On period spaces for p-divisible groups

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Abstract

In their book Rapoport and Zink constructed rigid analytic period spaces for Fontaine's filtered isocrystals, and period morphisms from moduli spaces of p-divisible groups to some of these period spaces. We determine the image of these period morphisms, thereby contributing to a question of Grothendieck. We give examples showing that only in rare cases the image is all of the Rapoport-Zink period space.

Résumé

Dans leur livre, Rapoport et Zink ont construits des espaces des périodes, rigides analytiques pour les isocristaux filtrés de Fontaine. Egalement ils ont construits des morphismes des périodes entre des espaces modulaires des groupes de Barsotti-Tate et certaines de leurs espaces des périodes. Dans cet article nous déterminons l'image des morphismes des périodes, contribuant ainsi à une question de Grothendieck. Nous présentons des examples montrant que l'image ne coïncide que rarement avec tout l'espace des périodes de Rapoport-Zink.

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1 A question of Grothendieck

Fix a Barsotti-Tate group $\mathbb{X}$ over $\mathbb{F}_p^{\rm alg}$ of height h and dimension d. Let $W := W(\mathbb{F}_p^{\rm alg})$ be the ring of Witt vectors and let $K_0 := W[\frac{1}{p}]$. Let $\mathcal{O}_K$ be a complete discrete valuation ring with residue field $\mathbb{F}_p^{\rm alg}$ and fraction field K of characteristic 0. To every Barsotti-Tate group X over $\mathcal{O}_K$ with $\mathbb{X} \cong X \otimes_{\mathcal{O}_K} \mathbb{F}_p^{\rm alg} =: X_{\mathbb{F}_p^{\rm alg}}$ the theory of Grothendieck-Messing [19] associates an extension

$$0 \longrightarrow (\mathrm{Lie}\, X^\vee)^\vee_K \longrightarrow \mathbb{D}(\mathbb{X})_K \longrightarrow \mathrm{Lie}\, X_K \longrightarrow 0$$

where $\mathbb{D}(\mathbb{X})_K$ is the crystal of Grothendieck-Messing evaluated on K. We denote by $\mathcal{F}$ the Grassmannian of $(h-d)$-dimensional subspaces of $\mathbb{D}(\mathbb{X})_{K_0}$.

Problem 1.1. *(A. Grothendieck [12, Remarque 3, p. 435]) Describe the subset of $\mathcal{F}$ formed by the points $(\mathrm{Lie}\, X^\vee)^\vee_K$ where X is any deformation of $\mathbb{X}$ over any complete discrete valuation ring $\mathcal{O}_K$ with residue field $\mathbb{F}_p^{\rm alg}$ and fraction field K of characteristic 0.*

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This problem is yet unsolved. However, a contribution was made by Rapoport and Zink [20] in the following way. Set $D := \mathbb{D}(\mathbb{X})_{K_0}$ and $\varphi_D := \mathbb{D}(\mathrm{Frob}_{\mathbb{X}})_{K_0}$. Let $L_K \in \mathcal{F}$ be a point defined over a complete, rank one valued extension K of K_0 (which not necessarily is discrete). View L_K as a K-subspace of $D_K := D \otimes_{K_0} K$. One defines the *Newton slope* $t_N(D, \varphi_D, L_K) := \mathrm{ord}_p(\det \varphi_D)$ and the *Hodge slope* $t_H(D, \varphi_D, L_K) := \dim_K L_K - \dim_K D_K$. Following Fontaine [11] and Rapoport-Zink [20, 1.18], the point $L_K \in \mathcal{F}$ is called *weakly admissible* if

$$\begin{array}{rclcll} t_N(D, \varphi_D, L_K) & = & t_H(D, \varphi_D, L_K) & = & -d & \quad\text{and} \\ t_N(D', \varphi_D|_{D'}, L_K \cap D'_K) & \geq & t_H(D', \varphi_D|_{D'}, L_K \cap D'_K) & & & \end{array}$$

for all φ_D-stable K_0-subspaces $D' \subset D$. Let $\mathcal{F}^{\mathrm{rig}}$ be the rigid analytic space, and $\mathcal{F}^{\mathrm{an}}$ the K_0-analytic space in the sense of Berkovich [3, 4] associated with $\mathcal{F}$.

