SIMPLICITY OF THE AUTOMORPHISM GROUPS OF SOME HRUSHOVSKI CONSTRUCTIONS

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ABSTRACT. we show that the automorphism groups of certain countable structures obtained using the Hrushovski amalgamation method are simple groups. The structures we consider are the 'uncollapsed' structures of infinite Morley rank obtained by the ab initio construction and the (unstable) $\aleph_0$-categorical pseudoplanes. The simplicity of the automorphism groups of these follows from results which generalize work of Lascar and of Tent and Ziegler.
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1. INTRODUCTION

In this paper, we show that the automorphism groups of certain countable structures obtained using the Hrushovski amalgamation method are simple groups. This answers a question raised in [10] (Question (iii) of the Introduction there). The structures we consider are the 'uncollapsed' structures of infinite Morley rank obtained by the ab initio construction in [7] and the (unstable) $\aleph_0$-categorical pseudoplanes in [6]. The simplicity of the automorphism groups of these follows from some quite general results which should be of wider interest and applicability. Although much of the intuition (and some of the motivation) behind these results is model-theoretic, the paper requires no knowledge of model theory.

The methods we use have their origins in the paper [9] of Lascar and it will be helpful to recall some of the results from there. Suppose M is a countable saturated structure with a 0-definable strongly minimal subset D such that M is in the algebraic closure of D. Consider $G = \mathrm{Aut}(M/\mathrm{acl}(\emptyset))$, the automorphisms of M which fix every element (of M^{eq}) algebraic over $\emptyset$. Suppose $g \in G$ is *unbounded* (as defined below). Then ([9], Théorème 2) the conjugacy class g^G generates G. In particular if all non-identity elements of G are unbounded, then G is a simple group.

Here, *unbounded* means that for all $n \in \mathbb{N}$ there is a finite $X \subseteq D$ such that $\dim(gX/X) > n$, where dim is dimension in the strongly minimal set D. It is worth noting what this says in the 'classical' cases

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where $M = D$. If M is a pure set, so G is the full symmetric group $\mathrm{Sym}(M)$, then $g \in G$ is bounded if and only if it is finitary. If M is a countably infinite dimensional vector space over a countable division ring F, then G is the general linear group $\mathrm{GL}(\aleph_0, F)$ and $g \in G$ is bounded if and only if it is a scalar multiple of an element of G with fixed point space of finite codimension. So in these cases, Lascar's result implies the well known results that G modulo the bounded part is simple. If M is an algebraically closed field of characteristic zero (and of countably infinite transcendence rank), then it can be shown that all non-identity automorphisms are unbounded, so in this case G is simple (note that $\mathrm{acl}(\emptyset)$ is the algebraic closure of the prime field). Lascar's result has recently been used in [4] to give examples of simple groups with BN-pairs which do not arise from algebraic groups.

Topological methods are a key feature of Lascar's proof: the automorphism group $\mathrm{Aut}(M)$ is regarded as a topological group and arguments about Polish groups are used. Another key feature, arising from the model theory, is the use of a natural independence relation on M. These ideas were applied in other contexts in [10] and [13]. In [10], M is a homogeneous structure arising from a free amalgamation class of finite structures. Assuming $G = \mathrm{Aut}(M) \neq \mathrm{Sym}(M)$ is transitive on M, it is shown that G is simple. The free amalgamation here can be viewed as giving a notion of independence on M, and [13] formalizes this into the notion of a *stationary independence relation* on M ([13], Definition 2.1; cf. Definition 2.1 here). Generalizing Lascar's notion of unboundedness, [13] introduce the notion of $g \in \mathrm{Aut}(M)$ *moving almost maximally* (with respect to the independence relation). It is shown ([13], Corollary 5.4) that in this case, every element of G is a product of 16 conjugates of g.

We now describe the main results of the current paper. In the contexts of [10] and [13], algebraic closure in M is trivial. In Section 2 here, M is a countable structure and cl is an $\mathrm{Aut}(M)$-invariant closure operation on M; we are interested in $G = \mathrm{Aut}(M/\mathrm{cl}(\emptyset))$. We define (Definition 2.1) the notion of a stationary independence relation *compatible with* cl and observe (Theorem 2.5) that the above result of Tent and Ziegler also holds in this wider context.

In Section 3, we assume that the closure and independence are controlled by an integer-valued dimension function d. This is the case in the Hrushovski construction which interests us, and of course is also the case in the almost strongly minimal situation of Lascar (where the closure is algebraic closure and dimension is given by Morley rank). The main result here is Corollary 3.12: there is a natural notion of an automorphism g being 'cl^d-unbounded' and assuming that M is in the closure of a basic orbit (a condition similar to almost strong minimality), every element of G is the product of 96 conjugates of g or its inverse. So this can be seen as a generalization of ([9], Théorème 2).

An example here (Example 3.13) is where M is a countable, saturated differentially closed field of characteristic 0 and cl^d is given by differential dependence. So $\mathrm{cl}^d(\emptyset) = F$ contains the field of constants, and cl^d is strictly bigger than algebraic closure. It follows from Corollary 3.12 that $\mathrm{Aut}(M/F)$ is a simple group.

In Section 4 we apply these results to structures M_0 coming from the simplest form of the Hrushovski predimension construction. Unlike in the collapsed case, the closure operation given by the dimension function is strictly bigger than algebraic closure and the independence notion is weaker than non-forking. Nevertheless, we show (Corollary 4.8) that it is a stationary independence relation. In the rest of the section, under some restrictions on the predimension function, we verify the conditions needed to apply Corollary 3.12. We show that M_0 is in the d-closure of a basic orbit (Lemma 4.11) and that the only cl^d-bounded automorphism is the identity (Theorem 4.14). It follows that $\mathrm{Aut}(M_0/\mathrm{cl}^d(\emptyset))$ is simple.

In the final section, we look at two further variations of the Hrushovski construction. In 5.1 we consider the 'uncollapsed' generalized n-gons constructed by the third Author in [12]. Here, the result is similar to the result in [4]: the automorphism group is a simple group, so this gives new examples of simple groups with a BN-pair. In Section 5.2 we consider the ω-categorical structures M_f constructed by Hrushovski in [6] using an integer-valued predimension. Here the closure is algebraic closure and is locally finite. However, the novelty is that in order to obtain stationarity, we work with an independence relation which is stronger than d-independence. The main result (Corollary 5.10) is that (under some mild restrictions on the control function f) if M_f is the algebraic closure of a basic orbit, then $\mathrm{Aut}(M_f)$ is simple. It seems plausible that the condition of being in the algebraic closure of a basic orbit should hold fairly generally, but the details of checking it even in special cases are quite involved.

Notation: Throughout, M will denote a countable first-order structure; we will not distinguish notationally between the structure and its domain. We denote by $\mathrm{Aut}(M)$ the group of automorphisms of M and if $X \subseteq M$, then $\mathrm{Aut}(M/X)$ is the subgroup consisting of automorphisms which fix every element of X. We also use an alternative notation for this: if $H \leq G$ is a group of permutations on M and $X \subseteq M$ we let $H_X = \{h \in H : h(x) = x \text{ for all } x \in X\}$. If a is a tuple of elements from M then the H-*orbit* of a is $\{ha : h \in H\}$. The $\mathrm{Aut}(M/X)$-orbit of a is denoted by $\mathrm{orb}(a/X)$ (and is sometimes called the *locus* of a over X). If $A, B \subseteq M$ and c is a tuple in M, then we will often use notation such as AB and Ac in place of $A \cup B$ and $A \cup \{c\}$. We write $A \subseteq_{fin} B$ to indicate that A is a finite subset of B.

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2. Stationary independence relations

In this section we use ideas from Lascar's paper [9] to generalise some of the results from [13]. We shall assume familiarlity with these papers and only sketch the modifications which are required to produce the generalisations. The treatment is axiomatic: examples can be found in the applications later in the paper.

Suppose M is a countable structure and $G = \mathrm{Aut}(M)$ is its automorphism group. Let cl be a closure operation on M which is G-invariant and finitary. So for all $g \in G$ and $X \subseteq M$ we have $\mathrm{cl}(gX) = g(\mathrm{cl}(X))$ and $\mathrm{cl}(X) = \bigcup\{\mathrm{cl}(Y) : Y \subseteq_{fin} X\}$. We shall also assume that the closure operation subsumes definable closure, in the sense that if $X \subseteq M$ is finite and $a \in M$ is fixed by all elements of G which fix all elements of X, then $a \in \mathrm{cl}(X)$. Let $\mathcal{X} = \{\mathrm{cl}(A) : A \subseteq_{fin} M\}$ consist of the closures of finite sets in M and let $\mathcal{F}$ consist of all maps $f : X \to Y$ with $X, Y \in \mathcal{X}$ which extend to automorphisms. We refer to the latter as *partial automorphisms* of M. So of course, $\mathcal{X}$ is countable but $\mathcal{F}$ need not be (if cl is not locally finite).

Now, as in Definition 2.1 of [13] we suppose that $\mathop{\smile\!\!\!\!\!|}$ is an invariant stationary independence relation between elements of $\mathcal{X}$, or more generally between subsets of elements of $\mathcal{X}$, which is compatible with the closure operation cl. More precisely we have the following modification of Definition 2.1 of [13].

Definition 2.1. We say that $\mathop{\smile\!\!\!\!\!|}$ is a *stationary independence relation compatible with* cl if for $A, B, C, D \in \mathcal{X}$ and finite tuples a, b:

(1) (Compatibility) We have $a \mathop{\smile\!\!\!\!\!|}_b C \Leftrightarrow a \mathop{\smile\!\!\!\!\!|}_{\mathrm{cl}(b)} C$ and
$$a \mathop{\smile\!\!\!\!\!|}_B C \Leftrightarrow e \mathop{\smile\!\!\!\!\!|}_B C \text{ for all } e \in \mathrm{cl}(a, B) \Leftrightarrow \mathrm{cl}(a, B) \mathop{\smile\!\!\!\!\!|}_B C.$$
(2) (Invariance) If $g \in G$ and $A \mathop{\smile\!\!\!\!\!|}_B C$, then $gA \mathop{\smile\!\!\!\!\!|}_{gB} gC$.
(3) (Monotonicity) If $A \mathop{\smile\!\!\!\!\!|}_B CD$, then $A \mathop{\smile\!\!\!\!\!|}_B C$ and $A \mathop{\smile\!\!\!\!\!|}_{BC} D$.
(4) (Transitivity) If $A \mathop{\smile\!\!\!\!\!|}_B C$ and $A \mathop{\smile\!\!\!\!\!|}_{BC} D$, then $A \mathop{\smile\!\!\!\!\!|}_B CD$.
(5) (Symmetry) If $A \mathop{\smile\!\!\!\!\!|}_B C$, then $C \mathop{\smile\!\!\!\!\!|}_B A$.
(6) (Existence) There is $g \in G_B$ with $g(A) \mathop{\smile\!\!\!\!\!|}_B C$.
(7) (Stationarity) Suppose $A_1, A_2, B, C \in \mathcal{X}$ with $B \subseteq A_i$ and $A_i \mathop{\smile\!\!\!\!\!|}_B C$. Suppose $h : A_1 \to A_2$ is the identity on B and $h \in \mathcal{F}$. Then there is some $k \in \mathcal{F}$ which contains $h \cup \mathrm{id}_C$ (where id_C denotes the identity map on C).

Henceforth, we shall assume that $\mathop{\smile\!\!\!\!\!\!|}\,$ is a stationary independence relation on M compatible with cl.

Remarks 2.2. By compatibility, $A \mathop{\smile\!\!\!\!\!\!|}_X \mathrm{cl}(X)$ for all finite X. Moreover, using existence and stationarity (and the fact that cl subsumes definable closure), if $A \in \mathcal{X}$ and $b \in M$, then $b \mathop{\smile\!\!\!\!\!\!|}_A b \Leftrightarrow b \in A$.

As in Section 2 of Lascar's paper [9], we topologise G by taking basic open sets of the form $O(f) = \{g \in G : g \supseteq f\}$, for $f \in \mathcal{F}$. It should be stressed that in general this is not the 'usual' automorphism group topology (where pointwise stabilisers of finite sets form a base of open neighbourhoods of the identity). It is complete metrizable, but not necessarily separable, so we cannot apply Polish group arguments directly to G. However, as in [9], we will work in separable closed subgroups to avoid this difficulty.

Suppose $\mathcal{S} \subseteq \mathcal{F}$ and let

$$G(\mathcal{S}) = \{g \in G : g|X \in \mathcal{S} \text{ for all } X \in \mathcal{X}\}.$$

Then $G(\mathcal{S})$ is a closed subset of G, and if $\mathcal{S}$ is countable, it is separable. Moreover, if $\mathcal{S}$ satisfies conditions (1-7) on page 241 of [9], then $G(\mathcal{S})$ is a subgroup of G. Thus, if $\mathcal{S}$ is countable and satisfies these conditions then $G(\mathcal{S})$ is a Polish subgroup of G. The conditions just say that $\mathcal{S}$: contains the identity maps; is closed under inverses, restrictions and compositions, and allows extension of domain (and codomain). It is clear that any countable $\mathcal{S}_0 \subseteq \mathcal{F}$ can be extended to a countable $\mathcal{S}$ satisfying these conditions. In particular, $G(\mathcal{S})$ can be taken to include any desired countable subset of G.

Lemma 2.3. *Suppose $\mathcal{S}_0$ is a countable subset of $\mathcal{F}$. Then there is a countable $\mathcal{S}$ with $\mathcal{S}_0 \subseteq \mathcal{S}$ such that $G(\mathcal{S})$ is a group and the conditions in Definition 2.1 hold with G replaced by $G(\mathcal{S})$ and $\mathcal{F}$ replaced by $\mathcal{S}$.*

Proof. First, note that we can assume (by extending $\mathcal{S}_0$) that Lascar's conditions (1-7) hold and for all $B \in \mathcal{X}$, the group $G(\mathcal{S}_0)_B$ has the same orbits on finite tuples from M as G_B. This gives Existence when G is replaced by $G(\mathcal{S}_0)$, by taking a finite set of generators for A and using the compatibility of $\mathop{\smile\!\!\!\!\!\!|}\,$ and cl.

