CRYSTALLINE CHEBOTARËV DENSITY THEOREMS

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Abstract. We formulate a conjectural analogue of Chebotarëv's density theorem for convergent F-isocrystals over a smooth geometrically irreducible curve defined over a finite field using the Tannakian formalism. We prove this analogue for several large classes, including direct sums of isoclinic convergent F-isocrystals and semi-simple convergent F-isocrystals which have an overconvergent extension and such that the semi-simplification of their pull-back to a sufficient small non-empty open sub-curve has abelian monodromy. In order to prove the latter, we also prove an overconvergent analogue of Chebotarëv's density theorem for semi-simple overconvergent F-isocrystals.

Contents

1. Introduction

Let U be a smooth, geometrically irreducible, non-empty curve over a finite field $\mathbb{F}_q$ having q elements and characteristic p. Let $k = \mathbb{F}_q(U)$ be the function field of U, let $\bar{k}$ be a separable closure of k, and let $|U|$ be the set of closed points in U. For every $x \in |U|$ let $\mathbb{F}_x$, $\deg(x)$ and q_x denote the residue field of x, its degree over $\mathbb{F}_q$ and its cardinality, respectively.

For every abelian variety A defined over k let $A[p^\infty]$ denote its p-divisible group, let $A[p^\infty]^{\text{ét}}$ denote the maximal étale p-divisible quotient of $A[p^\infty]$ and let $V_p(A)$ denote the p-adic $\mathrm{Gal}(\bar{k}/k)$-representation $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, A[p^\infty]^{\text{ét}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The dimension of the $\mathbb{Q}_p$-vector space underlying $V_p(A)$ is the p-rank $r(A)$ of A. It is known that $0 \leq r(A) \leq \dim(A)$ where $\dim(A)$ denotes the dimension of A. Recall that an abelian variety A is called ordinary if $r(A) = \dim(A)$.

As a motivation for our investigations we will have a look at the following

Theorem 1.1. *Let A and B be two ordinary abelian varieties over k. Then A and B are isogenous if and only if the p-adic $\mathrm{Gal}(\bar{k}/k)$-representations $V_p(A)$ and $V_p(B)$ are isomorphic.*

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We will give an easy l-adic proof of this theorem in Section 3 by switching to l-adic Tate modules and using Zarhin's isogeny theorem for them. Of course it would be aesthetically much more satisfying to prove Theorem 1.1 only utilizing p-adic objects, directly from de Jong's theorem [dJo99, Theorem 2.6] that the natural map

$$\mathrm{Hom}(A,B)\otimes_{\mathbb{Z}}\mathbb{Z}_p \to \mathrm{Hom}(A[p^\infty],B[p^\infty])$$

is an isomorphism for every pair of abelian varieties A and B over k. In particular, this implies that A and B are isogenous if and only if the p-divisible groups $A[p^\infty]$ and $B[p^\infty]$ are isogenous, and our theorem is a strengthening of this consequence in the case when A and B are both ordinary. In fact our l-adic proof can be largely adopted to the p-adic setting. However, there are two missing ingredients: a semi-simplicity statement for the overconvergent (rational) Dieudonné module of abelian varieties, which we settle in a separate paper [Pál15], and an analogue of Chebotarëv's density theorem for p-divisible groups, or more generally, for convergent F-isocrystals. The major aim of this article is to formulate such an analogue in the largest possible generality, and prove it in special cases. The most natural way to do that is through the Tannakian formalism. As an application we will give a p-adic proof of Theorem 1.1 in Section 13 before Theorem 13.6.

To describe our results let $\mathrm{Fr}_{q,U}$ denote the absolute q-Frobenius on U which is the identity on points and the q-power map on the structure sheaf. Let K be a finite totally ramified extension of $W(\mathbb{F}_q)[\frac{1}{p}]$ and fix an algebraic closure $\overline{K}$ of K. Since $\mathrm{Fr}_{q,U}$ is the identity on $\mathbb{F}_q$, we may choose on K the identity $F:=\mathrm{id}_K$ as a lift of $\mathrm{Fr}_{q,U}$. Let $F\text{-}\mathrm{Isoc}_K(U)$ denote the K-linear rigid abelian tensor category of K-linear convergent F-isocrystals on U; see [Cre92, Chapter 1] for details. If $\mathcal{F}$ is an object of $F\text{-}\mathrm{Isoc}_K(U)$ we let $\langle\langle\mathcal{F}\rangle\rangle$ denote the strictly full rigid abelian tensor sub-category of $F\text{-}\mathrm{Isoc}_K(U)$ generated by $\mathcal{F}$. We fix a base point $u\in U(\mathbb{F}_{q^e})$ and let K_e be the unramified field extension of K of degree e in $\overline{K}$. Pulling back to u defines a faithful fiber functor $\omega_u\colon\mathcal{F}\mapsto u^*\mathcal{F}$ which makes $F\text{-}\mathrm{Isoc}_K(U)$ into a Tannakian category and $\langle\langle\mathcal{F}\rangle\rangle$ into a Tannakian sub-category of $F\text{-}\mathrm{Isoc}_K(U)$; see Definitions A.1 and A.3 for explanations. Note that the fiber functor ω_u is non-neutral if $e\neq 1$. We let $\mathrm{Gr}(\mathcal{F}/U,u):=\mathrm{Aut}^{\otimes}(\omega_u|_{\langle\langle\mathcal{F}\rangle\rangle})$ be the smooth linear algebraic group over K_e consisting of the tensor automorphisms of $\omega_u\colon\mathcal{F}\mapsto u^*\mathcal{F}$; see Section 3 for the precise definition. For every closed point $x\in|U|$ the Frobenius $F_{\mathcal{F}}$ of $\mathcal{F}$ furnishes a conjugacy class $\mathrm{Frob}_x(\mathcal{F})$ in $\mathrm{Gr}(\mathcal{F}/U,u)(\overline{K})$; see Definition 3.1. The crystalline version of Chebotarëv's density theorem is the following

Conjecture 1.2. *For every subset $S\subset|U|$ of Dirichlet density one the set $\bigcup_{x\in S}\mathrm{Frob}_x(\mathcal{F})$ is Zariski-dense in $\mathrm{Gr}(\mathcal{F}/U,u)$.*

We follow Serre in the definition of Dirichlet density; see Definition 3.10. See Remark 5.6 why we do not expect a density statement for any other topology than the Zariski topology. When $\mathcal{F}$ has connected monodromy group we even expect the following

Conjecture 1.3. *If the monodromy group $\mathrm{Gr}(\mathcal{F}/U,u)$ is connected then for every subset $S\subset|U|$ of positive upper Dirichlet density the set $\bigcup_{x\in S}\mathrm{Frob}_x(\mathcal{F})$ is Zariski-dense in $\mathrm{Gr}(\mathcal{F}/U,u)$.*

The notion of positive upper Dirichlet density is a natural weakening of positive Dirichlet density (see Definition 3.11 for a precise definition). A variant of the conjecture above is the following

Conjecture 1.4. *For every subset $S\subset|U|$ of positive upper Dirichlet density the Zariski-closure of the set $\bigcup_{x\in S}\mathrm{Frob}_x(\mathcal{F})$ contains a connected component of the group $\mathrm{Gr}(\mathcal{F}/U,u)\times_{K_e}\overline{K}$.*

Note that the validity of Conjecture 1.4 for $\mathcal{F}$ trivially implies the validity of Conjecture 1.3 for $\mathcal{F}$ when $\mathrm{Gr}(\mathcal{F}/U,u)$ is connected. We will see later (see Proposition 6.7 below) that the validity of Conjecture 1.4 for $\mathcal{F}$ also implies the validity of Conjecture 1.2 for $\mathcal{F}$.

Let us continue by proving an application of crystalline Chebotarëv density. For every convergent F-isocrystal $\mathcal{F}$ on U and for every $x\in|U|$ let $\mathrm{Tr}(\mathrm{Frob}_x(\mathcal{F}))$ denote the common trace of all elements of $\mathrm{Frob}_x(\mathcal{F})$, considered as endomorphisms of the K_e-vector space $\omega_u(\mathcal{F})$.

Corollary 1.5. *Let $S\subset|U|$ be a subset of Dirichlet density one and let $\mathcal{F},\mathcal{G}$ be two convergent F-isocrystals of the same rank on U such that $\mathrm{Tr}(\mathrm{Frob}_x(\mathcal{F}))=\mathrm{Tr}(\mathrm{Frob}_x(\mathcal{G}))$ for every $x\in S$ and such that Conjecture 1.2 holds for the direct sum $\mathcal{F}^{ss}\oplus\mathcal{G}^{ss}$ of their semi-simplifications. Then the semi-simplifications $\mathcal{F}^{ss}$ and $\mathcal{G}^{ss}$ of $\mathcal{F}$ and $\mathcal{G}$ are isomorphic.*

Proof. By Lemma 3.3 below the Frobenius conjugacy class $\mathrm{Frob}_x(\mathcal{F})$ maps to $\mathrm{Frob}_x(\mathcal{F}^{ss})$ under the natural surjective map $\eta\colon \mathrm{Gr}(\mathcal{F},u) \twoheadrightarrow \mathrm{Gr}(\mathcal{F}^{ss},u)$ for every $x \in S$. For every $g \in \mathrm{Gr}(\mathcal{F},u)$ we have $\mathrm{Tr}(g) = \mathrm{Tr}(\eta(g))$ where we take traces with respect to the representations $\omega_u(\mathcal{F})$ and $\omega_u(\mathcal{F}^{ss})$, because in a suitable basis of $\omega_u(\mathcal{F})$ the kernel of η consists of unipotent upper triangular matrices by Lemma 3.8, and so the diagonal entries of g and $\eta(g)$ coincide. Therefore, we get that $\mathrm{Tr}(\mathrm{Frob}_x(\mathcal{F})) = \mathrm{Tr}(\mathrm{Frob}_x(\mathcal{F}^{ss}))$ for every $x \in S$. By repeating the same argument for $\mathcal{G}$ we get that $\mathrm{Tr}(\mathrm{Frob}_x(\mathcal{F}^{ss})) = \mathrm{Tr}(\mathrm{Frob}_x(\mathcal{G}^{ss}))$ for every $x \in S$.

Let ρ_1 and ρ_2 denote the representations of $\mathrm{Gr}(\mathcal{F}^{ss} \oplus \mathcal{G}^{ss}, u)$ on $\omega_u(\mathcal{F}^{ss})$ and $\omega_u(\mathcal{G}^{ss})$, respectively. Note that ρ_1 and ρ_2 correspond to the objects $\mathcal{F}^{ss} \otimes_K K_e$ and $\mathcal{G}^{ss} \otimes_K K_e$ of $F\text{-}\mathrm{Isoc}_K(U) \otimes_K K_e$ by Remark A.20. For every $x \in S$ the Frobenius conjugacy class $\mathrm{Frob}_x(\mathcal{F}^{ss} \oplus \mathcal{G}^{ss})$ maps by Lemma 3.3 to $\mathrm{Frob}_x(\mathcal{F}^{ss})$ and $\mathrm{Frob}_x(\mathcal{G}^{ss})$ under ρ_1 and ρ_2, respectively. Thus the trace functions of the representations ρ_1 and ρ_2 on the group $\mathrm{Gr}(\mathcal{F}^{ss} \oplus \mathcal{G}^{ss}, u)$ are equal on the Frobenius conjugacy classes $\mathrm{Frob}_x(\mathcal{F}^{ss} \oplus \mathcal{G}^{ss})$ for all $x \in S$. By assumption the latter are Zariski-dense in $\mathrm{Gr}(\mathcal{F}^{ss} \oplus \mathcal{G}^{ss}, u)$, so the trace functions of the representations ρ_1 and ρ_2 on the group $\mathrm{Gr}(\mathcal{F}^{ss} \oplus \mathcal{G}^{ss}, u)$ are equal.

Let $\Lambda \subset \mathrm{End}_{K_e}\big(\omega_u(\mathcal{F}^{ss} \oplus \mathcal{G}^{ss})\big)$ be the smallest K_e-linear subspace (viewed as a scheme) containing the image of $\mathrm{Gr}(\mathcal{F}^{ss} \oplus \mathcal{G}^{ss}, u)$. Then $\Lambda \otimes_{K_e} \overline{K}$ is the $\overline{K}$-linear span of $\mathrm{Gr}(\mathcal{F}^{ss} \oplus \mathcal{G}^{ss}, u)(\overline{K})$, because the latter is $\mathrm{Gal}(\overline{K}/K_e)$-invariant. Thus $\Lambda \otimes_{K_e} \overline{K}$ is a $\overline{K}$-algebra, and hence Λ is a K_e-algebra. Moreover, $\omega_u(\mathcal{F}^{ss})$ and $\omega_u(\mathcal{G}^{ss})$ are semi-simple Λ-modules, because every submodule invariant under $\mathrm{Gr}(\mathcal{F}^{ss} \oplus \mathcal{G}^{ss}, u)$ is also invariant under Λ. Finally, by their linearity the trace functions of Λ on both representations coincide, because they do on $\mathrm{Gr}(\mathcal{F}^{ss} \oplus \mathcal{G}^{ss}, u)(\overline{K})$. Therefore, by [Ser98, Lemma in § I.2.3 on p. I-11] the two representations are isomorphic and this implies that $\mathcal{F}^{ss} \otimes_K K_e \cong \mathcal{G}^{ss} \otimes_K K_e$ in $F\text{-}\mathrm{Isoc}_K(U) \otimes_K K_e$. Then it follows from Proposition A.21 that $\mathcal{F}^{ss} \cong \mathcal{G}^{ss}$ in $F\text{-}\mathrm{Isoc}_K(U)$. □

For an application to p-divisible groups X over U recall that the (rational) crystalline Dieudonné functor assigns to (the isogeny class of) X a convergent F-isocrystal $\mathbf{D}(X)$ on U. The functor $\mathbf{D}$ is fully faithful on the isogeny category of p-divisible groups by [dJo95, Main Theorem 1].

Corollary 1.6. *Let X and Y be two p-divisible groups over U which are semi-simple in the isogeny category. Let $S \in |U|$ be a subset of Dirichlet density one such that for every $s \in S$ the traces of the Frobenii on the rational Dieudonné modules are equal for s^*X and s^*Y. If Conjecture 1.2 holds for $\mathbf{D}(X) \oplus \mathbf{D}(Y)$ then X and Y are isogenous.*

Proof. For every $s \in |U|$ the rational Dieudonné module of the pullback s^*X of X to s equals $\omega_s(\mathbf{D}(X))$, which is (non-canonically) isomorphic to $\omega_u(\mathbf{D}(X))$, such that the Frobenius of s^*X is mapped to the conjugacy class $\mathrm{Frob}_s(\mathbf{D}(X))$. By the full faithfulness of $\mathbf{D}$ the F-isocrystals $\mathbf{D}(X)$ and $\mathbf{D}(Y)$ are semisimple, Corollary 1.5 shows that they are isomorphic, and hence X and Y are isogenous. □

Although currently we are unable to establish Conjectures 1.2, 1.3 and 1.4 in general, we can still prove them in many cases. We will start with an easy result explaining the relation to the classical Chebotarëv density theorem. Let $\mathrm{ord}_p\colon \overline{K}^\times \to \mathbb{Q}$ be the unique p-adic valuation such that $\mathrm{ord}_p(p) = 1$. When we will talk about slopes and Newton polygons, we will do so with respect to the valuation ord_p. Moreover for the sake of simple terminology we will say that $\alpha \in \overline{K}$ is an eigenvalue of $\mathrm{Frob}_x(\mathcal{F})$ if it is the eigenvalue of one and hence every element of $\mathrm{Frob}_x(\mathcal{F})$ acting on the K_e-vector space $\omega_u(\mathcal{F})$. By the Newton polygon of $\mathrm{Frob}_x(\mathcal{F})$ we will mean the Newton polygon of the semilinear Frobenius F on the fiber at x. If $x \in U(\mathbb{F}_{q^n})$ it equals $\frac{1}{n}$ times the common Newton polygon of the (K_n-linear) elements of $\mathrm{Frob}_x(\mathcal{F})$. Recall that an F-isocrystal $\mathcal{F}$ is called *isoclinic* if for every $x \in |U|$ the Newton polygon of $\mathrm{Frob}_x(\mathcal{F})$ has only one slope (which then is the same at all x). If $\mathcal{F}$ is isoclinic of slope zero, it is called *unit-root.* For those F-isocrystals Conjecture 1.2 is an easy consequence of the classical Chebotarëv density theorem.

Proposition 1.7. *Conjecture 1.2 holds for convergent unit-root F-isocrystals.*

We will prove a more general statement later, but we think that the proof is rather instructive, and it is also a good motivation for our conjectures. Therefore, we decided to present its proof here.

Proof of Proposition 1.7. Choose a geometric base point $\bar{u}$ above u and let $\pi_1^{\text{ét}}(U,\bar{u})$ be the étale fundamental group of U. By a result of R. Crew [Cre87, Theorem 2.1 and Remark 2.2.4] the full subcategory of $F\text{-}\mathrm{Isoc}_K(U)$ consisting of unit-root F-isocrystals is tensor equivalent to the category of continuous representations of $\pi_1^{\text{ét}}(U,\bar{u})$ on finite dimensional K-vector spaces; see Proposition 5.2 below. Let

$\rho\colon \pi_1^{\text{ét}}(U,\bar u)\to \mathrm{GL}_r(K)$ be a representation corresponding to a unit-root F-isocrystal $\mathcal{F}$. Then $\mathrm{Gr}(\mathcal{F}/U,u)$ is a closed subgroup of GL_{r,K_e} and by Corollary 5.4 below there is a finite field extension L of K_e such that $\mathrm{Gr}(\mathcal{F}/U,u)\times_{K_e} L$ equals the Zariski-closure of the image of ρ. Moreover, for all points $x\in|U|$ the $\mathrm{Gr}(\mathcal{F}/U,u)(\overline{K})$-conjugacy classes of $\rho(x_*\,\mathrm{Frob}_x^{-1})$ and $\mathrm{Frob}_x(\mathcal{F})$ coincide, where $\mathrm{Frob}_x^{-1}\in \mathrm{Gal}(\overline{\mathbb{F}}_x/\mathbb{F}_x)$ is the geometric Frobenius at x which maps $a\in\overline{\mathbb{F}}_x$ to a^{1/q_x} for $q_x=\#\mathbb{F}_x$.

To prove Conjecture 1.2 let $S\subset|U|$ be a subset of Dirichlet density one. By the Chebotarëv density theorem [Ser63, Theorem 7] the Frobenius conjugacy classes for the points $x\in S$ are dense in $\pi_1^{\text{ét}}(U,\bar u)$ with respect to the pro-finite topology. Since this topology is finer than the restriction of the Zariski topology from $\mathrm{Gr}(\mathcal{F}/U,u)$, the set $\bigcup_{x\in S}\mathrm{Frob}_x(\mathcal{F})$ is Zariski-dense in $\mathrm{Gr}(\mathcal{F}/U,u)$. □

Note that Conjecture 1.2 for convergent unit-root F-isocrystals on U is considerably weaker than the classical Chebotarëv density theorem for U; see Remark 5.6 for more explanations. Let us next describe cases for which we prove the conjecture. In Section 7 we use a theorem of Oesterlé [Oes82] to strengthen Proposition 1.7 to

Theorem 1.8. *Conjecture 1.4 holds for direct sums of isoclinic convergent F-isocrystals.*

In order to formulate our hardest result for the remaining cases where we can prove the conjectures in this article we make the following

Definition 1.9. Let $\mathcal{F}$ be a convergent F-isocrystal on U. We will say that $\mathcal{F}$ is *firm* if it is a successive extension of isoclinic convergent F-isocrystals and the monodromy group $\mathrm{Gr}(\mathcal{F}^{ss}/U,u)$ is abelian. We will say that $\mathcal{F}$ is *weakly firm* if it is a successive extension of isoclinic convergent F-isocrystals and the maximal quasi-torus of the monodromy group $\mathrm{Gr}(\mathcal{F}^{ss}/U,u)\times_{K_e}\overline{K}$ (or equivalently of $\mathrm{Gr}(\mathcal{F}/U,u)\times_{K_e}\overline{K}$) is abelian. This holds in particular, if $\mathrm{Gr}(\mathcal{F}/U,u)$ is connected. (For the definition of a maximal quasi-torus see Definition 8.6 below.) Since subgroups of abelian groups are abelian, we get that firm convergent F-isocrystals are weakly firm. If there is a non-empty open sub-curve $f\colon V\hookrightarrow U$ such that the restriction $\mathcal{F}|_V$ of $\mathcal{F}$ onto V is firm, resp. weakly firm we will say that $\mathcal{F}$ is *locally firm, resp. locally weakly firm (with respect to $V\subset U$)*. Note that in particular $f^*\mathcal{F}|_V$ has a slope filtration on V with isoclinic factors. If $\mathcal{F}$ is (weakly) firm, then its semi-simplification $\mathcal{F}^{ss}$ is a direct sum of isoclinic convergent F-isocrystals. Then by Proposition 10.11 below the natural morphism $\mathrm{Gr}(f^*\mathcal{F}/V,u)\to\mathrm{Gr}(\mathcal{F}/U,u)$ is an isomorphism. Therefore, (weakly) firm implies locally (weakly) firm.

Note that by the specialization theorem of Grothendieck and Katz [Kat79, Corollary 2.3.2] there is a non-empty open sub-curve $V\subset U$ on which the Newton polygon of $\mathcal{F}$ is constant, and by the slope filtration theorem [Kat79, Corollary 2.6.3] $\mathcal{F}|_V$ has a slope filtration with isoclinic subquotients. Therefore, the first condition in the definition of locally firm and locally weakly firm convergent F-isocrystals is not very restrictive. We show in Proposition 11.3 below that the categories of firm, weakly firm, locally firm and locally weakly firm convergent F-isocrystals on U are full Tannakian sub-categories of $F\text{-}\mathrm{Isoc}_K(U)$. In Proposition 11.1 we give examples for locally (weakly) firm convergent F-isocrystals. Our main result is the following

Theorem 1.10. *Let $\mathcal{F}$ be a semi-simple locally weakly firm convergent F-isocrystal on U which has an overconvergent extension. Let $\mathcal{G}\in\langle\langle\mathcal{F}\rangle\rangle$ and let $\mathcal{J}$ be a direct sum of semi-simple isoclinic convergent F-isocrystals on U. Then Conjectures 1.2, 1.3 and 1.4 hold true for $\mathcal{G}\oplus\mathcal{J}$.*

The proof of Theorem 1.10 given on page 69 consists of four main steps: using the theory of reductive groups and Theorem 1.8 above we first show that it is enough to show Conjecture 1.4 for $\mathcal{F}$ only. Then we prove an analogue of Conjecture 1.4 for the *overconvergent* monodromy group of the overconvergent extension of $\mathcal{F}$. Then, using group theory again, we show that $\mathcal{F}$ satisfies the hypotheses of the following

Theorem 1.11. *Let $\mathcal{F}$ be a semi-simple convergent F-isocrystal on U and let $f\colon V\hookrightarrow U$ be an open sub-curve containing u such that $f^*\mathcal{F}$ has a slope filtration on V with isoclinic subquotients. Assume that under the natural inclusion $\mathrm{Gr}(f^*\mathcal{F}/V,u)\subset\mathrm{Gr}(\mathcal{F}/U,u)$ every maximal quasi-torus of $\mathrm{Gr}(f^*\mathcal{F}/V,u)\times_{K_e}\overline{K}$ is also a maximal quasi-torus of the group $\mathrm{Gr}(\mathcal{F}/U,u)\times_{K_e}\overline{K}$. Then $\mathcal{F}$ satisfies Conjectures 1.2, 1.3 and 1.4.*

Once more using group theoretical methods, we finally show in Theorem 11.6 that the hypotheses on $\mathcal{F}$ in Theorem 1.11 above are actually equivalent to Conjecture 1.4 for semi-simple convergent F-isocrystals which are locally weakly firm with respect to $f\colon V \hookrightarrow U$. Theorem 1.11 is proven in the formulation of Theorem 10.4 below. Note that one of the distinguishing features of convergent F-isocrystals is that their monodromy groups in general shrink when we shrink the underlying curve U. This shows that F-isocrystals lack the type of rigidity which we have for l-adic and even p-adic Galois representations, where the monodromy group does not change when one shrinks the curve. R. Pink addressed this problem and asked whether the shrunken monodromy group always is a parabolic subgroup in the special case when $\mathcal{F}$ comes from a p-divisible group on U. We cannot prove this but we can prove the weaker Theorem 1.11 which is still sufficient for proving the Chebotarëv density conjectures for F-isocrystals appearing in Theorem 1.10. It also singles out Pink's problem as the most central one in the theory of convergent F-isocrystals.

In the last section we will also look at the analogous problem for *overconvergent* F-isocrystals, and prove the following theorem (see Theorem 13.2 and Corollary 13.3) using arguments inspired by our proof of Theorem 1.8. We think that this is a very interesting result on its own, but it also plays a crucial role in the proof of Theorem 1.10.

Theorem 1.12. *For every semi-simple overconvergent F-isocrystal the analogs of Conjectures 1.2, 1.3 and 1.4 hold true.*

We finish the introduction with a brief summary of the individual sections. In Section 2 we give the l-adic proof of Theorem 1.1. In Section 3 we give the precise definitions of Dirichlet density and of the Frobenius conjugacy class $\mathrm{Frob}_x(\mathcal{F})$, and we prove several elementary facts about the monodromy group. In Section 4 we collect properties of the algebraic envelope of a topological group and we treat constant F-isocrystals. Section 5 recalls Crew's theory [Cre87] of unit-root F-isocrystals and Section 6 discusses the group of connected components of the monodromy group $\mathrm{Gr}(\mathcal{F}/U,u)$. In Section 7 we prove our Chebotarëv density conjectures for direct sums of isoclinic F-isocrystals (Theorem 1.8). In Section 10 we formulate properties of the closed subgroup $\mathrm{Gr}(f^*\mathcal{F}/V) \subset \mathrm{Gr}(\mathcal{F}/U)$ which one might expect when one restricts a convergent F-isocrystal on U to an open sub-curve $f\colon V \hookrightarrow U$ and we prove in Theorem 10.4 that these properties imply our Chebotarëv density conjectures (Theorem 1.11). For this purpose we have to collect in Section 8 a few facts about semi-simple elements in non-connected linear algebraic groups, and study the notion of *maximal quasi-tori*, which is a good generalization of maximal tori in not necessarily connected groups. In Section 9 we study the intersections of conjugacy classes with maximal quasi-tori. In Section 10 we prove Theorem 1.11 and conduct a detailed investigation of the hypothesis of this theorem on maximal quasi-tori. In Section 11 we furnish a few useful conditions which guarantee that the direct sum of a locally firm convergent F-isocrystal with finitely many isoclinic convergent F-isocrystals satisfies the hypotheses of Theorem 1.11, and hence the Chebotarëv density. Finally, in Section 13 we treat the case of overconvergent F-isocrystals, and also derive Theorem 1.10. In Appendix A we briefly review the theory of (non-neutral) Tannakian categories and of representations of groupoids.