Theorem 1.2. *(Rapoport-Zink [20, Proposition 1.36])*
The set $\mathcal{F}^{\mathrm{rig}}_{wa} := \{\, L_K \in \mathcal{F}^{\mathrm{rig}} : L_K$ is weakly admissible$\,\}$ is an open rigid analytic subspace.

The space $\mathcal{F}^{\mathrm{rig}}_{wa}$ is an example for the *p-adic period domains* constructed more generally in [20, Proposition 1.36] for arbitrary filtered isocrystals. The proof of Rapoport and Zink even shows that $\mathcal{F}^{\mathrm{rig}}_{wa}$ is the rigid analytic space associated to an open K_0-analytic subspace $\mathcal{F}^{\mathrm{an}}_{wa} \subset \mathcal{F}^{\mathrm{an}}$; see [14, Proposition 1.3]. The period domain $\mathcal{F}^{\mathrm{rig}}_{wa}$ contains the set of Grothendieck's problem 1.1. To explain this let $\mathcal{N}ilp_W$ be the category of W-schemes on which p is locally nilpotent. For an $S \in \mathcal{N}ilp_W$ we set $\bar{S} := \mathrm{V}(p) \subset S$.

Theorem 1.3. *(Rapoport-Zink [20, Theorem 2.16]) The functor $\mathcal{G} : \mathcal{N}ilp_W \to \mathcal{S}ets$*

$$S \longmapsto \{\, \textit{isomorphism classes of pairs } (X, \rho) \textit{ where } X \textit{ is a Barsotti-Tate group over } S \textit{ and } \rho : \mathbb{X}_{\bar{S}} \to X_{\bar{S}} \textit{ is a quasi-isogeny} \,\}$$

is pro-representable by a formal scheme locally formally of finite type over W.

To $\mathcal{G}$ one can associate a rigid analytic space $\mathcal{G}^{\mathrm{rig}}$ by Berthelot's construction [5], and also a K_0-analytic space $\mathcal{G}^{\mathrm{an}}$. Rapoport and Zink [20, 5.16] construct a period morphism $\breve{\pi}_1^{\mathrm{rig}} : \mathcal{G}^{\mathrm{rig}} \to \mathcal{F}^{\mathrm{rig}}_{wa}$ as follows. By the theory of Grothendieck-Messing [19], the universal Barsotti-Tate group X over $\mathcal{G}$ gives rise to an extension

$$0 \longrightarrow (\mathrm{Lie}\, X^\vee)^\vee_{\mathcal{G}^{\mathrm{rig}}} \longrightarrow \mathbb{D}(X_{\bar{\mathcal{G}}})_{\mathcal{G}^{\mathrm{rig}}} \longrightarrow \mathrm{Lie}\, X_{\mathcal{G}^{\mathrm{rig}}} \longrightarrow 0$$

