS20, Proposition 4.6]). *Under the conditions of Theorem 5.4 let β, γ be sandwich terms such that the intersection of the minimal subgraphs $\Lambda^{min}_{A\beta}$ and $\Lambda^{min}_{A\gamma}$ is contained in a disjoint union of envelopes of rigid vertices. Then β and γ are independent over A.*

We will in fact prove a different result which holds for tuples of sandwich terms satisfying certain conditions, and prove afterwards that this implies Proposition 5.5. We use the special case of the last section in the proof.

Proposition 5.6 ([PS20, Proposition 4.7]). *Let $\mathbb{F}$ be a free group, let $A \subseteq \mathbb{F}$ be a set of parameters, and denote by $\mathbb{F}_A$ the smallest free factor of $\mathbb{F}$ containing A. Let T be a normalized pointed cyclic JSJ tree for $\mathbb{F}$ relative to A, and let Λ be the associated graph of groups. Let $\bar{\beta} = (\beta^1, \ldots, \beta^r)$ and $\bar{\gamma} = (\gamma^1, \ldots, \gamma^s)$ be tuples of sandwich terms.*

Let Δ be a connected subgraph of groups of Λ which contains v_A and whose intersection with $\Lambda^{min}_{\mathbb{F}_A}$ is connected. Assume that

- *the minimal subgraph of any element β^j of $\bar{\beta}$ lies in Δ, and*
- *the intersection of Δ with any block of $\Lambda^{min}_{A\bar{\gamma}}$ is contained in a disjoint union of envelopes of rigid vertices.*

Then the tuples $\bar{\beta}$ and $\bar{\gamma}$ are independent over A.

Proof. Pick a maximal subtree of $\Delta \cap \Lambda^{min}_{\mathbb{F}_A}$, which is possible since the intersection is connected, and extend it to a maximal subtree of $\Lambda^{min}_{\mathbb{F}_A}$. Lift it to a subtree T^0 of T which contains v_A, and extend this to a Bass-Serre presentation $(T^0, T^1, \{t_e\}_{e \in E_1})$ for Λ which contains a connected lift of Δ and in which the lifts of edges with trivial edge group are adjacent to v_A. To see that this is possible recall that $\Delta \cap \Lambda^{min}_{\mathbb{F}_A}$ contains v_A and that Λ consists of $\Lambda^{min}_{\mathbb{F}_A}$ together with edges with trivial edge group incident to v_A in Λ. This furthermore shows that Δ consists of its intersection with $\Lambda^{min}_{\mathbb{F}_A}$ together with several edges with trivial edge group.

Let $T_{\Delta \cap \Lambda^{min}_{\mathbb{F}_A}}$ be the connected component of the inverse image of $\Delta \cap \Lambda^{min}_{\mathbb{F}_A}$ in T containing v_A, and let H_0 be the stabilizer of $T_{\Delta \cap \Lambda^{min}_{\mathbb{F}_A}}$. By construction the minimal subgraph of H_0 is contained in $\Delta \cap \Lambda^{min}_{\mathbb{F}_A}$.

Every element β^j of the tuple $\bar{\beta}$ is a sandwich term with minimal subgraph contained in Δ, therefore it can be written as a product of the form $\beta^j = g_j t_{e_j} \beta^j_0 t^{-1}_{e'_j} g'_j$ where

- $g_j, g'_j \in \mathrm{Stab}(v_A)$,
- e_j, e'_j are trivially stabilized edges of $T^1 - T^0$ joining v_A to a vertex in a translate of $T_{\Delta \cap \Lambda^{min}_{\mathbb{F}_A}}$,
- $\beta^j_0 \in H_0$.

Note that the images of e_j and e'_j in Λ are edges of Δ, and that if the path $[v_A, \beta^j \cdot v_A]$ in T does not contain any trivially stabilized edge, and is thus contained in a translate of $T^{min}_{\mathbb{F}_A}$, then the elements t_{e_j} and $t_{e'_j}$ in the product are the identity element.

Moreover, the edges $\{e_j, e'_j \mid j = 1 \ldots, r\}$ do not lie in any translate of $T^{min}_{A\bar{\gamma}}$: Suppose on the contrary that there exist edges e_j, e'_j which lie in a translate of $T^{min}_{A\bar{\gamma}}$. Then their images in Λ are contained in the intersection of Δ with $\Lambda^{min}_{A\bar{\gamma}}$, and thus the edges e_j, e'_j are each contained in an envelope of a rigid vertex by our hypotheses. Hence, by definition they lie in a translate of $T^{min}_{\mathbb{F}_A}$. Remember that $T^{min}_{\mathbb{F}_A}$ is the pointed cyclic JSJ tree for $\mathbb{F}_A$ relative to A and thus does not contain any trivially stabilized edges, which gives a contradiction.

In particular, by setting $\mathbb{F}'$ as the fundamental group of the connected graph of groups $\Lambda^{min}_{\mathbb{F}_A} \cup \Lambda^{min}_{A\bar{\gamma}}$ this means that $\mathbb{F}$ admits a free product decomposition $\mathbb{F} = \mathbb{F}' * \mathbb{F}''$ where $\mathbb{F}_A \subseteq \mathbb{F}', \bar{\gamma} \in \mathbb{F}'$ and $t_{e_j}, t_{e'_j} \in \mathbb{F}''$ for all $j = 1, \ldots, r$. Thus, by Theorem 4.17 we get

$$t_{e_1} t_{e'_1} \ldots t_{e_r} t_{e'_r} \mathop{\smile\!\!\!\!\!\mid}_{\varnothing} \mathbb{F}_A \bar{\gamma}.$$

Let $\bar{\beta}_0 = (\beta_0^1, \ldots, \beta_0^r, g_1, g'_1, \ldots, g_r, g'_r)$. Then the minimal subgraph $\Lambda^{min}_{A\bar{\beta}_0}$ lies in $\Delta \cap \Lambda^{min}_{\mathbb{F}_A}$ because β_0^j lies in H_0 for each j, the elements g_j, g'_j lie in $\mathrm{Stab}(v_A)$ for each j, and the minimal subgraph of H_0 and the vertex v_A lie in $\Delta \cap \Lambda^{min}_{\mathbb{F}_A}$. In particular, the minimal subgraph $\Lambda^{min}_{A\bar{\beta}_0}$ lies in Δ, and thus its intersection with any block of $\Lambda^{min}_{A\bar{\gamma}}$ is contained in a disjoint union of envelopes of rigid vertices by our hypotheses. Furthermore, because the minimal subgraph $\Lambda^{min}_{A\bar{\beta}_0}$ lies in $\Lambda^{min}_{\mathbb{F}_A}$, the tuple $\bar{\beta}_0$ lies in $\mathbb{F}_A$.