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2. Isogenies of Ordinary Abelian Varieties

We will give two proofs of Theorem 1.1, an l-adic one in this section and a p-adic one in Section 13 before Theorem 13.6.

l-adic proof of Theorem 1.1. If $f\colon A \to B$ is an isogeny then $V_pf\colon V_p(A) \to V_p(B)$ is an isomorphism. Conversely assume that $V_p(A) \cong V_p(B)$. By the specialization theorem of Grothendieck and Katz [Kat79, Theorem 2.3.1] we may replace U by a non-empty open sub-curve such that for every $x \in |U|$ the abelian varieties A and B have good ordinary reduction at x. In particular the p-adic representations $V_p(A)$ and $V_p(B)$ are unramified at x for every such x. Let $k_U \subset \bar{k}$ be the maximal Galois extension of k unramified at every $x \in |U|$ and let $\mathrm{Fr}_x \subset \mathrm{Gal}(k_U/k)$ denote the Frobenius conjugacy class corresponding to x for every $x \in |U|$. By the above we may consider $V_p(A)$ and $V_p(B)$ as continuous representations of $\mathrm{Gal}(k_U/k)$. Let $\alpha_{x1}, \alpha_{x2}, \ldots, \alpha_{xd}$ denote the common eigenvalues of the actions of elements of Fr_x on $V_p(A) \cong V_p(B)$ where d is the common dimension of A and B. For every $x \in |U|$ let A_x and B_x denote the reductions of A and B over x, respectively. Then the eigenvalues of the action of the Frobenius element of $\mathrm{Gal}(\overline{\mathbb{F}}_x/\mathbb{F}_x)$ on

$V_p(A_x)$ and $V_p(B_x)$ are $\alpha_{x1},\ldots,\alpha_{xd}$ for every $x\in|U|$. Choose a prime number l different from p. Because A_x and B_x are ordinary abelian varieties of dimension d over $\mathbb{F}_x$ we get that the eigenvalues of the action of the Frobenius element of $\mathrm{Gal}(\overline{\mathbb{F}}_x/\mathbb{F}_x)$ on $V_l(A_x)$ and $V_l(B_x)$ are $\alpha_{x1},\ldots,\alpha_{xd},q_x/\alpha_{x1},\ldots,q_x/\alpha_{xd}$ for every $x\in|U|$. Indeed, by a classical theorem of Manin [Dem72, Chapter V.5, Corollary on p. 88] the eigenvalues of the Frobenius acting on the Dieudonné module associated with the p-divisible group $A_x[p^\infty]$ are the reciprocal roots of the L-function $L(A_x,t)$ of A_x where we take into account the multiplicities. In particular the eigenvalues of the Frobenius acting on $A_x[p^\infty]^{\text{ét}}$ are exactly those reciprocal roots of $L(A_x,t)$ which have p-adic valuation zero, again taking into account the multiplicities. Let $p(t)$ be the product of all linear factors of $L(A_x,t)$ whose reciprocal root has slope one. Since A is isogenous to its dual, we have the functional equation $L(A_x,t)=t^{2d}L(A_x,q/t)$. Therefore, $t^dp(q/t)$ divides $L(A_x,t)$. The reciprocal roots of $t^dp(q/t)$ have slope one, therefore $p(t)$ and $t^dp(q/t)$ are relatively prime. We get that $p(t)t^dp(q/t)$ divides $L(A_x,t)$. These polynomials are monic and have the same degree, therefore they are equal.

Note that the l-adic representations $V_l(A)$ and $V_l(B)$ are unramified at x and by the above the common eigenvalues of the actions of the elements of Fr_x are $\alpha_{x1},\ldots,\alpha_{xd},q_x/\alpha_{x1},\ldots,q_x/\alpha_{xd}$ for every $x\in|U|$. Since the union $\bigcup_{x\in|U|}\mathrm{Fr}_x$ is dense in $\mathrm{Gal}(k_U/k)$ by the Chebotarëv density theorem [Vil06, Theorem 11.2.20], the traces of the actions of γ on $V_l(A)$ and $V_l(B)$ are equal for every $\gamma\in\mathrm{Gal}(\bar{k}/k)$. Because by a theorem of Zarhin [Zar74a, Theorem 1.5] the l-adic representations $V_l(A)$ and $V_l(B)$ are semi-simple we get that they must be isomorphic; see [Ser98, Lemma in § I.2.3 on p. I-11]. Hence by Zarhin [Zar74a, Theorem 1.5] the abelian varieties A and B are isogenous. □

Remark 2.1. The claim of Theorem 1.1 is false when the abelian varieties are not assumed to be ordinary. Indeed let E be a supersingular elliptic curve defined over $\mathbb{F}_q$ and let E' be a twist of E by a continuous quadratic character χ of $\mathrm{Gal}(\bar{k}/k)$. The curve E has everywhere good reduction while E' has bad reduction at the places where χ is ramified. This follows from the criterion of Néron-Ogg-Shavarevich [ST68, § 1, Theorem 1] because the l-adic Tate modules satisfy $T_l(E')=T_l(E)\otimes\chi$; see for example [ST68, § 5, Proof of Theorem 8]. Since the set of places of bad reduction for an abelian variety A over k is an isogeny invariant of A ([ST68, § 1, Corollary 3]) we get that E is not isogenous to E' in the latter case while the zero-dimensional representations $V_p(E)$ and $V_p(E')$ are obviously isomorphic. Also note that for every abelian variety A over k the direct products $A\times E$ and $A\times E'$ are not isogenous (for example by Poincaré's reducibility theorem). However, the p-adic representations $V_p(A\times E)$ and $V_p(A\times E')$ are both isomorphic to $V_p(A)$ hence the ordinariness condition is necessary in every dimension.

3. Basic Definitions and Properties

We describe in complete detail our basic setup for the convenience of the reader, possibly at the price of some repetition. Let U be a smooth, geometrically irreducible, non-empty curve over $\mathbb{F}_q$, and let F be the q-Frobenius on U. Let K be a finite totally ramified extension of $W(\mathbb{F}_q)[\frac{1}{p}]$ and let $\overline{K}$ be an algebraic closure of K. Since F is the identity on $\mathbb{F}_q$, we may choose on K the identity $F=\mathrm{id}_K$ as a lift of F. For every $n\in\mathbb{N}$ let $U_n:=U\otimes_{\mathbb{F}_q}\mathbb{F}_{q^n}$, let $K_n\subset\overline{K}$ be the unramified extension of K of degree n, and let F be the Frobenius of K_n over K. Then F^n is the identity on K_n. Let $F^n\text{-}\mathrm{Isoc}_{K_n}(U_n)$ denote the K_n-linear rigid tensor category of K_n-linear convergent F^n-isocrystals on U_n; see [Cre92, Chapter 1] for details. Let $F\text{-}\mathrm{Isoc}_K(U)$ simply denote $F^1\text{-}\mathrm{Isoc}_K(U)$. If $\mathcal{F}$ is an object of $F^n\text{-}\mathrm{Isoc}_{K_n}(U_n)$ we let $\langle\!\langle\mathcal{F}\rangle\!\rangle$ denote the Tannakian sub-category of $F^n\text{-}\mathrm{Isoc}_{K_n}(U_n)$ generated by $\mathcal{F}$; see Definition A.3. There is a functor

$$(\,.\,)^{(n)}\colon\ F\text{-}\mathrm{Isoc}_K(U)\longrightarrow F^n\text{-}\mathrm{Isoc}_{K_n}(U_n),\quad \mathcal{F}\mapsto\mathcal{F}^{(n)}\tag{3.1}$$

which is given by pulling back under $U_n\to U$, that is, by tensoring the coefficients from K to K_n, and replacing the Frobenius $F_{\mathcal{F}}$ of $\mathcal{F}$ by $F^n_{\mathcal{F}}:=F_{\mathcal{F}}\circ\mathrm{Fr}^*_{q,U}F_{\mathcal{F}}\circ\ldots\circ\mathrm{Fr}^{(n-1)*}_{q,U}F_{\mathcal{F}}$, where $\mathrm{Fr}_{q,U}\colon U\to U$ is the absolute q-Frobenius of U.

We fix a base point $u\in U(\mathbb{F}_{q^e})=U_e(\mathbb{F}_{q^e})$. The pullback $u^*\mathcal{F}$ of an F^e-isocrystal $\mathcal{F}$ to u supplies a functor ω_u from $F^e\text{-}\mathrm{Isoc}_{K_e}(U_e)$ to the category of F^e-isocrystals on $\mathrm{Spec}\,\mathbb{F}_{q^e}$ with values in K_e. The latter is simply the category of finite dimensional K_e-vector spaces together with a K_e-linear automorphism coming from the Frobenius F^e. The fiber functor ω_u makes $F^e\text{-}\mathrm{Isoc}_{K_e}(U_e)$ into a neutral Tannakian category. For $\mathcal{F}\in F^e\text{-}\mathrm{Isoc}_{K_e}(U_e)$ let $\mathrm{Gr}(\mathcal{F},u):=\mathrm{Gr}(\mathcal{F}/U_e,u):=\mathrm{Aut}^{\otimes}(\omega_u|_{\langle\!\langle\mathcal{F}\rangle\!\rangle})$ denote the monodromy group of $\mathcal{F}$ with respect to the fiber functor ω_u; see [DM82, Theorem 2.11]. By [DM82, Proposition 2.20(b)] it is a linear algebraic group over K_e and $\langle\langle\mathcal{F}\rangle\rangle$ is tensor equivalent to the category of K_e-rational representations of $\mathrm{Gr}(\mathcal{F}/U_e, u)$. By Cartier's theorem $\mathrm{Gr}(\mathcal{F}, u)$ is smooth; see for example [Wat79, § 11.4].

On the category $F\text{-}\mathrm{Isoc}_K(U)$ we can still consider the fiber functor $\mathcal{F} \mapsto u^*\mathcal{F}$ to K_e-vector spaces. It factors through the functor $(\,.\,)^{(e)}$ from (3.1) as $u^* = \omega_u \circ (\,.\,)^{(e)}$. This makes $F\text{-}\mathrm{Isoc}_K(U)$ into a K-linear Tannakian category, but u^* is non-neutral when $e > 1$. Let $\mathcal{F} \in F\text{-}\mathrm{Isoc}_K(U)$ and let $\langle\langle\mathcal{F}\rangle\rangle$ be again the Tannakian sub-category of $F\text{-}\mathrm{Isoc}_K(U)$ generated by $\mathcal{F}$. Then $\langle\langle\mathcal{F}\rangle\rangle$ is tensor equivalent to the K_e-rational representations of a K_e/K-groupoid $\mathfrak{Gr}(\mathcal{F}/U, u)$; see Definition A.5 and Theorem A.11. In this article we are only interested in its kernel group $\mathrm{Gr}(\mathcal{F}, u) := \mathrm{Gr}(\mathcal{F}/U, u) := \mathfrak{Gr}(\mathcal{F}/U, u)^{\Delta}$, which equals the automorphism group of the fiber functor $u^*\colon\ F\text{-}\mathrm{Isoc}_K(U) \to K_e$-vector spaces and is a smooth linear algebraic group over K_e by Theorem A.11(a) and Proposition A.13. We will use this fact frequently. We explain the relation of $\mathrm{Gr}(\mathcal{F}/U, u)$ with Crew's monodromy group [Cre92] in Remark 4.10 and Proposition 4.11. Since $u^* = \omega_u \circ (\,.\,)^{(e)}$ the tensor functor $(\,.\,)^{(e)}$ induces (by [Mil92, Proposition A.12 and Example A.13]) a morphism of linear algebraic groups over K_e

$$h_e(\mathcal{F})\colon\ \mathrm{Gr}(\mathcal{F}^{(e)}/U_e, u) \longrightarrow \mathrm{Gr}(\mathcal{F}/U, u)\,,$$

which we study further in Lemma 6.3 and Remark 6.4. From now on we will also just write ω_u for the fiber functor u^* on $F\text{-}\mathrm{Isoc}_K(U)$.

The group $\mathrm{Gr}(\mathcal{F}/U, u)$ is independent of the base point u in the following sense. Let $u' \in U(\mathbb{F}_{q^{e'}})$ be another base point. By [DM82, Theorem 3.2] there is a (non-canonical) isomorphism of fiber functors $\alpha = \alpha_{u',u}\colon \omega_{u'} \otimes_{K_{e'}} \overline{K} \xrightarrow{\sim} \omega_u \otimes_{K_e} \overline{K}$ over the algebraic closure $\overline{K}$. Every other isomorphism differs from α by composition with an element $g \in \mathrm{Aut}^{\otimes}(\omega_u \otimes_{K_e} \overline{K}) = \mathrm{Gr}(\mathcal{F}/U, u)(\overline{K})$. The isomorphism α induces an isomorphism of algebraic groups $\alpha_*\colon\ \mathrm{Gr}(\mathcal{F}/U, u') \times_{K_{e'}} \overline{K} \xrightarrow{\sim} \mathrm{Gr}(\mathcal{F}/U, u) \times_{K_e} \overline{K}$ over $\overline{K}$ and the isomorphism $(g \circ \alpha)_*$ induced by $g \circ \alpha$ differs from α_* by conjugation with g. In this way we may move the base point whenever it is convenient.

To introduce the Frobenius conjugacy classes let $\mathcal{F} \in F\text{-}\mathrm{Isoc}_K(U)$ and fix a base point $u \in U(\mathbb{F}_{q^e})$. Let $x \in |U|$ be a closed point with residue field $\mathbb{F}_{q^n}$ and choose a point $y \in U(\mathbb{F}_{q^n})$ above x. Let $\tilde{n}$ be the least common multiple of n and e. Then $K_{\tilde{n}}$ is the compositum of K_n and K_e. Since $y = (\mathrm{Fr}_{q,U})^n \circ y$ as morphisms $\mathrm{Spec}\,\mathbb{F}_{q^n} \to U$, the Frobenius $F^n_{\mathcal{F}}\colon (\mathrm{Fr}_{q,U})^{n*}\mathcal{F}^{(n)} \xrightarrow{\sim} \mathcal{F}^{(n)}$ of the F^n-isocrystal $\mathcal{F}^{(n)}$ induces an automorphism $y^*F^n_{\mathcal{F}}$ of the fiber functor ω_y, that is an element of $\mathrm{Gr}(\mathcal{F}^{(n)}/U_n, y)(K_n)$. We denote by $\mathrm{Frob}_y(\mathcal{F})$ the $\mathrm{Gr}(\mathcal{F}/U, y)(\overline{K})$-conjugacy class of its image $h_n(\mathcal{F})(y^*F^n_{\mathcal{F}})$ under $h_n(\mathcal{F})$ in $\mathrm{Gr}(\mathcal{F}/U, y)(\overline{K})$. Choose an isomorphism of fiber functors $\alpha = \alpha_{y,u}\colon \omega_y \otimes_{K_n} \overline{K} \xrightarrow{\sim} \omega_u \otimes_{K_e} \overline{K}$ and the induced isomorphism of algebraic groups $\alpha_*\colon\ \mathrm{Gr}(\mathcal{F}/U, y) \times_{K_n} \overline{K} \xrightarrow{\sim} \mathrm{Gr}(\mathcal{F}/U, u) \times_{K_e} \overline{K}$ over $\overline{K}$ as above. Note that $\mathrm{Gal}(\overline{K}/K_{\tilde{n}})$ operates on the set of these isomorphisms. Since any other isomorphism α'_* differs from α_* by conjugation with an element in $\mathrm{Gr}(\mathcal{F}/U, u)(\overline{K})$, the conjugacy class $\alpha_*\big(\mathrm{Frob}_y(\mathcal{F})\big) \subset \mathrm{Gr}(\mathcal{F}/U, u)(\overline{K})$ is independent of α and hence also invariant under $\mathrm{Gal}(\overline{K}/K_{\tilde{n}})$. We claim that, moreover, it is invariant under $\mathrm{Gal}(K_{\tilde{n}}/K_e)$ and only depends on the closed point x of U lying below y. Indeed, there is a point $\tilde{y} \in U(\mathbb{F}_{q^n})$ above x with $\mathrm{Fr}_{q,U} \circ \tilde{y} = y$ and $\tilde{y}^*\mathrm{Fr}^*_{q,U}\mathcal{F} = y^*\mathcal{F}$. The isomorphism $F_{\mathcal{F}}\colon \mathrm{Fr}^*_{q,U}\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ of the F-isocrystal $\mathcal{F}$ induces an isomorphism $\tilde{y}^*F_{\mathcal{F}}\colon y^*\mathcal{F} \xrightarrow{\sim} \tilde{y}^*\mathcal{F}$ under which the K_n-linear automorphisms $F^n_{\mathcal{F}}$ on the fibers at y and $\tilde{y}$ are mapped to each other. So $F_{\mathcal{F}}$ maps $(\alpha_{\tilde{y},u})_*\big(\mathrm{Frob}_{\tilde{y}}(\mathcal{F})\big)$ onto $(\alpha_{y,u})_*\big(\mathrm{Frob}_y(\mathcal{F})\big)$, which therefore only depends on x and not on y. We denote this conjugacy class by $\mathrm{Frob}_x(\mathcal{F})$. Finally, the Galois group $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) = \mathrm{Gal}(K_n/K)$ surjects onto $\mathrm{Gal}(K_{\tilde{n}}/K_e)$ and the isomorphism $F_{\mathcal{F}}$ yields the $\mathrm{Gal}(K_{\tilde{n}}/K_e)$-invariance of $\mathrm{Frob}_x(\mathcal{F})$. We therefore may identify it with a subset of the K_e-group scheme $\mathrm{Gr}(\mathcal{F}/U, u)$.

Definition 3.1. The subset $\mathrm{Frob}_x(\mathcal{F}) \subset \mathrm{Gr}(\mathcal{F}/U, u)$ defined above is called the *(stable) Frobenius conjugacy class* of the K-linear convergent F-isocrystal $\mathcal{F}$ on U at the closed point $x \in U$.

Remark 3.2. The subset $\mathrm{Frob}_x(\mathcal{F}) \subset \mathrm{Gr}(\mathcal{F}/U, u)$ is not closed in general, but on its Zariski-closure $\overline{\mathrm{Frob}_x(\mathcal{F})}$ the characteristic polynomial is constant and K_e-rational when $u \in U(\mathbb{F}_{q^e})$. More precisely, for each element $g \in \mathrm{Gr}(\mathcal{F}/U, u)(L)$ for a field extension L/K_e we let $\chi_g \in L[T]$ be the characteristic polynomial of g viewed as an endomorphism of the L-vector space $\omega_u(\mathcal{F}) \otimes_{K_e} L$. Since all elements of $\mathrm{Frob}_x(\mathcal{F})$ are conjugate over $\overline{K}$ and since the characteristic polynomial is continuous with respect to the Zariski topology, it is constant on $\overline{\mathrm{Frob}_x(\mathcal{F})}$. Since moreover $\mathrm{Frob}_x(\mathcal{F})$ is $\mathrm{Gal}(\overline{K}/K_e)$-invariant, its characteristic polynomial is K_e-rational.

Lemma 3.3. *Let $\mathcal{F} \in F\text{-Isoc}_K(U)$ and let $\mathcal{G} \in \langle\langle\mathcal{F}\rangle\rangle$. Then there are canonical epimorphisms of group schemes $\mathrm{Gr}(\mathcal{F}/U,u) \twoheadrightarrow \mathrm{Gr}(\mathcal{G}/U,u)$ and their identity components $\mathrm{Gr}(\mathcal{F}/U,u)^\circ \twoheadrightarrow \mathrm{Gr}(\mathcal{G}/U,u)^\circ$. Under these the Frobenius conjugacy class $\mathrm{Frob}_x(\mathcal{F})$ is mapped onto $\mathrm{Frob}_x(\mathcal{G})$ for every closed point $x \in U$.*

Proof. The epimorphism of groups $\mathrm{Gr}(\mathcal{F}/U,u) \twoheadrightarrow \mathrm{Gr}(\mathcal{G}/U,u)$ comes from Corollary A.16(a). Its compatibility with the Frobenius conjugacy classes follows directly from their definition. That $\mathrm{Gr}(\mathcal{F}/U,u)^\circ$ surjects onto $\mathrm{Gr}(\mathcal{G}/U,u)^\circ$ follows from [Bor91, I.1.4 Corollary]. □

As a direct consequence of Lemma 3.3 we obtain the following

Lemma 3.4. *If $\mathcal{F}$ is a convergent F-isocrystal on U for which one of the Conjectures 1.2 or 1.3 or 1.4 holds, then this conjecture also holds for the semi-simplification $\mathcal{F}^{ss}$ and more generally for every $\mathcal{G} \in \langle\langle\mathcal{F}\rangle\rangle$.* □

The lemma has the following partial converse.

Lemma 3.5. *Let $\mathcal{F}$ be a convergent F-isocrystal on U and let $\mathcal{G} \in \langle\langle\mathcal{F}\rangle\rangle$ be such that the epimorphism $\pi\colon \mathrm{Gr}(\mathcal{F}/U,u) \twoheadrightarrow \mathrm{Gr}(\mathcal{G}/U,u)$ has* finite *kernel. If one of the Conjectures 1.3 or 1.4 holds for $\mathcal{G}$, then this conjecture also holds for $\mathcal{F}$.*

Remark. Note that the lemma might be false for Conjecture 1.2.

Proof of Lemma 3.5. Let $S \subset |U|$ be a subset of positive upper Dirichlet density and let C be a connected component of $\mathrm{Gr}(\mathcal{G},u) \times_{K_e} \overline{K}$ in which $\bigcup_{x\in S} \mathrm{Frob}_x(\mathcal{G})$ is Zariski dense. (Note for Conjecture 1.3 that if $\mathrm{Gr}(\mathcal{F},u)$ is connected, then $\mathrm{Gr}(\mathcal{G},u)$ is also connected.) Let $C_1,\ldots,C_n$ be the connected components of $\mathrm{Gr}(\mathcal{F},u) \times_{K_e} \overline{K}$ which map to C. By the finiteness assumption on the kernel of π, the dimensions of all C_i are the same as the dimension of C. If $\bigcup_{x\in S} \mathrm{Frob}_x(\mathcal{F})$ is not Zariski-dense in C_i for all i, then the Zariski-closure Z_i of $C_i \cap \bigcup_{x\in S} \mathrm{Frob}_x(\mathcal{F})$ has dimension strictly less than $\dim(C_i)$ for all i. The images $\pi(Z_i)$ are closed in C, because π is a finite morphism, and their union contains $\bigcup_{x\in S} \mathrm{Frob}_x(\mathcal{G})$ by Lemma 3.3. Since the latter is Zariski-dense in C and C is irreducible, we must have $C = \pi(Z_i)$ for one i. But this contradicts the dimension estimate $\dim \pi(Z_i) = \dim(Z_i) < \dim(C_i) = \dim(C)$, and proves the lemma. □

Proposition 3.6. *Let $\mathcal{F},\mathcal{G} \in F\text{-Isoc}_K(U)$, and let $u \in U(\mathbb{F}_{q^e})$. Then*

(a) $\mathrm{Gr}(\mathcal{F}\otimes\mathcal{G},u)$ *and* $\mathrm{Gr}(\mathcal{H}om(\mathcal{F},\mathcal{G}),u)$ *are quotients of* $\mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)$.

(b) the strictly full sub-category $\langle\langle\mathcal{F}\rangle\rangle \cap \langle\langle\mathcal{G}\rangle\rangle$ of $F\text{-Isoc}_K(U)$ consisting of all convergent F-isocrystals $\mathcal{H}$ which are both isomorphic to an object of $\langle\langle\mathcal{F}\rangle\rangle$ and to an object of $\langle\langle\mathcal{G}\rangle\rangle$, is a Tannakian sub-category.

(c) $\mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)$ *is a closed subgroup of* $\mathrm{Gr}(\mathcal{F},u) \times_{K_e} \mathrm{Gr}(\mathcal{G},u)$ *which sits in a cartesian diagram of epimorphisms*

$$\begin{tikzcd}[column sep=small] & \mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u) \arrow[dl, two heads] \arrow[dr, two heads] & \\ \mathrm{Gr}(\mathcal{F},u) \arrow[dr, two heads] & \square & \mathrm{Gr}(\mathcal{G},u)\,. \arrow[dl, two heads] \\ & \mathrm{Gr}(\langle\langle\mathcal{F}\rangle\rangle \cap \langle\langle\mathcal{G}\rangle\rangle,u) & \end{tikzcd} \tag{3.2}$$

(d) Let L be any field extension of K_e. In diagram (3.2) any maximal torus (respectively maximal split torus, respectively Borel subgroup) T in $G := \mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)^\circ \times_{K_e} L$ equals the connected component of unity $(T_1 \times_{T_0} T_2)^\circ$ of the fiber product of its images T_1 in $G_1 := \mathrm{Gr}(\mathcal{F},u)^\circ \times_{K_e} L$ and T_2 in $G_2 := \mathrm{Gr}(\mathcal{G},u)^\circ \times_{K_e} L$ over its image T_0 in $\mathrm{Gr}(\langle\langle\mathcal{F}\rangle\rangle\cap\langle\langle\mathcal{G}\rangle\rangle,u)^\circ \times_{K_e} L$. In this situation, where T is a maximal (split) torus, let $W = W(T,G) := N_G(T)/Z_G(T)$ and $W_i = W(T_i,G_i)$ be the (relative) Weyl groups. Then the natural map $W \hookrightarrow W_1 \times W_2$ is injective.