of locally free sheaves on $\mathcal{G}^{\mathrm{rig}}$. The quasi-isogeny $\rho : X_{\bar{\mathcal{G}}} \to \mathbb{X}_{\bar{\mathcal{G}}}$ induces by the crystalline nature of $\mathbb{D}(\,.\,)$ an isomorphism $\mathbb{D}(\rho)_{\mathcal{G}^{\mathrm{rig}}} : \mathbb{D}(X_{\bar{\mathcal{G}}})_{\mathcal{G}^{\mathrm{rig}}} \xrightarrow{\sim} \mathbb{D}(\mathbb{X})_{\mathcal{G}^{\mathrm{rig}}}$ and the image $\mathbb{D}(\rho)_{\mathcal{G}^{\mathrm{rig}}}(\mathrm{Lie}\, X^\vee)^\vee_{\mathcal{G}^{\mathrm{rig}}}$ defines a $\mathcal{G}^{\mathrm{rig}}$-valued point of $\mathcal{F}^{\mathrm{rig}}$. By [20, 5.27] the induced morphism $\mathcal{G}^{\mathrm{rig}} \to \mathcal{F}^{\mathrm{rig}}$ factors through $\mathcal{F}^{\mathrm{rig}}_{wa}$. This is the period morphism $\breve{\pi}_1^{\mathrm{rig}}$. The same construction also gives a K_0-analytic period morphism $\breve{\pi}_1^{\mathrm{an}} : \mathcal{G}^{\mathrm{an}} \to \mathcal{F}^{\mathrm{an}}$.

Theorem 1.4. *(Rapoport-Zink, Colmez-Fontaine, Breuil, Kisin)*
The set of Grothendieck's Problem 1.1 is contained in the subset $\mathcal{F}^{\mathrm{rig}}_{wa}$. The latter also equals the image of the period morphism $\breve{\pi}_1^{\mathrm{rig}} : \mathcal{G}^{\mathrm{rig}} \to \mathcal{F}^{\mathrm{rig}}$ in the sense of sets.

Proof. Let $L_K = (\mathrm{Lie}\, X^\vee)^\vee_K$ be a point in Grothendieck's set, given by a Barsotti-Tate group X over $\mathcal{O}_K$ with $\mathbb{X} \cong X_{\mathbb{F}_p^{\mathrm{alg}}}$. Over $\mathcal{O}_K/(p)$ this isomorphism lifts by rigidity [9, Appendix] to a quasi-isogeny $\rho : \mathbb{X}_{\mathcal{O}_K/(p)} \to X_{\mathcal{O}_K/(p)}$ and (X,ρ) gives a point of $\mathcal{G}(\mathcal{O}_K)$. By construction $\breve{\pi}_1^{\mathrm{rig}}(X,\rho)_K = L_K$. So the point L_K belongs to the image of $\breve{\pi}_1^{\mathrm{rig}}$, which in turn lies in $\mathcal{F}_{wa}^{\mathrm{rig}}$. It remains to show that every K-valued point $L_K \in \mathcal{F}_{wa}^{\mathrm{rig}}$ for K/K_0 finite lies in the image of $\breve{\pi}_1^{\mathrm{rig}}$. By the theorem of Colmez-Fontaine [8] the filtered isocrystal (D,φ_D,L_K) is admissible, i.e. arises from a crystalline p-adic Galois representation V. By Breuil [6, Theorem 1.4] there is a Barsotti-Tate group X over $\mathcal{O}_K$ and an isomorphism $T_pX_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong V$ if $p > 2$. Kisin [18, Corollary 2.2.6] extended Breuil's theorem to $p = 2$ and reproved the Colmez-Fontaine Theorem. Fontaine's functor D_{cris} transforms the isomorphism $T_pX_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong V$ into an isomorphism $\bar\rho_* : \big(\mathbb{D}(X)_{K_0}, \mathbb{D}(\mathrm{Frob}_X)_{K_0}, (\mathrm{Lie}\, X^\vee)_K^\vee\big) \xrightarrow{\sim} (D,\varphi_D,L_K)$ of filtered isocrystals. This defines a quasi-isogeny $\bar\rho : \mathbb{X} \to X_{\mathbb{F}_p^{\mathrm{alg}}}$ which again by rigidity lifts to a quasi-isogeny $\rho : \mathbb{X}_{\mathcal{O}_K/(p)} \to X_{\mathcal{O}_K/(p)}$. So L_K lies in the image of the period morphism. □