We can further extend $\mathcal{S}_0$ so that the Stationarity condition holds; alternating this with a step to ensure that (1-7) hold we obtain, after a countable number of steps, a set $\mathcal{S}$ in which (1-7) hold and the Stationarity condition holds. $\square$

Definition 2.4. We say that $g \in G$ *moves almost maximally* if for all $B \in \mathcal{X}$ and elements $a \in M$ there is a' in the G_B-orbit of a such that

$$a' \mathop{\smile\!\!\!\!\!\!|}_B ga'.$$

Following the proof of Corollary 5.4 in [13], we then have:

Theorem 2.5. *Suppose M is a countable structure with a stationary independence relation compatible with a closure operation* cl. *Suppose that $G = \mathrm{Aut}(M)$ fixes every element of* $\mathrm{cl}(\emptyset)$. *If $g \in G$ moves almost maximally, then every element of G is a product of 16 conjugates of g.*

Proof. Let $k \in G$ and let $\mathcal{S}_0 \subseteq \mathcal{F}$ be any countable set which contains the restrictions of k, g to all elements of $\mathcal{X}$. Extend $\mathcal{S}_0$ to a countable set $\mathcal{S}$ as in the above Lemma. So $g, k \in G(\mathcal{S})$ and $G(\mathcal{S})$ is a Polish group acting on M; furthermore, $\mathop{\smile\!\!\!\!\!\!|}\,$ is an invariant stationary independence relation *with respect to this group.*

For the rest of the proof only automorphisms in $G(\mathcal{S})$ will be considered.

The proof then just consists of checking that the argument in [13] works. We make some remarks about various parts of this.

(1) By stationarity and the assumption that G fixes every element of $\mathrm{cl}(\emptyset)$, the set $\mathcal{S}$ has the *joint embedding property.* This means that if $h_i : X_i \to Y_i$ are in $\mathcal{S}$ (for $i = 1, 2$) there are $f, h \in \mathcal{S}$ with $f^{-1}h_1 f, h_2 \subseteq h$. Indeed, by Existence we can assume (after applying a suitable f) that $X_1, Y_1 \mathop{\smile\!\!\!\!\!\!|}\, X_2, Y_2$. By Stationarity we can then extend h_i to g_i which is the identity on $X_j \cup Y_j$ (for $j \neq i$). Note that this uses the fact that h_i fixes every element of $\mathrm{cl}(\emptyset)$. Then $g_1 g_2$ extends h_1 and h_2, as required.

Once we have this, it follows that if U, V are non-empty open subsets of $G(\mathcal{S})$ then there is $f \in G(\mathcal{S})$ such that $f(V) \cap U \neq \emptyset$. Thus Theorem 8.46 of [2] applies, as in the proof of Theorem 2.7 of [13].

(2) The part of the proof in [13] which requires the most adaptation is in the use of Lemma 3.6 in the proof of Proposition 3.4. So we give a reformulation of this lemma, and outline its proof.

Suppose $g \in G$ moves maximally and $X, Y \in \mathcal{X}$ with $gX = Y$. Suppose $X \subseteq W \in \mathcal{X}$ and $Y \subseteq Z \in \mathcal{X}$ are such that W and Z are independent over $X; Y$ (write $W \mathop{\smile\!\!\!\!\!\!|}_{(X;Y)} Z$ for this: the definitions are as in [13]). Suppose $h : W \to Z$ is a partial automorphism (in $\mathcal{S}$) which extends $g|X$. Then there is $a \in G_{\mathrm{cl}(XY)}$ such that $g^a(w) = h(w)$ for all $w \in W$.

To see this, let w be a finite tuple with $\mathrm{cl}(w) = W$ and let $w' \in \mathrm{orb}(w/X)$ be moved maximally by g. So w', gw' are independent over $X; Y$ and in particular $w' \mathop{\smile\!\!\!\!\!\!|}_X Y$. Also $w \mathop{\smile\!\!\!\!\!\!|}_X Y$, so by stationarity there is $a_1 \in G_{\mathrm{cl}(XY)}$ with $a_1(w) = w'$. So g^{a_1} moves w maximally over X. Let $Z' = \mathrm{cl}(g^{a_1}(w))$. Thus $W \mathop{\smile\!\!\!\!\!\!|}_{(X;Y)} Z'$.

So $W, Y \mathop{\smile\!\!\!\!\!\!|}_Y Z$ and $W, Y \mathop{\smile\!\!\!\!\!\!|}_Y Z'$. We have partial automorphisms (in $\mathcal{S}$) $h : W \to Z$ and $h' : W \to Z'$ with $h'(w) = g^{a_1}(w)$ for $w \in W$. Note that $h(x) = h'(x)$ for $x \in W$. Let $k = h'h^{-1} : Z \to Z'$. Then $k(y) = y$ for all $y \in Y$. So by stationarity, there is $a_2 \in G_{\mathrm{cl}(WY)}$ which extends k. It is then easy to check that $a = a_1 a_2$ has the required properties.

□

3. Stationary independence relations with a dimension function

Suppose M is a countable structure and $G = \mathrm{Aut}(M)$. In this section we consider an independence relation arising from a dimension function on M.

Definition 3.1. We say that an integer-valued function d defined on finite subsets (or tuples) from M is a *dimension function* if for all $X, Y \subseteq_{fin} M$:

(1) $d(gX) = d(X)$ for all $g \in G$;
(2) $0 \leq d(X) \leq d(X \cup Y) \leq d(X) + d(Y) - d(X \cap Y)$.

For finite $X, Y \subseteq M$ we define $d(X/Y) = d(XY) - d(Y)$ and for arbitrary $Z \subseteq M$ we let $d(X/Z) = \min(d(X/Y) : Y \subseteq_{fin} Z)$. We obtain a finitary closure operation cl^d on M by setting $\mathrm{cl}^d(Z) = \{a \in M : d(a/Z) = 0\}$. Let $\mathcal{X} = \{\mathrm{cl}^d(X) : X \subseteq_{fin} M\}$ and for $A, B, C \in \mathcal{X}$, write $A \mathop{\smile\!\!\!\!\!\!\mid}^d_B C \Leftrightarrow d(A/BC) = d(A/B)$ (where the dimension of an arbitrary set is the maximum of the dimensions of its finite subsets).

If d is a dimension function on M, then it is easy to check that cl^d is a closure operation and $\mathop{\smile\!\!\!\!\!\!\mid}^d$ satisfies (1-5) of Definition 2.1. Note that we may assume $d(\emptyset) = 0$. We refer to cl^d and $\mathop{\smile\!\!\!\!\!\!\mid}^d$ as d-closure and d-independence. For the rest of this section we assume that these also satisfy (6) (Existence) in Definition 2.1. When we also require $\mathop{\smile\!\!\!\!\!\!\mid}^d$ to satisfy (7) (Stationarity), we shall say that $\mathop{\smile\!\!\!\!\!\!\mid}^d$ is stationary.

Definition 3.2. Suppose $b \in M$ and $A \in \mathcal{X}$. We say that b is *basic* over A if $b \notin A$ and whenever $A \subseteq C \in \mathcal{X}$ and $d(b/C) < d(b/A)$, then $b \in C$.

Remarks 3.3. As d is integer-valued and non-negative, if $d(b/A) = 1$, then b is basic over A. It is clear that if $b \notin A$ there is some $A \subseteq C \in \mathcal{X}$ such that b is basic over C. It is less clear that there should be such a C with $d(b/C) = 1$, which is why we are working with this notion.

Suppose $A \in \mathcal{X}$ and $D \subseteq M$ is such that the elements of $D \setminus A$ are basic over A. We claim that d-closure over A on D gives a pregeometry on D. So we need to verify the exchange condition: if $c_1, c_2 \in D$ and $c_1 \in \mathrm{cl}^d(A, c_2) \setminus A$, then $c_2 \in \mathrm{cl}^d(c_1, A)$. By assumption, $d(c_1, c_2/A) = d(c_2/A)$. So $d(c_2/Ac_1) = d(c_1, c_2/A) - d(c_1/A) < d(c_2/A)$, whence $d(c_2/Ac_1) = 0$ (as c_2 is basic over A), as required.

If $X \subseteq D$ is finite, we write $\dim_A(X)$ for the dimension of X with respect to this pregeometry. It is easy to show that if $c_1, \ldots, c_r \in D$ then $\dim_A(c_1, \ldots, c_r) = r$ if and only if $c_1, \ldots, c_r$ are d-independent over A (meaning that $d(c_1, \ldots, c_r/A) = \sum_i d(c_i/A)$).

Note that if $B \in \mathcal{X}$ contains A then all elements of $D \setminus B$ are basic over B, so we can also consider $\dim_B$ on D.

Definition 3.4. We say that M (with dimension function d) is *monodimensional* if for every $A \in \mathcal{X}$ and basic G_A -orbit D there is $A \subseteq B \in \mathcal{X}$ with $M = \mathrm{cl}^d(B, D \setminus B)$.

Remark: The terminology is chosen by association with the model-theoretic notion of unidimensionality. The structures we consider in the next section are not unidimensional, which is why we feel obliged to invent a different terminology.

If $\mathop{\smile\!\!\!\!\!\!|}^d$ is stationary, we can check monodimensionality on a single basic orbit.

Lemma 3.5. *Suppose $\mathop{\smile\!\!\!\!\!\!|}^d$ is stationary, $A \in \mathcal{X}$ and D is a basic G_A-orbit.*

1. *If $A \subseteq B \in \mathcal{X}$ then $D \setminus B$ is a basic G_B-orbit.*
2. *If $\mathrm{cl}^d(A, D) = M$ then M is monodimensional.*
3. *Suppose that for every $c \in M \setminus A$ there is a finite tuple b of elements of D such that $c \not\mathop{\smile\!\!\!\!\!\!|}^d_A b$. Then M is monodimensional.*

Proof. (1) If $b_1, b_2 \in D \setminus B$ then $b_i \mathop{\smile\!\!\!\!\!\!|}^d_A B$. So by stationarity, b_1, b_2 are in the same G_B-orbit.

(2) By (1), it suffices to show that if E is another basic G_A-orbit, then $\mathrm{cl}^d(B, E \setminus B) = M$ for some $A \subseteq B \in \mathcal{X}$. Let $e \in E$ and choose $c_1, \ldots, c_r \in D$ independent over A with $e \in \mathrm{cl}^d(c_1, \ldots, c_r, A)$ and r as small as possible. As cl^d over A gives a pregeometry on $D \cup E$, we may assume (by exchange) that $c_1 \in \mathrm{cl}^d(e, c_2, \ldots, c_r, A)$. Let $B = \mathrm{cl}^d(c_2, \ldots, c_r, A)$. So $c_1 \in \mathrm{cl}^d(B, e) \setminus B$ whence (by (1)) $\mathrm{cl}^d(B, E \setminus B) \supseteq \mathrm{cl}^d(B, D \setminus B) = M$.

(3) We show by induction on $r = d(c/A)$ that $c \in \mathrm{cl}^d(A, D)$. The induction is over all A, D. If $r = 0$, there is no problem. Otherwise we can find a finite tuple e in D with $c \not\mathop{\smile\!\!\!\!\!\!|}^d_A e$. So $d(c/A, e) < d(c/A)$. Let $B = \mathrm{cl}^d(A, e)$. By induction and (1) there is a finite tuple e' in $D \setminus B$ such that $c \in \mathrm{cl}^d(B, e')$, as required. □

The following notion of boundedness is less natural than Lascar's. We shall connect it with a more natural notion later in this section.

Definition 3.6. Suppose $A \in \mathcal{X}$. We say that $h \in G$ is *unbounded over A* if for all $A \subseteq C \in \mathcal{X}$ and $b \in M$ which is basic over C, there is $b' \in \mathrm{orb}(b/C)$ with $hb' \mathop{\smile\!\!\!\!\!\!|}^d_C b'$ (or equivalently, $b' \notin \mathrm{cl}^d(C, hb')$). We say that h is *unbounded* if it is unbounded over some $A \in \mathcal{X}$, otherwise, it is *bounded.*

Note that if h is unbounded over A and $A \subseteq B \in \mathcal{X}$, then h is unbounded over B.

Proposition 3.7. *Suppose $A \in \mathcal{X}$ is such that there is a G_A-invariant set D where the elements of $D \setminus A$ are basic over A and $\mathrm{cl}^d(D, A) = M$. Let $h \in G$ be unbounded over A.*

(1) *If $A \subseteq B \in \mathcal{X}$ and c is a finite tuple in M, then there is $c' \in \mathrm{orb}(c/B)$ with $hc' \mathop{\smile\!\!\!\!\!\!|}^d_B c'$.*
(2) *If $\mathop{\smile\!\!\!\!\!\!|}^d$ is stationary, and $h \in G_A$, then every element of $\mathrm{Aut}(M/A)$ is a product of 16 conjugates of h.*

Proof. (1) First, we show that this holds for c an n-tuple of elements of D with $\dim_B(c) = n$. If $n = 1$, this is just the definition of unboundedness of h. If $n > 1$ and $c = (c_1, \ldots, c_n)$ then write $e = (c_1, \ldots, c_{n-1})$. Inductively, there is $e' \in \mathrm{orb}(e/B)$ with $he' \mathop{\smile\!\!\!\!\!\!|}^d_B e'$. Let f' be such that $c' = (e', f') \in \mathrm{orb}(c/B)$, $f' \notin \mathrm{cl}^d(h^{-1}e', h^{-1}B, e')$ and (using the unboundedness) $f' \notin \mathrm{cl}^d(e', B, he', hf')$. From the first of these, $hf' \notin \mathrm{cl}^d(e', B, he')$ and so, from the second, $\dim_B(f', hf', he', e') = 2 + \dim_B(he', e') = 2 + 2(n-1) = 2n$. Thus $\dim_B(c', hc') = 2n$ and therefore $hc' \mathop{\smile\!\!\!\!\!\!|}^d_B c'$, as required.