Remark. Note that $T_1 \times_{T_0} T_2$ can be disconnected, as one sees for example by taking the n-th power map $[n]\colon \mathbb{G}_m \to \mathbb{G}_m$ for both maps $T_i \to T_0$.

Proof of Proposition 3.6. (a) follows from Lemma 3.3, because $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F}\otimes\mathcal{G}$ and $\mathcal{H}om(\mathcal{F},\mathcal{G})$ are objects of $\langle\langle\mathcal{F}\oplus\mathcal{G}\rangle\rangle$.

(b) follows from the obvious facts that tensor products, direct sums, duals, internal Hom-s and subquotients of objects in $\langle\langle\mathcal{F}\rangle\rangle \cap \langle\langle\mathcal{G}\rangle\rangle$ again lie in $\langle\langle\mathcal{F}\rangle\rangle \cap \langle\langle\mathcal{G}\rangle\rangle$.

(c) By Proposition A.13 the object $\mathcal{F}\oplus\mathcal{G}$ corresponds to a faithful representation of $\mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)$ which factors through $\mathrm{Gr}(\mathcal{F},u)\times_{K_e}\mathrm{Gr}(\mathcal{G},u)$, because a tensor automorphism of $\omega_u(\mathcal{F}\oplus\mathcal{G})$ is trivial as soon as its restrictions to $\omega_u(\mathcal{F})$ and $\omega_u(\mathcal{G})$ are trivial. Since $\mathcal{F}$ and $\mathcal{G}$ are objects of $\langle\!\langle\mathcal{F}\oplus\mathcal{G}\rangle\!\rangle$, the two upper arrows in diagram (3.2) are epimorphisms by Corollary A.16(a). Consider the kernels $N_1=\ker\big(\mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)\twoheadrightarrow\mathrm{Gr}(\mathcal{F},u)\big)$ and $N_2=\ker\big(\mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)\twoheadrightarrow\mathrm{Gr}(\mathcal{G},u)\big)$ and the linear algebraic group $G:=\mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)/N_1N_2$. We claim that the diagram

$$\begin{tikzcd}
& \mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u) \arrow[dl, two heads, "\pi_1"'] \arrow[dr, two heads, "\pi_2"] & \\
\mathrm{Gr}(\mathcal{F},u) \arrow[dr, two heads, "\rho_1"'] & \square & \mathrm{Gr}(\mathcal{G},u) \arrow[dl, two heads, "\rho_2"] \\
& G &
\end{tikzcd}$$

is cartesian. Since that diagram is commutative, we obtain a morphism from $\mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)$ to the fiber product $\mathrm{Gr}(\mathcal{F},u)\times_G\mathrm{Gr}(\mathcal{G},u)$, which is a closed immersion, because $\mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)\to\mathrm{Gr}(\mathcal{F},u)\times_{K_e}\mathrm{Gr}(\mathcal{G},u)$ is one. Consider an algebraically closed field L and an L-valued point $(g_1,g_2)\in\mathrm{Gr}(\mathcal{F},u)\times_G\mathrm{Gr}(\mathcal{G},u)$ with $\rho_1(g_1)=\rho_2(g_2)$. Since π_i is surjective, there are elements $\tilde g_i\in\mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)(L)$ with $\pi_i(\tilde g_i)=g_i$. The equation $\rho_1\pi_1(\tilde g_1)=\rho_1(g_1)=\rho_2(g_2)=\rho_2\pi_2(\tilde g_2)=\rho_1\pi_1(\tilde g_2)$ shows that $\tilde g_1^{-1}\tilde g_2$ lies in $\ker(\rho_1\pi_1)=N_1N_2$. So there are elements $n_i\in N_i$ with $\tilde g_1^{-1}\tilde g_2=n_1n_2^{-1}$. The element $\tilde g_1n_1=\tilde g_2n_2\in\mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)(L)$ satisfies $\pi_i(\tilde g_in_i)=\pi_i(\tilde g_i)=g_i$ for $i=1,2$ as desired. This proves that $\mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)$ is isomorphic to the fiber product $\mathrm{Gr}(\mathcal{F},u)\times_G\mathrm{Gr}(\mathcal{G},u)$.

It remains to identify G with $\mathrm{Gr}(\langle\!\langle\mathcal{F}\rangle\!\rangle\cap\langle\!\langle\mathcal{G}\rangle\!\rangle,u)$. Since N_1 arises from the epimorphism of groupoids $\mathfrak{Gr}(\mathcal{F}\oplus\mathcal{G},u)\twoheadrightarrow\mathfrak{Gr}(\mathcal{F},u)$ it is invariant under the conjugation action of $\mathfrak{Gr}(\mathcal{F}\oplus\mathcal{G},u)$ on $\mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)$ from (A.2). The same is true for N_2, and hence also for N_1N_2. Corollary A.16(b) applied to the epimorphism $\mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)\twoheadrightarrow G$ shows that G is the monodromy group $\mathrm{Gr}(\mathcal{K},u)$ of an object $\mathcal{K}\in\langle\!\langle\mathcal{F}\oplus\mathcal{G}\rangle\!\rangle$, that is, G is the kernel group of $\mathfrak{Gr}(\mathcal{K},u)$. Since $\mathfrak{Gr}(\mathcal{K},u)$ is also a quotient of $\mathfrak{Gr}(\mathcal{F},u)$, respectively of $\mathfrak{Gr}(\mathcal{G},u)$, the object $\mathcal{K}$ is both isomorphic to an object of $\langle\!\langle\mathcal{F}\rangle\!\rangle$ and to an object of $\langle\!\langle\mathcal{G}\rangle\!\rangle$ by Proposition A.14(a), that is, it belongs to $\langle\!\langle\mathcal{F}\rangle\!\rangle\cap\langle\!\langle\mathcal{G}\rangle\!\rangle$. This yields an epimorphism $\mathrm{Gr}(\langle\!\langle\mathcal{F}\rangle\!\rangle\cap\langle\!\langle\mathcal{G}\rangle\!\rangle,u)\twoheadrightarrow\mathrm{Gr}(\mathcal{K},u)=G$ by Corollary A.16(a). Conversely, since $\langle\!\langle\mathcal{F}\rangle\!\rangle\cap\langle\!\langle\mathcal{G}\rangle\!\rangle$ is contained both in $\langle\!\langle\mathcal{F}\rangle\!\rangle$ and $\langle\!\langle\mathcal{G}\rangle\!\rangle$ the map $\mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)\twoheadrightarrow\mathrm{Gr}(\langle\!\langle\mathcal{F}\rangle\!\rangle\cap\langle\!\langle\mathcal{G}\rangle\!\rangle,u)$ factors over $\mathrm{Gr}(\mathcal{F},u)$ and over $\mathrm{Gr}(\mathcal{G},u)$. So its kernel contains N_1 and N_2. This provides the epimorphism in the other direction $G\twoheadrightarrow\mathrm{Gr}(\langle\!\langle\mathcal{F}\rangle\!\rangle\cap\langle\!\langle\mathcal{G}\rangle\!\rangle,u)$ and shows that both are isomorphisms.

(d) The natural maps $T\hookrightarrow T_1\times_{T_0}T_2\hookrightarrow\mathrm{Gr}(\mathcal{F}\oplus\mathcal{G},u)$ are closed immersions. Since a product of (split) tori is a (split) torus (respectively, a product of connected solvable groups is connected solvable) and $(T_1\times_{T_0}T_2)^\circ\subset T_1\times_{K_e}T_2$ is a closed connected subgroup, it is a (split) torus by [Bor91, III.8.4 Corollary] (respectively a connected solvable group by [Hum75, 17.3 Lemma]) . But since T is a maximal such group, we must have $T=(T_1\times_{T_0}T_2)^\circ$ as desired.

To prove the injectivity of the natural map $W\to W_1\times W_2$ let $n\in N_G(T)$ be mapped to the identity, that is $\pi_i(n)\in Z_{G_i}(T_i)$ for $i=1,2$. Thus, for every $t\in T$ the element $ntn^{-1}t^{-1}\in G$ maps to the identity in G_i. Since $G\hookrightarrow G_1\times_L G_2$ is a closed immersion, this shows that $ntn^{-1}t^{-1}=1$ in G for every $t\in T$ and $n\in Z_G(T)$. The injectivity follows. □

For the next results we need the following well known

Lemma 3.7. *Let G be a linear algebraic group over an algebraically closed field L of characteristic zero.*

- *(a) Let $g\in G(L)$ and let n be a positive integer such that g^n is semi-simple, then g is semi-simple.*
- *(b) All unipotent elements of $G(L)$ are contained in the identity component G° of G. In particular all unipotent groups in characteristic zero are connected.*

Proof. (a) If $g=g_sg_u$ is the multiplicative Jordan decomposition, where $g_s,g_u\in G(L)$ are the semi-simple and unipotent parts of g, respectively, then $g^n=g_s^ng_u^n$ is the multiplicative Jordan decomposition of g^n. Consider a faithful representation $G\subset\mathrm{GL}_r$. Then g_u is conjugate in GL_r to a unipotent upper triangular matrix (use [Bor91, IV.11.10 Theorem]), and hence if $g_u\neq1$ it has infinite order as $\mathrm{char}(L)=0$. Therefore, g^n is semi-simple, that is $g_u^n=1$, if and only if $g_u=1$ and g is semi-simple.

(b) If $g\in G(L)$ is a unipotent element, then its image in G/G° is of finite order and unipotent by [Bor91, I.4.4 Theorem], whence trivial by (a). □

Lemma 3.8. *Let $\mathcal{F} \in F\text{-Isoc}_K(U)$. Then $\mathcal{F}$ is semi-simple if and only if the category $\langle\langle\mathcal{F}\rangle\rangle$ is semi-simple, if and only if the identity component $\mathrm{Gr}(\mathcal{F},u)^\circ$ is reductive. More generally, let $\mathcal{F}^{ss}$ be the semi-simplification of $\mathcal{F}$. Then $\alpha\colon \mathrm{Gr}(\mathcal{F},u) \twoheadrightarrow \mathrm{Gr}(\mathcal{F}^{ss},u)$ is the maximal reductive quotient of $\mathrm{Gr}(\mathcal{F},u)$, that is, $\ker\alpha$ is the unipotent radical of $\mathrm{Gr}(\mathcal{F},u)$. In particular, α induces an isomorphism on the groups of connected components.*

Proof. In the neutral situation where $K_e = K$, the first statement is proven in (the proof of) [DM82, Proposition 2.23 and Remark 2.28]. In the non-neutral situation, Proposition A.22 tells us that $\mathcal{F}$ is semi-simple if and only if $\mathcal{F} \otimes_K K_e$ is semi-simple in $F\text{-Isoc}_K(U) \otimes_K K_e$. Assume now that this is the case. By Remark A.20 the Tannakian category $\langle\langle\mathcal{F} \otimes_K K_e\rangle\rangle = \langle\langle\mathcal{F}\rangle\rangle \otimes_K K_e$ is equivalent to the category $\mathrm{Rep}_{K_e}\mathrm{Gr}(\mathcal{F},u)$ of K_e-rational representations of the K_e-group $\mathrm{Gr}(\mathcal{F},u)$. By the neutral situation discussed above it follows that $\mathrm{Gr}(\mathcal{F},u)^\circ$ is reductive. Conversely, the latter implies that $\langle\langle\mathcal{F} \otimes_K K_e\rangle\rangle$ is semi-simple. If now $\mathcal{G} \in \langle\langle\mathcal{F}\rangle\rangle$, then $\mathcal{G}\otimes_K K_e \in \langle\langle\mathcal{F}\otimes_K K_e\rangle\rangle$, and hence $\mathcal{G}$ is semi-simple by Proposition A.22. This proves the first statement also in the non-neutral situation.

We prove the rest. In a suitable basis the representation ρ of $G := \mathrm{Gr}(\mathcal{F},u)$ on $\omega_u(\mathcal{F})$ can be written in block matrix form such that the diagonal block entries are representations corresponding to the simple constituents of $\mathcal{F}$. Therefore, the kernel of α lies in the subgroup of upper triangular matrices with ones on the diagonal. So it is a unipotent group and as such connected by Lemma 3.7. Being also normal, it is contained in the unipotent radical. On the other hand the unipotent radical is mapped to $\{1\}$ in $\widetilde{G} := \mathrm{Gr}(\mathcal{F}^{ss},u)$, because the latter is a reductive group by our first statement. □

Proposition 3.9. *Let $\mathcal{F} \in F\text{-Isoc}_K(U)$ be a semi-simple convergent F-isocrystal on U. Let Z be the center (respectively the connected component of the center) of the connected component $G^\circ := \mathrm{Gr}(\mathcal{F}/U,u)^\circ$ and let $[G^\circ,G^\circ]$ be the derived group of G°. Then Z and $[G^\circ,G^\circ]$ are characteristic subgroups of G°. Let $\mathcal{S},\mathcal{T} \in \langle\langle\mathcal{F}\rangle\rangle$ be the convergent F-isocrystals whose monodromy groups are $\mathrm{Gr}(\mathcal{S}/U,u) = G/Z$ and $\mathrm{Gr}(\mathcal{T}/U,u) = G/[G^\circ,G^\circ]$; see Remark A.17 and Corollary A.16(b). Then $\mathrm{Gr}(\langle\langle\mathcal{F}\rangle\rangle \cap \langle\langle\mathcal{G}\rangle\rangle,u)$ is a finite group and in the diagram*

$$G \;=\; \mathrm{Gr}(\mathcal{F}/U,u) \;\twoheadrightarrow\; \mathrm{Gr}(\mathcal{S}\oplus\mathcal{T}/U,u) \;\xrightarrow{\;\sim\;}\; G/Z \times_{\mathrm{Gr}(\langle\langle\mathcal{F}\rangle\rangle\cap\langle\langle\mathcal{G}\rangle\rangle,u)} G/[G^\circ,G^\circ] \tag{3.3}$$

there is a natural isomorphism on the right, and the kernel of the surjection on the left is finite and contained in the center of G°.

Proof. Since $\mathcal{F}$ is semi-simple, G is reductive by Lemma 3.8. The group G°/Z is semi-simple by [Bor91, IV.11.21 Proposition] and $G^\circ/[G^\circ,G^\circ]$ is a torus by [Bor91, IV.14.11 Corollary and III.10.6 Theorem]. The group $\mathrm{Gr}(\langle\langle\mathcal{F}\rangle\rangle\cap\langle\langle\mathcal{G}\rangle\rangle,u)^\circ$ is both a quotient of G°/Z and $G^\circ/[G^\circ,G^\circ]$, and hence is semi-simple and a torus by [Bor91, IV.14.11 Corollary, III.8.4 Corollary and III.8.5 Proposition]. Therefore, $\mathrm{Gr}(\langle\langle\mathcal{F}\rangle\rangle \cap \langle\langle\mathcal{G}\rangle\rangle,u)^\circ$ is trivial by [Bor91, IV.11.21 Proposition] and $\mathrm{Gr}(\langle\langle\mathcal{F}\rangle\rangle \cap \langle\langle\mathcal{G}\rangle\rangle,u)$ is a finite group. The isomorphism on the right was established in Proposition 3.6(c). Finally the kernel of the map (3.3) is contained in the center of G° by construction and its connected component is contained in $Z^\circ \cap [G^\circ,G^\circ]$ which is a finite group by [Bor91, IV.14.2 Proposition]. □

In the rest of this section we want to make a few remarks on Dirichlet density.

Definition 3.10. We will say that a subset $S \subset |U|$ has *Dirichlet density* ε for some real number $0 \le \varepsilon \le 1$ (in the sense of Serre) if

$$\lim_{s\to 1^+} \frac{-\sum_{x\in S} q^{-\deg(x)s}}{\log(s-1)} = \varepsilon.$$

Moreover we will say that S has *positive Dirichlet density* if it has Dirichlet density ε for some positive ε.

Definition 3.11. The *upper Dirichlet density* $\overline{\delta}(S)$ of a subset $S \subset |U|$ is

$$\overline{\delta}(S) = \limsup_{s\to 1^+} \frac{-\sum_{x\in S} q^{-\deg(x)s}}{\log(s-1)}.$$

Note that the limit superior on the right hand side always exists and it is between 0 and 1, and it is equal to the Dirichlet density of the set S, if the latter exists. In particular the upper Dirichlet density of S is 0 if and only if S has Dirichlet density 0. We will say that S has *positive upper Dirichlet density* if $\overline{\delta}(S) > 0$. Trivially $\overline{\delta}(R) \le \overline{\delta}(S)$ when R is a subset of S.

The key property of upper Dirichlet density is the following easy to prove

Lemma 3.12. *Let $S \subset |U|$ be a subset, and assume that*
$$S = S_1 \cup S_2 \cup \cdots \cup S_n,$$
where the sets S_i are pair-wise disjoint. Then
$$\overline{\delta}(S) \leq \overline{\delta}(S_1) + \overline{\delta}(S_2) + \cdots + \overline{\delta}(S_n).$$

Proof. Note that
$$\frac{-\sum_{x\in S} q^{-\deg(x)s}}{\log(s-1)} = \frac{-\sum_{x\in S_1} q^{-\deg(x)s}}{\log(s-1)} + \frac{-\sum_{x\in S_2} q^{-\deg(x)s}}{\log(s-1)} + \cdots + \frac{-\sum_{x\in S_n} q^{-\deg(x)s}}{\log(s-1)},$$
so by the sub-additivity of the limit superior we get the lemma. □

We will also need the following

Lemma 3.13. *Let $S \subset |U|$ be a subset of Dirichlet density one, and let $R \subset |U|$ be a subset of positive upper Dirichlet density. Then $R \cap S$ also has positive upper Dirichlet density.*

Proof. Let $S^c \subset |U|$ be the complement of S in $|U|$. Since
$$\sum_{x\in|U|} q^{-\deg(x)s} = \sum_{x\in S} q^{-\deg(x)s} + \sum_{x\in S^c} q^{-\deg(x)s},$$
and
$$\lim_{s\to 1^+} \frac{-\sum_{x\in|U|} q^{-\deg(x)s}}{\log(s-1)} = 1$$
by the prime number theorem for U, we get that S^c has Dirichlet density zero. Therefore, $R \cap S^c$ has Dirichlet density zero, too. Since R is the disjoint union of $R \cap S$ and $R \cap S^c$, the claim follows from Lemma 3.12. □

The following well known property of the (upper) Dirichlet density obstructs the technique of replacing U by a finite étale Galois covering.

Example 3.14. Let $f\colon V \to U$ be a finite étale Galois covering of degree n, where n is a prime number, let $S \subset |U|$ be a subset of upper Dirichlet density $\bar{\delta}(S)$, and let $S' = f^{-1}(S) \subset |V|$ be the preimage of S under f.

(a) Assume that $n|\deg(x)$ for all $x \in S$. Then there are exactly n points x' of V lying above each $x \in S$, and they have degree $\deg(x') = \deg(x)/n$ we compute
$$\sum_{x'\in S'} (q^n)^{-\deg(x')s} \; = \; n \cdot \sum_{x\in S} q^{-\deg(x)s}.$$
Therefore, S' has upper Dirichlet density $\bar{\delta}(S') = n \cdot \bar{\delta}(S)$.

(b) Assume that $n \nmid \deg(x)$ for all $x \in S$. Then there is exactly one point x' of V lying above each $x \in S$, and it has degree $\deg(x') = \deg(x)$ we compute
$$\sum_{x'\in S'} (q^n)^{-\deg(x')s} \; = \; \sum_{x\in S} q^{-n\deg(x)s}.$$
When $s \to 1^+$ this sum converges in $\mathbb{R}$, whereas $\log(s-1)$ goes to ∞. Therefore, S' has upper Dirichlet density $\bar{\delta}(S') = 0$.

This is of course analogous to the situation for number fields, where one says that a subset $S \subset |M|$ of the set $|M|$ of places of a number field M has Dirichlet density δ if
$$\liminf_{m\to\infty} \frac{\#\{x \in S\colon N(x) \leq m\}}{\#\{x \in |M|\colon N(x) \leq m\}} \; = \; \delta\,,$$
where $N(x) := \#\mathcal{O}_M/x$ denotes the norm of x. Let N/M be a Galois extension of number fields whose degree n is a prime number. Then the set S of places in M which split completely in N has Dirichlet density $\delta(S) = \frac{1}{n}$, whereas its preimage $S' \subset |N|$ has Dirichlet density $\delta(S') = 1$. The reason is that above every place $x \in S$ there are exactly n places in S' with the same norm as x.

On the other hand the set S of places in M which are inert in N has Dirichlet density $\delta(S) = \frac{n-1}{n}$, whereas its preimage $S' \subset |N|$ has Dirichlet density $\delta(S') = 0$. The reason is that above every place $x \in S$ there is exactly one place x' in S' whose norm is $N(x') = N(x)^n$.

For applications to isoclinic F-isocrystals we will need the following effective form of the classical Chebotarëv density theorem for function fields.

Notation 3.15. By a *finite Galois group G of the curve U* we will mean a quotient $\pi_1^{\text{ét}}(U,\bar{u}) \twoheadrightarrow G$ by an open normal subgroup. For every such G and for every $x \in |U|$ let $\mathrm{Fr}_x^G \subset G$ denote the image under the quotient map of the conjugacy class of the geometric Frobenius Frob_x^{-1} at x in $\pi_1^{\text{ét}}(U,\bar{u})$ which maps $a \in \overline{\mathbb{F}}_x$ to a^{1/q_x}, where $q_x = \#\mathbb{F}_x$. For every such G let G^c denote the maximal constant quotient of G, that is, the largest quotient of G which can be pulled back from $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. In particular, G^c is finite cyclic, and hence abelian. Let G^{geo} be the kernel of the quotient map $G \to G^c$.

The aforementioned effective version is the following

Theorem 3.16. *Let $S \subset |U|$ be a set of positive upper Dirichlet density. Then there is a positive constant $\varepsilon > 0$ such that for every finite Galois group G of U there is an infinite subset $R \subset \mathbb{N}$ such that for every $n \in R$ the union in G of the Frobenius conjugacy classes Fr_x^G for all x in $\{x \in S\colon \deg(x) = n\}$ has cardinality at least $\varepsilon \cdot \#G^{geo}$.*

We will first prove a couple of lemmas. For every positive integer n let $P(n)$ denote the number of closed points of U of degree n. By the prime number theorem for function fields [Ros02, Theorem 5.12], which is an easy consequence of the Weil bounds, we have

$$P(n) = \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right). \tag{3.4}$$

Lemma 3.17. *Let $T \subset |U|$ be a set of positive upper Dirichlet density $\overline{\delta}(T)$. There is an infinite subset $R \subset \mathbb{N}$ such that for every $n \in R$ we have:*

$$\#\{x \in T\colon \deg(x) = n\} \geq \frac{\overline{\delta}(T) \cdot P(n)}{2}.$$

Proof. Assume that the claim is false. Then there is a positive integer m such that

$$\#\{x \in T\colon \deg(x) = n\} < \frac{\overline{\delta}(T) \cdot P(n)}{2}$$

for every $n > m$. Thus for every $s \in \mathbb{R}$ with $1 < s < 2$ we have:

$$\sum_{x \in T} q^{-\deg(x)s} \leq \sum_{x \in T\colon \deg(x) \leq m} q^{-\deg(x)s} + \frac{\overline{\delta}(T)}{2} \sum_{n>m} P(n) q^{-ns}.$$

Since $\log(s-1)$ is negative for such s, we get from the above that

$$\overline{\delta}(T) \leq \limsup_{s \to 1^+} \frac{-\sum_{\deg(x) \leq m} q^{-\deg(x)s}}{\log(s-1)} + \frac{\overline{\delta}(T)}{2} \limsup_{s \to 1^+} \frac{-\sum_{n>m} P(n) q^{-ns}}{\log(s-1)}.$$

The first limit on the right hand side is zero, while the second limit is

$$\limsup_{s \to 1^+} \frac{-\sum_{n>m} \frac{q^{(1-s)n}}{n} + O\left(\frac{q^{(1/2-s)n}}{n}\right)}{\log(s-1)} = \limsup_{s \to 1^+} \left(\frac{\log(1-q^{(1-s)n})}{\log(s-1)} - \sum_{n>m} \frac{O\left(\frac{q^{(1/2-s)n}}{n}\right)}{\log(s-1)} \right)$$

by the prime number theorem (3.4) for U. By L'Hôpital's rule this limit is 1. However, the resulting inequality $\overline{\delta}(T) \leq \overline{\delta}(T)/2$ is a contradiction to $\overline{\delta}(T) > 0$. □

Let G be a Galois group of U and let $D \subset G$ be a conjugacy class. Then the image of D under $G \to G^c$ is an element in G^c which we will denote by D^c. Let $c(G)$ denote the order of G^c. Then there is a unique isomorphism $\iota_G\colon G^c \xrightarrow{\sim} \mathbb{Z}/c(G)$ such that for every $x \in |U|$ the Frobenius $\mathrm{Fr}_x^{G_c}$ at x in G^c maps to $k \bmod c(G)$ if and only if $\deg(x) \equiv k \bmod c(G)$.