Denote by $\mathbb{F}_{A\bar{\gamma}}$ the smallest free factor of $\mathbb{F}$ containing $A\bar{\gamma}$. By Theorem 2.16 the groups $\mathbb{F}_A$ and $\mathbb{F}_{A\bar{\gamma}}$ are elementary substructures of $\mathbb{F}$. Since $\mathbb{F}_A \subseteq \mathbb{F}_{A\bar{\gamma}}$, the group $\mathbb{F}_A$ is also an elementary substructure of $\mathbb{F}_{A\bar{\gamma}}$, and thus by Theorem 2.16 a free factor of $\mathbb{F}_{A\bar{\gamma}}$. Remember that $\beta_0 \in \mathbb{F}_A$ and that the minimal subgraph $\Lambda^{min}_{A\bar{\beta}_0}$ lies in Δ whose intersection with $\Lambda^{min}_{A\bar{\gamma}}$ is contained in a disjoint union of envelopes of rigid vertices by the hypotheses. Therefore, we can apply Proposition 5.3 to $\mathbb{F}_{A\bar{\gamma}}$ and the tuples $\bar{\beta}_0$ and $\bar{\gamma}$ and get

$$\bar{\beta}_0 \mathop{\smile\!\!\!\!\!\mid}_{A} \bar{\gamma}.$$

On the other hand, we proved that $t_{e_1} t_{e'_1} \ldots t_{e_r} t_{e'_r} \mathop{\smile\!\!\!\!\!\mid}_{\varnothing} \mathbb{F}_A \bar{\gamma}$. Monotonicity of Fact 2.13 implies that $t_{e_1} t_{e'_1} \ldots t_{e_r} t_{e'_r} \mathop{\smile\!\!\!\!\!\mid}_{\varnothing} A\bar{\beta}_0\bar{\gamma}$ because A and $\bar{\beta}_0$ are contained in $\mathbb{F}_A$. Now the Facts finite basis and vii) of 2.13 imply $t_{e_1}t_{e'_1}\dots t_{e_r}t_{e'_r} \perp_{A\bar{\beta}_0} \bar{\gamma}$, so by normality of Fact 2.13 we have

$$A\bar{\beta}_0 t_{e_1}t_{e'_1}\dots t_{e_r}t_{e'_r} \perp_{A\bar{\beta}_0} \bar{\gamma}.$$

Because $\bar{\beta}_0$ is independent from $\bar{\gamma}$ over A, normality also gives us $A\bar{\beta}_0 \perp_A \bar{\gamma}$. We apply transitivity to $A \subseteq A\bar{\beta}_0 \subseteq A\bar{\beta}_0 t_{e_1}t_{e'_1}\dots t_{e_r}t_{e'_r}$ and γ, which implies

$$A\bar{\beta}_0 t_{e_1}t_{e'_1}\dots t_{e_r}t_{e'_r} \perp_A \bar{\gamma}.$$

Remember that $\bar{\beta} = (\beta^1, \dots, \beta^r)$ where β^j can be written as a product of the form $\beta^j = g_j t_{e_j} \beta_0^j t_{e'_j}^{-1} g'_j$, and that $\bar{\beta}_0 = (\beta_0^1, \dots, \beta_0^r, g_1, g'_1, \dots, g_r, g'_r)$. Fact vi) implies $\operatorname{acl}(A\bar{\beta}_0 t_{e_1}t_{e'_1}\dots t_{e_r}t_{e'_r}) \perp_{\operatorname{acl}(A)} \operatorname{acl}(A\gamma)$. Since $\beta_1, \dots, \beta_r$, and therefore $\bar{\beta}$, are elements of $\operatorname{acl}(A\bar{\beta}_0 t_{e_1}t_{e'_1}\dots t_{e_r}t_{e'_r})$, we get by monotonicity $\bar{\beta} \perp_A \bar{\gamma}$. □

Remark 5.7 ([PS16, Remark 4.8])**.** Proposition 5.5 follows from Proposition 5.6: Let β, γ be sandwich terms such that the intersection of the minimal subgraphs $\Lambda_{A\beta}^{min}$ and $\Lambda_{A\gamma}^{min}$ is contained in a disjoint union of envelopes of rigid vertices. To show that β and γ satisfy the conditions of Proposition 5.6 let P be a possibly trivial path in $\Lambda_{\mathbb{F}_A}^{min}$ from v_A to $\operatorname{imp}(\beta) \cup \operatorname{imp}(\gamma)$. Recall that the imprint of a sandwich term g is the image in Λ of the middle segment of $[v_A, g \cdot v_A]$ which lies in a translate of $T_{\mathbb{F}_A}^{min}$. Without loss of generality the endpoint of P lies in $\operatorname{imp}(\beta)$. Set $\Delta = P \cup \Lambda_{A\beta}^{min}$.

Then Δ fulfils the hypotheses of Proposition 5.6: Apart from the last one all hypotheses follow immediately. To see that the intersection of Δ with any block of $\Lambda_{A\gamma}^{min}$ is contained in a disjoint union of envelopes of rigid vertices first notice that this is true for the intersection of $\Lambda_{A\beta}^{min}$ and $\Lambda_{A\gamma}^{min}$ by our hypothesis. Furthermore, since P is a path in $\Lambda_{\mathbb{F}_A}^{min}$ from v_A to $\operatorname{imp}(\beta) \cup \operatorname{imp}(\gamma)$ with endpoint in $\operatorname{imp}(\beta)$, its intersection with $\Lambda_{A\gamma}^{min}$ can only contain v_A, which is a rigid vertex, and the end vertex of P, which lies in $\operatorname{imp}(\beta)$ and thus in $\Lambda_{A\beta}^{min}$. So the intersection of $\Delta = P \cup \Lambda_{A\beta}^{min}$ and $\Lambda_{A\gamma}^{min}$ is contained in a disjoint union of envelopes of rigid vertices and we can apply Proposition 5.6 and get $\beta \perp_A \gamma$.

Now we prove that tuples $\bar{b}, \bar{c}$ which fulfil the hypotheses of Theorem 5.4 consist of elements that can be written as products of sandwich terms whose minimal subgraphs have 'almost disjoint' blocks.

Proposition 5.8 ([PS20, Proposition 4.9]). *Let $\bar{b}, \bar{c}$ be tuples in a non-abelian finitely generated free group $\mathbb{F}$, and let $A \subseteq \mathbb{F}$ be a set of parameters.*

Suppose there exists a normalized pointed cyclic JSJ decomposition Λ for $\mathbb{F}$ relative to A in which the intersection of any two blocks of the minimal subgraphs $\Lambda^{min}_{A\bar{b}}$ and $\Lambda^{min}_{A\bar{c}}$ is contained in a disjoint union of envelopes of rigid vertices.

Then there exist finite sets B and C of sandwich terms such that

- *the intersection of a block of Λ^{min}_{AB} with a block of Λ^{min}_{AC} is contained in a disjoint union of envelopes of rigid vertices, and*
- *each element b^i of $\bar{b}$ (respectively c^j of $\bar{c}$) can be written as a product of elements of B (respectively C).*

Proof. Let T be the tree corresponding to Λ, and let b^i be an element of the tuple $\bar{b}$. Consider the path $[v_A, b^i \cdot v_A]$ in T, it lies in $T^{min}_{A\bar{b}}$. We subdivide it into finitely many subpaths, splitting $[v_A, b^i \cdot v_A]$ at every translate of v_A, that is, every subpath is of the form $[g \cdot v_A, h \cdot v_A]$ for $g, h \in \mathbb{F}$ and does not contain any translates of v_A other than its endpoints. Remember that T is normalized and every trivially stabilized edge in T is incident to a translate of v_A. Thus, each subpath contains at most two trivially stabilized edges which must appear as first and/or last edge of the path.

Then the translate of the subpath $[g \cdot v_A, h \cdot v_A]$ by g^{-1}, which is $[v_A, g^{-1}h \cdot v_A]$, also contains at most two trivially stabilized edges which must appear as first and/or last edge, and the middle segment lies in a translate of $T^{min}_{\mathbb{F}_A}$ because T is normalized. Therefore, the element $\beta := g^{-1}h$ is a sandwich term, and $[v_A, \beta \cdot v_A]$ lies in a translate of $T^{min}_{A\bar{b}}$ because $[g \cdot v_A, h \cdot v_A]$ lies in $T^{min}_{A\bar{b}}$. Hence, $\Lambda^{min}_{A\beta}$ lies in a block of $\Lambda^{min}_{A\bar{b}}$. By construction, b^i is a product of such terms β. Let B be the set of all these β for every $b^i \in \bar{b}$. Then the second assertion follows for $\bar{b}$.