Now suppose $b \in M$. By assumption on D, there is a tuple $c \in D^n$ such that $b \in \mathrm{cl}^d(c, B)$. Clearly we can take c to be d-independent over B. Let $B_1 = \mathrm{cl}^d(B, hB)$. By Extension, there is $b_1c_1 \in \mathrm{orb}(bc/B)$ with $c_1 \mathop{\smile\!\!\!\!\!\!|}^d_B B_1$.

By the above, we can find $b_2c_2 \in \mathrm{orb}(b_1c_1/B_1)$ with $c_2 \mathop{\smile\!\!\!\!\!\!|}^d_{B_1} hc_2$. Then $b_2 \mathop{\smile\!\!\!\!\!\!|}^d_{B_1} hc_2$. Moreover, as $b_2 \in \mathrm{cl}^d(c_2, B)$ we have $hb_2 \in \mathrm{cl}^d(hc_2, hB) \subseteq \mathrm{cl}^d(hc_2, B_1)$. Thus $b_2 \mathop{\smile\!\!\!\!\!\!|}^d_{B_1} hb_2$.

We also have $c_2 \mathop{\smile\!\!\!\!\!\!|}^d_B B_1$, so $b_2 \mathop{\smile\!\!\!\!\!\!|}^d_B B_1$, therefore $b_2 \mathop{\smile\!\!\!\!\!\!|}^d_B hb_2$. As $b_2 \in \mathrm{orb}(b/B)$, this completes the proof of (1).

(2) This follows from (1) and Theorem 2.5. □

Remark 3.8. Suppose $c \in M$ and $B \subseteq M$. If h is any automorphism of M, then $h(\mathrm{orb}(c/B))$ is the translate of this G_B-orbit by h. It is a G_{hB}-orbit, and depends only on the restriction of h to B. So the notation $h(\mathrm{orb}(c/B))$ also makes sense if h is a partial automorphism with B in its domain.

Theorem 3.9. *Suppose $\mathop{\smile\!\!\!\!\!\!|}^d$ is stationary and $A \in \mathcal{X}$ is such that there is a G_A-invariant set D where the elements of $D \setminus A$ are basic over A and $\mathrm{cl}^d(D, A) = M$. Suppose $g \in \mathrm{Aut}(M/\mathrm{cl}^d(\emptyset))$ is an unbounded automorphism of M. Then every element of $\mathrm{Aut}(M/\mathrm{cl}^d(\emptyset))$ is a product of 96 conjugates of $g^{\pm 1}$.*

Proof. By enlarging A if necessary, we can assume that g is unbounded over a subset of A. We first show that there is $\tilde{h} \in \mathrm{Aut}(M/\mathrm{cl}^d(\emptyset))$ such that the commutator $g_1 = [g, \tilde{h}] = g^{-1}\tilde{h}^{-1}g\tilde{h}$ is in G_A and is unbounded (over A). We build $\tilde{h}$ by back-and-forth as the union of a chain of partial automorphism (with domains and images in $\mathcal{X}$).

Note that if h is a partial automorphism which fixes all points of $A \cup gA$, then $g^{-1}h^{-1}gh(a) = a$ for all $a \in A$. So we start the construction of $\tilde{h}$ with such a partial automorphism. There is no problem extending this to an automorphism, the issue is to ensure the unboundedness of g_1. We enforce this in the 'forth' step in the construction.

Suppose that the partial automorphism h has been defined and $B = \mathrm{dom}(h)$. Suppose $C \subseteq B$, $C \in \mathcal{X}$ and a is basic over C. We want to find $a' \in \mathrm{orb}(a/C)$ so that (once $\tilde{h}$ is defined) $g_1 a' \mathop{\not\downarrow}_C a'$, that is, $a' \notin \mathrm{cl}^d(g_1 a', C)$. It will suffice to do this with $C = B$.

So suppose that a is basic over B. We may assume (by Existence) that $a \notin \mathrm{cl}^d(B, gB)$. By unboundedness of g there is $b \in h(\mathrm{orb}(a/B))$ such that $gb \mathop{\downarrow}^d_{hB} b$. Extend h to h' with $h'a = b$.

By Existence, there is $c \in h'^{-1}(\mathrm{orb}(gb/hB, b))$ with $c \mathop{\downarrow}^d_{B,a} gB, ga$. Extend h' to h'' with $h''(c) = gb$. As $gb \mathop{\downarrow}^d_{hB} b$ we have (applying h''^{-1}) that $c \mathop{\downarrow}^d_B a$. Thus, by Transitivity, $c \mathop{\downarrow}^d_B gB, ga$, so $c \mathop{\downarrow}^d_{B,gB} ga$. Then $g^{-1}c \mathop{\downarrow}^d_{g^{-1}B,B} a$. As a is basic over B and $a \notin \mathrm{cl}^d(B, g^{-1}B)$, we have $g^{-1}B \mathop{\downarrow}^d_B a$. It follows that $g^{-1}c \mathop{\downarrow}^d_B a$, that is,

$$g^{-1}h''^{-1}gh''a \mathop{\downarrow}^d_B a$$

as required.

It now follows from Proposition 3.7 that every element of G_A is a product of 32 conjugates of $g^{\pm 1}$. Thus, to prove the Theorem, it will suffice to show that $\mathrm{Aut}(M/\mathrm{cl}^d(\emptyset))$ is a product of 3 conjugates of $H_1 = G_A$.

By Existence, there is $A' \in \mathrm{orb}(A/\mathrm{cl}^d(\emptyset))$ with $A' \mathop{\downarrow}^d A$. So $H_2 = G_{A'}$ is a conjugate of H_1. Let $k \in \mathrm{Aut}(M/\mathrm{cl}^d(\emptyset))$. By Existence again, there is $f_1 \in H_1$ with $f_1 A' \mathop{\downarrow}^d A, kA$. By Stationarity, there is $f_2 \in \mathrm{Aut}(M/f_1 A')$ with $f_2|A = k|A$. Thus $f_2^{-1}k \in H_1$ and so $k \in f_2 H_1$. But $f_2 \in f_1 H_2 f_1^{-1}$, so $k \in H_1 H_2 H_1$, as required. $\square$

We now give a more natural interpretation of boundedness when M is monodimensional. Note that the following does not require stationarity of $\mathop{\downarrow}^d$.

Proposition 3.10. *Suppose M is monodimensional and suppose $g \in G$ is bounded. Then there is $E \in \mathcal{X}$ such that $g(B) = B$ for all $B \in \mathcal{X}$ which contain E.*

Proof. There is $C \in \mathcal{X}$ and a basic b over C such that for all $b' \in \mathrm{orb}(b/C)$ we have $b' \in \mathrm{cl}^d(C, gb')$, so $g^{-1}b' \in \mathrm{cl}^d(g^{-1}C, b')$. By extendidng C if necessary, we can assume by monodimensionality that

$\mathrm{cl}^d(C, \mathrm{orb}(b/C)) = M$. There are $b_1, \ldots, b_k \in \mathrm{orb}(b/C)$ with $g^{-1}C \subseteq \mathrm{cl}^d(C, b_1, \ldots, b_k) = E$. So

$$g^{-1}E = \mathrm{cl}^d(g^{-1}C, g^{-1}b_1, \ldots, g^{-1}b_k) \subseteq \mathrm{cl}^d(g^{-1}C, b_1, \ldots, b_k) \subseteq E.$$

As $d(E) = d(g^{-1}E)$ we obtain $g^{-1}E = E$. Let $b_1 \in \mathrm{orb}(b/C)$ be such that $b_1 \mathop{\smile\!\!\!\!\!\!\,|}^d_C E$. Then b_1 is basic over E and for all $b' \in \mathrm{orb}(b_1/E)$ we have that g^{-1} stabilizes $\mathrm{cl}^d(E, b')$ (and therefore g stabilizes it also).

Now, given any $B \supseteq E$ in $\mathcal{X}$ we can find a tuple $\bar{b}$ of elements of $\mathrm{orb}(b_1/E)$ such that $B_1 = \mathrm{cl}^d(E, \bar{b}) \supseteq B$. Then (by Extension) we can find $B_2 \in \mathrm{orb}(B_1/B)$ with $B_2 \mathop{\smile\!\!\!\!\!\!\,|}^d_B B_1$: in particular $B_1 \cap B_2 = B$. By the previous paragraph, g stabilizes both B_1 and B_2, so $gB = B$. $\square$

Definition 3.11. We say that $g \in Aut(M)$ is cl^d-*bounded* if there is some $E \in \mathcal{X}$ such that g stabilizes setwise all $B \in \mathcal{X}$ which contain E.

It is easy to see that the cl^d-bounded automorphisms form a normal subgroup of $\mathrm{Aut}(M)$. The following follows from the above two results and can be seen as a generalisation of Theorem 2 of [9] (the almost strongly minimal case where there is a strongly minimal set definable over the empty set).

Corollary 3.12. *Suppose $\mathop{\smile\!\!\!\!\!\!\,|}^d$ is stationary and $A \in \mathcal{X}$ is such that there is a basic $\mathrm{Aut}(M/A)$-orbit D with $\mathrm{cl}^d(A, D) = M$. Suppose $g \in \mathrm{Aut}(M/\mathrm{cl}^d(\emptyset))$ is not cl^d-bounded. Then every element of $\mathrm{Aut}(M/\mathrm{cl}^d(\emptyset))$ is a product of 96 conjugates of $g^{\pm 1}$.* $\square$

Example 3.13. Suppose M is a countable, saturated differentially closed field of characteristic 0. If a is a tuple of elements of M, let $d(a)$ denote the differential transcendence degree of a over $\emptyset$. This gives a closure operation cl^d which satisfies exchange. It follows from ([3], Corollary 2.6) that $\mathop{\smile\!\!\!\!\!\!\,|}^d$ is a stationary equivalence relation. The elements of differential transcendence degree 1 form a single orbit D under $G = \mathrm{Aut}(M/\mathrm{cl}^d(\emptyset))$ and clearly $\mathrm{cl}^d(D) = M$, so Corollary 3.12 applies. By ([3], Proposition 2.9), the only cl^d-bounded automorphism of M is the identity, so $\mathrm{Aut}(M/\mathrm{cl}^d(\emptyset))$ is a simple group. In fact, because we can use Proposition 3.7 with $A = \mathrm{cl}^d(\emptyset)$, if $1 \neq g \in G$, then every element of G is a product of 16 conjugates of g.

4. The ab initio Hrushovski constructions

4.1. **The structures.** The Hrushovski construction which originated in [7] admits many extensions and variations, and can be presented at various levels of generality. But to fix notation, we consider the following basic case, and comment on generalizations later. The article [14] is a convenient general reference for these constructions.

Suppose $r \geq 2$ and $m, n \geq 1$ are fixed coprime integers. We work with the class $\mathcal{C}$ of finite r-uniform hypergraphs, which we regard as structures in a language with a single r-ary relation symbol $R(x_1,\dots,x_r)$ whose interpretation is invariant under permutation of coordinates and satisfies $R(x_1,\dots,x_r) \to \bigwedge_{i<j}(x_i \neq x_j)$. If $B \in \mathcal{C}$ consider the *predimension*

$$\delta(B) = n|B| - m|R[B]|$$

where $R[B]$ denotes the set of hyperedges on B (i.e $\{\{b_1,\dots,b_r\} : B \models R(b_1,\dots,b_r)\}$). For $A \subseteq B$, we write $A \leq B$ iff for all $A \subseteq B' \subseteq B$ we have $\delta(A) \leq \delta(B')$, and let $\mathcal{C}_0 = \{B \in \mathcal{C} : \emptyset \leq B\}$. The following is standard (cf. ([7], Lemma 1), for example).

Lemma 4.1. *Suppose $A, B \subseteq C \in \mathcal{C}$.*

(1) $\delta(A \cup B) \leq \delta(A) + \delta(B) - \delta(A \cap B)$.
(2) *If $A \leq B$ and $X \subseteq B$ then $A \cap X \leq X$.*
(3) *If $A \leq B \leq C$, then $A \leq C$.*

We let $\bar{\mathcal{C}}_0$ be the set of structures all of whose finite substructure are in $\mathcal{C}_0$. If $C \subseteq B \in \bar{\mathcal{C}}_0$ we write $C \leq B$ iff $X \cap C \leq X$ for all finite $X \subseteq B$. (This agrees with what was previously defined, by the above lemma). If $A, B \subseteq_{fin} C \in \mathcal{C}_0$ then we define $\delta(A/B) = \delta(A \cup B) - \delta(B)$. Note that this is equal to $|A \setminus B| - |R[A \cup B] \setminus R[B]|$ and this makes sense for arbitrary B (allowing the value $-\infty$, if necessary). Then $B \leq A \cup B$ iff $\delta(A'/B) \geq 0$ for all $A' \subseteq A$.