Unfortunately an open rigid analytic subspace of $\mathcal{F}^{\mathrm{rig}}$, like the image of $\breve{\pi}_1^{\mathrm{rig}}$, is not uniquely determined by its underlying set of points. This however is true for K_0-analytic spaces. Therefore it is natural to ask: What is the image of $\breve{\pi}_1^{\mathrm{an}}$? Examples of Genestier-Lafforgue and the author (see Example 3.6 below) show that this image is in general smaller than $\mathcal{F}_{wa}^{\mathrm{an}}$. As [20, 1.37 and 5.53] indicate, Rapoport and Zink were aware of this phenomenon. We determine the image of $\breve{\pi}_1^{\mathrm{an}}$ in Section 3. But before we need to recall some of the rings of Fontaine Theory.

2 Some of Fontaine's rings

Let K/K_0 be a complete, rank one valued field extension, let C be the completion of an algebraic closure of K, and let $\mathcal{O}_C$ be its valuation ring. Starting out from C various rings are defined in Fontaine Theory. For details on their construction see Colmez [7]. We follow his notation.

We let $\widetilde{\mathbf{E}}^+ := \big\{\, u = (u^{(n)})_{n\in\mathbb{N}_0} : \ u^{(n)} \in \mathcal{O}_C, (u^{(n+1)})^p = u^{(n)} \,\big\}$. With the multiplication $uv := \big(u^{(n)}v^{(n)}\big)_{n\in\mathbb{N}_0}$, the addition $u+v := \big(\lim_{m\to\infty}(u^{(m+n)}+v^{(m+n)})^{p^m}\big)_{n\in\mathbb{N}_0}$, and the valuation $v_{\mathbf{E}}(u) := v_C(u^{(0)})$, $\widetilde{\mathbf{E}}^+$ becomes a complete valuation ring of rank one with algebraically closed fraction field, called $\widetilde{\mathbf{E}}$, of characteristic p. Fix primitive p^n-th roots of unity $\varepsilon^{(n)}$ such that $\varepsilon := (1,\varepsilon^{(1)},\varepsilon^{(2)},\ldots)$ is an element of $\widetilde{\mathbf{E}}^+$.

Let $\widetilde{\mathbf{A}} := W(\widetilde{\mathbf{E}})$ and consider the automorphism $\varphi = W(\mathrm{Frob}_p)$ of $\widetilde{\mathbf{A}}$. For an element $u \in \widetilde{\mathbf{E}}$ let $[u] \in \widetilde{\mathbf{A}}$ be its Teichmüller representative.

If $x = \sum_{i=0}^\infty p^i[x_i] \in \widetilde{\mathbf{A}}$ then we set $w_k(x) := \min\{\, v_{\mathbf{E}}(x_i) : i \le k \,\}$. For $r > 0$ let $\widetilde{\mathbf{A}}^{(0,r]} := \big\{\, x \in \widetilde{\mathbf{A}} : \lim\limits_{k\to+\infty} w_k(x) + \frac{k}{r} = +\infty \,\big\}$ and let $\widetilde{\mathbf{B}}^{(0,r]} := \widetilde{\mathbf{A}}^{(0,r]}[\frac{1}{p}]$.

On $\widetilde{\mathbf{B}}^{(0,r]}$ there is a valuation defined for $x = \sum_{i\gg-\infty}^\infty p^i[x_i]$ as
$v^{(0,r]}(x) \ := \ \min\{\, w_k(x) + \frac{k}{r} : k \in \mathbb{Z} \,\} \ = \ \min\{\, v_{\mathbf{E}}(x_i) + \frac{i}{r} : i \in \mathbb{Z} \,\}$.

Let $\widetilde{\mathbf{B}}^{]0,r]}$ be the Fréchet completion of $\widetilde{\mathbf{B}}^{(0,r]}$ with respect to the family of semi-valuations $v^{[s,r]}(x) := \min\{\, v^{(0,s]}(x), v^{(0,r]}(x) \,\}$ for $0 < s \le r$. The logarithm $t := \log[\varepsilon]$ converges to an element in $\widetilde{\mathbf{B}}^{]0,1]}$.