Theorem 3.18. *For every $\kappa > 0$ there is an $N(\kappa) = N(\kappa, G) \in \mathbb{N}$ such that for every $n > N(\kappa)$ and for every conjugacy class $C \subset G$ we have*

$$\#\{x \in U \colon \mathrm{Fr}_x^G \subset C,\ \deg(x) = n\} \leq \begin{cases} \frac{c(G)(1+\kappa)P(n)\#C}{\#G} & , \text{ if } n \equiv \iota_G(C^c) \bmod c(G), \\ 0 & , \text{ otherwise.} \end{cases}$$

Proof. We use [Vil06, Proposition 11.2.16] which says that

$$\#\{x \in U \colon \mathrm{Fr}_x^G \subset C,\ \deg(x) = n\} \ = \ \frac{c(G)\#C}{\#G}\,\frac{q^n}{n} + O(\#C\, q^{n/2})$$

if $n \equiv \iota_G(C^c) \bmod c(G)$ and zero otherwise. The theorem then follows from the prime number theorem (3.4). □

Proof of Theorem 3.16. We claim that $\varepsilon = \frac{\overline{\delta}(S)}{4}$ will do. Now assume that the claim is false, and let G be a Galois group of U which violates the assertion of the theorem. By Lemma 3.17 there is an infinite subset $R \subset \mathbb{N}$ such that for every $n \in R$ we have:

$$\#\{x \in S \colon \deg(x) = n\} \geq \frac{\overline{\delta}(S)P(n)}{2}. \tag{3.5}$$

For every $n \in \mathbb{N}$ let F_n be the union in G of the Frobenius conjugacy classes Fr_x^G for all x in the set $\{x \in S \colon \deg(x) = n\}$. It decomposes into a disjoint union of conjugacy classes C in G. We apply Theorem 3.18 for $\kappa = 1$ to each component C of F_n and add. By shrinking R, if necessary this tells us that

$$\#\{x \in S \colon \deg(x) = n\} \leq \#\{x \in U \colon \mathrm{Fr}_x^G \subset F_n,\ \deg(x) = n\} \leq \frac{2c(G)P(n)\#F_n}{\#G} \tag{3.6}$$

for every $n \in R$. Since we assumed that G is a counter-example, by shrinking R further, if this is necessary, we may assume that

$$\#F_n < \frac{\overline{\delta}(S)\#G^{geo}}{4} = \frac{\overline{\delta}(S)\#G}{4c(G)}$$

for all $n \in R$. By applying the inequality (3.6) to F_n we get that

$$\#\{x \in S \colon \deg(x) = n\} \leq \frac{2c(G)P(n)\#F_n}{\#G} < \frac{2c(G)P(n)}{\#G} \cdot \frac{\overline{\delta}(S)\#G}{4c(G)} = \frac{\overline{\delta}(S)P(n)}{2},$$

but this contradicts (3.5). □

4. Constant F-Isocrystals

Definition 4.1. For any topological group $\mathbf{G}$ and any topological field L we let $\mathrm{Rep}^{\mathrm{c}}_L \mathbf{G}$ be the neutral Tannakian category of continuous representations of $\mathbf{G}$ on finite dimensional L-vector spaces equipped with the forgetful fiber functor ω_f which sends a representation $\rho\colon \mathbf{G} \to \mathrm{Aut}_L(W)$ to the L-vector space W. We define the *L-linear algebraic envelope* $\mathbf{G}^{L\text{-alg}}$ of $\mathbf{G}$ as the Tannakian fundamental group $\mathrm{Aut}^{\otimes}(\omega_f)$ of $\mathrm{Rep}^{\mathrm{c}}_L \mathbf{G}$. By definition, ω_f induces a tensor equivalence between $\mathrm{Rep}^{\mathrm{c}}_L \mathbf{G}$ and the category $\mathrm{Rep}_L \mathbf{G}^{L\text{-alg}}$ of algebraic representations of $\mathbf{G}^{L\text{-alg}}$.

We recall the following lemma from [Ser93, p. 66]; compare also with Saavedra [Saa72, Chapter V.0.3.1] and [DM82, Example (2.33)].

Lemma 4.2. *For every continuous finite-dimensional L-linear representation $\rho\colon \mathbf{G} \to \mathrm{GL}_{n,L}$ of $\mathbf{G}$, the monodromy group $\mathrm{Aut}^{\otimes}(\omega_f|_{\langle\langle\rho\rangle\rangle})$ of ρ considered as an object of the category $\mathrm{Rep}^{\mathrm{c}}_L \mathbf{G}$, is canonically isomorphic to the Zariski-closure of the image $\rho(\mathbf{G}) \subset \mathrm{GL}_{n,L}(L)$ of $\mathbf{G}$. Therefore, we may describe $\mathbf{G}^{L\text{-alg}}$ as the limit of the Zariski-closures of the images $\mathrm{im}(\rho)$ over the diagram of all continuous finite-dimensional L-linear representations ρ of $\mathbf{G}$ (in some suitably large universe).* □

Example 4.3. When $L = \overline{K}$ then the L-linear algebraic envelope of $\mathbb{Z}$ is $\mathbb{G}_{m,L}^{\kappa} \times \mathbb{G}_{a,L} \times \widehat{\mathbb{Z}}$ where κ is the cardinality of L. We leave the verification of this fact to the reader. What we only need is Theorem 4.8 below.

But before let us record the following well known

Theorem 4.4. *Let L be a topological field. Then every short exact sequence*

$$1 \longrightarrow \mathbf{G}_1 \xrightarrow{\mathbf{q}} \mathbf{G}_2 \xrightarrow{\mathbf{p}} \mathbf{G}_3 \longrightarrow 1 \tag{4.1}$$

of compact topological groups induces a sequence

$$\mathbf{G}_1^{L\text{-alg}} \xrightarrow{q} \mathbf{G}_2^{L\text{-alg}} \xrightarrow{p} \mathbf{G}_3^{L\text{-alg}} \longrightarrow 1$$

of their L-linear algebraic envelopes, see Definition 4.1, in which the composition $p \circ q$ is trivial and $\ker(p) = q(\mathbf{G}_1^{L\text{-alg}})^{\mathrm{norm}}$ *is the smallest closed normal subgroup containing $q(\mathbf{G}_1^{L\text{-alg}})$.*

The proof of Theorem 4.4 will use the following criterion:

Theorem 4.5 ([LP17, Theorem 2.4])**.** *Let*

$$G_1 \xrightarrow{q} G_2 \xrightarrow{p} G_3 \longrightarrow 1 \tag{4.2}$$

be a sequence of affine group schemes over L such that p is faithfully flat. Assume that:

(a) if $V \in \mathrm{Rep}_L G_2$, then $q^*(V)$ is trivial in $\mathrm{Rep}_L G_1$ if and only if $V \cong p^*(W)$ for some $W \in \mathrm{Rep}_L G_3$,
(b) for any $V \in \mathrm{Rep}_L G_2$, if $W_0 \subset q^*(V)$ is the maximal trivial sub-object in $\mathrm{Rep}_L G_1$, then there exists $W \subset V \in \mathrm{Rep}_L G_2$ such that $q^*(W) = W_0 \subset q^*(V)$.

Then in sequence (4.2) *the composition $p \circ q$ is trivial and* $\ker(p) = q(G_1)^{\mathrm{norm}}$ *is the smallest closed normal subgroup containing $q(G_1)$.*

Now we are ready to give the

Proof of Theorem 4.4. We apply Theorem 4.5. Property (a) follows at once from the definition.

To prove property (b) let $V \in \mathrm{Rep}_L^{\mathrm{c}} \mathbf{G}_2$ be a representation and let $W_0 \subset \mathrm{Res}^{\mathbf{G}_2}_{\mathbf{G}_1} V$ be the L-linear subspace on which $\mathbf{G}_1$ acts trivially. Since $\mathbf{G}_1$ is a normal subgroup of $\mathbf{G}_2$ the subspace $W_0 \subset V$ is stable under $\mathbf{G}_2$. This proves property (b), because the $\mathbf{G}_1$-representation W_0 equals the restriction to $\mathbf{G}_1$ of the $\mathbf{G}_2$-representation W_0. □

Corollary 4.6. *In the situation of Theorem 4.4 let $\rho\colon \mathbf{G}_2 \to \mathrm{GL}_n(L)$ be a representation and let $\tilde{\rho} := \rho|_{\mathbf{G}_1}\colon \mathbf{G}_1 \to \mathrm{GL}_n(L)$ be the restriction of ρ to $\mathbf{G}_1$. Let $\mathcal{C} \subset \mathrm{Rep}_L \mathrm{Aut}^{\otimes}(\omega_f|_{\langle\langle\rho\rangle\rangle})$ be the full sub-category consisting of those objects on which the representation induced by ρ factors through $\mathbf{G}_3$. Then $\mathcal{C}$ is a Tannakian sub-category and the homomorphism ρ induces a commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{G}_1 \arrow[r, "\mathbf{q}"] \arrow[d, "\tilde{\rho}"] & \mathbf{G}_2 \arrow[r, "\mathbf{p}"] \arrow[d, "\rho"] & \mathbf{G}_3 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Aut}^{\otimes}(\omega_f|_{\langle\langle\tilde{\rho}\rangle\rangle}) \arrow[r, "q"] & \mathrm{Aut}^{\otimes}(\omega_f|_{\langle\langle\rho\rangle\rangle}) \arrow[r, "p"] & \mathrm{Aut}^{\otimes}(\omega_f|_{\mathcal{C}}) \arrow[r] & 1
\end{tikzcd}$$

with exact rows in which the three vertical maps have Zariski-dense image.

Proof. Clearly $\mathcal{C}$ is closed under the formation of direct sums, tensor products, duals, internal Hom-s and subquotients, and hence is a Tannakian sub-category. We apply Theorem 4.5 and argue as in the proof of Theorem 4.4. It remains to show that the morphism $q\colon \mathrm{Aut}^{\otimes}(\omega_f|_{\langle\langle\tilde{\rho}\rangle\rangle}) \to \mathrm{Aut}^{\otimes}(\omega_f|_{\langle\langle\rho\rangle\rangle})$ is a closed immersion which identifies $q(\mathrm{Aut}^{\otimes}(\omega_f|_{\langle\langle\tilde{\rho}\rangle\rangle}))$ with a normal subgroup of $\mathrm{Aut}^{\otimes}(\omega_f|_{\langle\langle\rho\rangle\rangle})$. That q is a closed immersion follows from [DM82, Proposition 2.21], because by definition every representation $W \in \langle\langle\tilde{\rho}\rangle\rangle$ is isomorphic to a subquotient of a representation $\mathrm{Res}^{\mathbf{G}_2}_{\mathbf{G}_1} V$ where $V \in \langle\langle\rho\rangle\rangle$. By Lemma 4.2, $\mathrm{Aut}^{\otimes}(\omega_f|_{\langle\langle\tilde{\rho}\rangle\rangle})$ is the Zariski-closure of the image $\tilde{\rho}(\mathbf{G}_1) \subset \mathrm{GL}_n(L)$. Since $q(\mathbf{G}_1)$ is a normal subgroup in $\mathbf{G}_2$, this Zariski-closure is a normal subgroup of the Zariski-closure of $\rho(\mathbf{G}_2) \subset \mathrm{GL}_n(L)$. The latter equals $\mathrm{Aut}^{\otimes}(\omega_f|_{\langle\langle\rho\rangle\rangle})$ by Lemma 4.2. □

To formulate the next Theorem 4.8 we recall the following

Definition 4.7. An F-isocrystal on $\mathrm{Spec}\,\mathbb{F}_q$ is by definition a pair (W, f) consisting of a finite dimensional K-vector space W together with a K-linear automorphism $f \in \mathrm{Aut}_K(W)$, its *Frobenius*. It can be pulled back under the structure morphism $\pi\colon U \to \mathrm{Spec}\,\mathbb{F}_q$ to a convergent F-isocrystal $\pi^*(W, f)$ on U, and any convergent F-isocrystal on U arising in this way is called *constant*. We denote by $F\text{-}\mathrm{Const}_K(U)$

the category of K-linear constant convergent F-isocrystals on U. And by $F\text{-Isoc}_K(\operatorname{Spec}\mathbb{F}_q)$ the category of K-linear F-isocrystals on $\operatorname{Spec}\mathbb{F}_q$. The category $F\text{-Isoc}_K(\operatorname{Spec}\mathbb{F}_q)$ is tensor equivalent to the category $\operatorname{Rep}^{\mathrm{c}}_K \mathbb{Z}$, where $\mathbb{Z}$ carries the discrete topology, by sending a representation $\rho\colon \mathbb{Z} \to \mathrm{GL}_r(K)$ to $(K^{\oplus r}, \rho(1)) \in F\text{-Isoc}_K(\operatorname{Spec}\mathbb{F}_q)$.

Theorem 4.8. *Let $\mathcal{C} = \pi^*(W, f) \in F\text{-Isoc}_K(U)$ be a constant F-isocrystal. Then the following holds.*

(a) The functor π^* induces an isomorphism between the Tannakian sub-categories $\langle\langle(W, f)\rangle\rangle \subset F\text{-Isoc}_K(\mathbb{F}_q)$ and $\langle\langle\mathcal{C}\rangle\rangle \subset F\text{-Isoc}_K(U)$. In particular, every convergent F-isocrystal $\mathcal{F}$ in $\langle\langle\mathcal{C}\rangle\rangle$ is constant.

(b) The monodromy group $\operatorname{Gr}(\mathcal{C}/U, u)$ of $\mathcal{C}$ is isomorphic to $\operatorname{Gr}((W, f)/\mathbb{F}_q, u)$. This group equals the Zariski-closure of $f^{\mathbb{Z}}$ in $\operatorname{Aut}_{K_e}(W \otimes_K K_e)$. In particular, it is commutative and isomorphic to $T \times_{K_e} \mathbb{G}^{\varepsilon}_{a,K_e}$ where T is an extension of a finite (étale) abelian group (scheme) T/T° (over K_e) by a torus T° and $\varepsilon = 0$ or 1.

(c) For every $x \in |U|$ the set $\operatorname{Frob}_x(\mathcal{C})$ consists of the single element $f^{\deg(x)}$.

(d) The categories $F\text{-Isoc}_K(\operatorname{Spec}\mathbb{F}_q)$ and $F\text{-Const}_K(U)$ are tensor equivalent to the category $\operatorname{Rep}^{\mathrm{c}}_K \mathbb{Z}$, where $\mathbb{Z}$ carries the discrete topology. In particular, they are neutral K-linear Tannakian categories. The Tannakian fundamental K_e/K-groupoids $\pi_1^{F\text{-Isoc}}(\mathbb{F}_q, u)$ and $\pi_1^{F\text{-Const}}(U, u)$ are equal to the neutral K_e/K-groupoid associated with the K-linear algebraic envelope $\mathbb{Z}^{K\text{-alg}}$ of $\mathbb{Z}$, see Definitions A.12 and A.7.

Proof. (a) Before we start the proof let us make the following obvious remark: an F-isocrystal is constant if and only if it is trivial as an isocrystal, i.e. it is generated by its horizontal sections. In particular, an F-isocrystal is constant if and only if it is constant as an F^n-isocrystal. Now we prove the claim. Since π^* is a tensor functor we only need to see that every sub-object of $\pi^*(W, f)$ is constant. This is trivial if the curve has a rational point, because we can pull back to this point and use that this functor is fully faithful. The general case follows by applying the previous case to F^n-isocrystals, and then use that the image of the sub-object actually must be invariant under F, too, by its uniqueness.

(d) The tensor equivalences follow from (a) and Definition 4.7. The last assertion follows from Lemma 4.2.

(b) The isomorphy of monodromy groups follows from (a). Since the category of F-isocrystals over an $\mathbb{F}_q$-linear point is just the category of linear representations of $\mathbb{Z}$, the second claim is an immediate consequence of the Tannakian formalism. Since the monodromy group is the Zariski-closure of $f^{\mathbb{Z}}$, it is commutative and is the direct product of its unipotent radical by the subgroup consisting of its semi-simple elements, see [Bor91, I.4.7 Theorem]. The latter is an extension of a finite étale abelian group scheme over K_e by a torus, see [Bor91, III.8.12 Proposition], and the unipotent radical is isomorphic to $\mathbb{G}^{\varepsilon}_{a,K_e}$ by [Ser88, Chapter VIII, § 2.7, Corollary]. Since the image of f must be Zariski-dense in the quotient $\mathbb{G}^{\varepsilon}_{a,K_e}$, we have $\varepsilon = 0$ or $\varepsilon = 1$.

(c) follows from the definition of $\operatorname{Frob}_x(\mathcal{C})$ from (before) Definition 3.1. The set $\operatorname{Frob}_x(\mathcal{C})$ consists of only one element $f^{\deg(x)}$ here, because this element is K-rational and $\operatorname{Gr}(\mathcal{C}/U, u)$ is commutative. □

Definition 4.9. For every convergent F-isocrystal $\mathcal{F} \in F\text{-Isoc}_K(U)$ let $\langle\langle\mathcal{F}\rangle\rangle_{const}$ be the full Tannakian sub-category of constant F-isocrystals in $\langle\langle\mathcal{F}\rangle\rangle$ and let $\mathbf{W}(\mathcal{F}, u)$ denote the fundamental group of $\langle\langle\mathcal{F}\rangle\rangle_{const}$ with respect to the fiber functor ω_u. Note that $\mathbf{W}(\mathcal{F}, u)$ is a quotient of the monodromy group $\pi_1^{F\text{-Isoc}}(\mathbb{F}_q, u) \times_K K_e = \pi_1^{F\text{-Const}}(U, u) \times_K K_e = \mathbb{Z}^{K_e\text{-alg}}$. Let $\beta\colon \operatorname{Gr}(\mathcal{F}/U, u) \twoheadrightarrow \mathbf{W}(\mathcal{F}, u)$ be the homomorphism induced by the inclusion $\langle\langle\mathcal{F}\rangle\rangle_{const} \subset \langle\langle\mathcal{F}\rangle\rangle$; see Corollary A.16(a). We call the kernel

$$\operatorname{Gr}(\mathcal{F}/U, u)^{geo} \;:=\; \ker\bigl(\operatorname{Gr}(\mathcal{F}/U, u) \twoheadrightarrow \mathbf{W}(\mathcal{F}, u)\bigr)$$

the *geometric monodromy group* of $\mathcal{F}$. This terminology is motivated by Corollary 5.8 below.

Remark 4.10. Let $\operatorname{Isoc}_K(U)$ be the category of K-linear convergent isocrystals on U. If $u \in U(\mathbb{F}_q)$ R. Crew [Cre92] has defined and studied the monodromy group $\operatorname{DGal}(\mathcal{F}, u)$ and the Weil group $W^{\mathcal{F}}(U/K, u)$ of any convergent isocrystal $\mathcal{F} \in \operatorname{Isoc}_K(U)$. The former is a linear algebraic group over K defined as the monodromy group of the neutral Tannakian category generated by $\mathcal{F}$ in $\operatorname{Isoc}_K(U)$ with respect to the fiber functor ω_u. The latter is the semi-direct product of $\mathbb{Z}$ with the former, where $1 \in \mathbb{Z}$ operates on $\operatorname{DGal}(\mathcal{F}, u)$ by conjugation with the Frobenius $u^*F_{\mathcal{F}}$. It is natural to expect that in our setting $\operatorname{DGal}(\mathcal{F}, u)$ plays the role of the geometric monodromy group from Definition 4.9. However, we are only able to prove this in the semi-simple case; see Proposition 4.11 below.

For every convergent F-isocrystal $\mathcal{F} \in F\text{-Isoc}_K(U)$ let $\mathcal{F}^\sim$ denote the underlying convergent isocrystal and let $\langle\!\langle \mathcal{F}^\sim \rangle\!\rangle$ denote the tannakian sub-category generated by $\mathcal{F}^\sim$ in $\text{Isoc}_K(U)$. Let $\alpha\colon \text{DGal}(\mathcal{F},u) \to \text{Gr}(\mathcal{F},u)$ be the homomorphism induced by the forgetful functor $(\,.\,)^\sim\colon \langle\!\langle \mathcal{F} \rangle\!\rangle \to \langle\!\langle \mathcal{F}^\sim \rangle\!\rangle$. It is a closed immersion by Proposition A.14(b).

Proposition 4.11. *Assume that $\mathcal{F}^\sim$ is semi-simple. Then there is a canonical diagram with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & \text{DGal}(\mathcal{F},u) \arrow[r] \arrow[d, equal] & W^{\mathcal{F}}(U/K,u) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \text{DGal}(\mathcal{F},u) \arrow[r, "\alpha"] & \text{Gr}(\mathcal{F}/U,u) \arrow[r, "\beta"] & \mathbf{W}(\mathcal{F},u) \arrow[r] & 0\,, \end{tikzcd}$$

where $\mathbf{W}(\mathcal{F},u)$ was defined in Definition 4.9. In particular, $\text{DGal}(\mathcal{F},u)$ *is canonically isomorphic to the geometric monodromy group* $\text{Gr}(\mathcal{F}/U,u)^{geo}$.

Proof. The upper sequence is exact by definition of the group $W^{\mathcal{F}}(U/K,u)$. We next prove the exactness of the lower sequence. Since $\langle\!\langle \mathcal{F} \rangle\!\rangle_{const}$ is a sub-category of $\langle\!\langle \mathcal{F} \rangle\!\rangle$ the map β is surjective and faithfully flat. Therefore, by [EHS07, Theorem A.1] we only have to check the following:

(i) For an object $\mathcal{G}$ of $\langle\!\langle \mathcal{F} \rangle\!\rangle$ the object $\mathcal{G}^\sim$ of $\langle\!\langle \mathcal{F}^\sim \rangle\!\rangle$ is trivial if and only if $\mathcal{G}$ is an object of $\langle\!\langle \mathcal{F} \rangle\!\rangle_{const}$.
(ii) Let $\mathcal{G}$ be an object of $\langle\!\langle \mathcal{F} \rangle\!\rangle$, and let $\mathcal{H}_0 \subset \mathcal{G}^\sim$ denote the largest trivial subobject. Then there exists $\mathcal{H} \subset \mathcal{G}$ with $\mathcal{H}_0 = \mathcal{H}^\sim$.
(iii) Every object $\mathcal{G}$ of $\langle\!\langle \mathcal{F}^\sim \rangle\!\rangle$ is a subobject of an object of the form $\mathcal{H}^\sim$ for some object $\mathcal{H}$ of $\langle\!\langle \mathcal{F} \rangle\!\rangle$.

Condition (i) trivially holds: an F-isocrystal is constant if and only if it is trivial as an isocrystal. Next we show (ii). The maximal trivial convergent sub-isocrystal $\mathcal{H}_0$ of a convergent F-isocrystal $\mathcal{G}$ is generated by horizontal sections of $\mathcal{G}$. Since the Frobenius $F_{\mathcal{G}}$ respects horizontal sections, the isocrystal $\mathcal{H}_0$ underlies a convergent F-isocrystal. Finally we prove (iii). Because the image of $\langle\!\langle F \rangle\!\rangle$ under $(\,.\,)^\sim$ is closed under direct sums, tensor products and duals, there is an object $\mathcal{H}$ of $\langle\!\langle F \rangle\!\rangle$ such that $\mathcal{G}$ is a subquotient of $\mathcal{H}^\sim$. Since $\mathcal{F}^\sim$ is semi-simple, so is every object in $\langle\!\langle F^\sim \rangle\!\rangle$. Therefore, $\mathcal{G}$ is isomorphic to a subobject of $\mathcal{H}^\sim$.

To prove the commutativity of the diagram we consider the morphism $\mathbb{Z} \to \text{Gr}(\mathcal{F}/U,u)$ which sends $1 \in \mathbb{Z}$ to the element $u^*F_{\mathcal{F}} \in \text{Gr}(\mathcal{F}/U,u)(K)$. This extends to a morphism $W^{\mathcal{F}}(U/K,u) \to \text{Gr}(\mathcal{F}/U,u)$ because $W^{\mathcal{F}}(U/K,u) = \text{DGal}(\mathcal{F},u) \rtimes \mathbb{Z}$ is defined as the semi-direct product where $1 \in \mathbb{Z}$ operates on $\text{DGal}(\mathcal{F},u)$ by conjugation with $u^*F_{\mathcal{F}}$ inside $\text{Gr}(\mathcal{F}/U,u)$; see [Cre92, § 5]. □

5. Unit-Root F-Isocrystals

We begin our discussion of unit root F-isocrystal with the following useful criterion.

Lemma 5.1. *Let $\mathcal{F}$ be a convergent F-isocrystal with finite monodromy group. Then $\mathcal{F}$ is a unit-root F-isocrystal.*

Proof. Let $N \in \mathbb{N}$ be the order of the group $\text{Gr}(\mathcal{F},u)$. Then $(\text{Frob}_x)^N = 1$ for every $x \in |U|$. This implies that the Newton polygon of Frob_x has slope zero for all x, that is $\mathcal{F}$ is unit-root. □

To recall Crew's result on unit-root F-isocrystals, fix a geometric base point $\bar{u} \in U(\overline{\mathbb{F}}_q)$ above the base point $u \in U(\mathbb{F}_{q^e})$ and let $\pi_1^{\text{ét}}(U,\bar{u})$ be the étale fundamental group. Let $K^{\text{un}} = \bigcup_n K_n$ be the maximal unramified extension of K (and K_e) in $\overline{K}$ and let $\widehat{K}^{\text{un}}$ be its p-adic completion. Let $F\text{-UR}_K(U) \subset F\text{-Isoc}_K(U)$ be the Tannakian sub-category of convergent unit-root F-isocrystals on U. It is tensor equivalent to the category of K_e-rational representations of the K_e/K-groupoid $\text{Aut}_K^\otimes(\omega_u)$. Let $\pi_1^{F\text{-UR}}(U,u)^\Delta$ be the kernel group of the K_e/K-groupoid $\pi_1^{F\text{-UR}}(U,u) := \text{Aut}_K^\otimes(\omega_u)$, see Definition A.5 and Theorem A.11. That is, $\pi_1^{F\text{-UR}}(U,u)^\Delta$ equals the affine group scheme over K_e of tensor automorphisms of the fiber functor ω_u on $F\text{-UR}_K(U)$.