The class $\bar{\mathcal{C}}_0$ has the following amalgamation property: suppose $B, C \in \bar{\mathcal{C}}_0$ have a common substructure A and $A \leq B$. Then the *free amalgam* $F = B \coprod_A C$ of B and C over A, consisting of the disjoint union of B and C over A with only the relations on B and on C, is in $\bar{\mathcal{C}}_0$ and $C \leq F$. Using this and a standard Fraïssé-style construction, we obtain the following well-known result, which is sometimes referred to as the *ab initio* case of the Hrushovski construction:

Theorem 4.2. *There is a unique countable $M_0 \in \bar{\mathcal{C}}_0$ having the properties: M_0 is a union of a chain of finite $\leq$-substructures; if $X \leq M_0$ is finite and $X \leq A \in \mathcal{C}_0$, then there is an embedding $\alpha : A \to M_0$ which is the identity on X and $\alpha(A) \leq M_0$. Moreover, if $A_1, A_2 \leq M_0$ are finite and $h : A_1 \to A_2$ is an isomorphism, then h extends to an automorphism of M_0.* □

The structure M_0 is the *generic structure* for the class $(\mathcal{C}_0, \leq)$. The property in the 'Moreover' statement is referred to as $\leq$-*homogeneity* of M_0. It is easy to see that every countable structure in $\bar{\mathcal{C}}_0$ can be embedded as a $\leq$-substructure of M_0.

As usual, we have two closure operations and a dimension function on M_0 (indeed, on any structure in $\bar{\mathcal{C}}_0$). If X is a finite subset of M_0, there is a smallest subset Y with $X \subseteq Y \leq M_0$. This Y is finite and we denote it by $\mathrm{cl}_0(X)$. The dimension $d(X)$ of X (in M_0) is defined to be $\delta(\mathrm{cl}_0(X))$. The d-closure of X is $\mathrm{cl}^d(X) = \{a \in M_0 : d(X \cup \{a\}) =$

$d(X)\}$. In general, this will not be finite. Let $\mathcal{X} = \{\mathrm{cl}^d(X) : X \subseteq_{fin} M_0\}$.

For tuples a, b, c in M_0 we define $a \mathop{\smile\!\!\!\!\!\!\;|}^d_b c$ to mean $d(a/b) = d(a/bc)$ (as in the previous section); similarly for sets in $\mathcal{X}$. This is not the same as non-forking independence. The following is well-known.

Lemma 4.3. (1) *If $A, B, C \in \mathcal{X}$ then $A \mathop{\smile\!\!\!\!\!\!\;|}^d_B C$ if and only if the following three conditions hold:* $\mathrm{cl}^d(AB) \cap \mathrm{cl}^d(BC) = B$*;* $\mathrm{cl}^d(AB)$*,* $\mathrm{cl}^d(BC)$ *are freely amalgamated over B; and* $\mathrm{cl}^d(AB) \cup \mathrm{cl}^d(BC) \leq M_0$.

(2) *The relation $\mathop{\smile\!\!\!\!\!\!\;|}^d$ satisfies the Compatibility, Invariance, Monotonicity, Transitivity and Symmetry properties in Definition 2.1.*

4.2. **Extending the homogeneity.** We will show that if $A_1, A_2 \in \mathcal{X}$ and $h : A_1 \to A_2$ is an isomorphism, then h extends to an automorphism of M_0.

We need the following notion from [7]. Suppose $Z \subset Y \in \bar{\mathcal{C}}_0$ and $Y \setminus Z$ is finite. We say that the extension $Z \subset Y$ is *simply algebraic* if $\delta(Y/Z) = 0$ and whenever $Z \subset Z_1 \subset Y$, then $\delta(Y/Z_1) < 0$. So $Z \leq Y$, but $Z_1 \not\leq Y$ for all $Z \subset Z_1 \subset Y$. We write *sa* for simply algebraic. The extension is *minimally* simply algebraic (msa) if the extension $Z_0 \subset Z_0 \cup (Y \setminus Z)$ is not simply algebraic for all proper subsets Z_0 of Z. In this case Z is finite and more generally, if $Z \subset Y$ is simply algebraic, there is finite subset Y_1 of Y which contains $Y \setminus Z$ and is such that $Y_1 \cap Z \subset Y_1$ is msa. Moreover, Y is the free amalgam of Z and Y_1 over $Z_1 = Y_1 \cap Z$.(In fact, Z_1 consists of the points in Z which are in some R-relation containing a point of $Y \setminus Z$.) In this case, we say that Y has *base* Z_1 and *type* (Z_1, Y_1) over Z.

If $A \leq M_0$ and $B \subseteq M_0$ is an sa extension of A, then $B \leq M_0$. Moreover, any collection $\{B_i : i \in I\}$ of (distinct) sa extensions of A in M_0 is in free amalgamation over A and $\bigcup_{i \in I} B_i \leq M_0$ (Lemma 2 of [7]). If $Z_1 \subseteq A$ and $Z_1 \subset Y_1$ is msa, then the *multiplicity* $\mathrm{mult}(Z_1, Y_1/A)$ is the number of distinct minimal extensions of A of type (Z_1, Y_1) in M_0. So this is the maximum cardinality of $\{B_i : i \in I\}$ where each B_i is a sa extension of A of type (Z_1, Y_1).

Note that $\mathrm{cl}^d(A) = A$ iff each such multiplicity is zero. Indeed, $\mathrm{cl}^d(A)$ is the union of all subsets of M_0 which can be obtained from A by a finite chain of successive sa extensions. The (full) amalgamation property for $\mathcal{C}_0$ shows that if A is finite, then all multiplicities over A are infinite.

Definition 4.4. Suppose $A_1, A_2 \leq M_0$ and $k : A_1 \to A_2$ is an isomorphism. We say that k is *potentially extendable* if for every $Z_1 \subseteq A_1$ and msa $Z_1 \subset Y_1$ we have $\mathrm{mult}(Z_1, Y_1/A_1) = \mathrm{mult}(Z_2, Y_2/A_2)$, where $Z_2 = k(Z_1)$, and $k|Z_1$ extends to an isomorphism between Y_1 and Y_2.

Evidently, if k as above extends to an automorphism of M_0, then k is potentially extendable. Moreover, there are isomorphisms $k : A_1 \to A_2$ with $A_i \leq M_0$ which are not potentially extendable.

Lemma 4.5. *If $A_1, A_2 \leq M_0$ are such that $d(A_i)$ is finite and $k : A_1 \to A_2$ is potentially extendable, then k can be extended to an automorphism of M_0.*

Proof. For $i = 1, 2$, let A'_i be the union of all sa extensions of A_i in M_0. By the above, $A'_i \leq M_0$ and A'_i is the free amalgam over A of the various sa extensions. So by the condition on the multiplicities, k extends to an isomorphism $k' : A'_1 \to A'_2$.

We claim that k' is potentially extendable. Indeed, suppose $Z_1 \subseteq A'_1$ is finite and $Z_1 \subset Y_1$ is msa. If $Z_1 \subseteq A_1$ then by construction of A'_1 we have $\mathrm{mult}(Z_1, Y_1/A'_1) = 0$. So it will suffice to show that if $Z_1 \not\subseteq A_1$ then there are only finitely many copies of Y_1 over Z_1 in A'_1 (because it then follows that $\mathrm{mult}(Z_1, Y_1/A'_1)$ is infinite, and the same will be true for the corresponding msa extension of $k'(Z_1)$ over A'_2).

To see this, note that as A'_1 is a free amalgam over A_1, any point in $A'_1 \setminus A_1$ is contained in only finitely many instances of the relation R. But, in any msa extension, every point in the base is in some instance of the relation R which also contains a non-base point. As any two msa extensions with the same base are disjoint over the base, it follows that Z_1 is the base of only finitely many msa extensions contained in A'_1.

This shows that k' is potentially extendable, so we can repeat the argument and adjoin to A'_1 all sa extensions of A'_1 and extend k'. Continuing in this way, we see that we can extend k to $h : B_1 \to B_2$, where $B_i = \mathrm{cl}^d(A_i)$. Evidently h is potentially extendable (as all multiplicities over its domain and image are zero).

Now, suppose we have $c \in M_0$. It will be enough to show how to extend h to a potentially extendable map which has c in its domain (for then we can proceed by a back-and-forth argument to build up an automorphism extending the original k). We may assume $c \notin B_1$. Let $S_0 \subseteq B$ be finite and such that $\mathrm{cl}^d(S_0) = B_1$ and let $S = \mathrm{cl}_0(c, S_0) \cap B_1$. Then $S \leq M_0$ is finite and $\mathrm{cl}_0(c, S) \cap \mathrm{cl}^d(S) = S$. Furthermore, $C = \mathrm{cl}_0(c, S)$ and B_1 are freely amalgamated over S, and $C \cup B_1 \leq M_0$.

Let $T = h(S)$ and $T \leq D \in \mathcal{C}_0$ be such that $h|S$ extends to an isomorphism $C \to D$. We claim that we can find a copy D_1 of D over T such that D_1, B_2 are freely amalgamated over T and $D_1 \cup B_2 \leq M_0$. In fact, take any copy $D_1 \leq M_0$ of D over T in M_0: this exists, by the characteristic property in Theorem 4.2. We have $cl^d(T) \cap D_1 = T$ (because the same is true of $S \leq C$), so $D_1 \cap B_2 = T$. The other properties follow as $d(D_1/T) = d(D_1/B_2)$.

So now we can extend h to $h' : B_1 \cup C \to B_2 \cup D$ and to finish, we need to show that h' is potentially extendable. But this is a similar argument to what was done previously. If $Z_1 \subset B_1 \cup C$ and $Z_1 \subset Y_1$ is msa, then either $Z_1 \subseteq B_1$, in which case $\mathrm{mult}(Z_1, Y_1/B_1) = 0$, or $Z_1 \cap (C \setminus B_1) \neq \emptyset$. But points in $C \setminus B_1$ are in only finitely many relations within $B_1 \cup C$, so in this latter case $B_1 \cup C$ contains only finitely many copies of Y_1 over Z_1. Thus $\mathrm{mult}(Z_1, Y_1/B_1)$ is infinite. The same argument also holds with B_2 and D_1, so we are finished. $\square$

Lemma 4.6. *Suppose $A = \mathrm{cl}^d(A)$ and $C = \mathrm{cl}^d(C)$ have finite d-dimension and are such that A, C are freely amalgamated over $B = A \cap C$ and $A \cup C \leq M_0$. Then for every msa $Z \subset Y$ with $Z \subseteq A \cup C$ and $Z \not\subseteq A$ and $Z \not\subseteq C$, there are only finitely many copies of Y over Z in $A \cup C$. In particular, $\mathrm{mult}(Z, Y/A \cup C)$ is infinite.*

Proof. The proof of Hrushovski's algebraic amalgamation lemma (Lemma 3 of [7]) shows that there are at most $\delta(Z)$ copies of Y over Z which are contained in $A \cup C$. $\square$

Corollary 4.7. *We have the following additional homogeneity properties of M_0.*

1. *(d-homogeneity:) Suppose $A_1, A_2 \subseteq M_0$ are d-closed and of finite d-dimension. Suppose $h : A_1 \to A_2$ is an isomorphism. Then h extends to an automorphism of M_0.*
2. *(d-stationarity:) Suppose $A_1, A_2, C \subseteq M_0$ are d-closed and of finite d-dimension. Suppose that for each i we have that $A_i \cup C \leq M_0$ and A_i, C are freely amalgamated over $B = A_i \cap C$. If $h : A_1 \to A_2$ is an isomorphism which is the identity on B, then h extends to an automorphism of M_0 which fixes every element of C pointwise.*

Proof. (1) As the A_i are d-closed, h is potentially extendable. So by Lemma 4.5, it extends to an automorphism of M_0.

(2) Let $k : A_1 \cup C \to A_2 \cup C$ be the union of h with the identity map on C. By the freeness, this is an isomorphism. By Lemma 4.6, it is potentially extendable. So by Lemma 4.5, it extends to an automorphism of M_0. $\square$

Corollary 4.8. *The relation $\mathop{\smile\!\!\!\!\!\!|}^{d}$ is a stationary independence relation on M_0 compatible with cl^d.*

Proof. We have already verified everything apart from the Existence property. Given $A, B, C \in \mathcal{X}$ we need to show that there is $g \in G_B$ with $gA \mathop{\smile\!\!\!\!\!\!|}^{d}_{B} C$. By taking d-closures over B, we may assume that $B \subseteq A, C$. Let F be the free amalgam of A, C over B and let A' denote the copy of A inside F. So there is an isomorphism $h : A \to A'$ which is the identity on B. By the construction of M_0 we can assume that $F \leq M_0$. Then $A' \mathop{\smile\!\!\!\!\!\!|}^{d}_{B} C$ and h extends to an automorphism g of M_0 by d-homogeneity. $\square$

4.3. **Bounded automorphisms.** We shall show that, under a mild restriction on the parameters n, m, r, the structure M_0 has no non-trivial bounded automorphisms. To see that some restriction is necessary, consider the case where $r = 2$ and $n = m = 1$. Then M_0 is a graph each of whose connected components consists of an infinite tree with infinite valency, or a single cycle with a collection of such trees attached. Points in the first type of component have d-dimension 1, and those in the second type form the d-closure of the empty set. It is clear that there are non-trivial automorphisms which stabilise each component (and fix every element in $\mathrm{cl}^d(\emptyset)$), and these are obviously bounded.

For the rest of this section we assume that n, m are coprime, if $r = 2$ then $n > m$, and if $r \geq 3$ then $n \geq m$. The following is straightforward for the case $m = 1$. The proof for the general case is surprisingly delicate and makes use of some well known properties of *Beatty sequences* (Lemma 4.10).

Lemma 4.9. *There is $X \subseteq Y \in \mathcal{C}_0$ such that:*
(1) *$\delta(Y/X) = -1$ and $|X| \geq 2$.*
(2) *If $U \subseteq Y$ and $X \not\subseteq U$, then $U \cap X \leq U$.*
(3) *If $X \subseteq Z \subset Y$, then $\delta(Z/X) \geq 0$.*

Proof. Suppose first that $m = 1$. If $r = 2$, take $X = \{x_0, \ldots, x_n\}$ with no relations on it and Y is X together with an extra point y, where $R(y, x_i)$ holds for all i. If $r \geq 3$, do the same, but X also includes an $(r-2)$-tuple $\bar{z}$, and $R(\bar{z}, y, x_i)$ holds.