Let $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger := \bigcup_{r>0} \widetilde{\mathbf{B}}^{]0,r]}$. The homomorphism φ gives rise to a bicontinuous isomorphism $\varphi : \widetilde{\mathbf{B}}^{]0,r]} \xrightarrow{\sim} \widetilde{\mathbf{B}}^{]0,r/p]}$ and thus induces an automorphism of $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$. It satisfies $\varphi(t) = p\,t$.

Finally there is a homomorphism $\theta : \widetilde{\mathbf{B}}^{(0,1]} \to C$ sending $\sum_{i\gg-\infty}^\infty p^i[x_i]$ to $\sum_{i\gg-\infty}^\infty p^i x_i^{(0)}$ which extends by continuity to $\widetilde{\mathbf{B}}^{]0,1]}$. The element t lies in the kernel of θ.

Definition 2.1. A *φ-module over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$* is a finite free $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$-module $\mathcal{M}$ together with a φ-linear automorphism $\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$.

The following structure theorem was proved by Kedlaya [17, Theorem 4.5.7].

Theorem 2.2. *Every φ-module $\mathcal{M}$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ is isomorphic to $\bigoplus_i \mathcal{M}_{c_i,d_i}$ where $\mathcal{M}_{c,d} = (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}})^{\oplus d}$, $\varphi_{\mathcal{M}_{c,d}} = \begin{pmatrix} 0 & & p^c \\ 1 & \ddots & \\ & 1 & 0 \end{pmatrix} \cdot \varphi$ for $c, d \in \mathbb{Z}$ with $(c,d) = 1$ and $d > 0$.*

One defines the *degree of* $\mathcal{M}$ as $\deg \mathcal{M} := \sum_i c_i$.

3 The construction of $\mathcal{F}_a^{\mathrm{an}}$

As above let $D = \mathbb{D}(\mathbb{X})_{K_0}$ and $\varphi_D = \mathbb{D}(\mathrm{Frob}_{\mathbb{X}})_{K_0}$. Let $(\mathcal{D}^{]0,1]}, \varphi_{\mathcal{D}}) := (D, \varphi_D) \otimes_{K_0} \widetilde{\mathbf{B}}^{]0,1]}$ and consider the morphism $1 \otimes \theta : \mathcal{D}^{]0,1]} \to D \otimes_{K_0} C$. By a variant of a construction of Berger [1, §II] every point $L = L_K \in \mathcal{F}^{\mathrm{an}}$, with values in a field K as in Section 2, defines a φ-module over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ as follows.

Proposition 3.1. *([14, Proposition 4.1])*
There exists a uniquely determined $\widetilde{\mathbf{B}}^{]0,1]}$-submodule $t\,\mathcal{D}^{]0,1]} \subset \mathcal{M}_L^{]0,1]} \subset \mathcal{D}^{]0,1]}$ such that $(1 \otimes \theta)\big(\mathcal{M}_L^{]0,1]}\big) = L_K \otimes_K C$ and $\varphi_{\mathcal{D}} : \mathcal{M}_L^{]0,1]} \xrightarrow{\sim} \mathcal{M}_L^{]0,1]} \otimes_{\widetilde{\mathbf{B}}^{]0,1]}} \widetilde{\mathbf{B}}^{]0,1/p]}$ is an isomorphism. In particular $\mathcal{M}_L^{]0,1]}$ defines a φ-module $\mathcal{M}_L := \mathcal{M}_L^{]0,1]} \otimes_{\widetilde{\mathbf{B}}^{]0,1]}} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ over $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$.

The following results are proved in [14].

Theorem 3.2. *([14, Theorem 4.4])*
$\deg \mathcal{M}_L = t_N(D, \varphi_D, L_K) - t_H(D, \varphi_D, L_K)$.