Similarly, each term in the tuple $\bar{c}$ can be written as a product of sandwich terms γ such that $\Lambda^{min}_{A\gamma}$ lies in a block of $\Lambda^{min}_{A\bar{c}}$. Let C be the set of all these γ for every term in $\bar{c}$. Then the second assertion follows for $\bar{c}$.

The first assertion follows by our hypothesis on the minimal subgraphs of $A\bar{b}$ and $A\bar{c}$ and by definition of B and C. $\square$

The idea of the proof of Theorem 5.4 is to divide the sets of sandwich terms B and C obtained above so that their minimal subgraphs are alternately contained in a growing chain of connected subgraphs of Λ and then to use Proposition 5.6 inductively to prove the result. We first show how this works on an example. Note that in this example we changed the names of $\bar{b}$ and $\bar{c}$ of Example 4.10 in [PS20] because then it is easier to see that this is the same construction as in the proof of Theorem 5.4.

Example 5.9 ([PS16, Example 4.10])**.** Let $\mathbb{F}$ be a non-abelian finitely generated free group, let A be a subset of $\mathbb{F}$, and let Λ be a normalized pointed cyclic JSJ decomposition as in Figure 7 where A is contained in U_1. Note that $\Lambda^{min}_{\mathbb{F}_A}$ is exactly the path between U_1 and U_4 which contains no edge with trivial edge group. Let $b = (s_2u_2s_2^{-1})(s_3u_3u_4u_3s_3^{-1})$ and $c = (t_3v_3v_2t_2^{-1})(t_4v_4t_4^{-1})$ where u_i, v_i are elements of the vertex group U_i, and s_i (respectively t_i) is a Bass-Serre element corresponding to the edge with trivial edge group on the right-hand side (respectively left-hand side) joining the base vertex to the vertex stabilized by U_i.

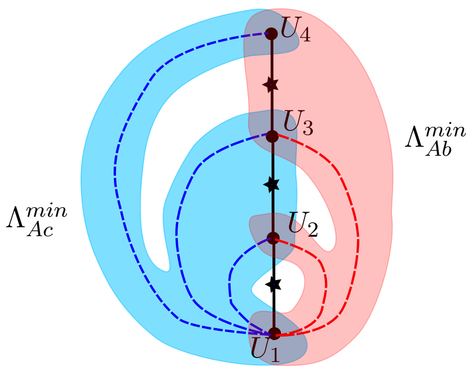


Figure 7: The normalized pointed cylic JSJ decomposition in Example 5.9. (Dot vertices represent rigid non-Z-type vertices, stars represent Z-type vertices, dashed edges have an associated trivial edge group and full ones have an associated infinite cyclic edge group.) (Modified from Source: [PS20, p. 16])

Decompose b and c as products of sandwich terms, as $b = \beta_1\beta_2$ and $c = \gamma_1\gamma_2$ where $\beta_1 = s_2u_2s_2^{-1}, \beta_2 = s_3u_3u_4u_3s_3^{-1}$ and $\gamma_1 = t_3v_3v_2t_2^{-1}, \gamma_2 = t_4v_4t_4^{-1}$. Then the minimal subgraphs of β_1 and β_2 are exactly the blocks of Λ^{min}_{Ab}, the minimal subgraphs of γ_1 and γ_2 are exactly the blocks of Λ^{min}_{Ac}, and the intersection between any two blocks of Λ_{Ab}^{min} and Λ_{Ac}^{min}, even the intersection of Λ_{Ab}^{min} and Λ_{Ac}^{min} itself, is a disjoint union of rigid vertices.

Let Δ be the subgraph of groups consisting of the vertices with vertex groups U_1 and U_2, the path in $\Lambda_{\mathbb{F}_A}^{min}$ between them, and the trivially stabilized edge with associated Bass-Serre element s_2. Let $\bar{\beta} = (\beta_1)$ and $\bar{\gamma} = (\gamma_1)$. Then we can apply Proposition 5.6 with $\Delta, \bar{\beta}$, and $\bar{\gamma}$ and get $\beta_1 \mathop{\smile\!\!\!\!\!\!\,|}_A \gamma_1$. Normality of Fact 2.13 implies $A\beta_1 \mathop{\smile\!\!\!\!\!\!\,|}_A \gamma_1$.

Now let Δ be the subgraph of groups consisting of the path in $\Lambda_{\mathbb{F}_A}^{min}$ connecting the vertices with vertex groups U_1 and U_3 as well as the trivially stabilized edges labelled by s_2, t_2 and t_3, and let $\bar{\beta} = (\beta_1, \gamma_1)$ and $\bar{\gamma} = (\beta_2)$. We apply Proposition 5.6 again with $\Delta, \bar{\beta}$, and $\bar{\gamma}$ and get $\beta_1\gamma_1 \mathop{\smile\!\!\!\!\!\!\,|}_A \beta_2$. Thus, by Fact vii) we have in particular $\gamma_1 \mathop{\smile\!\!\!\!\!\!\,|}_{A\beta_1} \beta_2$, and using normality gives $\gamma_1 \mathop{\smile\!\!\!\!\!\!\,|}_{A\beta_1} A\beta_1\beta_2$. Now we can apply transitivity with $A \subseteq A\beta_1 \subseteq A\beta_1\beta_2$ and γ_1 and get that $\gamma_1 \mathop{\smile\!\!\!\!\!\!\,|}_A A\beta_1\beta_2$, thus $A\gamma_1 \mathop{\smile\!\!\!\!\!\!\,|}_A \beta_1\beta_2$.

Finally, let Δ be the complement in Λ of the edge with Bass-Serre element t_4, and let $\bar{\beta} = (\beta_1, \beta_2, \gamma_1)$ and $\bar{\gamma} = (\gamma_2)$. Then Proposition 5.6 gives $\beta_1\beta_2\gamma_1 \mathop{\smile\!\!\!\!\!\!\,|}_A \gamma_2$, so by Fact vii) and normality we get $\beta_1\beta_2 \mathop{\smile\!\!\!\!\!\!\,|}_{A\gamma_1} A\gamma_1\gamma_2$. Applying transitivity with $A \subseteq A\gamma_1 \subseteq A\gamma_1\gamma_2$ and $\beta_1\beta_2$ gives $\beta_1\beta_2 \mathop{\smile\!\!\!\!\!\!\,|}_A \gamma_1\gamma_2$, which implies $b \mathop{\smile\!\!\!\!\!\!\,|}_A c$.

Now we generalize this example to get partitions of B and C which fulfil a certain disjointness property.

Proposition 5.10 ([PS20, Proposition 4.11])**.** *Let $\mathbb{F}$ be a non-abelian finitely generated free group, let $A \subseteq \mathbb{F}$ be a set of parameters, and let Λ be a normalized pointed cyclic JSJ decomposition for $\mathbb{F}$ relative to A.*

Suppose $\{v_A\} = \Delta_0 \subseteq \Delta_1 \subseteq \ldots \subseteq \Delta_s = \Lambda$ is a chain of connected subgraphs of groups of Λ such that the intersection of each Δ_i with $\Lambda_{\mathbb{F}_A}^{min}$ is connected. Let B, C be finite sets of sandwich terms such that no trivially stabilized edge is associated to both an element β of B and an element γ of C.