So now suppose that $n > m > 1$. We will suppose that $r = 2$: a similar argument to that used above will then allow us to deduce the general case.

Write

$$n = ma + c \quad \text{with } 0 < c < m.$$

So m, c are coprime and we can find $\ell, b \in \mathbb{Z}$ with

$$\ell m - cb = 1.$$

We can take $0 < b < m$ (take an inverse of $-c$ modulo m) and it then follows that $0 < \ell \leq b, c$. Note that

$$nb - m(ab + \ell) = -1.$$

We now assume that $b > 2$ and describe the construction of Y (the cases $b = 1, 2$ will be considered at the end).

Let X consist of $(a-1)b + \ell$ points (with no edges). Let $Y = X \cup \{y_0, \ldots, y_{b-1}\}$ with $ab + \ell$ edges as follows:
(i) the vertices $y_0, \ldots, y_{b-1}$ form a b-cycle (with $R(y_i, y_{i+1})$ holding, where the indices are read modulo b);
(ii) each vertex y_i is adjacent to at least $(a-1)$ of the vertices in X;

(iii) each vertex in X is adjacent to exactly one vertex in $Y \setminus X$.

Thus there are a further ℓ edges of Y to be specified. These will be of the form (x_i, y_i) for i in some subset $I \subseteq \{0, \ldots, b-1\}$ of size ℓ (and distinct $x_i \in X$). The subset I is chosen so that (3) of the Lemma holds. Once we have this, the rest of the Lemma follows. Indeed, first note that as Y is a cycle with some extra edges freely amalgamated over its vertices, then $Y \in \mathcal{C}_0$. By construction $\delta(Y/X) = nb - m(ab+\ell) = -1$, so (1) holds. For (2) suppose $\emptyset \neq A \subseteq X$. We claim that $X \setminus A \leq Y \setminus A$, and then (2) follows (by Lemma 4.1(2)). To see the claim, note that $\delta((Y \setminus A)/(X \setminus A)) = -1 + m|A| > 0$, and if $Z \subset Y \setminus X$ then $\delta(Z/(X \setminus A)) \geq \delta(Z/X) \geq 0$, by (3).

To prove (3) (for suitable choice of I) it will suffice (by free amalgamation) to show that if $Z \subset Y \setminus X$ is connected, then $\delta(Z/X) \geq 0$. Let $q = |\{i \in I : b_i \in Z\}|$ and $s = p + q = |Z|$.
Then

$$\delta(Z/X) = sn - m(qa + p(a-1) - p + q - 1) = sc - m(q-1).$$

Thus

$$\delta(Z/X) \geq 0 \Leftrightarrow \frac{q-1}{s} \leq \frac{c}{m}. \tag{1}$$

So we need to construct I of size ℓ so that for any s consecutive elements of $0, \ldots, b-1$ (read modulo b, and with $s < b$), the number of elements q in I satisfies the above inequality. The construction uses the following.

Lemma 4.10. *There is a sequence $(a_i)_{i \in \mathbb{Z}}$ with $a_i \in \{0,1\}$ having the following properties:*

(1) $a_{i+b} = a_i$ *for all i;*
(2) *for all i, $\sum_{i+1 \leq j \leq i+b} a_j = \ell$;*
(3) *for all i, s we have*

$$\frac{1}{s}(-1 + \sum_{i+1 \leq j \leq i+s} a_j) \leq \frac{\ell}{b}.$$

Proof of Lemma: Let $\theta = \ell/b$ and note that $0 < \theta < 1$. The Beatty sequence $(\beta_i(\theta))_{i \in \mathbb{Z}}$ is defined as follows. For $i \in \mathbb{Z}$ let

$$\beta_i(\theta) = \lfloor i\theta \rfloor$$

(where $\lfloor x \rfloor$ is the largest integer $\leq x$). Let

$$a_i = \beta_i(\theta) - \beta_{i-1}(\theta).$$

It is easy to see that $a_i \in \{0,1\}$ and $a_{i+b} = a_i$. For part (3) of the Lemma, note that

$$\begin{aligned}\frac{1}{s}(-1+\sum_{i+1\leq j\leq i+s} a_j) &= \frac{1}{s}(\beta_{i+s}(\theta)-\beta_i(\theta)-1)\\ &= \frac{1}{s}(\lfloor (i+s)\theta\rfloor - \lfloor i\theta\rfloor -1) \leq \frac{i+s}{s}\theta - \frac{\lfloor i\theta\rfloor+1}{s}\\ &< \frac{i+s}{s}\theta - \frac{i\theta}{s} = \theta.\end{aligned}$$

A similar calculation shows that

$$\frac{1}{s}(1+\sum_{i+1\leq j\leq i+s} a_j) > \theta.$$

Thus for all $i \in \mathbb{Z}$, we have $\frac{1}{s}\sum_{i+1\leq j\leq i+s} a_j \to \theta$ as $s \to \infty$. The periodicity in (1) then implies (2). $\square_{Lemma}$

Returning to the construction of Y, we let (a_i) be the above sequence and let:

$$I = \{i \in \{0,\ldots,b-1\} : a_i = 1\}.$$

Verifying equation 1 amounts to showing that if $0 < s < b$ and $i < b$, then $\frac{q-1}{s} \leq \frac{c}{m}$, where $q = \sum_{i+1\leq j\leq i+sa_j} a_j$. Suppose for a contradiction that $(q-1)/s > c/m$. Recall that $\ell m - cb = 1$, so $\frac{\ell}{b} = \frac{c}{m} + \frac{1}{bm}$. By (3) of the Lemma, $(q-1)/s \leq \ell/b$, so by assumption, we have:

$$\frac{c}{m} < \frac{q-1}{s} \leq \frac{\ell}{b} = \frac{c}{m} + \frac{1}{bm}.$$

Thus

$$0 < \frac{q-1}{s} - \frac{c}{m} < \frac{1}{bm}.$$

But

$$\frac{q-1}{s} - \frac{c}{m} = \frac{(q-1)m-cs}{sm} \geq \frac{1}{sm} > \frac{1}{bm}$$

as $s < b$. This is a contradiction. So $(q-1)/s \leq c/m$ and therefore by equation 1, $\delta(Z/X) \geq 0$, as required.

This completes the proof that Y satisfies the properties of Lemma 4.9.

For the remaining cases $b = 1, 2$ we use a similar (but easier) construction with $Y \setminus X$ of size b. We leave the details to the Reader. $\square$

Lemma 4.11. *Suppose $A \in \mathcal{X}$ and $u_0 \in M_0 \setminus A$ is basic over A. Let $D = \mathrm{orb}(u_0/A)$. Then $\mathrm{cl}^d(A,D) = M_0$.*

Proof. Suppose $c \in M_0 \setminus A$. By Lemma 3.5 (3), it will suffice to show that there is a finite tuple e in D with $c \not\perp^d_A e$.

Let $A_0 \leq A$ be finite with $d(A_0) = A$. Let $C = \mathrm{cl}_0(cA_0)$. We can assume that $C \cap A = A_0$. Similarly let $B = \mathrm{cl}_0(u_0A_0)$ and note we can also assume that $B \cap A = A_0$ (if it is bigger, then replace A_0 by the intersection; this will not affect the condition on C).

Let $X \subseteq Y$ be as in Lemma 4.9 and $k = |X|$. Note that we can assume that there are no relations on the set X. Let Z be the free amalgam of C and $k-1$ copies $B_2, \ldots, B_k$ of B over A_0. Let $x_1 = c$ and for $i = 2, \ldots, k$ let $x_i \in B_i \setminus A_0$ be the copy of u_0 inside B_i. Identify the x_i with the points of X and let E consist of the free amalgam $Z \coprod_X Y$ of Z and Y over X.

Claim: We have $C, B_i \leq E$.

Note that once we have the claim, it follows (as $\emptyset \leq C$) that $E \in \mathcal{C}_0$, so we can assume that $E \leq M_0$. Then $x_2, \ldots, x_k \in D$ and $d(c/A_0, x_2, \ldots, x_k) = d(c/A_0) - 1$, so $c \not\perp^d_A x_2, \ldots, x_k$.

We now prove the claim. By the symmetry of the sitaution, it is enough to show $C \leq E$. Let $C \subseteq F \subseteq E$. Then F is the free amalgam $F \cap Z \coprod_{F \cap X} F \cap Y$. If $X \not\subseteq F$ then $F \cap X \leq F \cap Y$ (by (2)) so $F \cap Z \leq Z$. As $C \leq F \cap Z$ we obtain $C \leq F$. If $X \subseteq F$ and $Y \not\subseteq F$, then similarly (using (3)) we have $X = F \cap X \leq F \cap Y$, so again $C \leq F$.

So now suppose $Y \subseteq F$. Note that $\delta(F \cap Z) \geq d_Z(XC)$ (the dimension in Z of $X \cup C$). So

$$\delta(F) \geq d_Z(XC) + \delta(Y/X) = d_Z(C) + d_Z(X/C) - 1 \geq \delta(C) + k - 2 \geq \delta(C).$$

(Here we have used $C \leq Z$ and (1).) $\square$

Corollary 4.12. *If $g \in \mathrm{Aut}(M_0/\mathrm{cl}^d(\emptyset))$ is bounded, then there is $E \in \mathcal{X}$ such that $g(\mathrm{cl}^d(Eb)) = \mathrm{cl}^d(Eb)$ for all $b \in M_0$.*

Proof. This follows from the above and Proposition 3.10. $\square$

Remarks 4.13. The class $\mathcal{C}_0$ contains some msa extension $X \subset Y$. If we change the structure on X to some other structure in $\mathcal{C}_0$, then then result is still a msa extension in $\mathcal{C}_0$. Furthermore, by 'duplicating' the points in X if necessary, we can obtain a msa extension with the property that if $r, r' \in R[Y]$ are distinct and both involve points of $Y \setminus X$ and X, then $r \cap r' \cap X = \emptyset$. To do this, replace X by the disjoint union of non-empty $r \cap X$ (for $r \in R[Y] \setminus R[X]$). Then each element of the new X is in exactly one relation in $R[Y] \setminus R[X]$.

Theorem 4.14. *If $g \in \mathrm{Aut}(M_0/\mathrm{cl}^d(\emptyset))$ is bounded, then g is the identity.*

Proof. Let $E \in \mathcal{X}$ be as in the Corollary: so $g(\mathrm{cl}^d(Eb)) = \mathrm{cl}^d(Eb)$ for all $b \in M_0$. Let $A \leq E$ be finite and $d(A) = d(E)$.

Step 1: If $b \in M_0$ is such that $Ab \leq M_0$ and $\delta(b/A) = n$, then $gb = b$.

Case 1: $r \geq 3$, $m = n = 1$. Note that E is infinite, so we may take A to be of size at least $r - 3$. By using elements of A for the first $r - 3$ coordinates in R, we can assume without loss that $r = 3$.

Take c with $c \mathop{\smile\!\!\!\!\!\mid}\nolimits^{d}_{A} b$ of the same type as b over E. By the boundedness condition on g we have $c, gc \mathop{\smile\!\!\!\!\!\mid}\nolimits^{d}_{A} b, gb$. So there are finite $C, B \leq M_0$ with $c, gc \in C$, $b, gb \in B$, $C \cup B \leq M_0$; by enlarging A if necessary we can assume that $E \cap C = A = E \cap B$, and so C, B are freely amalgamated over A.

There is $f \in M_0$ with $R(c, b, f)$ and $CBf \leq M_0$. Note that $d(f/A) = 1$ and $gf \in \mathrm{cl}^d(fA)$, so there is a finite $A \leq F \leq M_0$ with $\delta(F/A) = 1$ and $f, gf \in F$. Note that $\delta(C/F) = 1$ (otherwise it is zero and then $b \in \mathrm{cl}^d(cA)$). So $\delta(C \cap F/A) = 0$ and therefore (as $C \cap E = A$) $C \cap F = A$. Similarly $B \cap F = A$.

Suppose that $\{c, e, b\} \neq \{gc, ge, gb\}$. Then on $C \cup E \cup B$, there are at least 2 extra relations beyond those in the free amalgam over A. So

$$\delta(CEB/A) \leq \delta(C/A) + \delta(E/A) + \delta(B/A) - 2 = 1.$$

But this contradicts $d(cb/A) = 2$. Thus, in particular, $gb = b$.

Case 2: $r \geq 2$, $n > m$. By using elements of A for the first $r - 2$ coordinates, we can assume $r = 2$. Let $B = \mathrm{cl}_0(A, gA, b, gb)$ and suppose for a contradiction that $gb \neq b$.

Let $Ab \leq C$ be a simply algebraic extension in M_0 with base U containing b. We can assume that b is in exactly one relation in C. Let $D = C \setminus (Ab)$; so $U \leq U \cup D$ is msa. As $gA \subseteq E$, we can assume that $g(U \cap A) \subseteq A$. We can also assume that $D \cap (B \cup g^{-1}B) = \emptyset$. Then $gD \cap B = \emptyset$. So both $B \leq B \cup D$ and $B \leq B \cup gD$ are simply algebraic extensions (based on U and $gU = g(U \cap A)gb$ respectively). As $gb \neq b$, we must have $gb \notin U$, so $D \neq gD$. As the extensions are minimal, it follows that $D \cap gD = \emptyset$.

Note that $\delta(A) + n = \delta(Ab) = \delta(C) = \delta(AD) + n - m$. So $\delta(AD) = \delta(A) + m$. In particular, $AD \leq C \leq M_0$, so $d(AD) = d(A) + m$. Let $V = \mathrm{cl}_0(A, D, gD)$. We show that $b, gb \notin V$. Note that $V \subseteq \mathrm{cl}^d(AD)$ (by boundedness of g) so $d(V) = d(AD) = d(A) + m$. But $d(Ab) = d(A) + n > d(A) + m$, so $b \notin V$. As $\mathrm{cl}^d(V)$ is g-invariant, we then obtain $gb \notin V$.