Consider the subset $\mathcal{F}_a^{\mathrm{an}} := \{\, L \in \mathcal{F}^{\mathrm{an}} : \mathcal{M}_L \cong \mathcal{M}_{0,1}^{\oplus h} \,\}$ of the K_0-analytic space $\mathcal{F}^{\mathrm{an}}$.

Theorem 3.3. *([14, Theorem 5.2])*
The set $\mathcal{F}_a^{\mathrm{an}}$ is an open K_0-analytic subspace of $\mathcal{F}_{wa}^{\mathrm{an}}$.

Remark 3.4. (on the proof of Theorem 3.3. See also Remark 3.7 below.)
The inclusion $\mathcal{F}_a^{\mathrm{an}} \subset \mathcal{F}_{wa}^{\mathrm{an}}$ is seen as follows. If $D' \subset D$ is a φ_D-stable K_0-subspace then $(D', \varphi_D|_{D'}, L_K \cap D_K)$ defines by Proposition 3.1 a φ-submodule $\mathcal{M}'_L \subset \mathcal{M}_L$. Since $\mathcal{M}_{0,1}^{\oplus h}$ is "semistable" of slope zero we conclude by [17, Lemma 3.4.8] that

$$t_N(D', \varphi_D|_{D'}, L_K \cap D'_K) \;-\; t_H(D', \varphi_D|_{D'}, L_K \cap D'_K) \;=\; \deg \mathcal{M}'_L \;\geq\; \deg \mathcal{M}_L \;=\; 0$$

with equality if $D' = D$. Hence $\mathcal{F}_a^{\mathrm{an}} \subset \mathcal{F}_{wa}^{\mathrm{an}}$. On the other hand Berger's proof [1] of the Colmez-Fontaine Theorem shows that this inclusion induces a bijection on rigid analytic points (with K/K_0 finite), or more generally points of $\mathcal{F}_{wa}^{\mathrm{an}}$ with with values in a discretely valued field K.

Theorem 3.5. *$\mathcal{F}_a^{\mathrm{an}}$ is the image of the period morphism $\breve{\pi}_1^{\mathrm{an}} : \mathcal{G}^{\mathrm{an}} \to \mathcal{F}_{wa}^{\mathrm{an}}$.*

Proof. Let $L_K \in \mathcal{F}^{\mathrm{an}}$ be a K-valued point in the image of $\breve{\pi}_1^{\mathrm{an}}$ with K/K_0 not necessarily finite. So $L_K = \mathbb{D}(\rho)_K(\mathrm{Lie}\,X^\vee)_K^\vee$ for a Barsotti-Tate group X over $\mathcal{O}_K$ and a quasi-isogeny $\rho : \mathbb{X}_{\mathcal{O}_K/(p)} \to X_{\mathcal{O}_K/(p)}$. Then the Tate module T_pX_K of X induces an injection

$\mathcal{M}_{0,1}^{\oplus h} \cong T_pX_K \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \hookrightarrow \mathcal{M}_L$ which must be an isomorphism by reasons of degree. This was proved in [14, Proposition 6.1]. Thus $L_K \in \mathcal{F}_a^{\mathrm{an}}$.