For every $\beta_0 \in B$ denote by $\mathrm{Block}_B(\beta_0)$ the block of Λ_{AB}^{min} containing $\Lambda_{A\beta_0}^{min}$, and for every $\gamma_0 \in C$ denote by $\mathrm{Block}_C(\gamma_0)$ the block of Λ_{AC}^{min} containing $\Lambda_{A\gamma_0}^{min}$.

Suppose B and C can be partitioned as $B = \bigsqcup_{i=0}^{\lfloor (s-1)/2 \rfloor} B_i$ and $C = \bigsqcup_{i=0}^{\lfloor s/2 \rfloor} C_i$ such that

- *any two terms $\beta, \beta' \in B$ such that $\mathrm{Block}_B(\beta) = \mathrm{Block}_B(\beta')$ are in the same subset B_i, and any two terms $\gamma, \gamma' \in C$ such that $\mathrm{Block}_C(\gamma) = \mathrm{Block}_C(\gamma')$ are in the same subset C_i;*

- *for each i and each $\beta \in B_i$ the block* $\mathrm{Block}_B(\beta)$ *lies in Δ_{2i+1}, and its intersection with Δ_{2i} is contained in a disjoint union of envelopes of rigid vertices;*
- *for each i and each $\gamma \in C_i$ the block* $\mathrm{Block}_C(\gamma)$ *lies in Δ_{2i}, and its intersection with Δ_{2i-1} is contained in a disjoint union of envelopes of rigid vertices.*

Then $B \mathop{\smile\!\!\!\!\!\!\mid}_A C$.

Note that we assume that the intersection of every Δ_i with $\Lambda^{min}_{\mathbb{F}_A}$ is connected, which is not the case in Proposition 4.11 of [PS20]. However, we want to apply Proposition 5.6 to these Δ_i, so we need the above assumption. In the proof of Theorem 5.4, where we need Proposition 5.10, the intersection of $\Lambda^{min}_{\mathbb{F}_A}$ with every graph in the constructed chain of subgraphs is connected by construction, so we can make the above assumption without any issues.

Proof. We prove this by induction on the length of the chain of subgraphs of groups. If $s = 1$, then the minimal subgraph of any term in C is contained in $\{v_A\}$. Hence, we can apply Proposition 5.6 to $\Delta = \{v_A\}$, interpreting B and C as tuples of sandwich terms, and get $B \mathop{\smile\!\!\!\!\!\!\mid}_A C$.

Suppose the Proposition holds for chains of length at most s. We want to prove it for a chain $\{v_A\} = \Delta_0 \subseteq \Delta_1 \subseteq \ldots \subseteq \Delta_s \subseteq \Delta_{s+1} = \Lambda$. First we consider the case $s = 2k$. Since we have a chain of length $2k+1$, the hypotheses give us partitions $B = \bigsqcup_{i=0}^{k} B_i$ and $C = \bigsqcup_{i=0}^{k} C_i$. We get the following:

i) By our hypotheses the minimal subgraphs of all terms contained in $B_0 \ldots B_{k-1} C_0 \ldots C_k$ lie in $\Delta_{2k} = \Delta_s$, and any block of $\Lambda^{min}_{AB_k}$ intersects Δ_s in at most a disjoint union of envelopes of rigid vertices. Thus, the hypotheses of Proposition 5.6 are satisfied for Δ_s and the tuples $B_0 \ldots B_{k-1} C_0 \ldots C_k$ and B_k. We get $B_0 \ldots B_{k-1} C_0 \ldots C_k \mathop{\smile\!\!\!\!\!\!\mid}_A B_k$, and Fact vii) implies $C_0 \ldots C_k \mathop{\smile\!\!\!\!\!\!\mid}_{AB_0 \ldots B_{k-1}} B_k$. Hence, by normality of Fact 2.13 it holds
$$C_0 \ldots C_k \mathop{\smile\!\!\!\!\!\!\mid}_{AB_0 \ldots B_{k-1}} AB_0 \ldots B_{k-1} B_k.$$

ii) The induction hypothesis can now be applied to the sets $\bigsqcup_{i=0}^{k-1} B_i$ and C and the chain $\{v_A\} = \Delta_0 \subseteq \Delta_1 \subseteq \ldots \subseteq \Delta_{s-1} \subseteq \Delta_{s+1} \subseteq \Lambda$ where we skip Δ_s, and we get $B_0 \ldots B_{k-1} \mathop{\smile\!\!\!\!\!\!\mid}_A C_0 \ldots C_k$. Hence, by normality it holds
$$AB_0 \ldots B_{k-1} \mathop{\smile\!\!\!\!\!\!\mid}_A C_0 \ldots C_k.$$

Now we apply transitivity to $A \subseteq AB_0 \dots B_{k-1} \subseteq AB_0 \dots B_{k-1}B_k$ and $C_0 \dots C_k$ and get $B_0 \dots B_k \mathop{\smile\!\!\!\!\!\!|}_A C_0 \dots C_k$, as wanted.

In the odd case, where $s = 2k+1$, we get partitions $B = \bigsqcup_{i=0}^{k} B_i$ and $C = \bigsqcup_{i=0}^{k+1} C_i$. Analogous to i) we get $B_0 \dots B_k \mathop{\smile\!\!\!\!\!\!|}_{AC_0 \dots C_k} AC_0 \dots C_k C_{k+1}$, and analogous to ii) we get $AC_0 \dots C_k \mathop{\smile\!\!\!\!\!\!|}_A B_0 \dots B_k$. Finally, we can apply transitivity to $A \subseteq AC_0 \dots C_k \subseteq AC_0 \dots C_k C_{k+1}$ and $B_0 \dots B_k$ and get $B_0 \dots B_k \mathop{\smile\!\!\!\!\!\!|}_A C_0 \dots C_{k+1}$, as wanted. □

Finally we can prove Theorem 5.4.

Proof of Theorem 5.4. Let B and C be the sets of sandwich terms given by Proposition 5.8. We build a sequence of connected subgraphs of the form $\{v_A\} = \Delta_0 \subseteq \Delta_1 \subseteq \dots \subseteq \Delta_s = \Lambda$ satisfying the conditions of Proposition 5.10 as follows:

Set $\Delta_0 = \{v_A\}$ and suppose we have built Δ_i such that the conditions of Proposition 5.10 are satisfied for all Δ_j with $j \leq i$. If Δ_i contains all the blocks $\mathrm{Block}_B(\beta)$ and $\mathrm{Block}_C(\gamma)$ for all $\beta \in B$ and $\gamma \in C$, then we set $\Delta_{i+1} = \Lambda$ and we are done.

If not, consider all the blocks $\mathrm{Block}_B(\beta)$ which are not contained in Δ_i and which fulfil the following property: there is a path in $\Lambda^{min}_{\mathbb{F}_A}$ between Δ_i and $\mathrm{Block}_B(\beta)$ which does not intersect any block $\mathrm{Block}_C(\gamma) \nsubseteq \Delta_i$ in more than a disjoint union of envelopes of rigid vertices. If they exist, we build Δ_{i+1} by adding to Δ_i these blocks as well as the paths joining them to Δ_i. Let $B_{\lfloor i/2 \rfloor}$ be the set of all sandwich terms of B whose minimal subgraphs we added to Δ_i.