Thus $B \cup V$ has at least 2 more relations in it than in the free amalgam of B, V over $B \cap V$ (a relation from D to b and a relation from gD to gb: neither of these is in the free amalgam, by the previous paragraph). So

$$\delta(BV) \leq \delta(B) + \delta(V) - \delta(B \cap V) - 2m \leq \delta(B) + \delta(V) - \delta(A) - 2m.$$

Now, $\delta(V) = d(A) + m$. So $\delta(BV) \leq \delta(B) - m$. But this is a contradiction as $m \geq 1$ and $B \leq M_0$.

Step 2: If $c \in M_0$ then $gc = c$.

Case 1: $r \geq 3$, $m = n = 1$. As before, we may assume that $r = 3$. It remains to show that if $c \in E$ then $gc = c$. As g fixes all elements of $\mathrm{cl}^d(\emptyset)$, we may assume $c \notin \mathrm{cl}^d(\emptyset)$. We may also assume $gc, c \in A$.

There exist $e, f \in M_0$ with $Aef \leq M_0$ and $R[Aef] = R[A] \cup \{\{c, e, f\}\}$. Then $Ae, Af \leq Aef$, so by Step 1, e, f are fixed by g. It then follows that c is fixed by g (otherwise $\{gc, e, f\} \notin R$), as required.

Case 2: $r \geq 2$, $n > m$. As before, we may assume that $r = 2$. Let $C = \mathrm{cl}_0(A, c)$. Suppose $s \in \mathbb{N}$. There exist $b_0 = c, b_1, b_2, \ldots, b_s \in M_0$ such that $R(b_{i-1}, b_i)$ (and no other relations hold on $C \cup \{b_1, \ldots, b_s\}$ outside C), and $Cb_1 \ldots b_s \leq M_0$. It is easy to see that for $t \leq s$ we have $Cb_1 \ldots b_t \leq M_0$, $d(b_t/Cb_1 \ldots b_{t-1}) = n - m$. Moreover, if s is large enough, then $Cb_s \leq M_0$, so $Ab_s \leq M_0$ and $d(b_s/A) = n$. (For this, take $s \geq n/(n-m)$.) It follows from Step 1 that $gb_s = b_s$.

We now show that if $0 \leq t < s$ and b_{t+1} is fixed by g, then so is b_t. It follows that c is fixed by g, as required. So suppose b_t is not fixed by g. Note that $R(b_t, b_{t+1}) \wedge R(gb_t, b_{t+1})$. Also, using the boundedness of g we have:

$$n-m = d(b_{t+1}/Cb_1 \ldots b_t) = d(b_{t+1}/Cb_1 \ldots b_t gb_1 \ldots gb_t) \leq d(b_{t+1}/b_t gb_t).$$

In particular, $b_{t+1} \notin \mathrm{cl}_0(b_t, gb_t)$ and

$$d(b_{t+1}/b_t gb_t) \leq \delta(b_{t+1}/\mathrm{cl}_0(b_t, gb_t)) \leq n - 2m,$$

because of the edges from b_{t+1} to b_t, gb_t. This is a contradiction (as $m \geq 1$).

□

Corollary 4.15. *Suppose either that $r = 2$ and $n > m$, or that $r \geq 3$ and $n \geq m$. Then $\mathrm{Aut}(M_0/\mathrm{cl}^d(\emptyset))$ is a simple group. In fact, if $g \in \mathrm{Aut}(M_0/\mathrm{cl}^d(\emptyset))$ is not the identity then every element of $\mathrm{Aut}(M_0/\mathrm{cl}^d(\emptyset))$ can be written as a product of 96 conjugates of $g^{\pm 1}$*

Proof. This follows from Corollary 3.12, Lemma 4.8, Lemma 4.11 and Theorem 4.14. □

Remarks 4.16. We have been working with symmetric structures in a signature with a single r-ary relation. More generally, suppose we have a signature with relations R_i of arity r_i (for $i \in I$). Suppose n, m_i are non-negative integers with $n \geq 1$. We define the predimension of a finite structure A to be

$$\delta(A) = n|A| - \sum_{i \in I} m_i |R_i[A]|.$$

Let $\mathcal{C}_0$ consist of such A with $\delta(A') \geq 0$ for all $A' \subseteq A$. Then we can form the generic structure M_0 for $(\mathcal{C}_0, \leq)$ exactly as before. If there is some i such that $m_i \neq 0$ is coprime to n, $r_i = 2$ and $n > m_i$, or $r_i \geq 3$ and $n \geq m_i$, then Corollary 4.15 holds. The argument is the same: for all of the constructions in the proof, just work with R_i in place of R. It should also be clear that our assumption that R is symmetric is not essential.

5. Further applications

5.1. **Generalized polygons.** For $n \geq 3$, a *generalized n-gon* is a bipartite graph Γ of diameter n and girth $2n$. It is *thick* if each vertex has valency at least 3. In [12], Hrushovski's amalgamation method from [7] was adapted to produce thick generalized n-gons of finite Morley rank. These are almost strongly minimal and in [4], Lascar's result ([9], Théorème 2) was applied to show that their autmorphism groups are simple. This gives new examples of simple groups having a BN-pair which are not algebraic groups.

As with Hrushovski's original construction, an intermediate stage in the construction produces ω-stable generalized n-gons Γ_n of infinite Morley rank. In this subsection we observe that we can use Corollary 4.15 in place of Lascar's result to show that these generalized n-gons also have simple automorphism group. As in [4], $\mathrm{Aut}(\Gamma_n)$ is transitive on ordered $2n$-cycles in Γ_n, so is also an example of a (non-algebraic) simple group with a spherical BN-pair of rank 2.

We describe very briefly the construction of Γ_n from Section 3 of [12]. Work with a signature which has a unary predicate symbol P and a binary relation symbol R and consider bipartite graphs as structures in this signature, where P picks out the vertices in one part of the partition and R gives the adjacency relation. Vertices in P are called points and those not in P are called lines. Fix a natural number $n \geq 3$.

For a finite (bipartite) graph A define

$$\delta(A) = (n-1)|A| - (n-2)|R[A]|.$$

As in the previous section, let $\mathcal{C}_0$ consist of the finite bipartite graphs A with $\delta(B) \geq 0$ for all $B \subseteq A$. If $C \subseteq A$ write $C \leq A$ to mean $\delta(B) \geq \delta(C)$ whenever $C \subseteq B \subseteq A$.

Consider the class $\mathcal{K}_n$ of finite bipartite graphs A which satisfy:

(1) the graph A has no $2m$-cycle, for $m < n$;
(2) if $B \subseteq A$ contains a $2m$-cycle for $m > n$, then $\delta(B) \geq 2n + 2$.

The following is from ([12], Corollary 3.13 and Theorem 3.15):

Lemma 5.1. *We have $\mathcal{K}_n \subseteq \mathcal{C}_0$ and $(\mathcal{K}_n, \leq)$ is an amalgamation class.*

Let Γ_n be the generic structure for the class $(\mathcal{K}_n, \leq)$ (cf. Theorem 4.2). So Γ_n is a countable generalized n-gon which is $\leq$-homogeneous. Lemmas 4.5, 4.6 and Corollary 4.7 hold (essentially because of $\leq$-homogeneity and the fact that $\mathcal{K}_n \subseteq \mathcal{C}_0$). As in Corollary 4.8, we have:

Corollary 5.2. *The relation $\mathop{\smile\!\!\!\!\!\!|}^{d}$ is a stationary independence relation on Γ_n compatible with cl^d.*

Proof. If $X \subseteq Y, Z \in \mathcal{K}_n$ is d-closed in Y, Z, then the proof of Theorem 3.15 in [12] shows that the free amalgam of Y and Z over X is in $\mathcal{K}_n$. It follows that the class $\mathcal{X}$ of d-closures of finite sets in Γ_n has the free amalgamation property, and so the proof of Corollary 4.7 gives what we want here. □

Theorem 5.3. *The group* $\mathrm{Aut}(\Gamma_n)$ *is a simple group. In fact, if* $1 \neq g \in \mathrm{Aut}(\Gamma_n)$*, then every element of* $\mathrm{Aut}(\Gamma_n)$ *is a product of 96 conjugates of* $g^{\pm 1}$*.*

Proof. It follows from ([12], Corollary 3.13) that $\mathrm{cl}^d(\emptyset) = \emptyset$ for Γ_n. To prove the theorem, we shall apply Corollary 3.12. So we first find a suitable basic orbit D and then show that there are no non-trivial bounded automorphisms. The first part is essentially as in the proof of ([12], Theorem 4.6), but we give a few details.

If $x \in \Gamma_n$, let $D(x)$ denote the set of vertices adjacent to x. Then by the $\leq$-homogeneity, $D(x)$ is a basic orbit over x. If $x, y \in \Gamma_n$ are at distance n, then there is a bijection definable over x, y from $D(x)$ to $D(y)$ ([11], 1.3). Suppose $x_0, \ldots, x_{2n-1}$ is a $2n$-cycle in Γ_n with $x_0 \in P$. Then Γ_n is in the definable closure of $D(x_0), D(x_1), x_2, \ldots, x_{2n-1}$ (see [11], 1.6). If n is odd, there is a vertex z at distance n from both x_0 and x_1 and therefore Γ_n is in the definable closure of $D(x_0), x_1, \ldots, x_{2n-1}, z$. So if we let $A = \{x_0, \ldots, x_{2n-1}, z\}$ and $D = \{c \in D(x_0) : d(c/A) = 1\}$, then D is a basic orbit over A and $\Gamma_n = \mathrm{cl}^d(A, D)$.

So now suppose n is even. As in the previous paragraph, it will suffice to show that there is a line ℓ and a finite set A with $D(\ell) \subseteq \mathrm{cl}^d(D(x_0), A)$, because $D(x_1)$ is in the definable closure of $D(\ell)$ and some finite set. Let $p_3 \in P$ be at distance n from x_0 and let $\ell \notin P$ be at distance $n-1$ from x_0, p_3. If $k \in D(x_0)$ there is a unique path of length $n-1$ from k to p_3. Let a denote the vertex adjacent to k on this path. There is then a unique path of length $n-1$ from a to ℓ. Let $\phi(k)$ denote the vertex on this path adjacent to ℓ. So we have a definable map $\phi : D(x_0) \to D(\ell)$. It can be seen (by considering the paths involved in this definition of ϕ) that that $d(k/x_0, p_3, \ell, \phi(k)) = 0$ for all $k \in D(x_0)$. Thus, if $d(k/x_0, p_3, \ell) = 1$, then $d(\phi(k)/x_0, p_3, \ell) = 1$. It follows that the image of ϕ contains $D(\ell) \setminus \mathrm{cl}^d(x_0, p_3, \ell)$, so $D(\ell) \subseteq \mathrm{cl}^d(D(x_0), x_0, p_3, \ell)$, as required.

To show that there are no non-trivial bounded automorphisms, one uses that same proof as in ([4], Proposition 6.3), replacing acl there by cl^d. □

5.2. **$\aleph_0$-categorical structures.** We recall briefly a variation on the construction method of Section 4.1 which gives rise to $\aleph_0$-categorical structures. The original version of this is in [6] where it is used to provide a counterexample to Lachlan's conjecture, and in [8] where it is used to construct a non-modular, supersimple $\aleph_0$-categorical structure. The book [15] (Section 6.2.1) is a convenient reference for this. Generalizations and reworkings of the method (particulalrly relating to simple theories) can be found in [1]. For the rest of this subsection, assume that $m, n, r, \delta, (\mathcal{C}_0, \leq)$ etc. are as in Section 4.1.

In this version of the construction, d-closure is uniformly locally finite. Suppose $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is a continuous, increasing function with $f(x) \to \infty$ as $x \to \infty$. Let

$$\mathcal{C}_f = \{A \in \mathcal{C}_0 : \delta(X) \geq f(|X|) \ \forall X \subseteq A\}.$$

Note that if $X \subseteq A \in \mathcal{C}_f$ then

$$\delta(X) \geq \delta(\mathrm{cl}^d(X)) \geq f(|\mathrm{cl}^d(X)|)$$

so $|\mathrm{cl}^d_A(X)| \leq f^{-1}(\delta(X)) \leq f^{-1}(n|X|)$.

If $B \subseteq A \in \mathcal{C}_f$ and $\mathrm{cl}^d_A(B) = B$, then we write $B \leq_d A$. For suitable choice of f (call these *good* f), $(\mathcal{C}_f, \leq_d)$ has the free $\leq_d$-amalgamation property: if $A_0 \leq_d A_1, A_2 \in \mathcal{C}_f$ then $A_i \leq_d A_1 \coprod_{A_0} A_1 \in \mathcal{C}_f$. In this case we have an associated countable *generic structure* M_f. So M_f is $\leq_d$-homogeneous and the set $\mathcal{X}$ of finite d-closed subsets of M_f is (up to isomorphism) $\mathcal{C}_f$. As d-closure is uniformly locally finite, the structure M_f is $\aleph_0$-categorical (by the Ryll - Nardzewski Theorem). Algebraic closure in M_f is equal to d-closure.

Remarks 5.4. To construct good functions, we can take f which are piecewise smooth and where the right derivative f' satisfies $f'(x) \leq 1/x$ and is non-increasing, for $x \geq 1$. The latter condition implies that $f(x + y) \leq f(x) + yf'(x)$ (for $y \geq 0$). It can be shown that under these conditions, $\mathcal{C}_f$ has the free $\leq_d$-amalgamation property. Also note that if $f'(x) \leq 1/x$ for all $x \geq x_0$, then for $y \geq x \geq x_0$ we have $f(y) \leq f(x) + \log(y - 1) - \log(x - 1)$.