Conversely let $L_K \in \mathcal{F}_a^{\mathrm{an}}$. Then the morphism $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}})^{\oplus h} \xrightarrow{\sim} \mathcal{M}_{L_K} \hookrightarrow D \otimes_{K_0} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ is represented, with respect to a K_0-basis of D, by a matrix $M \in M_h(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}})$ with $tM^{-1} \in M_h(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}})$. Then in fact $M, tM^{-1} \in M_h(\widetilde{\mathbf{B}}^+_{\mathrm{rig}}) \subset M_h(\mathbf{B}^+_{\mathrm{cris}})$ by [2, Proposition I.4.1]. For notation see [2, §I.1] or [14, §2]. So M defines an isomorphism $\mathbf{B}_{\mathrm{cris}}^{\oplus h} \xrightarrow{\sim} D \otimes_{K_0} \mathbf{B}_{\mathrm{cris}}$ compatible with Frobenius, which maps $(\mathbf{B}^+_{\mathrm{cris}})^{\oplus h}$ onto the preimage of $L_K \otimes_K C$ under the map $\mathrm{id} \otimes \theta : D \otimes_{K_0} \mathbf{B}^+_{\mathrm{cris}} \to D \otimes_{K_0} C$. This means that $(D, \varphi_D, L_K)^\vee$ is admissible in the sense of Faltings' [10, Definition 1]. Note that Faltings uses contravariant Dieudonné modules. By [10, Theorems 9 and 14], $L_K = \mathbb{D}(\rho)_K (\mathrm{Lie}\, X^\vee)_K^\vee$ for a Barsotti-Tate group X over $\mathcal{O}_K$ and a quasi-isogeny $\rho : \mathbb{X}_{\mathcal{O}_K/(p)} \to X_{\mathcal{O}_K/(p)}$, hence L_K lies in the image of $\breve{\pi}_1^{\mathrm{an}}$. ☐

There are many examples showing that only in rare cases $\mathcal{F}_a^{\mathrm{an}} = \mathcal{F}_{wa}^{\mathrm{an}}$. We mention one here. Similar examples are due to A. Genestier and V. Lafforgue.

Example 3.6. Let $D = K_0^{\oplus 5}$ and $\varphi_D = \begin{pmatrix} 0 & & p^{-3} \\ 1 & \ddots & \\ & \ddots\ 1 & 0 \end{pmatrix} \cdot \varphi$. Then $h = 5$, $d = 3$ and $\mathcal{F} = \mathrm{Grass}(2,5)$. Since the isocrystal (D, φ_D) is simple $\mathcal{F}_{wa}^{\mathrm{an}} = \mathcal{F}^{\mathrm{an}}$. Let $L = L_K \in \mathcal{F}^{\mathrm{an}}$. Then

$$\mathcal{M}_{-3,5} \;=\; \mathcal{D} \;\supset\; \mathcal{M}_L \;\supset\; t\mathcal{D} \;\cong\; \mathcal{M}_{2,5}\,.$$

By Theorem 2.2, $\mathcal{M}_L \cong \bigoplus_i \mathcal{M}_{c_i,d_i}$ with $\sum_i d_i = \mathrm{rk}\,\mathcal{M}_L = 5$ and $\sum_i c_i = \deg \mathcal{M}_L = 0$. Moreover by [17, Lemma 3.4.8] all the weights c_i/d_i must lie between $-3/5$ and $2/5$. So either $\mathcal{M}_L \cong \mathcal{M}_{0,1}^{\oplus 5}$ or $\mathcal{M}_L \cong \mathcal{M}_{-1,2} \oplus \mathcal{M}_{1,3}$.

Now one easily checks that $\mathrm{Hom}_\varphi\big(\mathcal{M}_{-1,2}, \mathcal{D}\big) \;=\; \mathrm{Hom}_\varphi\big(\mathcal{M}_{-1,2}, \mathcal{M}_{-3,5}\big) \;=$

$$\left\{ A \;=\; \begin{pmatrix} \varphi^5(x) & x \\ \varphi^{11}(x) & \varphi^6(x) \\ \varphi^{17}(x) & \varphi^{12}(x) \\ \varphi^{23}(x) & \varphi^{18}(x) \\ \varphi^{29}(x) & \varphi^{24}(x) \end{pmatrix} \;:\; x \;=\; \sum_{\nu \in \mathbb{Z}} p^\nu \varphi^{-10\nu}([u]),\ u \in \mathbf{E},\ 0 < v_{\mathbf{E}}(u) < \infty \right\}.$$

The bad situation $\mathcal{M}_L \cong \mathcal{M}_{-1,2} \oplus \mathcal{M}_{1,3}$ occurs if and only if L_K is generated by the columns of such a matrix $\theta(A) \in C^{5\times 2}$, since then the homomorphism A factors through $\mathcal{M}_L$ and this forbids $\mathcal{M}_L \cong \mathcal{M}_{0,1}^{\oplus 5}$ by [17, Lemma 3.4.8]. Since obviously such L_K exist, this proves that the inclusion $\mathcal{F}_a^{\mathrm{an}} \subset \mathcal{F}_{wa}^{\mathrm{an}}$ is strict.