Proposition 5.2. *The category $F\text{-UR}_K(U)$ is canonically tensor equivalent to the category $\text{Rep}_K^{\text{c}}\,\pi_1^{\text{ét}}(U,\bar{u})$, such that the fiber functor ω_u on $F\text{-UR}_K(U)$ and the forgetful fiber functor ω_f on $\text{Rep}_K^{\text{c}}\,\pi_1^{\text{ét}}(U,\bar{u})$ become canonically isomorphic over $\widehat{K}^{\text{un}}$. In particular, $\pi_1^{F\text{-UR}}(U,u)^\Delta \times_{K_e} \widehat{K}^{\text{un}}$ is canonically isomorphic to the base-change to $\widehat{K}^{\text{un}}$ of the K-linear algebraic envelope of the topological group $\pi_1^{\text{ét}}(U,\bar{u})$.*

Proof. The tensor equivalence of categories was established by Crew [Cre87, Theorem 2.1 and Remark 2.2.4]. As this equivalence is natural (see loc. cit.), it commutes with the pull-back to the base point $u \in U(\mathbb{F}_{q^e})$. We explicitly describe the tensor equivalence at u between $\mathrm{Rep}^{\mathrm{c}}_K \pi_1^{\text{ét}}(u,\bar{u})$ and $F\text{-UR}_K(\mathbb{F}_{q^e})$; see [Cre87, p. 119]. The objects in the latter category are pairs $(\mathcal{F}, F_{\mathcal{F}})$ consisting of a K_e-vector space $\mathcal{F}$ and an F-semi linear automorphism $F_{\mathcal{F}}$ of $\mathcal{F}$. Also $\pi_1^{\text{ét}}(u,\bar{u}) = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^e}) \cong \widehat{\mathbb{Z}}$. The tensor equivalence associates a Galois representation $\rho\colon \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^e}) \to \mathrm{Aut}_K(W)$ with a unit-root F-isocrystal $(\mathcal{F}, F_{\mathcal{F}})$ in such a way that there is a canonical Galois and F-equivariant isomorphism

$$\alpha\colon W \otimes_K \widehat{K}^{\mathrm{un}} \xrightarrow{\ \sim\ } \mathcal{F} \otimes_{K_e} \widehat{K}^{\mathrm{un}} \tag{5.1}$$

where $\gamma \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^e}) = \mathrm{Gal}(K^{\mathrm{un}}/K_e) = \mathrm{Aut}^{\mathrm{cont}}_{K_e}(\widehat{K}^{\mathrm{un}})$ acts on the left hand side as $\rho(\gamma)\otimes\gamma$ and on the right hand side as $\mathrm{id}_{\mathcal{F}} \otimes \gamma$, and where Frobenius F acts on the left hand side as $\mathrm{id}_W \otimes F$ and on the right hand side as $F_{\mathcal{F}} \otimes F$. The isomorphism α allows to recover $(\mathcal{F}, F_{\mathcal{F}})$ as $(W \otimes_K \widehat{K}^{\mathrm{un}}, \mathrm{id}\otimes F)^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^e})}$ and W as $(\mathcal{F}\otimes_{K_e} \widehat{K}^{\mathrm{un}})^{F=\mathrm{id}}$, and yields a canonical isomorphism of fiber functors $\alpha\colon \omega_f \otimes_K \widehat{K}^{\mathrm{un}} \xrightarrow{\sim} \omega_u \otimes_{K_e} \widehat{K}^{\mathrm{un}}$. The latter induces an isomorphism of $\widehat{K}^{\mathrm{un}}$-group schemes $\alpha_*\colon \mathrm{Aut}^{\otimes}(\omega_f) \times_K \widehat{K}^{\mathrm{un}} \xrightarrow{\sim} \pi_1^{F\text{-UR}}(U,u)^{\Delta} \times_{K_e} \widehat{K}^{\mathrm{un}}$ and so the last statement follows directly from Lemma 4.2. □

Remark 5.3. We can compute the difference between ω_u and $\omega_f \otimes_K K_e$, that is the torsor $\mathrm{Isom}^{\otimes}(\omega_f \otimes_K K_e, \omega_u)$ and the corresponding cohomology class in $\check{\mathrm{H}}^1\big((\mathrm{Spec}\, K_e)_{fpqc}, \mathrm{Aut}^{\otimes}(\omega_f)\big)$; see [DM82, Theorem 3.2]. It is given by the 1-cocycle $h := pr_2^*\alpha^{-1} \circ pr_1^*\alpha \in \mathrm{Aut}^{\otimes}(\omega_f)(\widehat{K}^{\mathrm{un}} \otimes_{K_e} \widehat{K}^{\mathrm{un}})$ where $pr_i^*\colon \widehat{K}^{\mathrm{un}} \to \widehat{K}^{\mathrm{un}} \otimes_{K_e} \widehat{K}^{\mathrm{un}}$ is the inclusion into the i-th factor. The image $(g_\gamma)_\gamma \in \prod_{\gamma\in\mathrm{Gal}(K^{\mathrm{un}}/K_e)} \mathrm{Aut}^{\otimes}(\omega_f)(\widehat{K}^{\mathrm{un}})$ of h under the morphism $\widehat{K}^{\mathrm{un}} \otimes_{K_e} \widehat{K}^{\mathrm{un}} \to \prod_{\gamma\in\mathrm{Gal}(K^{\mathrm{un}}/K_e)} \widehat{K}^{\mathrm{un}}$, $x\otimes y \mapsto (\gamma(x)y)_\gamma$ is given by

$$g_\gamma \;=\; \alpha^{-1}\circ\gamma^*\alpha \;=\; \alpha^{-1}\circ\big((\mathrm{id}_{\mathcal{F}}\otimes\gamma)\circ\alpha\circ(\mathrm{id}_W\otimes\gamma)^{-1}\big) \;=\; \rho^{\mathrm{univ}}(u_*\gamma)\otimes 1_{\widehat{K}^{\mathrm{un}}}\,,$$

where we identify $\mathrm{Gal}(K^{\mathrm{un}}/K_e) = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^e}) = \pi_1^{\text{ét}}(u,\bar{u})$, where $u_*\colon \pi_1^{\text{ét}}(u,\bar{u}) \hookrightarrow \pi_1^{\text{ét}}(U,\bar{u})$ is the natural inclusion, and where $\rho^{\mathrm{univ}}\colon \pi_1^{\text{ét}}(U,\bar{u}) \to \mathrm{Aut}^{\otimes}(\omega_f)(K)$ is the homomorphism corresponding to the fact that every element of $\pi_1^{\text{ét}}(U,\bar{u})$ acts as a tensor automorphism of ω_f defined over K.

Note that although $\big(\rho^{\mathrm{univ}}(u_*\gamma)\big)_\gamma \in \prod_{\gamma\in\mathrm{Gal}(K^{\mathrm{un}}/K_e)} \mathrm{Aut}^{\otimes}(\omega_f)(K)$ it does in general not lie in the image of the homomorphism $K^{\mathrm{un}} \otimes_{K_e} K^{\mathrm{un}} \hookrightarrow \prod_{\gamma\in\mathrm{Gal}(K^{\mathrm{un}}/K_e)} K^{\mathrm{un}}$, $x\otimes y\mapsto(\gamma(x)y)_\gamma$ which equals the union of $\prod_{\gamma\in\mathrm{Gal}(L/K_e)} L$ embedded diagonally into $\prod_{\gamma\in\mathrm{Gal}(K^{\mathrm{un}}/K_e)} K^{\mathrm{un}}$ over all finite Galois extensions $L\subset K^{\mathrm{un}}$ of K_e. Namely $\big(\rho^{\mathrm{univ}}(u_*\gamma)\big)_\gamma$ lies in $\prod_{\gamma\in\mathrm{Gal}(L/K_e)} \mathrm{Aut}^{\otimes}(\omega_f)(L)$ if and only if $\rho^{\mathrm{univ}}(u_*\gamma)=1$ for all $\gamma\in\mathrm{Gal}(K^{\mathrm{un}}/L)$.

To formulate the consequence for the individual monodromy groups $\mathrm{Gr}(\mathcal{F},u)$ let $\widehat{\overline{K}}$ be the p-adic completion of $\overline{K}$. Moreover, for each $x\in|U|$ let $\bar{x}$ be a geometric base point of U lying above x and choose an isomorphism of groups $\pi_1^{\text{ét}}(U,\bar{x}) \xrightarrow{\sim} \pi_1^{\text{ét}}(U,\bar{u})$. It is unique up to conjugation in $\pi_1^{\text{ét}}(U,\bar{u})$. Let $\mathrm{Frob}_x^{-1} \in \mathrm{Gal}(\overline{\mathbb{F}}_x/\mathbb{F}_x) = \pi_1^{\text{ét}}(x,\bar{x})$ be the *geometric Frobenius* which maps $a\in\overline{\mathbb{F}}_x$ to a^{1/q_x}, where $q_x = \#\mathbb{F}_x$. It is the inverse of the *arithmetic Frobenius* $\mathrm{Frob}_x\colon a\mapsto a^{q_x}$. Then the conjugacy class of $x_*\,\mathrm{Frob}_x^{-1}$ in $\pi_1^{\text{ét}}(U,\bar{u})$ is well defined.

Corollary 5.4. *Let $\mathcal{F}$ be a convergent unit-root F-isocrystal on U and let $\rho\colon \pi_1^{\text{ét}}(U,\bar{u}) \to \mathrm{Aut}_K(W)$ be the representation corresponding to $\mathcal{F}$ under the tensor equivalence from Proposition 5.2. Then the categories $\langle\langle\mathcal{F}\rangle\rangle \subset F\text{-UR}_K(U)$ and $\langle\langle\rho\rangle\rangle \subset \mathrm{Rep}^{\mathrm{c}}_K \pi_1^{\text{ét}}(U,\bar{u})$ are tensor equivalent and there is a finite field extension L of K_e and an isomorphism $\beta\colon \omega_f|_{\langle\langle\rho\rangle\rangle}\otimes_K L \xrightarrow{\sim} \omega_u|_{\langle\langle\mathcal{F}\rangle\rangle}\otimes_{K_e} L$ of tensor functors on these categories. In particular $\mathrm{Gr}(\mathcal{F},u)\times_{K_e} L$ is the Zariski-closure of the image of $\beta_*\circ\rho\colon \pi_1^{\text{ét}}(U,\bar{u}) \to \mathrm{Aut}_{K_e}(u^*\mathcal{F})(L)$ and for all points $x\in|U|$ the $\mathrm{Gr}(\mathcal{F},u)(\overline{K})$-conjugacy classes of $\beta_*\circ\rho(x_*\,\mathrm{Frob}_x^{-1})$ and $\mathrm{Frob}_x(\mathcal{F})$ coincide.*

Remark 5.5. The field L and the isomorphism β are not canonical. We do not know whether one can find such a field L contained in K^{un}. If L and β are replaced by L' and β' then $h := \beta'\circ\beta^{-1} \in \mathrm{Gr}(\mathcal{F}/U,u)(LL')$ and so the map $\beta_*\circ\rho\colon \pi_1^{\text{ét}}(U,\bar{u}) \to \mathrm{Aut}_{K_e}(u^*\mathcal{F})(LL')$ is only canonical up to conjugation by h.

Proof of Corollary 5.4. By Proposition 5.2 the categories $\langle\langle\mathcal{F}\rangle\rangle$ and $\langle\langle\rho\rangle\rangle$ are tensor equivalent. Since $\mathrm{Aut}^{\otimes}(\omega_f|_{\langle\langle\rho\rangle\rangle})$ is a closed subgroup of $\mathrm{Aut}_K(\omega_f(\rho))$ the $\mathrm{Aut}^{\otimes}(\omega_f|_{\langle\langle\rho\rangle\rangle})$-torsor $\mathrm{Isom}^{\otimes}(\omega_f|_{\langle\langle\rho\rangle\rangle}\otimes_K K_e,\, \omega_u|_{\langle\langle\mathcal{F}\rangle\rangle})$ is a scheme of finite type over K_e by [EGA, IV$_2$, Proposition 2.7.1] and therefore has a point over a finite field extension L of K_e. This point defines the isomorphism $\beta\colon \omega_f|_{\langle\langle\rho\rangle\rangle}\otimes_K L \xrightarrow{\sim} \omega_u|_{\langle\langle\mathcal{F}\rangle\rangle}\otimes_{K_e} L$ of tensor functors and an isomorphism of algebraic groups $\beta_*\colon \operatorname{Aut}^{\otimes}(\omega_f|_{\langle\langle\rho\rangle\rangle}) \times_K L \xrightarrow{\sim} \operatorname{Gr}(\mathcal{F}/U,u) \times_{K_e} L$, $g \mapsto \beta \circ g \circ \beta^{-1}$. So the statement about the latter group follows from Lemma 4.2.

To prove the equality of conjugacy classes note that the tensor isomorphism α from (5.1) satisfies $\alpha \circ \big(\rho(u_* \operatorname{Frob}_u^{-1}) \otimes \operatorname{id}_{\widehat{K}^{\mathrm{un}}}\big) \circ \alpha^{-1} = u^* F_{\mathcal{F}}$. If $h := \beta \circ \alpha^{-1} \in \operatorname{Gr}(\mathcal{F},u)(\widehat{\overline{K}})$ then $\beta_* \circ \rho(u_* \operatorname{Frob}_u^{-1}) \cdot h = h \cdot \alpha \circ \big(\rho(u_* \operatorname{Frob}_u^{-1}) \otimes \operatorname{id}_{\widehat{K}^{\mathrm{un}}}\big) \circ \alpha^{-1} = h \cdot u^* F_{\mathcal{F}}$. Since $\beta_* \circ \rho(u_* \operatorname{Frob}_u^{-1})$ and $u^* F_{\mathcal{F}}$ lie in $\operatorname{Gr}(\mathcal{F},u)(\overline{K})$ this is an equation for h with coefficients in $\overline{K}$ which has a solution in $\widehat{\overline{K}}$. By Hilbert's Nullstellensatz it thus has a solution $h \in \operatorname{Gr}(\mathcal{F},u)(\overline{K})$, too. This proves that the $\operatorname{Gr}(\mathcal{F},u)(\overline{K})$-conjugacy classes of $\beta_* \circ \rho(u_* \operatorname{Frob}_u^{-1})$ and $u^* F_{\mathcal{F}}$ coincide for $x = u$. The equality for general x follows from this by replacing u by x and arguing as above. $\square$

Remark 5.6. Note that Conjecture 1.2 for convergent unit-root F-isocrystals on U, which we proved in Proposition 1.7, is considerably weaker than the classical Chebotarëv density theorem for U to the same extent as the pro-finite topology on $\pi_1^{\text{ét}}(U,\bar{u})$ is finer than the Zariski topology on in its K_e-linear algebraic envelope. Namely, the classical Chebotarëv density theorem says that the Frobenii of a set S of Dirichlet density 1 are dense in $\pi_1^{\text{ét}}(U,\bar{u})$ for the pro-finite topology, see [Ser63, Theorem 7]. If $\mathcal{F}$ is a unit-root F-isocrystal the representation $\pi_1^{\text{ét}}(U,\bar{u}) \to \operatorname{Gr}(\mathcal{F},u)(L)$ corresponding to $\mathcal{F}$ by Corollary 5.4, where L is a finite extension of K_e, is continuous for the p-adic topology. So the Frobenii lie p-adically dense in the image of this representation, but this image itself is not p-adically dense in $\operatorname{Gr}(\mathcal{F},u)(L)$, since it is closed, but not the whole group in general. This image is only Zariski-dense by Corollary 5.4. Therefore, the stronger assertion that the set $\bigcup_{x\in S} \operatorname{Frob}_x(\mathcal{F})$ is p-adically dense in $\operatorname{Gr}(\mathcal{F},u)(\overline{K})$ is false in general. So it is unreasonable to expect a density statement for any topology other than the Zariski topology, even in the most simple case of constant F-isocrystals. This can be seen from the following

Example 5.7. Let $\mathcal{C}$ be the pullback to U of the F-isocrystal on $\mathbb{F}_q$ of rank 1 given by $(K, F = \pi^s)$ with $s \in \mathbb{Z}$. If $s \neq 0$ then $\operatorname{Gr}(\mathcal{F},u) = \mathbb{G}_{m,K_e}$. Indeed, $\operatorname{Gr}(\mathcal{F},u)$ is a closed subgroup of $\operatorname{Aut}_{K_e}(u^*\mathcal{C}) = \mathbb{G}_{m,K_e}$ which contains $\operatorname{Frob}_u(\mathcal{C}) = \{\pi^{es}\}$. Since the set $\pi^{\mathbb{Z}es}$ is infinite, the only such group is $\mathbb{G}_{m,K_e}$. However, the set $\bigcup_{x\in U} \operatorname{Frob}_x(\mathcal{F}) \subset \pi^{\mathbb{Z}es}$ is discrete in $\mathbb{G}_m(K_e)$ for the p-adic topology.

To formulate further corollaries, recall that the geometric fundamental group $\pi_1^{\text{ét}}(U,\bar{u})^{geo}$ is defined as the étale fundamental group $\pi_1^{\text{ét}}(U \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \bar{u})$ of $U \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. It sits in Grothendieck's fundamental exact sequence [SGA 1, IX, Théorème 6.1]

$$1 \longrightarrow \pi_1^{\text{ét}}(U,\bar{u})^{geo} \longrightarrow \pi_1^{\text{ét}}(U,\bar{u}) \longrightarrow \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \longrightarrow 1 \tag{5.2}$$

For the next corollary recall from Definition 4.9 the definition of the geometric monodromy group $\operatorname{Gr}(\mathcal{F}/U,u)^{geo}$ and of $\mathbf{W}(\mathcal{F},u)$ as the monodromy group of $\langle\langle\mathcal{F}\rangle\rangle_{const}$.

Corollary 5.8. *In the situation of Corollary 5.4 the homomorphism $\beta_* \circ \rho$ induces a commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\text{ét}}(U,\bar{u})^{geo} \arrow[r] \arrow[d, "\beta_*\circ\rho"] & \pi_1^{\text{ét}}(U,\bar{u}) \arrow[r] \arrow[d, "\beta_*\circ\rho"] & \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \operatorname{Gr}(\mathcal{F}/U,u)^{geo} \times_{K_e} L \arrow[r] & \operatorname{Gr}(\mathcal{F}/U,u) \times_{K_e} L \arrow[r] & \mathbf{W}(\mathcal{F},u) \times_{K_e} L \arrow[r] & 1
\end{tikzcd} \tag{5.3}$$

with exact rows in which the three vertical maps have Zariski-dense image. In particular, if $\rho^{geo} := \rho|_{\pi_1^{\text{ét}}(U,\bar{u})^{geo}}$ is the restriction of ρ, then β_* induces an isomorphism

$$\operatorname{Aut}^{\otimes}(\omega_f|_{\langle\langle\rho^{geo}\rangle\rangle}) \times_K L \xrightarrow{\sim} \operatorname{Gr}(\mathcal{F}/U,u)^{geo} \times_{K_e} L\,.$$

Proof. The category $\langle\langle\mathcal{F}\rangle\rangle_{const}$ has a tensor generator $\mathcal{C}$. Let $(K^{\oplus r}, f) \in F\text{-}\mathrm{Isoc}_K(\operatorname{Spec}\mathbb{F}_q)$ be an F-isocrystal on $\operatorname{Spec}\mathbb{F}_q$ such that $\mathcal{C}$ is the pullback of $(K^{\oplus r}, f)$ under the structure morphism $U \to \operatorname{Spec}\mathbb{F}_q$. Then $\operatorname{Gr}(\mathcal{C}/U,u) = \operatorname{Gr}((K^{\oplus r}, f)/\operatorname{Spec}\mathbb{F}_q, u)$ equals the Zariski-closure of $f^{\mathbb{Z}}$ in $\operatorname{GL}_{r,K_e}$ by Theorem 4.8(b). Since $\mathcal{F} \in F\text{-}\mathrm{Isoc}_K(U)$ is unit-root, also $\mathcal{C}$ is unit-root, and possibly after a change of basis, we can assume that $f \in \operatorname{GL}_r(\mathcal{O}_K)$, where $\mathcal{O}_K$ denotes the valuation ring of K. Since the group $\operatorname{GL}_r(\mathcal{O}_K)$ is pro-finite, the morphism $\mathbb{Z} \to \operatorname{GL}_r(\mathcal{O}_K)$, $n \mapsto f^n$ extends uniquely to a morphism $\rho_{const}\colon \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) = \widehat{\mathbb{Z}} \cdot \operatorname{Frob}_q \to \operatorname{GL}_r(\mathcal{O}_K)$, $\operatorname{Frob}_q \mapsto f$. Conversely, if $\mathcal{G} \in \langle\langle\mathcal{F}\rangle\rangle$ is an F-isocrystal on which the representation $\rho\colon \pi_1^{\text{ét}}(U,\bar{u}) \to \operatorname{Aut}(\omega_u(\mathcal{G}) \otimes_{K_e} L)$ factors through $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, then $\mathcal{G}$ is constant and belongs to $\langle\langle\mathcal{F}\rangle\rangle_{const}$

by Crew's result [Cre87, Theorem 2.1 and Remark 2.2.4] for $\operatorname{Spec}\mathbb{F}_q$. Therefore, the first two assertions follow from Corollary 4.6. □

Remark 5.9. We do not know, whether the image $\beta_*\circ\rho\big(\pi_1^{\text{ét}}(U,\bar u)^{geo}\big)$ equals the intersection of $\beta_*\circ\rho\big(\pi_1^{\text{ét}}(U,\bar u)\big)$ with $\mathrm{Gr}(\mathcal{F},u)^{geo}(L)$. To prove this, one needs to find a faithful representation ρ' of the group $\rho\big(\pi_1^{\text{ét}}(U,\bar u)\big)/\rho\big(\pi_1^{\text{ét}}(U,\bar u)^{geo}\big)$ on a finite dimensional K-vector space which belongs to the Tannakian subcategory $\langle\!\langle\rho\rangle\!\rangle\subset\mathrm{Rep}_K^{\mathrm{c}}\,\pi_1^{\text{ét}}(U,\bar u)$. Nevertheless, we can prove the following

Corollary 5.10. *In the situation of Corollary 5.8 there is a constant unit-root F-isocrystal $\mathcal{F}'$ on U, corresponding to a representation $\rho'\colon\pi_1^{\text{ét}}(U,\bar u)\twoheadrightarrow\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)\to\mathrm{GL}_r(K)$ for $r\in\mathbb{N}$, whose image $\mathrm{im}(\rho')$ is isomorphic to $\rho\big(\pi_1^{\text{ét}}(U,\bar u)\big)/\rho\big(\pi_1^{\text{ét}}(U,\bar u)^{geo}\big)$. If we replace $\mathcal{F}$ by $\mathcal{F}\oplus\mathcal{F}'$ and ρ by $\rho\oplus\rho'$, then in diagram* (5.3) *we have*

$$\beta_*\circ(\rho\oplus\rho')\big(\pi_1^{\text{ét}}(U,\bar u)^{geo}\big)\;=\;\beta_*\circ(\rho\oplus\rho')\big(\pi_1^{\text{ét}}(U,\bar u)\big)\cap\mathrm{Gr}(\mathcal{F}\oplus\mathcal{F}',u)^{geo}(L)\,.$$

Proof. We first construct ρ'. By Cartan's theorem, see [Ser92, Part II, § V.9, Corollary to Theorem 1 on page 155] or Theorem 7.1 below, the images $C:=\rho\big(\pi_1^{\text{ét}}(U,\bar u)\big)$ and $C^{geo}:=\rho\big(\pi_1^{\text{ét}}(U,\bar u)^{geo}\big)$ are Lie groups over $\mathbb{Q}_p$, and the quotient $C/C^{geo}=\rho\big(\pi_1^{\text{ét}}(U,\bar u)\big)/\rho\big(\pi_1^{\text{ét}}(U,\bar u)^{geo}\big)$ is again a Lie group over $\mathbb{Q}_p$ by [Ser92, Part II, § IV.5, Remark 2 after Theorem 1 on page 108]. If the quotient C/C^{geo} is finite, then it has a faithful representation ρ' on a finite dimensional K-vector space. So we now assume that C/C^{geo} is not finite. Note that $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)\cong\widehat{\mathbb{Z}}=\mathbb{Z}_p\times\prod_{\ell\neq p}\mathbb{Z}_\ell$ surjects onto C/C^{geo}. By the incompatibility of the ℓ-adic and the p-adic topologies, the image of $\prod_{\ell\neq p}\mathbb{Z}_\ell$ in C/C^{geo} is a finite subgroup H and thus has a faithful representation on a finite dimensional K-vector space V_1'. On the other hand, the map $\mathbb{Z}_p\hookrightarrow\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)\twoheadrightarrow C/C^{geo}$ is analytic by [Ser92, Part II, § V.9, Theorem 2] and its image is an at most one-dimensional Lie group over $\mathbb{Q}_p$ by [Ser92, Part II, § IV.5, Theorems 1 and 3 and Corollary to Theorem 2]. If it were zero dimensional, then it would be finite because it is compact, and this was excluded. So it is one-dimensional and the map from $\mathbb{Z}_p$ onto its image is a local isomorphism by [Ser92, Part II, § III.9, Theorem 2]. The kernel of this map is finite, and hence trivial, because $\mathbb{Z}_p$ is compact and torsion free. Therefore, we obtain an epimorphism $\varphi\colon\mathbb{Z}_p\times H\twoheadrightarrow C/C^{geo}$, which is even an isomorphism, because if an element (g,h) lies in the kernel, then $\varphi(g)=\varphi(h^{-1})$ is a torsion element of $\varphi(\mathbb{Z}_p)=\mathbb{Z}_p$, and hence trivial. Therefore, $g=1$, and since $\varphi|_H$ is injective also $h=1$. Now take a faithful representation of $\mathbb{Z}_p$ on a finite dimensional K-vector space V_2', for example in a unipotent group. The sum $V_1'\oplus V_2'$ is the desired representation $\rho'\colon\pi_1^{\text{ét}}(U,\bar u)\twoheadrightarrow\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)\to\mathrm{GL}_r(K)$. The convergent F-isocrystal $\mathcal{F}'$ on U corresponding to ρ' is constant by Crew's result [Cre87, Theorem 2.1 and Remark 2.2.4] for $\operatorname{Spec}\mathbb{F}_q$.