Assumption 5.5. Henceforth, we assume that if $r = 2$, then $n > m$ and if $r \geq 3$, then $n \geq m$. We suppose that f is a good function. We will assume that $f(0) = 0$ and $f(1) > 0$, therefore $\mathrm{cl}^d(\emptyset) = \emptyset$. We shall also assume that $f(1) = n$. Thus if $X \in \mathcal{C}_f$ and $|X| \geq 2$, then $\delta(X) \geq f(|X|) > n$. In particular $\{x\} \leq_d X$ for all $x \in X$.

Let $G = \mathrm{Aut}(M_f)$.

As before, we write $\mathop{\smile\!\!\!\!\!\vert}^d$ for d-independence in M_f. This is not stationary. If $A \leq_d C \in \mathcal{X}$ and $b_0 \in M_f$, then $\{b \in \mathrm{orb}(b_0/A) : b \mathop{\smile\!\!\!\!\!\vert}^d_A C\}$ need not be a single G_C-orbit: the orbits are determined by the d-closures $\mathrm{cl}^d(bC)$. Clearly $\mathrm{cl}^d(bC) \supseteq \mathrm{cl}^d(bA) \cup C$ and as in Lemma 4.3 it can be shown that $\mathrm{cl}^d(bA) \cap C = A$, $\mathrm{cl}^d(bA), C$ are freely amalgamated over A and $\mathrm{cl}^d(bA) \cup C \leq M_f$ if and only if $b \mathop{\smile\!\!\!\!\!\vert}^d_A C$.

Definition 5.6. Suppose $A \leq_d C \in \mathcal{X}$ and b is a tuple of elements of M_f. Write $b \perp_A C$ to mean that $b \mathop{\smile\!\!\!\!\!\vert}^d_A C$ and $\mathrm{cl}^d(bC) = \mathrm{cl}^d(bA) \cup C$. Note that in this case, $\mathrm{cl}^d(bC)$ is the free amalgam of $\mathrm{cl}^d(bA)$ and C over A.

The following is straightforward:

Lemma 5.7. *The relation $\perp$ is a stationary independence relation compatible with* cl^d. $\square$

We will use Theorem 2.5 to show that, under some restrictions, the group $G = \mathrm{Aut}(M_f)$ is simple. The proof is similar to that in the previous sections, but we need to make some modifications as the dimension function does not give rise to a stationary independence relation.

Suppose $A \in \mathcal{X}$ and $b \in M_f$. We shall continue to say that b is basic over A if $b \notin A$ and whenever $A \leq_d C \in \mathcal{X}$ and $d(b/C) < d(b/A)$, then $b \in C$. Recall also that M_f is monodimensional if for all basic orbits $D = \mathrm{orb}(b/A)$ (for $A \in \mathcal{X}$) there is $B \in \mathcal{X}$ with $A \subseteq B$ and $M_f = \mathrm{cl}^d(B, D \setminus B)$. In fact, in the examples below where we verify this, we will take $B = A$.

As before, we say that $g \in G$ is d-bounded over $A \in \mathcal{X}$ if there is $A \subseteq C \in \mathcal{X}$ and $b \in M_f$ which is basic over C such that for all $b' \in \mathrm{orb}(b/C)$ we have $gb' \in \mathrm{cl}^d(b'C)$.

Lemma 5.8. *Suppose M_f is monodimensional and $g \in \mathrm{Aut}(M_f)$ is d-bounded (over some element of $\mathcal{X}$). Then $g = 1$.*

Proof. By Proposition 3.10 there is $E \in \mathcal{X}$ such that g stabilizes every $B \in \mathcal{X}$ containing E. In particular, g fixes all $b \in M_f \setminus E$ for which $Eb \leq_d M_f$.

Let c, c' be distinct elements of M_f and $C = \mathrm{cl}^d(E, c, c')$. First suppose that $r > 2$. Consider the structure B consisting of c together with $r-1$ points $b_1, \ldots, b_{r-1}$ such that $R[B]$ is the single relation $\{c, b_1, \ldots, b_{r-1}\}$. Then $B \in \mathcal{C}_f$ and $c \leq_d B$. By Assumption 5.5, the free amalgam U of C and B over c is in $\mathcal{C}_f$, so we may suppose $U \leq_d M_f$. One calculates that $Eb_i \leq_d U$ for each i (this uses that $r > 2$), therefore the b_i are fixed by g. As g stabilizes E, C and U, it is then clear that $gc \neq c'$. But this holds for all $c' \neq c$, so in fact, $gc = c$.

Now suppose that $r = 2$ (and $n > m$). Take $b \perp C$. Suppose $c, e_1, \ldots, e_s, b$ is a simple path with endpoints c, b. If $s > m/(n-m)$ then $cb \leq_d ce_1 \ldots e_s b$. As $cb \leq_d Cb$ we may use free amalgamation over cb to find such a path with $U = Ce_1 \ldots e_s b \leq_d M_f$. Then $gb = b$ and g stabilizes E, C, U. There is a path from b to c whose internal vertices are in $U \setminus C$, but there is no such path to c'. So $gc \neq c'$, and it follows that $gc = c$. $\square$

Proposition 5.9. *Suppose M_f is monodimensional, $A \in \mathcal{X}$ and D is a basic orbit over A. Suppose $1 \neq g \in \mathrm{Aut}(M_f/A)$.*

(1) *If $c \in M_f$ and $A \subseteq B \in \mathcal{X}$, then there is $c' \in \mathrm{orb}(c/B)$ with $gc' \mathop{\perp}\limits^{d}_{B} c'$.*
(2) *There is $\tilde{h} \in G_A$ such that the commutator $\tilde{g} = [g, \tilde{h}]$ moves almost maximally over A with respect to $\perp$, that is, if $a' \in M_f$ and $A \subseteq X \in \mathcal{X}$, there is $a \in \mathrm{orb}(a'/X)$ such that $\tilde{g}a \perp_X a$.*

Proof. (1) This follows from Lemma 5.8 and Proposition 3.7.

(2) We build $\tilde{h}$ by a back-and-forth construction as in the first part of the proof of Theorem 3.9. During the 'forth' step we shall ensure that $\tilde{g}$ moves almost maximally with respect to $\perp$ (over A). So suppose we have constructed a partial automorphism $h : U \to V$ (fixing A) and X, a' are given. By extending h arbitrarily, we may assume that $U \supseteq X, gX, h^{-1}ghX$.

Claim 1: We can choose $a \in \mathrm{orb}(a'/X)$ such that $a \perp_X U, g^{-1}U$ and $ga \mathop{\smile\!\!\!\!\!\!\mid}^d_U a$.

To do this, take $a'' \in \mathrm{orb}(a'/X)$ with $a'' \perp_X U, g^{-1}U$ (by Extension). Then by (1), there is $a \in \mathrm{orb}(a''/\mathrm{cl}^d(U, g^{-1}U))$ with $ga \mathop{\smile\!\!\!\!\!\!\mid}^d_{U,g^{-1}U} a$. It follows from Transitivity (for $\mathop{\smile\!\!\!\!\!\!\mid}^d$) that $ga \mathop{\smile\!\!\!\!\!\!\mid}^d_U a$, as required.

Similarly, we can take $b \in h\mathrm{orb}(a'/U)$ with $b \perp_{hX} V, g^{-1}V$ and $gb \mathop{\smile\!\!\!\!\!\!\mid}^d_V b$. Extend h by setting $ha = b$.

Note that $h^{-1}\mathrm{orb}(gb/\mathrm{cl}^d(V, b))$ is an orbit over $\mathrm{cl}^d(U, a)$. We choose e in this with $e \perp_{U,a} ga$ and extend h further by setting $he = gb$.

We have that $\mathrm{cl}^d(e, U, a) \perp_{U,a} \mathrm{cl}^d(ga, U, a)$. Intersecting this d-closed free amalgam with $Y = \mathrm{cl}^d(U, e, ga)$ we obtain another d-closed free amalgam, so $e \perp_Z ga$, where $Z = \mathrm{cl}^d(U, a) \cap Y$.

Claim 2: We have $Z = U$, so $e \perp_U ga$.

By Claim 1 we have $d(ga, a/U) = d(ga/U) + d(a/U)$, and similarly $d(gb/V, b) = d(gb/V)$. So we have:

$$d(e/U, a, ga) = d(e/U, a) = d(gb/V, b) = d(gb/V) = d(e/U),$$

where the second and fourth of these come from applying h. It then follows that a, ga, U are d-independent over U, so $a \mathop{\smile\!\!\!\!\!\!\mid}^d_U ga, e$. In particular, $\mathrm{cl}^d(u, a) \cap \mathrm{cl}^d(U, ga, e) = U$.

Claim 3: We have $e \perp_{gX} ga$.

By Claim 1, $U \perp_{gX} ga$ so $\mathrm{cl}^d(U, ga) = U \coprod_{gX} E_2$, where $E_2 = \mathrm{cl}^d(gX, ga)$.

By choice of b we have $gb \perp_{ghX} gV, V$, so (applying h^{-1}) $e \perp_{h^{-1}ghX} U$. Thus $\mathrm{cl}^d(U, e) = U \coprod_{h^{-1}ghX} E_1$ where $E_1 = \mathrm{cl}^d(h^{-1}ghX, e)$.

Let $A_i = E_i \cap U$. So $A_1 = h^{-1}ghX$ and $A_2 = gX$. Let $W = \mathrm{cl}^d(A_1, A_2)$. By Claim 2, $U \cup E_1 \cup E_2 \leq_d M_f$. We also have $W \cup E_1 \cup E_2 \leq_d U \cup E_1 \cup E_2$, so $E_1 \perp_W E_2$, that is:

$$E_1 \perp_{A_1, A_2} E_2.$$

As $a \perp_X g^{-1}U$, we have (applying g) $E_2 \perp_{A_2} U$. So $E_2 \perp_{A_2} E_1$. By Transitivity we obtain $E_1 \perp_{A_2} E_2$, which gives the claim.

By applying g^{-1} to Claim 3 we obtain:

$$[g, h]a \perp_X a$$

which is what we wanted to do in this step of the construction.

□

Corollary 5.10. *Suppose M_f is monodimensional and $1 \neq g \in \mathrm{Aut}(M_f)$. Then every element of $\mathrm{Aut}(M_f)$ is a product of 192 conjugates of $g^{\pm 1}$.*

Proof. Note that $\mathrm{cl}^d(\emptyset) = \emptyset$ so $G = \mathrm{Aut}(M_f)$. Let $A \in \mathcal{X}$ be such that there is a basic orbit D over A. It is easy to show that there is a non-identity commutator g_1 of g which fixes every element of A. By Proposition 5.9, by taking a further commutator with an element of G_A we obtain some $g_2 \in G_A$ which moves almost maximally over A (with respect to $\perp$). It follows from Theorem 2.5 that every element of G_A is a product of 16 conjugates of g_2. As g_2 is a product of 4 conjugates of $g^{\pm 1}$, it follows that every element of G_A is a product of 64 conjugates of $g^{\pm 1}$. As in the final part of the proof of Theorem 3.9, G is the product of three conjugates of G_A: hence the result. □

We believe that under the conditions of Assumption 5.5, the structure M_f should be monodimensional. However, proving this appears to require an extremely technical argument and we only have a full proof in some special cases.

Example 5.11. Suppose that $r \geq 3$ and $m = n = 1$; so $\delta(A) = |A| - |R[A]|$. Suppose f is as in Remarks 5.4 and also that Assumption 5.5 holds.

If $A \in \mathcal{X}$ and $b \in M_f \setminus A$ then $d(b/A) = 1$ so b is basic over A. Let $D = \mathrm{orb}(b/A)$. We show that $M_f = \mathrm{cl}^d(A, D)$.

Step 1. There is $c \in \mathrm{cl}^d(A, D)$ with $c \perp A$.

Let $B = \mathrm{cl}^d(A, b)$ and let F be the free amalgam of copies $B_1, \ldots, B_{r-1}$ of B over A, with $b_i \in B_i$ being the copy of b inside B_i. Let $E = F \cup \{c\}$ where $R(b_1, \ldots, b_{r-1}, c)$ holds and this is the only relation in E involving c. We show that:

(i) $E \in \mathcal{C}_f$;
(ii) $B_i \leq_d E$;
(iii) $Ac \leq_d E$.

Note that once we have this, it follows that we may assume $E \leq_d M_f$ and so (by (ii)) $b_1, \ldots, b_{r-1} \in D$. Moreover, $c \in \mathrm{cl}^d(A, b_1, \ldots, b_{r-1})$ and (by (iii)) $A \perp c$, which finishes Step 1.

For (i), note of course that $F \in \mathcal{C}_f$. Let $Y \subseteq E$. We want to show that $\delta(Y) \geq f(|Y|)$. We may assume that $c, b_1, \ldots, b_{r-1} \in Y$ and $Y \leq_d E$. In the following, if $C \subseteq E$, let $Y_C = Y \cap C$.

If $Y_A = \emptyset$ then Y is obtained by free amalgamation over the b_i from $\{b_1, \ldots, b_{r-1}, c\}$ and the Y_{B_i}, so is in $\mathcal{C}_f$. So we may assume that $Y_A \neq \emptyset$. Also, if $|Y_{B_i} \setminus A| = 1$ for all i, then as $d(b_i/A) = 1$, there are no relations between Y_A and $\{b_1, \ldots, b_{r-1}, c\}$ and Y is again a free amalgam. So we may also assume that $2 \leq |Y_{B_1} \setminus A| \geq |Y_{B_i} \setminus A|$. In particular, $|B_1| \geq 3$.