Remark 3.7. (on the proof of Theorem 3.3.) One can explain the idea for the proof of Theorem 3.3 by means of this example. Namely the complement $\mathcal{F}^{\mathrm{an}} \smallsetminus \mathcal{F}_a^{\mathrm{an}}$ is the image of the continuous map from the compact set $\{u \in \widetilde{\mathbf{E}}^+ :\ 1 \le v_{\mathbf{E}}(u) \le p^{10}\}$ given by $u \mapsto \theta(A)$.

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References

[1] L. Berger: *Équations différentielles p-adiques et (φ, N)-modules filtrés*, Preprint on `arXiv:math.NT/0406601`, version from June 29th, 2004.

[2] L. Berger: Limites de représentations cristallines, *Compos. Math.* **140** (2004), no. 6, 1473–1498.

[3] V.G. Berkovich: *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs **33**, Amer. Math. Soc., Providence 1990.

[4] V.G. Berkovich: Étale cohomology for non-Archimedean analytic spaces, *Publ. Math. I.H.E.S.* **78** (1993), 5–161.

[5] P. Berthelot: *Cohomologie rigide et cohomologie rigide à support propre*, prepublication 96-03, IRMA, Rennes, 1996.

[6] C. Breuil: Groupes p-divisibles, groupes finis et modules filtrés, *Ann. of Math.* **152** (2000), 489–549.

[7] P. Colmez: *Espaces Vectoriels de dimension finie et répresentations de de Rham*, Preprint 2003, available at `http://www.math.jussieu.fr/~colmez`.

[8] P. Colmez, J.-M. Fontaine: Construction des représentations p-adiques semi-stables, *Invent. Math.* **140** (2000), no. 1, 1–43.

[9] V.G. Drinfeld: Coverings of p-adic symmetric domains, *Funct. Anal. Appl.* **10** (1976), 107–115.

[10] G. Faltings: *Coverings of P-adic Period Domains*, Preprint 2007.

[11] J.-M. Fontaine: Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate, in *Journées de géometrie algebrique de Rennes (III), Astérisque* **65** (1979), 3–80.

[12] A. Grothendieck: Groupes de Barsotti-Tate et cristaux, *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pp. 431–436, Gauthier-Villars, Paris, 1971.

[13] U. Hartl: *Period Spaces for Hodge Structures in Equal Characteristic*, Preprint on `arXiv:math.NT/0511686`.

[14] U. Hartl: *On a Conjecture of Rapoport and Zink*, Preprint on `arXiv:math.NT/0605254`.

[15] U. Hartl: *A Dictionary between Fontaine-Theory and its Analogue in Equal Characteristic*, Preprint on `arXiv:math.NT/0607182`.

[16] J. de Jong: Étale Fundamental groups of non-Archimedean analytic spaces, *Comp. Math.* **97** (1995), 89–118.

[17] K. Kedlaya: Slope filtrations revisited, *Doc. Math.* **10** (2005), 447–525.

[18] M. Kisin: Crystalline Representations and F-Crystals, *Algebraic geometry and number theory*, pp. 459–496, Progr. Math., 253, Birkhäuser Boston, Boston, MA, 2006.

[19] W. Messing: *The Crystals Associated to Barsotti-Tate Groups*, LNM **264**, Springer-Verlag, Berlin etc. 1972.

[20] M. Rapoport, T. Zink: *Period Spaces for p-divisible Groups*, Ann. Math. Stud. **141**, Princeton University Press, Princeton 1996.

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