To prove the last statement, note that the inclusion "$\subset$" is trivial. To prove the converse inclusion "$\supset$" let $\widetilde C:=(\rho\oplus\rho')\big(\pi_1^{\text{ét}}(U,\bar u)\big)$ and $\widetilde C^{geo}:=(\rho\oplus\rho')\big(\pi_1^{\text{ét}}(U,\bar u)^{geo}\big)$. By construction of ρ' we have isomorphisms $C\xrightarrow{\sim}\widetilde C$ and $C^{geo}\xrightarrow{\sim}\widetilde C^{geo}$ given by $c\mapsto(c,c\bmod C^{geo})$. Therefore, every element $\beta_*(c,c\bmod C^{geo})\in\beta_*(\widetilde C)=\beta_*\circ(\rho\oplus\rho')\big(\pi_1^{\text{ét}}(U,\bar u)\big)$ which lies in the kernel $\mathrm{Gr}(\mathcal{F}\oplus\mathcal{F}',u)^{geo}$ of $\mathrm{Gr}(\mathcal{F}\oplus\mathcal{F}',u)\twoheadrightarrow\mathbf{W}(\mathcal{F}\oplus\mathcal{F}',u)$ is mapped to 1 in the quotient $\mathrm{Gr}(\mathcal{F}',u)=\mathrm{Aut}^{\otimes}(\omega_f|_{\langle\!\langle\rho'\rangle\!\rangle})$ of $\mathbf{W}(\mathcal{F}\oplus\mathcal{F}',u)$. This implies $\rho'(c)=1$ and $c\in C^{geo}$ as desired. □

Another consequence of Corollary 5.4 is the following

Corollary 5.11. *Let $\mathcal{F}\in F\text{-}\mathrm{UR}_K(U)$ be a convergent unit-root F-isocrystal on U and let $f\colon V\hookrightarrow U$ be a non-empty open sub-curve. Then the pullback functor $f^*\colon\mathcal{G}\mapsto f^*\mathcal{G}$ from $\langle\!\langle\mathcal{F}\rangle\!\rangle$ to $\langle\!\langle f^*\mathcal{F}\rangle\!\rangle$ is a tensor equivalence of Tannakian categories. In particular, if $u\in V(\mathbb{F}_{q^e})$ is a base point, the induced morphism of monodromy groups $\mathrm{Gr}(f^*\mathcal{F}/V,u)\to\mathrm{Gr}(\mathcal{F}/U,u)$ is an isomorphism.*

Proof. We use that $\pi_1^{\text{ét}}(f)\colon\pi_1^{\text{ét}}(U,\bar u)\twoheadrightarrow\pi_1^{\text{ét}}(V,\bar u)$ is an epimorphism by [SGA 1, V, Proposition 8.2]. Let $\rho\colon\pi_1^{\text{ét}}(U,\bar u)\to\mathrm{Aut}_K(W)$ be the representation corresponding to $\mathcal{F}$ under the tensor equivalence from Proposition 5.2. Then $f^*\rho:=\rho\circ\pi_1^{\text{ét}}(f)\colon\pi_1^{\text{ét}}(V,\bar u)\to\mathrm{Aut}_K(W)$ is the representation corresponding to $f^*\mathcal{F}$. Therefore, the functor f^* on $\langle\!\langle\mathcal{F}\rangle\!\rangle$ is fully faithful. Moreover, every subobject of $\langle\!\langle f^*\rho\rangle\!\rangle$ is of the form $\rho'\colon\pi_1^{\text{ét}}(V,\bar u)\to\mathrm{Aut}_K(W')$ for an invariant subspace $W'\subset W$. By the surjectivity of $\pi_1^{\text{ét}}(f)$ this subspace is also $\pi_1^{\text{ét}}(U,\bar u)$-invariant, and hence $\rho'=f^*(\rho_{W'})$ for the subobject $\rho_{W'}\colon\pi_1^{\text{ét}}(U,\bar u)\to\mathrm{Aut}_K(W')$ of ρ. Since f^* clearly is a tensor functor, the corollary is proven. □

6. Groups of Connected Components

We consider a finite étale Galois-covering $f\colon V\to U$ of smooth, geometrically irreducible curves and the pull back functor

$$f^*\colon\; F\text{-Isoc}_K(U)\longrightarrow F\text{-Isoc}_K(V)\,,\quad \mathcal{F}\mapsto f^*\mathcal{F}\,; \tag{6.1}$$

see for example [Cre92, p. 431]. In addition recall the functor $(\,.\,)^{(n)}$ from (3.1). Both functors possess right adjoints

$$\begin{aligned} f_*\,:\;& F\text{-Isoc}_K(V) &&\longrightarrow F\text{-Isoc}_K(U)\,,\quad \mathcal{G}\mapsto f_*\mathcal{G}\qquad\text{and}\\ (\,.\,)_{(n)}\,:\;& F^n\text{-Isoc}_{K_n}(U_n) &&\longrightarrow F\text{-Isoc}_K(U)\,,\quad \mathcal{G}\mapsto \mathcal{G}_{(n)}\,.\end{aligned} \tag{6.2}$$

For f_* see [Cre92, 1.7]. The functor $(\,.\,)_{(n)}$ can explicitly be described as follows. Let $pr\colon U_n\to U$ be the projection, take $\mathcal{G}_{(n)}:=\bigoplus_{i=0}^{n-1}F^{i*}pr_*\mathcal{G}=\bigoplus_{i=0}^{n-1}pr_*F^{i*}\mathcal{G}$ and let the Frobenius $F_{\mathcal{G}_{(n)}}\colon F^*\mathcal{G}_{(n)}\xrightarrow{\sim}\mathcal{G}_{(n)}$ be given by the matrix

$$\begin{pmatrix} 0 & & & pr_*F_{\mathcal{G}}\\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0\end{pmatrix},$$

where $F_{\mathcal{G}}\colon F^{n*}\mathcal{G}\xrightarrow{\sim}\mathcal{G}$ is the Frobenius of $\mathcal{G}$. Fix a normal basis $(b_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ of the field extension K_n/K, that is $K_n=\bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}}K\cdot b_i$ and $F(b_i)=b_{i+1}$ for the Frobenius $F\in\operatorname{Gal}(K_n/K)$. Let $\mathcal{K}$ be the pullback to U of the constant F-isocrystal on $\operatorname{Spec}\mathbb{F}_q$ given by $\bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}}K\cdot e_i$ with Frobenius $F(e_i)=e_{i+1}$. Then for the trivial F^n-isocrystal ${}^n\mathbb{1}_{U_n}$ on U_n one has $F^{i*}({}^n\mathbb{1}_{U_n})={}^n\mathbb{1}_{U_n}$ and

$$({}^n\mathbb{1}_{U_n})_{(n)}\;=\;pr_*(pr^*\mathcal{K})\;=\;\mathcal{K}\otimes_K K_n\;=\;\bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}}\mathcal{K}\cdot b_i\;\cong\;\mathcal{K}\otimes\mathcal{K}\,,$$

where the last isomorphism is given by sending b_i to e_i. The adjunction satisfies the projection formula $(\mathcal{F}^{(n)}\otimes\mathcal{G})_{(n)}\cong\mathcal{F}\otimes\mathcal{G}_{(n)}$, and $(\mathcal{F}^{(n)})_{(n)}\cong\mathcal{F}\otimes({}^n\mathbb{1}_{U_n})_{(n)}$, as well as $(\mathcal{G}_{(n)})^{(n)}=(\bigoplus_{i=0}^{n-1}F^{i*}\mathcal{G})\otimes_K K_n$. In particular, via the counit morphism of the adjunction, $\mathcal{G}$ is a quotient of $(\mathcal{G}_{(n)})^{(n)}$. Now we write $\mathcal{K}^{\oplus n}=(\bigoplus_{i,j}K\cdot e_i\otimes d_j)_U$ with Frobenius $F(e_i\otimes d_j)=e_{i+1}\otimes d_j$. Then there is an isomorphism of F-isocrystals

$$\psi\colon\mathcal{K}\otimes\mathcal{K}\xrightarrow{\sim}\mathcal{K}^{\oplus n},\quad e_i\otimes e_j\mapsto e_i\otimes d_{j-i}\,,\quad e_i\otimes e_{i+j}\leftarrow\!\shortmid e_i\otimes d_j\,. \tag{6.3}$$

Similarly, let $\mathcal{L}:=f_*\mathbb{1}_V$ where $\mathbb{1}_V$ is the trivial F-isocrystal on V. Then $f_*(f^*(\mathcal{F})\otimes\mathcal{G})\cong\mathcal{F}\otimes f_*\mathcal{G}$, and $f_*f^*\mathcal{F}=\mathcal{F}\otimes\mathcal{L}$. And if we set $\Gamma=\operatorname{Gal}(V/U)$ then $f^*f_*\mathcal{G}\cong\bigoplus_{\gamma\in\Gamma}\gamma^*\mathcal{G}$ and $f^*\mathcal{L}\cong\bigoplus_\Gamma\mathbb{1}_V$. So again, via the counit morphism of the adjunction, $\mathcal{G}$ is a quotient of $f^*f_*\mathcal{G}$. The formulas also yield isomorphisms of F-isocrystals $\psi\colon\mathcal{L}\otimes\mathcal{L}\xrightarrow{\sim}\bigoplus_\Gamma\mathcal{L}$ and $\psi'\colon\mathcal{L}\otimes\mathcal{L}^\vee=f_*(f^*\mathcal{L})^\vee\xrightarrow{\sim}f_*\bigoplus_\Gamma\mathbb{1}_V=\bigoplus_\Gamma\mathcal{L}$. Fix a base point $v\in V(\mathbb{F}_{q^e})$ and let $u:=f(v)\in U(\mathbb{F}_{q^e})$. Let $L:=\omega_u(\mathcal{L})$. Since $\omega_u(\mathcal{L})=\omega_v(f^*\mathcal{L})$ and $f^*\mathcal{L}=\bigoplus_{\gamma\in\Gamma}\mathbb{1}_V$, the K_e-vector space L possesses a basis $(e_\gamma)_{\gamma\in\Gamma}$ for which the isomorphisms ψ and ψ' on L have the description

$$\psi\colon\quad L\otimes L\xrightarrow{\sim}\bigoplus_{\delta\in\Gamma}L\cdot d_\delta,\quad e_\gamma\otimes e_\delta\mapsto e_\gamma\cdot d_{\gamma^{-1}\delta}\,,\quad e_\gamma\otimes e_{\gamma\delta}\leftarrow\!\shortmid e_\gamma\cdot d_\delta\,, \tag{6.4}$$

$$\psi'\colon\quad L\otimes L^\vee\xrightarrow{\sim}\bigoplus_{\delta\in\Gamma}L\cdot d'_\delta,\quad e_\gamma\otimes e_\delta^\vee\mapsto e_\gamma\cdot d'_{\gamma^{-1}\delta}\,,\quad e_\gamma\otimes e^\vee_{\gamma\delta}\leftarrow\!\shortmid e_\gamma\cdot d'_\delta\,, \tag{6.5}$$

where $(e_\gamma^\vee)_\gamma$ is the basis of $L^\vee$ which is dual to $(e_\gamma)_\gamma$.

Lemma 6.1. *The monodromy groups satisfy* $\operatorname{Gr}(\mathcal{K}/U,u)\cong\mathbb{Z}/n\mathbb{Z}$ *and* $\operatorname{Gr}(\mathcal{L}/U,u)\cong\operatorname{Gal}(V/U)$. *Moreover,* $\mathcal{K}=pr_*\mathbb{1}_{U_n}$ *for the projection* $pr\colon U_n\to U$ *and the trivial F-isocrystal* $\mathbb{1}_{U_n}$ *on* U_n.

Proof. We prove both assertions simultaneously using the isomorphisms (6.3) and (6.4), which take identical form if we set $\Gamma:=\mathbb{Z}/n\mathbb{Z}$ in (6.3). The isomorphisms in the lemma are explicitly given as follows.

Let R be a K_e-algebra without non-trivial idempotents. If $g\in\operatorname{Gr}(\mathcal{L}/U,u)(R)$ acts on $L\otimes_{K_e}R$ via $g(e_\gamma)=\sum_{\varepsilon\in\Gamma}g_{\varepsilon,\gamma}\cdot e_\varepsilon$ with $g_{\varepsilon,\gamma}\in R$ then it acts on $(L\otimes L)\otimes_{K_e}R$ via $g(e_\gamma\otimes e_{\gamma\delta})=\sum_{\varepsilon,\eta}g_{\varepsilon,\gamma}\cdot g_{\eta,\gamma\delta}\cdot e_\varepsilon\otimes e_\eta$

and on $(\bigoplus_{\delta\in\Gamma} L\cdot d_\delta)\otimes_{K_e} R$ via $g(e_\gamma\cdot d_\delta)=\sum_\varepsilon g_{\varepsilon,\gamma}\cdot e_\varepsilon\cdot d_\delta$. Since these actions have to be compatible with the isomorphism ψ it follows that

$$g_{\varepsilon,\gamma}\cdot g_{\eta,\gamma\delta} \;=\; g_{\varepsilon,\gamma} \quad\text{if}\quad \eta=\varepsilon\delta \qquad\text{and}\qquad g_{\varepsilon,\gamma}\cdot g_{\eta,\gamma\delta} \;=\; 0 \quad\text{if}\quad \eta\neq\varepsilon\delta\,.$$

For $\delta=1$ this implies in particular

$$(g_{\varepsilon,\gamma})^2 \;=\; g_{\varepsilon,\gamma} \qquad\text{and}\qquad g_{\varepsilon,\gamma}\cdot g_{\eta,\gamma} \;=\; 0 \quad\text{if}\quad \eta\neq\varepsilon\,.$$

Thus $g_{\varepsilon,\gamma}=0$ or 1, because these are the only idempotents in R. If $g_{\varepsilon,\gamma}=1$ then also $g_{\varepsilon\delta,\gamma\delta}=1$ for all δ and $g_{\eta\delta,\gamma\delta}=0$ for all $\eta\neq\varepsilon$ and all δ. Therefore, $g(e_\eta)=e_{\lambda\eta}$ for all η, where $\lambda:=\varepsilon\gamma^{-1}$. Mapping this g to $\lambda\in\Gamma$ defines an injective group homomorphism $\alpha\colon \operatorname{Gr}(\mathcal{L}/U,u)(R)\hookrightarrow\Gamma$.

To see that α is surjective we use that $\mathcal{K}$ and $\mathcal{L}$ are convergent unit-root F-isocrystals by Lemma 5.1. Under Crew's equivalence between unit-root F-isocrystals and representations of the fundamental group (Proposition 5.2) the trivial F-isocrystal $\mathbb{1}_V$ and the trivial F^n-isocrystal ${}^n\mathbb{1}_{U_n}$ correspond to the trivial representations $\pi_1^{\text{ét}}(V,\bar v)\to K^\times$ and $\pi_1^{\text{ét}}(U_n,\bar u)\to K_n^\times$. (Here we assume that $u\in U_n(\mathbb{F}_{q^e})$.) Moreover, under this equivalence the functors f^* and $(\,.\,)^{(n)}$ correspond to the functors

$$\begin{aligned}
\text{Res}\colon\quad & \operatorname{Rep}^{\mathrm c}_K\pi_1^{\text{ét}}(U,\bar u)\longrightarrow \operatorname{Rep}^{\mathrm c}_K\pi_1^{\text{ét}}(V,\bar v)\,, \qquad \rho\longmapsto \rho|_{\pi_1^{\text{ét}}(V,\bar v)} \qquad\text{and}\\
\text{Res}\otimes_K K_n\colon\quad & \operatorname{Rep}^{\mathrm c}_K\pi_1^{\text{ét}}(U,\bar u)\longrightarrow \operatorname{Rep}^{\mathrm c}_{K_n}\pi_1^{\text{ét}}(U_n,\bar u)\,, \qquad \rho\longmapsto \big(\rho|_{\pi_1^{\text{ét}}(U_n,\bar u)}\big)\otimes_K K_n\,.
\end{aligned}$$

For an open subgroup H of a compact group G the right and left adjoint to Res^G_H is the induction functor $\operatorname{Ind}^G_H\colon \operatorname{Rep}^{\mathrm c}_K H\to\operatorname{Rep}^{\mathrm c}_K G$ with

$$\operatorname{Ind}^G_H(\rho,W_\rho) \;:=\; \big\{\, r\colon G\to W_\rho \text{ continuous}\colon r(hg)=\rho(h)r(g)\;\forall\, h\in H, g\in G\,\big\}$$

for a continuous representation $\rho\colon H\to\operatorname{Aut}_K(W_\rho)$; see [NSW08, Footnotes on pp. 61 and 63]. Also the restriction of scalars from K_n to K is right adjoint to $\,.\otimes_K K_n$. So the right adjoints f_* and $(\,.\,)_{(n)}$ correspond to the right adjoints

$$\begin{aligned}
\text{Ind}\colon\quad & \operatorname{Rep}^{\mathrm c}_K\pi_1^{\text{ét}}(V,\bar v)\longrightarrow \operatorname{Rep}^{\mathrm c}_K\pi_1^{\text{ét}}(U,\bar u)\,, \qquad \rho\longmapsto \operatorname{Ind}^{\pi_1^{\text{ét}}(U,\bar u)}_{\pi_1^{\text{ét}}(V,\bar v)}\rho \qquad\text{and}\\
\text{Ind}\colon\quad & \operatorname{Rep}^{\mathrm c}_{K_n}\pi_1^{\text{ét}}(U_n,\bar u)\longrightarrow \operatorname{Rep}^{\mathrm c}_K\pi_1^{\text{ét}}(U,\bar u)\,, \qquad \rho\longmapsto \operatorname{Ind}^{\pi_1^{\text{ét}}(U,\bar u)}_{\pi_1^{\text{ét}}(U_n,\bar u)}\rho\,.
\end{aligned}$$

We find that $\mathcal{L}=f_*\mathbb{1}_V$ and $\mathcal{K}^{\oplus n}=({}^n\mathbb{1}_{U_n})_{(n)}$ correspond to the representations

$$\bigoplus_{\operatorname{Gal}(V/U)} K \qquad\text{and}\qquad \bigoplus_{\mathbb{Z}/n\mathbb{Z}} K_n \qquad\text{in}\qquad \operatorname{Rep}^{\mathrm c}_K\pi_1^{\text{ét}}(U,\bar u)$$

on which $\pi_1^{\text{ét}}(U,\bar u)/\pi_1^{\text{ét}}(V,\bar v)=\operatorname{Gal}(V/U)$, respectively $\pi_1^{\text{ét}}(U,\bar u)/\pi_1^{\text{ét}}(U_n,\bar u)=\mathbb{Z}/n\mathbb{Z}$, act as permutation representations. In particular, $\mathcal{K}=pr_*\mathbb{1}_{U_n}$. Moreover, by Corollary 5.4 the groups $\operatorname{Gr}(\mathcal{K}/U,u)\times_{K_e}\overline{K}$ and $\operatorname{Gr}(\mathcal{L}/U,u)\times_{K_e}\overline{K}$ equal the Zariski-closure of the image of $\pi_1^{\text{ét}}(U,\bar u)$. This proves the surjectivity of α. ☐

Let $\pi_1^{F\text{-Isoc}}(U,u)^\Delta$ be the automorphism group of the fiber functor $\omega_u\colon F\text{-Isoc}_K(U)\to\{K_e\text{-vector spaces}\}$. It is an affine group scheme over K_e and equals the kernel group of the K_e/K-groupoid $\pi_1^{F\text{-Isoc}}(U,u):=\operatorname{Aut}^{\otimes}_K(\omega_u)$ whose category of K_e-rational representations is tensor equivalent to $F\text{-Isoc}_K(U)$, see Definition A.5 and Theorem A.11. We again assume that $u\in U_n(\mathbb{F}_{q^e})$ and similarly define $\pi_1^{F\text{-Isoc}}(V,v)^\Delta$ and $\pi_1^{F^n\text{-Isoc}}(U_n,u)^\Delta$.

Lemma 6.2. *Let $f\colon V\to U$ be a finite étale Galois-covering of curves with Galois group $\Gamma:=\operatorname{Gal}(V/U)$, let $v\in V(\mathbb{F}_{q^e})$ and let $u:=f(v)\in U(\mathbb{F}_{q^e})$.*

(a) There is an exact sequence of affine group schemes over K_e

$$0\longrightarrow \pi_1^{F\text{-Isoc}}(V,v)^\Delta \xrightarrow{\;\alpha\;} \pi_1^{F\text{-Isoc}}(U,u)^\Delta \xrightarrow{\;\beta\;} \operatorname{Gal}(V/U)\longrightarrow 0\,,$$

where the morphism α is induced by the pullback functor $f^*\colon F\text{-Isoc}_K(U)\to F\text{-Isoc}_K(V)$, and β comes from the epimorphism $\pi_1^{F\text{-Isoc}}(U,u)^\Delta\twoheadrightarrow\operatorname{Gr}(\mathcal{L}/U,u)\cong\operatorname{Gal}(V/U)$ using Lemma 6.1.

(b) *For every $\mathcal{F}\in F\text{-Isoc}_K(U)$ the sequence in (a) induces the following exact sequence of affine group schemes over K_e*

$$0 \longrightarrow \mathrm{Gr}(f^*\mathcal{F}/V,v) \longrightarrow \mathrm{Gr}(\mathcal{F}/U,u) \longrightarrow G \longrightarrow 0\,, \tag{6.6}$$

where G is a finite group which is a quotient of $\mathrm{Gal}(V/U)$. *In particular if* $\mathrm{Gr}(\mathcal{F}/U,u)$ *is connected then* $\mathrm{Gr}(f^*\mathcal{F}/V,v) \xrightarrow{\sim} \mathrm{Gr}(\mathcal{F}/U,u)$.

Proof. (a) Explicitly α is given as follows. Note that $\omega_v(f^*\mathcal{F})=\omega_u(\mathcal{F})$ for all $\mathcal{F}\in F\text{-Isoc}_K(U)$. If $h\in\pi_1^{F\text{-Isoc}}(V,v)^\Delta$ then $\alpha(h)$ acts on $\omega_u(\mathcal{F})$ as $\alpha(h)|_{\mathcal{F}}:=h|_{f^*\mathcal{F}}$. Since $f^*\mathcal{L}=\bigoplus_{\gamma\in\Gamma}\gamma^*\mathbb{1}_V=\mathbb{1}_V^{\oplus\#\Gamma}$ is a direct sum of the trivial F^n-isocrystal $\mathbb{1}_V$, the group $\pi_1^{F\text{-Isoc}}(V,v)^\Delta$ maps to the kernel of β. Next every object $\mathcal{G}\in F\text{-Isoc}_K(V)$ is a quotient of $f^*\mathcal{F}$ for the object $\mathcal{F}=f_*\mathcal{G}\in F\text{-Isoc}_K(U)$, because $f^*f_*\mathcal{G}=\bigoplus_{\gamma\in\Gamma}\gamma^*\mathcal{G}$. Therefore, the map α is a closed immersion by Proposition A.14(b).

To prove exactness in the middle let $g\in\pi_1^{F\text{-Isoc}}(U,u)^\Delta$ lie in the kernel of β. We must show that $g=\alpha(h)$ for some $h\in\pi_1^{F\text{-Isoc}}(V,v)^\Delta$, and this means that for every $\mathcal{G}\in F\text{-Isoc}_K(V)$ we have to exhibit $h|_{\mathcal{G}}\in\mathrm{Aut}_{K_e}(\omega_v(\mathcal{G}))$. We reuse the technique from Lemma 6.1. For any such $\mathcal{G}$ we have a K_e-linear automorphism $g|_{f_*\mathcal{G}}$ of

$$\omega_u(f_*\mathcal{G}) \;=\; \omega_v(f^*f_*\mathcal{G}) \;=\; \bigoplus_{\gamma\in\Gamma}\omega_v(\gamma^*\mathcal{G})\,.$$

which we decompose as $g|_{f_*\mathcal{G}}=(h_{\varepsilon,\gamma})_{\varepsilon,\gamma}\colon\bigoplus_{\gamma\in\Gamma}\omega_v(\gamma^*\mathcal{G})\xrightarrow{\sim}\bigoplus_{\varepsilon\in\Gamma}\omega_v(\varepsilon^*\mathcal{G})$ for K_e-homomorphisms $h_{\varepsilon,\gamma}\colon\omega_v(\gamma^*\mathcal{G})\to\omega_v(\varepsilon^*\mathcal{G})$. To compute $g|_{f_*\mathcal{G}}$ we use the isomorphism

$$\psi_{\mathcal{G}}\colon \mathcal{L}\otimes f_*\mathcal{G} \;\xrightarrow{\sim}\; f_*\big((f^*\mathcal{L})\otimes\mathcal{G}\big) \;=\; f_*\big((\bigoplus_{\delta\in\Gamma}\delta^*\mathbb{1}_V)\otimes\mathcal{G}\big) \;=\; \bigoplus_{\delta\in\Gamma}f_*(\mathbb{1}_V\otimes\mathcal{G}) \;=\; \bigoplus_{\Gamma}f_*\mathcal{G}\,.$$

We fix bases $\underline{c}_\gamma$ of the K_e-vector space $\omega_v(\gamma^*\mathcal{G})$ and e_δ of the 1-dimensional K_e-vector space $\omega_v(\delta^*\mathbb{1}_V)$. We compute

$$\begin{aligned}
\omega_u(\mathcal{L}\otimes f_*\mathcal{G}) \;&=\; \omega_v(f^*\mathcal{L})\otimes_{K_e}\omega_v(f^*f_*\mathcal{G}) \;=\; \bigoplus_{\gamma,\delta\in\Gamma}\omega_v(\delta^*\mathbb{1}_V)\otimes_{K_e}\omega_v(\gamma^*\mathcal{G}) \;=\; \bigoplus_{\gamma,\delta\in\Gamma}\langle e_\delta\rangle_{K_e}\otimes_{K_e}\langle\underline{c}_\gamma\rangle_{K_e}\,,\\
\omega_u(\bigoplus_{\Gamma}f_*\mathcal{G}) \;&=\; \bigoplus_{\delta\in\Gamma}\omega_v(f^*f_*\mathcal{G}) \;=\; \bigoplus_{\gamma,\delta\in\Gamma}(K_e\cdot d_\delta)\otimes_{K_e}\omega_v(\gamma^*\mathcal{G}) \;=\; \bigoplus_{\gamma,\delta\in\Gamma}\langle d_\delta\rangle_{K_e}\otimes_{K_e}\langle\underline{c}_\gamma\rangle_{K_e}\,,
\end{aligned}$$

where the basis elements d_δ in the last line simply help to keep track of the summands for $\delta\in\Gamma$. As in (6.4) the isomorphism $\psi_{\mathcal{G}}$ on these fiber functors is given by

$$\omega_u(\psi_{\mathcal{G}})\colon\; \omega_u(\mathcal{L}\otimes f_*\mathcal{G}) \;\xrightarrow{\sim}\; \omega_u(\bigoplus_{\Gamma}f_*\mathcal{G})\,,\quad e_\delta\otimes\underline{c}_\gamma\mapsto d_{\gamma^{-1}\delta}\otimes\underline{c}_\gamma\,,\quad e_{\gamma\delta}\otimes\underline{c}_\gamma\mapsfrom d_\delta\otimes\underline{c}_\gamma\,.$$

Since $g|_{\mathcal{L}}=\beta(g)=\mathrm{id}$, that is $g(e_\delta)=e_\delta$ for all $\delta\in\Gamma$, we obtain

$$\begin{aligned}
g|_{\mathcal{L}\otimes f_*\mathcal{G}}(e_{\gamma\delta}\otimes a_\gamma\cdot\underline{c}_\gamma) \;&=\; \sum_\varepsilon e_{\gamma\delta}\otimes h_{\varepsilon,\gamma}(a_\gamma)\cdot\underline{c}_\varepsilon \qquad\text{and}\\
g|_{\bigoplus_\Gamma f_*\mathcal{G}}(d_\delta\otimes a_\gamma\cdot\underline{c}_\gamma) \;&=\; \sum_\varepsilon d_\delta\otimes h_{\varepsilon,\gamma}(a_\gamma)\cdot\underline{c}_\varepsilon\,.
\end{aligned}$$

The compatibility with the isomorphism $\psi_{\mathcal{G}}$ imposes for every γ and δ the condition

$$\sum_\varepsilon e_{\gamma\delta}\otimes h_{\varepsilon,\gamma}(a_\gamma)\cdot\underline{c}_\varepsilon \;=\; \sum_\varepsilon e_{\varepsilon\delta}\otimes h_{\varepsilon,\gamma}(a_\gamma)\cdot\underline{c}_\varepsilon$$

It follows that $h_{\varepsilon,\gamma}(a_\gamma)=0$ if $\gamma\neq\varepsilon$. In particular, we can define $h|_{\gamma^*\mathcal{G}}:=h_{\gamma,\gamma}=(g|_{f_*\mathcal{G}})\big|_{\omega_v(\gamma^*\mathcal{G})}$ and $h|_{\mathcal{G}}:=h_{1,1}=(g|_{f_*\mathcal{G}})\big|_{\omega_v(\mathcal{G})}$. As our argument is functorial in $\mathcal{G}$ this shows that indeed h is an element of $\pi_1^{F\text{-Isoc}}(V,v)^\Delta$.