Now we compute that

$$\delta(Y) = \delta(Y_F) = \delta(Y_{B_1}) + \sum_{i\geq 2} \delta(Y_{B_i}/Y_{B_1}) \leq \delta(Y_{B_1}) + (r-2).$$

Also

$$|Y| = 1 + |Y_{B_1}| + \sum_{i\geq 2} |Y_{B_i} \setminus A| \leq 1 + |Y_{B_1}| + (r-2)|Y_{B_1} \setminus A|.$$

As in Remarks 5.4

$$f(|Y|) \leq f(|Y_{B_1}|) + \log\left(\frac{|Y_{B_1}| + (r-2)|Y_{B_1} \setminus A|}{|Y_{B_1}| - 1}\right).$$

So to prove that $\delta(Y) \geq f(|Y|)$ it will suffice to show that

$$r - 2 \geq \log\left(\frac{|Y_{B_1}| + (r-2)|Y_{B_1} \setminus A|}{|Y_{B_1}| - 1}\right).$$

As $|Y_A| \geq 1$ and $|Y_{B_1} \setminus Y_A| \geq 2$ we have:

$$\frac{|Y_{B_1}| + (r-2)|Y_{B_1} \setminus A|}{|Y_{B_1}| - 1} \leq (r-1) + \frac{1}{2},$$

and the required inequality holds as $r \geq 3$. This completes the proof of (i).

We now verify (ii); without loss we take $i = 1$. Suppose $B_1 \subset Y \subseteq E$. We need to show that $\delta(B_1) < \delta(Y)$. We may assume that $Y \leq_d E$ and also that $b_1, \ldots, b_{r-1}, c \in Y$ (otherwise what we want follows from free amalgamation). But then $Y = E$ and $\delta(E) = \delta(B_1) + (r-2) > \delta(Y)$.

For (iii), suppose $Ac \subset Y \subseteq E$. If Y does not contain all of $b_1, \ldots, b_{r-1}$, then $\delta(Y) = \delta(Y_F) + 1 > \delta(A) + 1 = \delta(Ac)$. On the other hand, if Y contains all of $b_1, \ldots, b_{r-1}$, then $\delta(Y) \geq \delta(A) + (r-1) > \delta(Ac)$. This completes Step 1.

From Step 1 and Stationarity, it follows that $\mathrm{cl}^d(A, D) \supseteq \{e \in M_f : e \perp A\}$. So to show that $\mathrm{cl}^d(A, D) = M_f$ it will suffice to show:

Step 2. If $a \in M_f \setminus A$, there exist $e_1, \ldots, e_{r-1} \in M_f$ with $e_i \perp A$ and $a \in \mathrm{cl}^d(A, e_1, \ldots, e_{r-1})$.

To see this, let $C = \mathrm{cl}^d(A, a)$ and let F be the free amalgam of this over a with the structure on points $\{a, e_1, \ldots, e_{r-1}\}$ which has a single relation $R(a, e_1, \ldots, e_{r-1})$. As $A \leq_d F$, we can assume that $F \leq_d M_f$. Moreover, an easy calculation shows that $Ae_i \leq_d F$ and so $e_i \perp A$ for all i. But $a \in \mathrm{cl}^d(e_1, \ldots, e_{r-1})$ so we have completed Step 2.

Example 5.12. Suppose as in [6] that $r = 2$, $n = 2$ and $m = 1$. So we are considering graphs A and $\delta(A) = 2|A| - e(A)$ where $e(A)$ denotes the number of edges in A. We take $f(0) = 0$, $f(1) = 2$, $f(2) = 3$ and $f'(x) \leq 1/x$ non-increasing for $x \geq 2$ as in Remarks 5.4. So if $A \in \mathcal{C}_f$, then vertices and edges are d-closed in A. Moreover $f(x) \leq 3 + \log(x-1)$

for $x \geq 2$; more generally, $f(y) \leq f(x) + \log(y-1) - \log(x-1)$ for $2 \leq x \leq y$.

By free amalgamation, $\mathcal{C}_f$ contains paths P_ℓ of arbitrary length ℓ. One easily computes that if u, v are the endpoints of P_ℓ then $uv \leq_d P_\ell$ iff $\ell \geq 3$. In particular (using free amalgamation), $\mathcal{C}_f$ contains a 6-cycle, but need not contain shorter cycles.

The strategy for verifying monodimensionality is as in the previous example, but the details are considerably more complicated. Suppose $A \in \mathcal{X}$ and $\mathrm{orb}(b/A)$ is any G_A-orbit on $M_f \setminus A$. We shall show that there exist $b_0, \ldots, b_{s-1} \in \mathrm{orb}(b/A)$ and $c \in \mathrm{cl}^d(b_0, \ldots, b_{s-1}, A)$ such that $c \perp A$. So $\mathrm{cl}^d(A, \mathrm{orb}(b/A))$ contains $\{e : e \perp A\}$. We then observe that $\mathrm{cl}^d(A, \{e : e \perp A\}) = M_f$.

In order to do this, we construct various graphs and verify that they are in $\mathcal{C}_f$.

Step 1. Let $s \in \mathbb{N}$ be sufficiently large. Construct a graph with vertices $C = \{c_0, \ldots, c_{s-1}\}$ and $D = \{d_0, \ldots, d_{s-1}\}$ such that:

- $c_0, d_0, c_1, d_1, \ldots, c_{s-1}, d_{s-1}$ is a $2s$-cycle;
- the remaining edges on CD form a single s-cycle on D and CD has girth at least 6.

To do this, we can take adjacencies in D to be $d_i \sim d_{i+\ell}$ where the indices are read modulo s and ℓ is chosen coprime to s and $6 \leq \ell < s/12$.

Step 2. We have $CD \in \mathcal{C}_f$.

Note that as s is large, $\delta(CD) = s > 3 + \log(2s-1) \geq f(2s) = f(|CD|)$. Let $X \subset CD$. We need to show that $\delta(X) \geq f(|X|)$. We may assume that $X \leq_d CD$. Write $X_D = D \cap X$ and use similar notation throughout what follows. We have $X_D \subset D$, so

$$\delta(X_D) \geq 2|X_D| - (|X_D| - 1) = |X_D| + 1.$$

Consider the valencies of vertices in X_C within X. There are at most $|X_D| - 1$ of valency 2 and those of valency at most 1 contribute at least 1 to $\delta(X/X_D)$. Thus

$$|X_C| \leq \delta(X/X_D) + |X_D| - 1,$$

so

$$\delta(X) \geq |X_C| - |X_D| + 1 + \delta(X_D) \geq |X_C| + 2.$$

Also,

$$\delta(X) = 2|X_C| + 2|X_D| - e(X_C, X_D) - e(X_D) \geq \delta(X_D)$$

as $e(X_C, X_D)$, the number of edges between X_C and X_D, is at most $2|X_C|$. So

$$\delta(X) \geq \delta(X_D) \geq |X_D| + 1.$$

We therefore obtain:

$$\delta(X) \geq \frac{1}{2}(|X| + 3).$$

As $f(x) \leq 3 + 2\log(x-1)$, we have $\delta(X) \geq f(|X|)$ if $|X| \geq 7$. If $|X| \leq 6$ then X is either a 6-cycle or has no cycles, so is in $\mathcal{C}_f$.

Step 3. If $X \leq_d CD$ and X is the d-closure in CD of X_C, then $|X| \leq 4|X_C| - 3$.

This follows from the fact that $0 \geq \delta(X/X_C) \geq \frac{1}{2}(|X|+3) - 2|X_C|$.

Step 4. Let B consist of copies $B_0, \ldots, B_{s-1}$ of $B' = \mathrm{cl}^d(A, b)$ freely amalgamated over A, with b_i the copy of of b inside B_i. Let $E = B \cup C \cup D$ with edges as in B, $C \cup D$ and additional edges $b_i \sim c_i$ for $i = 0, \ldots, s-1$. Note that $\delta(E) = \delta(A) + s\delta(B'/A) = \delta(B)$ and $|E| = |A| + s|B' \setminus A| + 2s = |A| + s(|B' \setminus A| + 2)$. For sufficiently large s we have $\delta(E) \geq f(|E|)$ (by the logarithmic growth of f).

Suppose $Y \subset E$; we claim that $\delta(Y) \geq f(|Y|)$, so $E \in \mathcal{C}_f$. We may assume that $Y \leq_d E$. It is clear that E is the free amalgam of BC and CD over C and it is easy to check that $C \leq_d BC$. So $Y_C \leq_d Y_{BC}$.

Let Y'_C be the d-closure of Y_C inside CD. So $Y'_C \subseteq Y_{CD}$ and $Y'_C \cap C = Y_C$. Then $Y_B \cup Y'_C$ is a free amalgam over Y_C and $Y'_C \leq_d Y_B \cup Y'_C$. Moreover, $Y'_C \leq Y_{CD}$; so it will suffice to show that $Y_B \cup Y'_C \in \mathcal{C}_f$. Thus we may assume $Y'_C = Y_{CD}$. In particular, by Step 3, we may assume that $|Y_{CD}| \leq 4t - 3$, where $t = |Y_C|$. We can assume $t \geq 2$.

We may assume that $\delta(Y_{B_i}/Y_A) \leq 1$ for all i. Then we may further assume that $b_i \in Y$ iff $c_i \in Y$. (If $c_i \in Y$ and $b_i \notin Y$, then adding b_i into Y increases the size of Y without increasing δ; conversely if $b_i \in Y$ but c_i is not, then Y_{B_i} is freely amalgamated with the rest of Y over Y_A.) Similarly we can assume that if $Y_{B_i} \supset Y_A$ then $b_i \in Y_i$. It follows that $\delta(Y_B/Y_A) = t$.

Choose i such that $|Y_{B_i} \setminus Y_A|$ is as large as possible; say $i = 1$ and the size is k. Then

$$|Y| = |Y_B| + |Y_{CD}| \leq |Y_{B_1}| + (t-1)k + 4t - 3.$$

Also

$$\delta(Y) = \delta(Y_B) + \delta(Y_{CD}) - e(Y_B, Y_C) \geq (\delta(Y_{B_1}) + (t-1)) + (t+2) - t$$

using the inequality $\delta(Y_{CD}) \geq t + 2$ from Step 2, and so:

$$\delta(Y) \geq \delta(Y_{B_1}) + t + 1.$$

So it will suffice to show that

$$\delta(Y_{B_1}) + t + 1 \geq f(|Y_{B_1}| + (t-1)k + 4t - 3).$$

By the logarithmic nature of f, and $\delta(B_1) \geq f(|B_1|)$, this will follow from:

$$t + 1 \geq \log((t-1)(k+4)) - \log(|Y_{B_1}| - 1).$$

It is easily checked that this is the case (as $t \geq 2$ and $|Y_{B_1}| \geq k+1$). This finishes the proof that $E \in \mathcal{C}_f$.

Step 5. If $e \in D$, then $Ae \leq_d E$. To see this, let $Ae \subset X \subseteq E$. As E is a free amalgam over C

$$\delta(X) = \delta(X_{BC}/X_C) + \delta(X_{CD}).$$

It is straightforward to see that this is greater than $\delta(Ae) = \delta(A) + 2$.

Step 6. We have $B_i \leq_d E$. This follows from the the calculations in Step 4.

It follows that $A \leq_d E$, so we may assume that $E \leq_d M_f$. As $\delta(E) = \delta(B)$, we have $E = \mathrm{cl}^d(B)$. By Step 6, each b_i is in $\mathrm{orb}(b/A)$. By Step 5, we have that $A \perp e$ for $e \in D$. It follows that $\mathrm{cl}^d(A, \mathrm{orb}(b/A))$ contains $\{e \in M_f : e \perp A\}$.

To conclude, we show that $\mathrm{cl}^d(A, \{e : e \perp A\}) = M_f$. Let $x \in M_f \setminus A$ and $X = \mathrm{cl}^d(x, A)$. Using the above construction we can find $V \in \mathcal{C}_f$ and distinct $b_1, \ldots, b_s, y \in V$ such that $y \in \mathrm{cl}^d(b_1, \ldots, b_s)$ and y is not adjacent to any of the b_i. The latter implies that $yb_i \leq V$. Identify y with x and form the free amalgam U of V and X over x. This is in $\mathcal{C}_f$ so we may assume $U \leq_d M_f$. Using that $xb_i \leq V$, it is straightforward to check that $b_i \perp A$, and so $x \in \mathrm{cl}^d(A, \{e : e \perp A\})$, as required. It follows that M_f is monodimensional.

5.3. **Concluding remarks.** Hrushovski's paper [6] uses a further variation on the construction method of the previous subsection to produce stable, $\aleph_0$-categorical structures which are not one-based. In this variation of the construction, the predimension is given by

$$\delta(A) = |A| - \alpha|R[A]|$$

where $\alpha \in \mathbb{R}^{\geq 0}$ is *irrational.* For certain α one defines a control function $f_\alpha : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that $\mathcal{C}_{f_\alpha}$ is a free amalgamation class and the Fraïssé limit M_α is stable and $\aleph_0$-categorical. The details of this can be found in ([14], Example 5.3). Forking independence gives a stationary independence relation on M_α and it would be interesting to investigate simplicity (or otherwise) of $\mathrm{Aut}(M_\alpha)$ using Theorem 2.5.

In his paper [9], Lascar also proves a *small index property* for countable, saturated almost strongly minimal structures and it would be interesting to know whether these methods can be used to prove that such a property also holds for the structures M_0 and M_f (for good f) of Sections 4.1 and 5.2. More specifically, we ask:

- Suppose G is $\mathrm{Aut}(M_0)$ or $\mathrm{Aut}(M_f)$ and $H \leq G$ is of index less than $2^{\aleph_0}$ in G. Does there exist $A \in \mathcal{X}$ such that $H \geq G_A$?

In the case where $G = \mathrm{Aut}(M_0)$, it seems likely that Lascar's methods work, though we have not checked all of the details. For the case where $G = \mathrm{Aut}(M_f)$, the following problem is relevant:

- Suppose $A_i, B_i \leq_d M_f$ are finite and $h_i : A_i \to B_i$ is an isomorphism (for $i = 1, \ldots, n$). Do there exist $D \in \mathcal{X}$ with $A_i, B_i \leq_d D$ and $g_i \in \mathrm{Aut}(D)$ such that $g_i \supseteq h_i$ for all $i \leq n$?

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