If there do not exist blocks $\mathrm{Block}_B(\beta)$ with this property, then consider the blocks $\mathrm{Block}_C(\gamma)$ which are not contained in Δ_i and such that there exists a path in $\Lambda^{min}_{\mathbb{F}_A}$ between Δ_i and $\mathrm{Block}_C(\gamma)$ which does not intersect any block $\mathrm{Block}_B(\beta) \nsubseteq \Delta_i$ in more than a disjoint union of envelopes of rigid vertices. Blocks like this have to exist: If $\mathrm{Block}_B(\beta) \subseteq \Delta_i$ for all $\beta \in B$, this is clear because Δ_i then does not contain all blocks $\mathrm{Block}_C(\gamma)$ and the paths between these blocks and Δ_i cannot intersect any $\mathrm{Block}_B(\beta) \nsubseteq \Delta_i$ because there are none. If there exists a $\beta \in B$ with $\mathrm{Block}_B(\beta) \nsubseteq \Delta_i$, then every path in $\Lambda^{min}_{\mathbb{F}_A}$ joining it to Δ_i intersects a block $\mathrm{Block}_C(\gamma) \nsubseteq \Delta_i$ in more than a disjoint union of envelopes of rigid vertices. Thus, there exists an element $\gamma \in C$ with $\mathrm{Block}_C(\gamma) \nsubseteq \Delta_i$. Either there exists a path in $\Lambda^{min}_{\mathbb{F}_A}$ from Δ_i to $\mathrm{Block}_C(\gamma)$ with the desired property, which we wanted to show, or every path in $\Lambda^{min}_{\mathbb{F}_A}$ from Δ_i

to $\mathrm{Block}_C(\gamma)$ intersects a block $\mathrm{Block}_B(\beta') \nsubseteq \Delta_i$ in more than a disjoint union of envelopes of rigid vertices. In this case we apply the same argumentation as above until we find a block $\mathrm{Block}_C(\gamma')$ with the desired property. Now construct Δ_{i+1} similarly as in the previous case, and let $C_{\lfloor (i+1)/2 \rfloor}$ be the set of all sandwich terms of C whose minimal subgraphs we added to Δ_i.

We want to apply Proposition 5.10 to the chain of subgraphs and to B and C. Note that if we added blocks $\mathrm{Block}_B(\beta)$ to Δ_0, then for all $\beta \in B_i$ it holds that $\mathrm{Block}_B(\beta) \subseteq \Delta_{2i+1}$, and for all $\gamma \in C_i$ we have $\mathrm{Block}_C(\gamma) \subseteq \Delta_{2i}$. In this case we define the set C_0 to be the empty set. If we added blocks $\mathrm{Block}_C(\gamma)$ to Δ_0, then we change the names of B and C and get the same result.

To see that for every i and each $\beta \in B_i$ the intersection of $\mathrm{Block}_B(\beta)$ with Δ_{2i} is contained in a disjoint union of envelopes of rigid vertices remember that it consists of certain blocks $\mathrm{Block}_B(\beta')$ and $\mathrm{Block}_C(\gamma')$ and paths between these blocks and Δ_j for $j < 2i$. Let $\beta \in B_i$ for an arbitrary i. By Proposition 5.8 the intersection of $\mathrm{Block}_B(\beta)$ with $\mathrm{Block}_C(\gamma)$ is contained in a disjoint union of envelopes of rigid vertices for every $\gamma \in C$. By definition of the blocks the intersection of $\mathrm{Block}_B(\beta)$ with $\mathrm{Block}_B(\beta')$ can at most contain v_A and distinct Z-type vertices for every $\beta' \in B$. Thus, their intersection is also contained in a disjoint union of envelopes of rigid vertices. Let P be a path we added to Δ_j for $j < 2i$. If P is a path from Δ_j to a block $\mathrm{Block}_C(\gamma)$, then it intersects all blocks $\mathrm{Block}_B(\beta') \nsubseteq \Delta_j$ at most in a disjoint union of envelopes of rigid vertices by construction. Hence, this holds in particular for $\mathrm{Block}_B(\beta)$. If P is a path from Δ_j to a block $\mathrm{Block}_B(\beta')$, then it does not intersect $\mathrm{Block}_B(\beta)$ because otherwise we would have added $\mathrm{Block}_B(\beta)$ to Δ_j. Thus, we get the desired result. Analogous we get that for every i and each $\gamma \in C_i$ the intersection of $\mathrm{Block}_C(\gamma)$ with Δ_{2i-1} is contained in a disjoint union of envelopes of rigid vertices.

Finally, up to switching B and C, the partitions and the chain of subgraphs $\{v_A\} = \Delta_0 \subseteq \Delta_1 \subseteq \ldots \subseteq \Delta_s = \Lambda$ fulfil the conditions of Proposition 5.10, and we get $B \mathop{\smile\!\!\!\!\!\!\mid}_A C$, thus $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{c}$. Note that we may have to add empty sets as last items of the partitions to write B and C in the desired form. $\square$

The construction above is exactly the construction we used in Example 5.9 to prove independence.

Remark 5.11. We apply the construction of the above proof to Example 5.9. First we get sets $B = \{\beta_1, \beta_2\}$ and $C = \{\gamma_1, \gamma_2\}$, and let $\Delta_0 = \{v_A\}$. Then we define Δ_1 to be the subgraph of groups consisting of $\mathrm{Block}_B(\beta_1)$ and the path in $\Lambda^{min}_{\mathbb{F}_A}$ from Δ_0 to $\mathrm{Block}_B(\beta_1)$. The subgraph Δ_2 is defined as Δ_1 where we added $\mathrm{Block}_C(\gamma_1)$, the path in $\Lambda^{min}_{\mathbb{F}_A}$ from Δ_1 to $\mathrm{Block}_C(\gamma_1)$ is trivial. To Δ_2 we add $\mathrm{Block}_B(\beta_2)$ to get Δ_3, and Δ_4 is obtained by furthermore adding $\mathrm{Block}_C(\gamma_2)$. Finally we set $\Delta_5 = \Lambda$. This is exactly the chain of subgraphs we used in Example 5.9 to prove independence.

Furthermore, we get partitions $B = \bigsqcup_{i=0}^{\lfloor (5-1)/2 \rfloor} B_i$ and $C = \bigsqcup_{i=0}^{\lfloor 5/2 \rfloor} C_i$ where $B_0 = \{\beta_1\}, B_1 = \{\beta_2\}, B_2 = \varnothing$ and $C_0 = \varnothing, C_1 = \{\gamma_1\}, C_2 = \{\gamma_2\}$. Hence, the chain of subgraphs of groups $\{v_A\} = \Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_5 = \Lambda$ and B, C satisfy all conditions of Proposition 5.10, and we get $B \mathop{\smile\hspace{-0.9em}\mid}_A C$, and therefore $b \mathop{\smile\hspace{-0.9em}\mid}_A c$.

5.3 Proving existence of an adequate decomposition

The exposition in this subsection closely follows Section 5 of [PS20]. In this subsection we prove the second direction of Theorem 4.15, namely:

Theorem 5.12 ([PS20, Proposition 5.1])**.** *Let $\bar{b}, \bar{c}$ be tuples in a non-abelian finitely generated free group $\mathbb{F}$, and let $A \subseteq \mathbb{F}$ be a set of parameters. Suppose $\bar{b}$ and $\bar{c}$ are independent over A.*

Then there exists a normalized pointed cyclic JSJ decomposition Λ for $\mathbb{F}$ relative to A in which the intersection of any two blocks of the minimal subgraphs $\Lambda^{min}_{A\bar{b}}$ and $\Lambda^{min}_{A\bar{c}}$ is contained in a disjoint union of envelopes of rigid vertices.