It remains to show that $g=\alpha(h)$. Let $\mathcal{F}\in F\text{-Isoc}_K(U)$. Then $\alpha(h)|_{\mathcal{F}}=h|_{f^*\mathcal{F}}$ and we must show that this is equal to $g|_{\mathcal{F}}$. From $f_*f^*\mathcal{F}=\mathcal{F}\otimes\mathcal{L}$ we deduce $g|_{f_*f^*\mathcal{F}}=g|_{\mathcal{F}}\otimes\mathrm{id}|_{\mathcal{L}}$. Then $h|_{f^*\mathcal{F}}$ is defined as $h|_{f^*\mathcal{F}}:=(g|_{f_*f^*\mathcal{F}})\big|_{\omega_v(f^*\mathcal{F})}=(g|_{\mathcal{F}})\big|_{\omega_u(\mathcal{F})}$. This proves (a).

(b) The group scheme $\mathrm{Gr}(\mathcal{F}/U,u)$ is the image of the representation $\pi_1^{F\text{-Isoc}}(U,u)^\Delta\to\mathrm{Aut}_{K_e}(\omega_u(\mathcal{F}))$ corresponding to $\mathcal{F}$ and likewise for $f^*\mathcal{F}$ by Propositions A.13 and A.14(a). Since $\omega_u(\mathcal{F})=\omega_v(f^*\mathcal{F})$, the group $\mathrm{Gr}(f^*\mathcal{F}/V,v)$ is a closed normal subgroup of $\mathrm{Gr}(\mathcal{F}/U,u)$ and the quotient G is a quotient of $\mathrm{Gal}(V/U)$.

If $\mathrm{Gr}(\mathcal{F}/U,u)$ is connected then its image in G will be zero. This proves the lemma. □

Lemma 6.3. *Let $n \in \mathbb{N}$ and assume that $u \in U_n(\mathbb{F}_{q^e})$.*

(a) The functor $[n]^*\colon F\text{-Isoc}_K(U_n) \to F^n\text{-Isoc}_{K_n}(U_n)$, $(\mathcal{F}, F_{\mathcal{F}}) \mapsto (\mathcal{F}, F_{\mathcal{F}}^n)$ given by passing to the n-th power of the Frobenius induces an isomorphism of group schemes $\pi_1^{F^n\text{-Isoc}}(U_n,u)^\Delta \xrightarrow{\sim} \pi_1^{F\text{-Isoc}}(U_n,u)^\Delta$ over K_e.

(b) There is an exact sequence of affine group schemes over K_e

$$0 \longrightarrow \pi_1^{F^n\text{-Isoc}}(U_n,u)^\Delta \xrightarrow{\ \alpha\ } \pi_1^{F\text{-Isoc}}(U,u)^\Delta \xrightarrow{\ \beta\ } \mathbb{Z}/n\mathbb{Z} \longrightarrow 0\,,$$

where the morphism α is induced by the functor $(\,.\,)^{(n)}$ from (3.1), and β comes from the epimorphism $\pi_1^{F\text{-Isoc}}(U,u)^\Delta \twoheadrightarrow \mathrm{Gr}(\mathcal{K}/U,u) \cong \mathbb{Z}/n\mathbb{Z}$ using Lemma 6.1.

(c) For every $\mathcal{F} \in F\text{-Isoc}_K(U)$ the sequence in (b) induces the following exact sequence of affine group schemes over K_e

$$0 \longrightarrow \mathrm{Gr}(\mathcal{F}^{(n)}/U_n,u) \longrightarrow \mathrm{Gr}(\mathcal{F}/U,u) \longrightarrow G \longrightarrow 0\,,$$

where G is a finite group which is a quotient of $\mathbb{Z}/n\mathbb{Z}$. In particular if $\mathrm{Gr}(\mathcal{F}/U,u)$ is connected then $\mathrm{Gr}(\mathcal{F}^{(n)}/U_n,u) \xrightarrow{\sim} \mathrm{Gr}(\mathcal{F}/U,u)$.

Remark 6.4. Note that Lemma 6.3(a) does not imply that the functor $[n]^*$ is an equivalence of categories. Namely, by Theorem A.11 the category $F\text{-Isoc}_K(U_n)$ is equivalent to the K_e-rational representations of the K_e/K-groupoid $\mathrm{Aut}_K^\otimes\big(\omega_u|\, F\text{-Isoc}_K(U_n)\big)$ and the category $F^n\text{-Isoc}_{K_n}(U_n)$ is equivalent to the K_e-rational representations of the K_e/K_n-groupoid $\mathrm{Aut}_{K_n}^\otimes\big(\omega_u|\, F^n\text{-Isoc}_{K_n}(U_n)\big)$. The latter is obtained as the fiber product

$$\mathrm{Aut}_{K_n}^\otimes\big(\omega_u|\, F^n\text{-Isoc}_{K_n}(U_n)\big) \;=\; \mathrm{Aut}_K^\otimes\big(\omega_u|\, F\text{-Isoc}_K(U_n)\big) \underset{\operatorname{Spec} K_e\otimes_K K_e}{\times} \operatorname{Spec} K_e \otimes_{K_n} K_e \tag{6.7}$$

by [Mil92, Proposition A.12], because $F^n\text{-Isoc}_{K_n}(U_n)$ is the base extension category $F\text{-Isoc}_K(U_n)\otimes_K K_n$. In particular, the kernel groups $\pi_1^{F^n\text{-Isoc}}(U_n,u)^\Delta$ and $\pi_1^{F\text{-Isoc}}(U_n,u)^\Delta$ of both groupoids coincide, because they are obtained as the pullback along the diagonal $\operatorname{Spec} K_e \to \operatorname{Spec} K_e \otimes_{K_n} K_e$ of (6.7).

On unit root F-isocrystals Crew's equivalence from Proposition 5.2 yields a commutative diagram of categories

$$\begin{tikzcd} F\text{-UR}_K(U_n) \arrow[rr, "{[n]^*}"] \arrow[d, "\cong"'] && F\text{-UR}_{K_n}(U_n) \arrow[d, "\cong"] \\ \mathrm{Rep}^{\mathrm{c}}_K\, \pi_1^{\text{ét}}(U_n,\bar u) \arrow[rr, "\otimes_K K_n"] && \mathrm{Rep}^{\mathrm{c}}_{K_n}\, \pi_1^{\text{ét}}(U_n,\bar u)\,. \end{tikzcd}$$

Note that the horizontal functors have right adjoints $\mathrm{Rep}^{\mathrm{c}}_{K_n}\, \pi_1^{\text{ét}}(U_n,\bar u) \to \mathrm{Rep}^{\mathrm{c}}_K\, \pi_1^{\text{ét}}(U_n,\bar u)$ given by restriction of scalars from K_n to K, and

$$[n]_*\colon\ F^n\text{-Isoc}_{K_n}(U_n) \;\to\; F\text{-Isoc}_K(U_n)$$

given by $[n]_*\mathcal{G} := \bigoplus_{i=0}^{n-1} F^{i*}\mathcal{G}$ with Frobenius $F_{[n]_*\mathcal{G}}\colon F^*([n]_*\mathcal{G}) \xrightarrow{\sim} [n]_*\mathcal{G}$ given by the matrix

$$\begin{pmatrix} 0 & & F_{\mathcal{G}} \\ 1 & \ddots & \\ & \ddots\ 1 & 0 \end{pmatrix},$$

where $F_{\mathcal{G}}\colon F^{n*}\mathcal{G} \xrightarrow{\sim} \mathcal{G}$ is the Frobenius of $\mathcal{G}$. In particular, the functor $(\,.\,)_{(n)}$ from (6.2) equals $pr_* \circ [n]_*$ for the projection $pr : U_n \to U$.

Proof of Lemma 6.3. (a) follows from Remark 6.4. Alternatively, it can be proven by a strategy similar to Lemma 6.2(a). Indeed, there is an exact sequence of affine K_e-group schemes

$$0 \longrightarrow \pi_1^{F^n\text{-Isoc}}(U_n,u)^\Delta \xrightarrow{\ [n]^*\ } \pi_1^{F\text{-Isoc}}(U_n,u)^\Delta \longrightarrow \mathrm{Gr}([n]_*\, {}^n\underline{\mathbb{1}}_{U_n}/U_n,u) \longrightarrow 0\,,$$

where ${}^n\underline{\mathbb{1}}_{U_n} \in F^n\text{-Isoc}_{K_n}(U_n)$ is the unit object. We show that $[n]_*\, {}^n\underline{\mathbb{1}}_{U_n}$ is trivial. Let $\lambda \in \mathbb{F}_{q^n}$ be a generator of the field extension $\mathbb{F}_{q^n}/\mathbb{F}_q$, that is $\mathbb{F}_{q^n} = \bigoplus_{j=0}^{n-1} \mathbb{F}_q \cdot \lambda^j$ and let $b \in \mathcal{O}_{K_n}$ be the q^n-th root of unity which reduces to λ modulo the maximal ideal $\mathfrak{m}_{K_n}$ of $\mathcal{O}_{K_n}$. (Use Hensel's Lemma for the existence and uniqueness of b.) Then an isomorphism $\mathbb{1}_{U_n}^{\oplus n} \xrightarrow{\sim} [n]_* {}^n\mathbb{1}_{U_n}$ is given by the matrix $\big(F^i(b^j)\big)_{i,j=0\ldots n-1}$, which is invertible, because its reduction modulo $\mathfrak{m}_{K_n}$ is an invertible Moore matrix, see for example [Gos96, § 1.3]. Therefore, $\mathrm{Gr}([n]_* {}^n\mathbb{1}_{U_n}/U_n,u)$ is trivial and $\pi_1^{F^n\text{-Isoc}}(U_n,u)^\Delta \xrightarrow{\sim} \pi_1^{F\text{-Isoc}}(U_n,u)^\Delta$ is an isomorphism.

(b) and (c) now follow from Lemma 6.2 by observing that $pr_*\mathbb{1}_{U_n} = \mathcal{K}$, see Lemma 6.1. □

Corollary 6.5. *Let $\mathcal{F}$ be a convergent F-isocrystal on U and let $u \in U(\mathbb{F}_{q^e})$. Then, after possibly enlarging e, there exists a finite étale Galois covering of curves $f\colon V \to U$ and a point $v \in V(\mathbb{F}_{q^e})$ with $f(v) = u$ such that $\mathrm{Gr}(f^*\mathcal{F}/V,v)$ equals the identity component $\mathrm{Gr}(\mathcal{F}/U,u)^\circ$ in sequence (6.6) and $\mathrm{Gal}(V/U)$ is isomorphic to the group of connected components of $\mathrm{Gr}(\mathcal{F}/U,u)$.*

Proof. Let $G := \mathrm{Gr}(\mathcal{F}/U,u)\big/\,\mathrm{Gr}(\mathcal{F}/U,u)^\circ$ be the quotient by the characteristic subgroup $\mathrm{Gr}(\mathcal{F}/U,u)^\circ \subset \mathrm{Gr}(\mathcal{F}/U,u)$. It corresponds to an object $\mathcal{G} \in \langle\langle\mathcal{F}\rangle\rangle$ with $G = \mathrm{Gr}(\mathcal{G}/U,u)$ by Remark A.17 and Corollary A.16(b). Since G is a finite group, $\mathcal{G}$ is a convergent unit-root F-isocrystal by Lemma 5.1. Let $\rho_{\mathcal{G}}\colon \pi_1^{\text{ét}}(U,\bar u) \twoheadrightarrow G$ be the representation of the fundamental group corresponding to $\mathcal{G}$ by Proposition 5.2 which is surjective onto G by Corollary 5.4. The kernel of $\rho_{\mathcal{G}}$ equals $\pi_1^{\text{ét}}(V,\bar v)$ for a finite étale Galois covering $f\colon V \to U$ and a lift $\bar v \in V(\overline{\mathbb{F}}_q)$ of $\bar u$, that is $G = \pi_1^{\text{ét}}(U,\bar u)/\pi_1^{\text{ét}}(V,\bar v) = \mathrm{Gal}(V/U)$. In particular, $\rho_{\mathcal{G}}|_{\pi_1^{\text{ét}}(V,\bar v)}$ is the trivial representation and, as it corresponds to $f^*\mathcal{G}$ as in the proof of Lemma 6.1, consequently $f^*\mathcal{G}$ is a direct sum of trivial F-isocrystals. At the expense of enlarging e there is a unique point $v \in V(\mathbb{F}_{q^e})$ below $\bar v$ and above u. It follows that $\mathrm{Gr}(f^*\mathcal{F}/V,v)$ maps to the kernel of $\mathrm{Gr}(\mathcal{F}/U,u) \twoheadrightarrow G = \mathrm{Gal}(V/U)$. From sequence (6.6) we conclude that $\mathrm{Gr}(f^*\mathcal{F}/V,v)$ is equal to that kernel, and hence equals $\mathrm{Gr}(\mathcal{F}/U,u)^\circ$. □

For the next corollary note that $\pi_1^{F\text{-Isoc}}(U,\bar u)^\Delta = \pi_1^{F\text{-Isoc}}(U,u)^\Delta \times_{K_e} K^{\mathrm{un}}$ and $\mathrm{Gr}(\mathcal{F}/U,\bar u) = \mathrm{Gr}(\mathcal{F}/U,u) \times_{K_e} K^{\mathrm{un}}$ for every point $u \in U(\mathbb{F}_{q^e})$ below $\bar u$ and for every $\mathcal{F} \in F\text{-Isoc}_K(U)$. Here $K^{\mathrm{un}} \subset \overline{K}$ is the maximal unramified extension of K.

Corollary 6.6. *Let $\bar u \in U(\overline{\mathbb{F}}_q)$ be a geometric base point. Then the exact sequence from Lemma 6.2(a) induces an exact sequence of affine group schemes over K^{un}*

$$0 \longrightarrow \pi_1^{F\text{-Isoc}}(U,\bar u)^{\Delta\circ} \longrightarrow \pi_1^{F\text{-Isoc}}(U,\bar u)^\Delta \longrightarrow \pi_1^{\text{ét}}(U,\bar u) \longrightarrow 0\,, \tag{6.8}$$

where $\pi_1^{F\text{-Isoc}}(U,\bar u)^{\Delta\circ}$ is the identity component. In particular, the pro-group of connected components of $\pi_1^{F\text{-Isoc}}(U,\bar u)^\Delta$ equals the étale fundamental group $\pi_1^{\text{ét}}(U,\bar u)$.

Proof. For every convergent F-isocrystal on U we obtain from Corollary 6.5 a finite étale Galois covering $f\colon V \to U$ and an exact sequence

$$0 \longrightarrow \mathrm{Gr}(\mathcal{F}/U,\bar u)^\circ \longrightarrow \mathrm{Gr}(\mathcal{F}/U,\bar u) \longrightarrow \mathrm{Gal}(V/U) \longrightarrow 0\,.$$

We now take the projective limit of these sequences over the diagram of the Tannakian sub-categories $\langle\langle\mathcal{F}\rangle\rangle$ of $F\text{-Isoc}_K(U)$. This limit is taken in the category of sheaves of groups on K^{un} for the étale topology. We claim that this limit is the sequence (6.8). First of all, by Lemma 3.3 the projective system of the $\mathrm{Gr}(\mathcal{F}/U,\bar u)^\circ$ consists of epimorphisms and so satisfies the Mittag-Leffler condition. Therefore, (6.8) is exact at $\pi_1^{\text{ét}}(U,\bar u) = \varprojlim_V \mathrm{Gal}(V/U)$. By the remark before the corollary the group $\pi_1^{F\text{-Isoc}}(U,\bar u)^\Delta$ is the projective system of the $\mathrm{Gr}(\mathcal{F}/U,\bar u)$, which can equivalently be taken in the category of affine group schemes over K^{un}. It remains to identify $\pi_1^{F\text{-Isoc}}(U,\bar u)^{\Delta\circ}$ with the limit of the projective system of the $\mathrm{Gr}(\mathcal{F}/U,\bar u)^\circ$. By construction this limit is representable by a closed subgroup scheme of $\pi_1^{F\text{-Isoc}}(U,\bar u)^\Delta$. Moreover, it is connected, because if d is an idempotent in its structure sheaf then d lies in the structure sheaf of some $\mathrm{Gr}(\mathcal{F}/U,\bar u)^\circ$ and satisfies $d^2 = d$ after maybe replacing $\langle\langle\mathcal{F}\rangle\rangle$ by a larger such category. Since $\mathrm{Gr}(\mathcal{F}/U,\bar u)^\circ$ is connected we have $d = 0$ or $d = 1$, whence the limit is connected and a closed subgroup scheme of $\pi_1^{F\text{-Isoc}}(U,\bar u)^{\Delta\circ}$. On the other hand, the limit contains $\pi_1^{F\text{-Isoc}}(U,\bar u)^{\Delta\circ}$ since the latter maps to the connected component of unity in $\pi_1^{\text{ét}}(U,\bar u)$, which is trivial. Therefore, this limit equals $\pi_1^{F\text{-Isoc}}(U,\bar u)^{\Delta\circ}$ and the corollary is proven. □

Proposition 6.7. *Let $\mathcal{F} \in F\text{-Isoc}_K(U)$ for which Conjecture 1.4 is true. Then Conjecture 1.2 also holds true for $\mathcal{F}$.*

Proof. let $S \subset |U|$ be a subset of Dirichlet density one. Let G denote the group of connected components of $\mathrm{Gr}(\mathcal{F}/U,u)$; it is a quotient of $\pi_1^{\text{ét}}(U,\bar{u})$ and equals the Galois group of a finite étale Galois cover of U by Corollary 6.5. For every conjugacy class $C \subset G$ let $R_C \subset |U|$ denote the set of those closed points $x \in |U|$ whose Frobenius class $\mathrm{Frob}_x(\mathcal{F})$ maps onto $C \subset G$. By Corollary 5.4 the image of $\mathrm{Frob}_x(\mathcal{F})$ in G coincides with the image of the conjugacy class of the Frobenius Frob_x^{-1} of x in $\pi_1^{\text{ét}}(U,\bar{u})$. Thus by the classical Chebotarëv density theorem [Ser63, Theorem 7], R_C has positive Dirichlet density. Therefore, $S_C = S \cap R_C$ has positive upper Dirichlet density by Lemma 3.13. By the validity of Conjecture 1.4 for $\mathcal{F}$ we get that the Zariski-closure of $\bigcup_{x\in S_C} \mathrm{Frob}_x(\mathcal{F})$ contains a connected component of $\mathrm{Gr}(\mathcal{F}/U,u)$. This connected component must map to a point in C. Since the Zariski-closure of $\bigcup_{x\in S_C} \mathrm{Frob}_x(\mathcal{F})$ is conjugation-invariant, we get that this set is equal to the union of all connected components mapping into C. The Zariski-closure of $\bigcup_{x\in S} \mathrm{Frob}_x(\mathcal{F})$ contains the union of the Zariski-closures of $\bigcup_{x\in S_C} \mathrm{Frob}_x(\mathcal{F})$ for varying C, and hence it must be the whole group $\mathrm{Gr}(\mathcal{F}/U,u)$. □

7. Chebotarëv for Direct Sums of Isoclinic F-Isocrystals

In this section we will prove Theorem 1.8 by working with p-adic analytic manifolds and Lie groups. First we recall the p-adic version of Cartan's theorem:

Theorem 7.1. *Let G be a linear algebraic group over $\mathbb{Q}_p$, and let $C \subset G(\mathbb{Q}_p)$ be a subgroup which is compact in the p-adic topology. Then C is a Lie subgroup of $G(\mathbb{Q}_p)$ over $\mathbb{Q}_p$.*

Proof. Since C is compact, it is closed in the Hausdorff space $G(\mathbb{Q}_p)$, so the claim follows from [Ser92, Part II, § V.9, Corollary to Theorem 1 on page 155]. □

Definition 7.2. We will recall what Serre calls a *standard group*; see [Ser92, Part II, § IV.8]. Let $F = (F_1, F_2, \ldots, F_N) \in \mathbb{Z}_p[\![x_1,\ldots,x_N,y_1,\ldots,y_N]\!]^N$ be a formal group law in N variables over $\mathbb{Z}_p$. Serre equips $(p\mathbb{Z}_p)^N$ with the structure of a Lie group over $\mathbb{Q}_p$. We need a re-normalization which identifies $\mathbb{Z}_p$ with $p\mathbb{Z}_p$ by multiplication with p. So we equip the p-adic analytic space $G_F := \mathbb{Z}_p^N$ of dimension N with the structure of a Lie group over $\mathbb{Q}_p$ where the multiplication is given by the formula

$$x \cdot_F y \ := \ \tfrac{1}{p} \cdot F(px,py) \quad \text{for } x,y \in G_F := \mathbb{Z}_p^N, \tag{7.1}$$

and the identity is $(0,0,\ldots,0)$.

Let us next recall the following

Lemma 7.3. *Let G be a connected linear algebraic group over a field L, and let n be an integer, which is not divisible by the characteristic of L. Then the n-th power map $[n]\colon G \to G$ (which is not a group homomorphism if G is not commutative) is a dominant morphism whose image contains an open subset. In particular, if $X \subset G(L)$ is a subset which is Zariski-dense in G, then its image $[n](X)$ is again Zariski-dense in G.*

Proof. The differential of $[n]$ at the neutral element of G equals the multiplication by n which is invertible. Therefore, $[n]$ is étale in an open neighborhood of the neutral element by [BLR90, § 2.2, Corollary 10], and hence the image under $[n]$ of this neighborhood is open by [EGA, IV$_2$, Théorème 2.4.6]. Since G is irreducible, this image is dense. The last assertion follows from this. □

As a further preparation we need the following

Theorem 7.4. *Let T be a commutative linear algebraic group over a field L of characteristic zero, whose identity component T° is the product $\mathbb{G}_{m,L}^r \times_L \mathbb{G}_{a,L}^n$ of a split torus $\mathbb{G}_{m,L}^r$ with an additive group scheme $\mathbb{G}_{a,L}^n$ for $r,n \geq 0$. Let $G \subset T(L)$ be an infinite cyclic Zariski-dense subgroup in T. Then $n = 0$ or $n = 1$, and for any connected component T^c of T, any infinite subset S of $G \cap T^c$ is still Zariski-dense in T^c.*

Proof. Clearly the image of G under the projection $T \to \mathbb{G}_{a,L}^n$ is Zariski-dense. Since this image lies in the at most 1-dimensional linear subspace generated by the image of a generator of G, the dimension n is either zero or one. By choosing an element h of S and considering the translate $S-h$ we can assume that $S \subset T^\circ$, and we must show that S is Zariski-dense in T°. Thus, by replacing G with $G \cap T^\circ = \ker(G \to T/T^\circ)$ which is still infinite cyclic and dense in T°, we may assume that $T = T^\circ$ is connected.

The $n = 0$ case can be easily deduced from the the Mordell-Lang conjecture for tori, proved by Michel Laurent. Indeed, by [Lau84, Théorème 2] the Zariski-closure of S is the finite union of finitely many translates of sub-tori of T. By shrinking S, if it is necessary, we may assume that this finite union consists of just one translate of a sub-torus $\widetilde{T}$. It will be enough to show that $\widetilde{T}=T$. Pick an element $h\in S$. Then the translate $S-h$ lies in $\widetilde{T}$, and hence the intersection $H=G\cap\widetilde{T}$ is a subgroup of G which contains the infinite set $S-h$. Since $G\cong\mathbb{Z}$ we get that H is a subgroup of G of finite index, say n. Then the n-th power map $x\mapsto nx$ on T maps G into H, and hence into $\widetilde{T}$. Because $G\cap T$ is Zariski-dense in T, Lemma 7.3 implies that H is dense in T too. We get that $T=\widetilde{T}$.