To prove this theorem, we construct an adequate JSJ decomposition for $\mathbb{F}$ relative to A. A key idea of the proof is the same as in the proof of the second direction of Theorem 3.4 in [PS16], which is Theorem 4.17 in this thesis, that deals with the special case where A is a free factor of $\mathbb{F}$, i.e. $\mathbb{F} = F * A$. It states that $\bar{b}$ and $\bar{c}$ are independent over A if and only if there exists a free product decomposition $\mathbb{F} = \mathbb{F}_{\bar{b}} * A * \mathbb{F}_{\bar{c}}$ of $\mathbb{F}$ with $\bar{b} \in \mathbb{F}_{\bar{b}} * A$ and $\bar{c} \in A * \mathbb{F}_{\bar{c}}$. In the case where $\bar{b}$ is independent from $\bar{c}$ over A, Perin and Sklinos consider the free group $\hat{\mathbb{F}} = F * A * F'$ where F' is a copy of F. By the first direction of the theorem, $\bar{b}$ is independent over A from the tuple $\bar{c}'$ of $A * F'$ corresponding to $\bar{c}$. Then they use stationarity of types and homogeneity of the free group to prove the existence of an adequate free product decomposition of $\mathbb{F}$.

To prove Theorem 5.12 we also consider a 'doubled' model $\hat{\mathbb{F}} = F * \mathbb{F}_A * F''$ where $\mathbb{F} = F * \mathbb{F}_A$ with $\mathbb{F}_A$ is the smallest free factor of $\mathbb{F}$ containing A. Similarly as in the before mentioned proof, we want to apply the first direction of the main theorem, which is Theorem 5.4, to $\hat{\mathbb{F}}$ to prove that $\bar{b}$ is independent from $\bar{c}'$, the tuple of $\mathbb{F}_A * F''$ corresponding to $\bar{c}$, over A. Thus, we need to construct a normalized pointed cyclic JSJ decomposition for $\hat{\mathbb{F}}$ relative to A in which the blocks of the minimal subgraphs of $\langle A, \bar{b}\rangle$ and $\langle A, \bar{c}'\rangle$ intersect at most in a disjoint union of envelopes of rigid vertices. In fact, we only need to show that the parts of the minimal subgraphs of $\langle A, \bar{b}\rangle$ and $\langle A, \bar{c}'\rangle$ lying in Λ_A, which is the pointed cyclic JSJ decomposition for $\mathbb{F}_A$ relative to A, do not intersect much. Proposition 5.13, which we do not prove in this thesis, helps to construct this normalized pointed cyclic JSJ decomposition. Then we continue along the lines of the proof in the case where A is a free factor of $\mathbb{F}$.

We only cite the following Proposition without proving it.

Proposition 5.13 ([PS20, Proposition 5.3]). *Let $\mathbb{F}$ be a non-abelian finitely generated free group, and let $A \subseteq \mathbb{F}$ be a set of parameters. Let $\bar{b}$ be a tuple in $\mathbb{F}$ such that $\mathbb{F}$ is freely indecomposable with respect to $A\bar{b}$. Denote by $\mathbb{F}_A$ the smallest free factor of $\mathbb{F}$ containing A, by $\Lambda_{\mathbb{F}_A}$ the pointed cyclic JSJ decomposition of $\mathbb{F}_A$ relative to A, and by T the corresponding tree.*

There are subgraphs $\Gamma^1_{A\bar{b}}, \ldots, \Gamma^r_{A\bar{b}}$ of $\Lambda_{\mathbb{F}_A}$ such that the following hold:

i) If $i \neq j$, then $\Gamma^i_{A\bar{b}}$ and $\Gamma^j_{A\bar{b}}$ intersect at most in a disjoint union of Z-type vertices.

ii) Suppose $e = (z, x)$ and $e' = (z, y)$ are adjacent edges of $\Gamma^i_{A\bar{b}}$ for some i, that x, y are neither Z-type nor surface-type vertices, and let the edges $\hat{e} = (\hat{z}, \hat{x}), \hat{e}' = (\hat{z}, \hat{y})$ be adjacent lifts of e, e' in T. Denote by s_x, s_y some generating tuples for the stabilizers of $\hat{x}, \hat{y}$ respectively. Then the conjugacy class of (s_x, s_y) lies in $\mathrm{acl}^{eq}(A\bar{b})$.

iii) Suppose v is a surface-type vertex of $\Gamma^i_{A\bar{b}}$ for some i, and let $\hat{v}$ be a lift of v in T. Then $\mathrm{acl}^{eq}(A\bar{b})$ contains the conjugacy class of an element in the stabilizer of $\hat{v}$ which corresponds to a non-boundary-parallel simple closed curve.

Moreover, there is a normalized pointed cyclic JSJ decomposition Λ for $\mathbb{F}$ relative to A such that the intersection of any block of the minimal subgraph $\Lambda^{min}_{A\bar{b}}$ with $\Lambda^{min}_{\mathbb{F}_A}$ (which is isomorphic to $\Lambda_{\mathbb{F}_A}$) is contained in $\Gamma^i_{A\bar{b}}$ for some i.

The following Proposition will be helpful in the proof of Theorem 5.12.

Proposition 5.14 ([PS20, Proposition 5.4]). *Let $\mathbb{F}$ be a non-abelian finitely generated free group, let $A \subseteq \mathbb{F}$ be a set of parameters, and let Λ be a normalized pointed cyclic JSJ decomposition for $\mathbb{F}$ relative to A. Let T be the tree corresponding to Λ.*

Suppose $e = (z, x)$ and $e' = (z, y)$ are two distinct edges of Λ with non-trivial edge group such that x, y are non-Z-type vertices, and let $\hat{e} = (\hat{z}, \hat{x}), \hat{e}' = (\hat{z}, \hat{y})$ be adjacent lifts of e, e' in T. Denote by s_x, s_y some generating tuples for the stabilizers of $\hat{x}, \hat{y}$ respectively. Then the conjugacy class of (s_x, s_y) is not in $\mathrm{acl}^{eq}(A)$.

Suppose v is a surface-type vertex of Λ, and let $\hat{v}$ be a lift of v in T, which means that the stabilizer of $\hat{v}$ is the fundamental group of a hyperbolic surface with boundary. Let g, h be two elements corresponding to non-boundary-parallel simple closed curves on this surface, i.e. they are elements of $\mathrm{Stab}(\hat{v})$. Then the conjugacy classes of g and h fork over A.

Note that we consider the edges and vertices in a normalized pointed cyclic JSJ decomposition, whereas in Proposition 5.4 of [PS20] they work with a normalized pointed cyclic JSJ tree. We want to say that at least one of two distinct non-Z-type vertices must have a stabilizer that is not infinite cyclic. This is only possible in a normalized pointed cyclic JSJ decomposition, not a tree, because we changed the definition of a Z-type vertex.

Proof. The edges e and e' are non-trivially stabilized and thus lie in $\Lambda^{min}_{\mathbb{F}_A}$ where $\mathbb{F}_A$ denotes the smallest free factor of $\mathbb{F}$ containing A. Therefore, we may assume, up to conjugation, that the tuple (s_x, s_y) lies in $\mathbb{F}_A$. By Lemma 2.9, slightly modified for the T^{eq}-construction, it is thus enough to show that the orbit of (s_x, s_y) under $\mathrm{Aut}_A(\mathbb{F}_A)$ contains infinitely many distinct conjugacy classes.

The vertices x and y are not of Z-type. This means that each one of them is either the base vertex or its vertex group is not infinite cyclic. At leat one of the vertices x, y is not the base vertex, thus we may assume without loss of generality that the stabilizer of $\hat{x}$ is not infinite cyclic. Note that in fact we may assume that it is not cyclic because it cannot be trivial since $\hat{e}$ is not trivially stabilized. Let τ_e be a Dehn twist about $\hat{e}$ by some element ε of $\mathrm{Stab}(\hat{e})$ such that τ_e restricts to a conjugation (and not the identity) on the group containing the stabilizer of $\hat{y}$. Since $\langle s_x \rangle$ is not abelian, the equation

$$\gamma(s_x, s_y)\gamma^{-1} = (s_x, \varepsilon^m s_y \varepsilon^{-m})$$

implies for every m that $\gamma = 1$. Hence, the pairs $\tau_e^m(s_x, s_y)$ lie in distinct conjugacy classes for ever m, which implies that the orbit of (s_x, s_y) under $\mathrm{Mod}_A(\mathbb{F}_A)$, and thus the orbit under $\mathrm{Aut}_A(\mathbb{F}_A)$, contains infinitely many conjugacy classes.

For the second part remember that Λ is a normalized pointed cyclic JSJ decomposition, thus the vertex v lies in $\Lambda_{\mathbb{F}_A}^{min}$ and we may assume, up to conjugation, that g and h lie in $\mathbb{F}_A$. By Theorem 4.18 the element g forks with h over A. Now Fact vi) implies that $\mathrm{acl}^{eq}(Ag)$ forks with $\mathrm{acl}^{eq}(Ah)$ over $\mathrm{acl}^{eq}(A)$, thus they fork over A. By monotonicity the conjugacy classes of g and h fork over A. □

Now we can prove Theorem 5.12 which finally proves the main result, Theorem 4.15.

Proof of Theorem 5.12. Assume $\bar{b} \perp_A \bar{c}$. We construct a normalized pointed cyclic JSJ decomposition for $\mathbb{F}$ relative to A in which any two blocks of the minimal subgraphs $\Lambda_{A\bar{b}}^{min}$ and $\Lambda_{A\bar{c}}^{min}$ intersect at most in a disjoint union of envelopes of rigid vertices.

Denote by $\mathbb{F}_A, \mathbb{F}_{A\bar{b}}$, and $\mathbb{F}_{A\bar{c}}$ the smallest free factors of $\mathbb{F}$ containing $A, A\bar{b}$ and $A\bar{c}$ respectively. Let $\Lambda_{\mathbb{F}_A}$ be the pointed cyclic JSJ decomposition for $\mathbb{F}_A$ relative to A, and let H be a subgroup of $\mathbb{F}$ such that $\mathbb{F} = H * \mathbb{F}_A$. Let $(\Gamma^j_{A\bar{b}})_{j=0}^r$ and $(\Gamma^j_{A\bar{c}})_{j=0}^s$ be the (possibly disconnected) subgraphs of $\Lambda_{\mathbb{F}_A}$ which can be obtained by applying Proposition 5.13 to $\mathbb{F}_{A\bar{b}}$ and $\mathbb{F}_{A\bar{c}}$ respectively. This Proposition also gives us normalized pointed cyclic JSJ decompositions $\tilde{\Lambda}_{A\bar{b}}$ for $\mathbb{F}_{A\bar{b}}$ relative to $A\bar{b}$ and $\tilde{\Lambda}_{A\bar{c}}$ for $\mathbb{F}_{A\bar{c}}$ relative to $A\bar{c}$ in which the intersection of every block of the minimal subgraph of $\langle A, \bar{b}\rangle$ (respectively $\langle A, \bar{c}\rangle$) with $\Lambda_{\mathbb{F}_A}$ lies in one of the subgraphs $\Gamma^j_{A\bar{b}}$ (respectively $\Gamma^j_{A\bar{c}}$). We can extend these JSJ decompositions to JSJ decompositions $\tilde{\Lambda}^+_{A\bar{b}}$ and $\tilde{\Lambda}^+_{A\bar{c}}$ for $\mathbb{F}$ relative to A by adding some trivially stabilized loops to the base vertex v_A. Note that $\tilde{\Lambda}^+_{A\bar{b}}$ and $\tilde{\Lambda}^+_{A\bar{c}}$ are normalized pointed cyclic JSJ decompositions for $\mathbb{F}$ relative to A in which he intersection of every block of the minimal subgraph of $\langle A, \bar{b}\rangle$ (respectively $\langle A, \bar{c}\rangle$) with $\Lambda_{\mathbb{F}_A}$ lies in one of the subgraphs $\Gamma^j_{A\bar{b}}$ (respectively $\Gamma^j_{A\bar{c}}$).

Suppose there exist i, j such that the intersection of $\Gamma^i_{A\bar{b}}$ and $\Gamma^j_{A\bar{c}}$ is not a disjoint union of envelopes of rigid vertices. There are three cases we need to consider. The intersection may be a union of envelopes of rigid vertices that is not disjoint, or the intersection may contain more than a disjoint union of envelopes of rigid vertices, i.e. it contains a surface-type vertex or two edges from the same cylinder.

Case 1: If the intersection is a union of envelopes of rigid vertices which is not disjoint, then there exist distinct rigid vertices x, y and envelopes H_x, H_y in $\Gamma^i_{A\bar{b}} \cap \Gamma^j_{A\bar{c}}$ of x and y respectively such that H_x and H_y intersect in a vertex z of Λ. Remember that envelopes do not contain surface-type vertices by Remark 4.10, so x, y and z cannot be of surface-type. Denote by $e = (x, z)$ and $e' = (y, z)$ the edges of H_x and H_y respectively incident to z. Remember that $\Gamma^i_{A\bar{b}}$ and $\Gamma^j_{A\bar{c}}$ lie in the pointed cyclic JSJ tree Λ_A, so in particular e and e' are not trivially stabilized. If both x and y are rigid non-Z-type vertices, then we know by item ii) of Proposition 5.13 that the conjugacy class of a tuple (s_x, s_y) is contained in the intersection of $\mathrm{acl}^{eq}(A\bar{b})$ and $\mathrm{acl}^{eq}(A\bar{c})$. By Proposition 5.14 the conjugacy class of (s_x, s_y) does not lie in $\mathrm{acl}^{eq}(A)$, a contradiction to $\bar{b}$ and $\bar{c}$ being independent over A. If at least one of the vertices x, y, assume that it is x, is of Z-type, then H_x only consists of the single edge e because all edges incident to x lie in the cylinder whose centre is x. Since $\Gamma^i_{A\bar{b}}$ and $\Gamma^j_{A\bar{c}}$ lie in the pointed cyclic JSJ tree Λ_A, the vertex z must be of non-Z-type, and y is either of Z-type or it is the base vertex v_A by Remark 4.7. In both cases the envelope H_y consists of the single edge e'. Now $H_x \cup H_y$ only consists of the edges e and e'. Since z cannot be of surface-type, it is a rigid non-Z-type vertex, so $H_x \cup H_y$ is an envelope of z and the intersection of $\Gamma^i_{A\bar{b}}$ and $\Gamma^j_{A\bar{c}}$ can be written as a disjoint union of envelopes of rigid vertices.

Case 2: If the intersection contains a surface-type vertex v, then by item iii) of Proposition 5.13 there exist elements g and g' corresponding to non-boundary-parallel simple closed curves on the surface associated to a lift of v whose conjugacy classes lie in $\mathrm{acl}^{eq}(A\bar{b})$ and $\mathrm{acl}^{eq}(A\bar{c})$ respectively. By Proposition 5.14 these conjugacy classes fork over A, a contradiction to $\bar{b}$ and $\bar{c}$ being independent over A.

Case 3: If the intersection contains two edges e, e' from the same cylinder, then they are adjacent and share a common Z-type vertex by Remark 3.17. Thus we can either apply the first or the second case.

In particular we know that the intersection of every two $\Gamma^i_{A\bar{b}}$ and $\Gamma^j_{A\bar{c}}$ is a disjoint union of envelopes of rigid vertices.

Now consider the group $\hat{\mathbb{F}} = H * \mathbb{F}_A * H'$, where H' is a copy of H, and the element $\bar{c}'$ of $\mathbb{F}_A * H'$ which is the image of $\bar{c}$ under the obvious isomorphism $H * \mathbb{F}_A \longrightarrow \mathbb{F}_A * H'$. We build a graph of groups $\hat{\Lambda}$ for $\hat{\mathbb{F}}$ by 'amalgamating' the graphs of groups $\tilde{\Lambda}^+_{A\bar{b}}$ and $\tilde{\Lambda}^+_{A\bar{c}}$ along $\Lambda_{\mathbb{F}_A}$, i.e. we add $\mathrm{rk}(H) + \mathrm{rk}(H')$ many trivially stabilized edges to $\Lambda_{\mathbb{F}_A}$, according to how they are attached in $\tilde{\Lambda}^+_{A\bar{b}}$ and $\tilde{\Lambda}^+_{A\bar{c}}$ respectively, and associate them with bases for H and H' respectively. This is possible because $\tilde{\Lambda}^+_{A\bar{b}}$ and $\tilde{\Lambda}^+_{A\bar{c}}$ are normalized pointed cyclic JSJ decompositions for $\mathbb{F}$ relative to A, i.e. they consist of $\Lambda_{\mathbb{F}_A}$ together with trivially stabilized edges to which the basis elements of H and H' respectively are associated.

Remember that we saw before that in $\tilde{\Lambda}^+_{A\bar{b}}$ (respectively $\tilde{\Lambda}^+_{A\bar{c}}$) the intersection of every block of the minimal subgraph of $\langle A, \bar{b}\rangle$ (respectively $\langle A, \bar{c}\rangle$) with $\Lambda_{\mathbb{F}_A}$ lies in one of the subgraphs $\Gamma^i_{A\bar{b}}$ (respectively $\Gamma^j_{A\bar{c}}$), and that any two $\Gamma^i_{A\bar{b}}$ and $\Gamma^j_{A\bar{c}}$ intersect in a disjoint union of envelopes of rigid vertices. Thus, by construction of $\hat{\Lambda}$ the intersection of any two blocks of the minimal subgraphs of $\langle A, \bar{b}\rangle$ and $\langle A, \bar{c}'\rangle$ in $\hat{\Lambda}$ is at most a disjoint union of envelopes of rigid vertices. Hence, by the first direction of Theorem 5.4 the tuples $\bar{b}$ and $\bar{c}'$ are independent over A. Fact vi) and monotonicity now imply that $\bar{b}$ is independent from $\bar{c}'$ over $\mathrm{acl}(A)$. Thus, the type $\mathrm{tp}(\bar{c}' / \mathrm{acl}(A)\bar{b})$ is a non-forking extension of $\mathrm{tp}(\bar{c}' / \mathrm{acl}(A))$ over $\mathrm{acl}(A)\bar{b}$.

We assumed that $\bar{b}$ is independent from $\bar{c}$ over A and hence over $\mathrm{acl}(A)$. Thus, the type $\mathrm{tp}(\bar{c} / \mathrm{acl}(A)\bar{b})$ is a non-forking extension of $\mathrm{tp}(\bar{c} / \mathrm{acl}(A))$ over $\mathrm{acl}(A)\bar{b}$. Note that $\bar{c}$ and $\bar{c}'$ have the same type over $\mathrm{acl}(A)$ because there exists an automorphism of $\hat{\mathbb{F}}$ fixing $\mathrm{acl}(A)$ and sending $\bar{c}$ to $\bar{c}'$. By Fact 2.14 every type over an algebraically closed set is stationary. Therefore, the tuples $\bar{c}$ and $\bar{c}'$ realize the same non-forking extension of $\mathrm{tp}(\bar{c} / \mathrm{acl}(A)) = \mathrm{tp}(\bar{c}' / \mathrm{acl}(A))$ over $\mathrm{acl}(A)\bar{b}$, and they have the same type over $\mathrm{acl}(A)\bar{b}$. Hence, they have the same type over $A\bar{b}$. Free groups are homogenous by Theorem 2.6, so by Remark 2.7 there exists an automorphism σ of $\hat{\mathbb{F}}$ fixing $A\bar{b}$ such that $\sigma(\bar{c}') = \bar{c}$. Hence, the tuples $\bar{c}$ and $\bar{c}'$ have the same orbit under $\mathrm{Aut}_{A\bar{b}}(\hat{\mathbb{F}})$. The group $\mathbb{F}_A$ is preserved by $\mathrm{Aut}_{A\bar{b}}(\hat{\mathbb{F}})$, so the automorphism σ sends the decomposition $\hat{\mathbb{F}} = H * \mathbb{F}_A * H'$ to a decomposition $\hat{\mathbb{F}} = \sigma(H) * \mathbb{F}_A * \sigma(H')$ where $\bar{b} \in \sigma(H) * \mathbb{F}_A$ since $\bar{b}$ was contained in $\mathbb{F} = H * \mathbb{F}_A$, and $\bar{c} \in \mathbb{F}_A * \sigma(H')$ since $\bar{c}'$ was contained in $\mathbb{F}_A * H'$. By Kurosh's Theorem 2.24 this induces a decomposition $\mathbb{F} = \mathbb{F}_{\bar{b}} * \mathbb{F}_A * \mathbb{F}_{\bar{c}}$ where $\bar{b} \in \mathbb{F}_{\bar{b}} * \mathbb{F}_A$ and $\bar{c} \in \mathbb{F}_A * \mathbb{F}_{\bar{c}}$.

This means that $\mathbb{F}_{A\bar{b}} = \mathbb{F}_{\bar{b}} * \mathbb{F}_A$ and $\mathbb{F}_{A\bar{c}} = \mathbb{F}_A * \mathbb{F}_{\bar{c}}$. Now we can build a JSJ decomposition for $\mathbb{F}$ relative to A by 'amalgamating' the decompositions $\tilde{\Lambda}_{A\bar{b}}$ and $\tilde{\Lambda}_{A\bar{c}}$ along $\Lambda_{\mathbb{F}_A}$ as we did before with $\tilde{\Lambda}^+_{A\bar{b}}$ and $\tilde{\Lambda}^+_{A\bar{c}}$. Analogous as in $\hat{\Lambda}$ we get that in this decomposition the intersection of every two blocks of the minimal subgraphs of $\langle A, \bar{b} \rangle$ and $\langle A, \bar{c} \rangle$ is contained in a disjoint union of envelopes of rigid vertices. $\square$

Conclusion and Prospect

This thesis followed [PS20] to characterize forking independence in non-abelian finitely generated free groups. It introduced some background on model theory, Bass-Serre theory, trees of cylinders, and modular groups needed to understand the notions and concepts used in the proof of the main theorem. Furthermore, it gave more detailed proofs and explanations than [PS20]. In some places it slightly changed [PS20] to specify some imprecisions. Moreover, it explained why the results that Perin and Sklinos presented in [PS16] are in fact special cases of the main theorem.

The characterization of forking independence in non-abelian finitely generated free groups presented in this thesis is a first step towards the characterization of forking independence in all models of the first order theory of free groups, and thus a first step towards an alternative and maybe more understandable proof of the stability of the first order theory of free groups. To obtain an alternative proof of the stability of the first order theory of free groups it still remains to characterize forking independence in the other models of the theory. Maybe it is possible to alter the definitions used for the characterization of forking independence in non-abelian finitely generated free groups to obtain an appropriate framework for characterizing it in other models of the first order theory of free groups.

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