We will prove the $n=1$ case by a different method (which nevertheless can be applied to the $n=0$ case as well). First we will need the following useful

Lemma 7.5. *Over a field L of characteristic zero, all closed subgroups Γ of $\mathbb{G}^r_{m,L}\times_L\mathbb{G}_{a,L}$ are of the form $\Gamma_s\times_L\mathbb{G}^\varepsilon_{a,L}$, where $\Gamma_s\subset\mathbb{G}^r_{m,L}$ is a closed subgroup and ε is either 0 or 1. In particular, the set of such subgroups is countable.*

Proof. Since Γ is commutative, it is the direct product $\Gamma=\Gamma_u\times\Gamma_s$ of the set Γ_u of its unipotent elements and the set Γ_s of its semi-simple elements, which are both closed subgroups; see [Hum75, § 15.5, Theorem]. The projections $\Gamma_u\to\mathbb{G}^r_{m,L}$ and $\Gamma_s\to\mathbb{G}_{a,L}$ are both zero, because 1 is the only element which is at the same time unipotent and semi-simple. Therefore, $\Gamma_u\subset\ker(\mathbb{G}^r_{m,L}\times_L\mathbb{G}_{a,L}\to\mathbb{G}^r_{m,L})=\mathbb{G}_{a,L}$ and $\Gamma_s\subset\ker(\mathbb{G}^r_{m,L}\times_L\mathbb{G}_{a,L}\to\mathbb{G}_{a,L})=\mathbb{G}^r_{m,L}$. Since Γ_u is connected by Lemma 3.7, there are only the possibilities $\Gamma_u=\{1\}$ or $\Gamma_u=\mathbb{G}_{a,L}$. This proves the first assertion.

For the last assertion we only have to show that the set of all closed subgroups $\Gamma_s\subset\mathbb{G}^r_{m,L}$ is countable. By [Bor91, III.8.2 Proposition] these subgroups correspond to quotients of the free abelian group $X^*(\mathbb{G}^r_{m,L})=\mathbb{Z}^r$ and so there are only countably many. □

Continuation of the Proof of Theorem 7.4. Assume now that the claim is false and let $H\subset T=\mathbb{G}^r_{m,L}\times_L\mathbb{G}_{a,L}$ be a proper hyper-surface such that S lies in H. Let $g\in G$ be a generator. Since the coefficients of the defining polynomial of H and the coordinates of g form a finite set I, we may assume without the loss of generality that L is finitely generated over $\mathbb{Q}$, by replacing it with the field generated by I, if this is necessary. Let V be a smooth irreducible variety over $\mathbb{Q}$ whose function field is L. By shrinking V, if this is necessary, we may assume that g extends to a section $\widetilde{g}$ of the projection map:

$$\mathbb{G}^r_{m,\mathbb{Q}}\times_{\mathbb{Q}}\mathbb{G}_{a,\mathbb{Q}}\times_{\mathbb{Q}}V\to V\,.$$

The projection onto the first two factors induces a morphism

$$\widetilde{g}\colon V\to\mathbb{G}^r_{m,\mathbb{Q}}\times_{\mathbb{Q}}\mathbb{G}_{a,\mathbb{Q}}\,. \tag{7.2}$$

By shrinking V further, we may also assume that the projection of $\widetilde{g}$ onto the factor $\mathbb{G}_{a,\mathbb{Q}}$ is nowhere zero on V. Similarly we may assume that H extends to a closed subscheme $\widetilde{H}$ of $\mathbb{G}^r_{m,\mathbb{Q}}\times_{\mathbb{Q}}\mathbb{G}_{a,\mathbb{Q}}\times_{\mathbb{Q}}V$ which is a proper hyper-surface in the fiber over any point of V.

Let Q be a closed point of V and let K be the residue field of Q. Then K is a number field. As usual we identify $\mathbb{G}_{a,\mathbb{Q}}$ with $\mathbb{A}^1_{\mathbb{Q}}$ and we embed $\mathbb{G}_{m,\mathbb{Q}}$ into $\mathbb{A}^2_{\mathbb{Q}}$ as the closed subscheme on which the product of the coordinates of $\mathbb{A}^2_{\mathbb{Q}}$ equals 1. We consider the coordinates of the point $\widetilde{g}(Q)\in(\mathbb{G}^r_{m,\mathbb{Q}}\times_{\mathbb{Q}}\mathbb{G}_{a,\mathbb{Q}})(K)\subset(\mathbb{A}^{2r}_{\mathbb{Q}}\times_{\mathbb{Q}}\mathbb{A}^1_{\mathbb{Q}})(K)$. For all but finitely many valuations μ of K all these coordinates of $\widetilde{g}(Q)$ are μ-adic units. Fix such a valuation μ and let K_μ be the completion of K with respect to μ. It is a finite extension of $\mathbb{Q}_\ell$ for a prime number ℓ (different from or equal to p). By continuity there is an open ball B around Q in $V(K_\mu)$ such that for every $P\in B$ all the coordinates of $\widetilde{g}(P)$ are still μ-adic units. We next prove the following

Claim. There is a $P\in B$ such that the group $\widetilde{g}(P)^{\mathbb{Z}}$ generated by $\widetilde{g}(P)$ is Zariski-dense in $\mathbb{G}^r_{m,K_\mu}\times_{K_\mu}\mathbb{G}_{a,K_\mu}$.

If this is false, the closure of the group $\widetilde{g}(P)^{\mathbb{Z}}$ is a proper closed subgroup $W\subsetneq\mathbb{G}^r_{m,K_\mu}\times_{K_\mu}\mathbb{G}_{a,K_\mu}$. Note that for every such subgroup W the locus $L_W\subset V_{K_\mu}:=V\otimes_{\mathbb{Q}}K_\mu$ where the map $\widetilde{g}$ from (7.2) factors through W is a Zariski-closed subset. Moreover, L_W does not contain any irreducible component of V_{K_μ}, because it does not contain any generic point η of V_{K_μ}. Namely, the residue field $\kappa(\eta)$ of η contains L, and so the element $\widetilde{g}(\eta)=g$ generates a Zariski-dense subgroup of $\mathbb{G}^r_{m,\kappa(\eta)}\times_{\kappa(\eta)}\mathbb{G}_{a,\kappa(\eta)}$ by assumption. We claim that the intersection $L_W(K_\mu)\cap B$ is a proper analytic subset of B which is nowhere dense in B (that is, has Baire category one), see [Bou98, Chapter 9, § 5.1, Definition 1]. Indeed, assume that there is a point $x\in B$ and a small open neighborhood around x which is contained in $L_W(K_\mu)\cap B$. Since V is smooth in x, an algebraic neighborhood of x in V is étale over some affine space $\mathbb{A}^d_{K_\mu}$ where

$d = \dim V$. By shrinking the small open neighborhood around x if necessary, we can assume that it maps isomorphically onto an open ball in $\mathbb{A}^d(K_\mu)$. The latter ball is contained inside the scheme theoretic image of $L_W \subset V_{K_\mu}$ in $\mathbb{A}^d_{K_\mu}$, which is a proper Zariski-closed subset of dimension $< d$. This is a contradiction and proves that $L_W(K_\mu) \cap B$ is nowhere dense in B. Now by Lemma 7.5 the set of proper closed subgroups $W \subsetneq \mathbb{G}^r_{m,K_\mu} \times_{K_\mu} \mathbb{G}_{a,K_\mu}$ is countable, so the union $\bigcup_W L_W(K_\mu) \cap B \subset B$ is meager (that is, has still has Baire category one) by [Bou98, Chapter 9, § 5.2, Definition 2] and cannot equal B by [Bou98, Chapter 9, § 5.3, Theorem 1 and Definition 3]. Every point in the complement of $\bigcup_W L_W(K_\mu) \cap B$ satisfies our claim.

Since all the coordinates of the generator $\widetilde{g}(P)$ of $\widetilde{g}(P)^{\mathbb{Z}}$ are ℓ-adic units, the ℓ-adic closure C of $\widetilde{g}(P)^{\mathbb{Z}}$ in $(\mathbb{G}^r_{m,K_\mu} \times_{K_\mu} \mathbb{G}_{a,K_\mu})(K_\mu)$ is a compact group. By [CGP10, Propositions A.5.1. and A.5.2] the Weil restriction $Y = \operatorname{Res}_{K_\mu/\mathbb{Q}_\ell} \mathbb{G}^r_{m,K_\mu} \times_{K_\mu} \mathbb{G}_{a,K_\mu}$ is a smooth linear algebraic group scheme over $\mathbb{Q}_\ell$, and $Y(\mathbb{Q}_\ell) = (\mathbb{G}^r_{m,K_\mu} \times_{K_\mu} \mathbb{G}_{a,K_\mu})(K_\mu)$ contains C. By Theorem 7.1 for $\mathbb{Q}_\ell$, the group C is a Lie group over $\mathbb{Q}_\ell$, and by [Ser92, Part II, § IV.8, Theorem] there is an open subgroup C_0 of C which is standard in the sense of Definition 7.2. This means that there is a commutative formal group law F in N variables over $\mathbb{Z}_\ell$ and an isomorphism

$$\psi\colon C_0 \xrightarrow{\sim} G_F$$

of Lie groups over $\mathbb{Q}_\ell$, where $G_F := \mathbb{Z}_\ell^N$ is equipped with the group law (7.1) given by F. Since C is compact the index $m := [C : C_0]$ is finite. Since C is topologically generated by $\widetilde{g}(P)$, the finite group C/C_0 is also generated by $\widetilde{g}(P)$, and hence is cyclic of order m. This implies that the group $\widetilde{g}(P)^{m\mathbb{Z}}$ generated by $\widetilde{g}(P)^m$ is contained in C_0 and C_0 is the ℓ-adic closure of $\widetilde{g}(P)^{m\mathbb{Z}}$. Via its logarithm map $\log_F$ the Lie group G_F is isomorphic to the additive Lie group $(\ell\,\mathbb{Z}_\ell^N, +)$. Since it is topologically generated by one element, its dimension N is 1.

Using again that the index of C_0 in C is finite, there is an $h \in C$ such that $S \cap hC_0$ is still infinite by the pigeonhole principle. Under the ℓ-adic analytic isomorphism $\log_F \circ \psi \circ h^{-1}\colon hC_0 \xrightarrow{\sim} \ell\,\mathbb{Z}_\ell$ the intersection $H \cap hC_0$ is mapped isomorphically onto an ℓ-adic analytic subset A of $\ell\,\mathbb{Z}_\ell$, that is A is locally in the ℓ-adic topology on $\ell\,\mathbb{Z}_\ell$ the zero locus of power series. The set $H \cap hC_0$ contains the infinite set $S \cap hC_0$, so A contains an infinite set. Since $\ell\,\mathbb{Z}_\ell$ is compact, this infinite set has an accumulation point y. In a neighborhood $U = y + \ell^n\mathbb{Z}_\ell$ of y for suitable $n \gg 0$ the power series defining A have infinitely many zeros. But this implies, that A contains U, because the zeros of a power series in one variable are ℓ-adically discrete by [Laz62, Proposition 2]. Since U is the translate of an open subgroup of $\ell\,\mathbb{Z}_\ell$, it follows that H contains a translate $h'C_0'$ of an open subgroup C_0' of C_0. Let $m' = [C : C_0']$ be the index. Since C contains $\widetilde{g}(P)^{\mathbb{Z}}$, it is also Zariski-dense in $\mathbb{G}^r_{m,K_\mu} \times_{K_\mu} \mathbb{G}_{a,K_\mu}$. By Lemma 7.3 we get that $C_0' = [m'](C)$ is Zariski-dense in $\mathbb{G}^r_{m,K_\mu} \times_{K_\mu} \mathbb{G}_{a,K_\mu}$, too. Therefore, the translate $h'C_0'$ of C_0' is still Zariski-dense in $\mathbb{G}^r_{m,K_\mu} \times_{K_\mu} \mathbb{G}_{a,K_\mu}$. So H contains a Zariski-dense subset, but this is a contradiction. □

Corollary 7.6. *Let L, T and G be as in Theorem 7.4. Let X be a linear algebraic group over L and let $\varphi\colon X \to T$ be a surjective morphism of algebraic groups over L. Assume that every connected component of $\ker(\varphi)$ contains an L-rational point. Let T^c be a connected component of T and let $H \subset \varphi^{-1}(T^c)$ be a Zariski-closed subset which does not contain any irreducible component of $\varphi^{-1}(T^c)$. Then the set of those $g \in G \cap T^c$ such that H contains a connected component of $\varphi^{-1}(g)$ is finite.*

Proof. Let $S \subset G \cap T^c$ be the set of all those elements for which H contains a connected component of $\varphi^{-1}(g)$. For every connected component $\ker(\varphi)^b$ of $\ker(\varphi)$ let $x_b \in \ker(\varphi)^b(L)$ be the L-rational point whose existence was assumed. Then $\ker(\varphi)^b = x_b \cdot \ker(\varphi)^\circ$. We claim that the closed subscheme $\bigcup_b x_b \cdot H$ contains $\varphi^{-1}(g)$ for every $g \in S$. Namely, let $g \in S$ and let C be a connected component of $\varphi^{-1}(g)$ which is contained in H. Let $x \in \varphi^{-1}(g)(\overline{L})$ and let $y \in C(\overline{L})$ be points with values in an algebraic closure $\overline{L}$ of L. Then $\varphi(xy^{-1}) = \varphi(x) \cdot \varphi(y)^{-1} = g \cdot g^{-1} = 1$, and so $xy^{-1} \in \ker(\varphi)$. Let $\ker(\varphi)^b = x_b \cdot \ker(\varphi)^\circ$ be the connected component containing xy^{-1}. By multiplying y on the left with the element $x_b^{-1} \cdot xy^{-1} \in \ker(\varphi)^\circ(\overline{L})$, and hence y^{-1} on the right with $yx^{-1} \cdot x_b$, we can assume that $xy^{-1} = x_b$ without changing that $y \in C$. Then $x = x_b \cdot y \in x_b \cdot C \subset x_b \cdot H$ proving the claim. In particular $\bigcup_b x_b \cdot H$ contains the Zariski-closure W of $\varphi^{-1}(S)$.

Now let V be the Zariski-closure of S in T^c. Since $\varphi^{-1}(S)$ is invariant under the translation by the closed subgroup $\ker(\varphi)$, the same holds for the Zariski-closure W of $\varphi^{-1}(S)$, and hence $W = \varphi^{-1}(\varphi(W))$. Therefore, $\varphi(W)$ is a Zariski-closed subset of T^c which contains S, so it contains V. We get that $\varphi^{-1}(V) \subset \varphi^{-1}(\varphi(W)) = W$. If S were infinite then $V = T^c$ by Theorem 7.4, and hence $\varphi^{-1}(T^c) \subset W$ is contained in $\bigcup_b x_b \cdot H$. Let X' be an irreducible component of $\varphi^{-1}(T^c)$. Then X' is contained in $\bigcup_b x_b \cdot H$, and since it is irreducible it is contained in $x_b \cdot H$ for one x_b. But this implies that H contains the irreducible component $x_b^{-1}X'$ of $\varphi^{-1}(T^c)$, which is a contradiction. Therefore, S is finite. $\square$

To prove Theorem 1.8 we will need the following result of Oesterlé [Oes82]. Let $\mathrm{ord}_p\colon \mathbb{Q}_p^\times \to \mathbb{Z}$ be the valuation on $\mathbb{Q}_p$ with $\mathrm{ord}_p(p) = 1$. Let $\mathbb{Z}_p\langle z_1, \ldots, z_N\rangle$ denote the integral Tate ring of restricted power series in N variables, i.e. the ring of formal power series whose coefficients converge to zero p-adically, and let $\mathbb{Q}_p\langle z_1, \ldots, z_N\rangle = \mathbb{Z}_p\langle z_1, \ldots, z_N\rangle \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ denote the Tate algebra over $\mathbb{Q}_p$ in N variables. Let $0 \neq f \in \mathbb{Q}_p\langle z_1, \ldots, z_N\rangle$ be a power series and let

$$Z(f) \;:=\; \{\, x = (x_1, \ldots, x_N) \in \mathbb{Z}_p^N\colon f(x) = 0 \,\} \;\subset\; \mathbb{Z}_p^N$$

be the analytic hyper-surface defined by f. Note that $\dim Z(f) = N-1$ by [BGR84, § 5.2.4, Proposition 1, § 5.2.2, Theorem 1, § 5.2.3, Proposition 3 and the Remark after § 6.1.2, Corollary 2]. For all $\nu \in \mathbb{N}_{>0}$ let $Z(f)_\nu$ denote the image of $Z(f)$ in $(\mathbb{Z}_p/p^\nu\mathbb{Z}_p)^N$. Assume that f is normalized in such a way that all its coefficients are in $\mathbb{Z}_p$, but not all are in $p\mathbb{Z}_p$. (Of course this is possible after multiplying f by a suitable constant.) The reduction of f modulo p is a non-zero polynomial in $z_1, \ldots, z_N$ with coefficients in $\mathbb{F}_p = \mathbb{Z}_p/(p)$. Let $\deg(f)$ denote the degree of this polynomial and call it the *Oesterlé degree* of f. Then Oesterlé [Oes82, Theorem 4] proves the following inequality on the cardinality of $Z(f)_\nu$:

$$\#Z(f)_\nu \;\leq\; \deg(f)p^{\nu(N-1)} \quad \text{for every } \nu > 0. \tag{7.3}$$

To apply this result we need a bound on the Oesterlé degree $\deg(f)$ of f. To this end consider the following situation. Let $|\cdot|\colon \mathbb{Q}_p \to p^{\mathbb{Z}} \cup \{0\} \subset \mathbb{R}_{\geq 0}$ denote the p-adic norm on $\mathbb{Q}_p$ satisfying $|p| = 1/p$. Let V be a finite-dimensional vector space over $\mathbb{Q}_p$ equipped with a p-adic ultra-metric norm $\|\cdot\|$. We assume that $\|\cdot\|$ is normalized such that $\|V\| = p^{\mathbb{Z}} \cup \{0\}$. Let $B \subset V$ be the unit sphere with respect to this norm:

$$B = \{v \in V\colon \|v\| = 1\}.$$

Note that by our normalization every vector $0 \neq v \in V$ has a multiple which lies in B. The norm $\|\cdot\|$ induces a norm on the dual space $V^\vee = \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p)$ which we will also denote by $\|\cdot\|$ by slight abuse of notation. It is defined for $c \in V^\vee$ by $\|c\| := \inf\{M \in \mathbb{R}\colon |c(v)| \leq M\|v\| \text{ for all } v \in V\} = \sup\{\, |c(v)|\colon v \in B\,\}$, see [Sch84, Proposition 13.5], and hence satisfies $|c(v)| \leq \|c\| \cdot \|v\|$ for every $v \in V$.

Lemma 7.7. *Let c_n, $n \in \mathbb{N}$ be an infinite sequence of $\mathbb{Q}_p$-linear maps $c_n : V \to \mathbb{Q}_p$ such that $\|c_n\| \leq 1$. Assume that for every $n \in \mathbb{N}$ there is an $x_n \in B$ such that $|c_n(x_n)| = 1$. Then there is an $x \in B$ such that $|c_n(x)| = 1$ for all but finitely many n.*

Proof. Using that B is compact, we may assume that the sequence x_n converges to some $x \in B$, by taking an infinite subsequence, if it is necessary. Then

$$|c_n(x) - c_n(x_n)| = |c_n(x - x_n)| \leq \|c_n\| \cdot \|x - x_n\| \leq \|x - x_n\|.$$

Since the right hand side converges to 0 as $n \to \infty$, we get that $|c_n(x)| = |c_n(x_n)| = 1$ for sufficiently large n. $\square$

The bound on the Oesterlé degree mentioned above will be provided by the following

Proposition 7.8. *Let V be a finite dimensional $\mathbb{Q}_p$-linear vector subspace of the ring $\mathbb{Q}_p\langle z_1, z_2, \ldots, z_N\rangle$. Then there is a constant c_V only depending on V such that for every non-zero $f \in V$ the Oesterlé degree $\deg(f)$ of f is at most c_V.*

Proof. Let $\|\cdot\|$ denote the Gauss norm on $\mathbb{Q}_p\langle z_1, \ldots, z_N\rangle$ and restrict it to V. Note that it is also the supremum norm, considered as functions on $\mathbb{Z}_p^N$. Let $B \subset V$ be the unit sphere as above. For every multi-index $\underline{k} = (k_1, \ldots, k_N) \in \mathbb{N}_0^N$ let $c_{\underline{k}}\colon V \to \mathbb{Q}_p$ be the $\underline{k}$-th coefficient, that is $f = \sum_{\underline{k}} c_{\underline{k}}(f)\, z_1^{k_1} \cdots z_N^{k_N}$ for every $f \in V$. Then $|c_{\underline{k}}(f)| \leq 1$ for every $f \in B$ and every $\underline{k}$ by definition of the Gauss norm. So $c_{\underline{k}}$ is a $\mathbb{Q}_p$-linear form on V of norm at most 1. Now assume that the claim is false. This means that there is an infinite sequence of multi-indices $\underline{k}_1, \underline{k}_2, \ldots, \underline{k}_n, \ldots$ such that for every $n \in \mathbb{N}$ there is an $f_n \in B$ such that $|c_{\underline{k}_n}(f_n)| = 1$. By Lemma 7.7 there is an $f \in B$ such that $|c_{\underline{k}_n}(f)| = 1$ for infinitely many n. But since $f \in \mathbb{Q}_p\langle z_1, z_2, \ldots, z_N\rangle$ the coefficients of f converge to 0, which is a contradiction. $\square$

For the rest of the section we use the following

Notation 7.9. Let L be a finite field extension of $\mathbb{Q}_p$ and let T be a commutative linear algebraic group over L which is the product of a split torus $\mathbb{G}_{m,L}^r$ for $r \geq 0$ with a finite abelian group and possibly an additional factor $\mathbb{G}_{a,L}$. Let $G \subset T(L)$ be a Zariski-dense subgroup in T, which is infinite cyclic, $G \cong \mathbb{Z}$, and let g be a generator of G. We write the group G additively as $G = \{ng \text{ for } n \in \mathbb{Z}\}$. Let Y and Z be linear algebraic groups over L such that Z is of the same kind as T and assume that there are two surjective homomorphisms $\varphi_1 \colon Y \to Z$ and $\varphi_2 \colon T \to Z$ of algebraic groups over L. Let $X := Y \times_Z T$ be the fiber product of these two maps. Note that X is a closed subgroup of $Y \times_L T$. We denote by $pr_1 \colon X \to Y$ and $pr_2 \colon X \to T$ the maps obtained as the restriction to X of the projection from $Y \times_L T$ onto the first and onto the second factor, respectively. Let $C \subset Y(L)$ be a compact, but not necessarily open subgroup with respect to the p-adic topology, let $Y^{geo} \subset Y$ be the kernel of φ_1 and set $C^{geo} = Y^{geo}(L) \cap C$. Finally let $F \subset (C \times G) \cap X(L)$ be a subset.

$$\begin{tikzcd}
F \subset (C\times G)\cap X(L) \subset & X = Y\times_Z T \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] \arrow[rr, hookrightarrow] & & Y\times_L T \\
C \subset Y(L) \subset Y \arrow[dr, twoheadrightarrow, "\varphi_1"'] & & T \supset T(L) \supset G \ni ng\,, \arrow[dl, twoheadrightarrow, "\varphi_2"] & C\times G \arrow[u, hookrightarrow] \\
& Z & &
\end{tikzcd}$$

Note that pr_1 induces an isomorphism between $\ker(pr_2) \subset X$ and Y^{geo} whose inverse is given as

$$Y^{geo} \xrightarrow{\sim} \ker(pr_2) \subset X \subset Y \times_Z T\,, \quad y \mapsto (y,1)\,. \tag{7.4}$$

Theorem 7.10. *Assume that the following hold:*

(a) the Zariski-closure of C^{geo} is Y^{geo},

(b) there is a positive constant $\varepsilon > 0$ such that for every open normal subgroup $D \subset C$ there is an infinite subset $S_D \subset \mathbb{N}$ such that for every $n \in S_D$ the image of $pr_1(F \cap pr_2^{-1}(ng)) \subset C$ under the quotient map $C \to C/D$ has cardinality at least $\varepsilon \cdot \#(C^{geo}/D \cap C^{geo})$.

Then the Zariski-closure of F contains a connected component of X.

We begin with the preparations to prove Theorem 7.10. Note that condition (a) and the isomorphism (7.4) imply that every connected component of $\ker(pr_2)$ contains an L-rational point. Let $(Y^{geo})^\circ$ be the connected component of identity of Y^{geo}. Since C is compact, and C^{geo} and $C^{geo} \cap (Y^{geo})^\circ$ are closed subgroups we get that C^{geo} and $C^{geo} \cap (Y^{geo})^\circ$ are also compact. Let $\widetilde{Y} = \mathrm{Res}_{L/\mathbb{Q}_p} Y$ be the Weil restriction of Y. By [CGP10, Propositions A.5.1. and A.5.2] it is a smooth linear algebraic group scheme over $\mathbb{Q}_p$. Then $C^{geo} \cap (Y^{geo})^\circ \subset C^{geo} \subset C \subset Y(L) = \widetilde{Y}(\mathbb{Q}_p)$, and C and C^{geo} and $C^{geo} \cap (Y^{geo})^\circ$ are Lie groups over $\mathbb{Q}_p$ by Theorem 7.1. Note that $C^{geo} \cap (Y^{geo})^\circ \subset C^{geo}$ has finite index, and hence is open, because it is the kernel of the homomorphism $C^{geo} \to Y^{geo}/(Y^{geo})^\circ$. By [Ser92, Part II, § IV.8, Theorem] there is an open subgroup C_0 of $C^{geo} \cap (Y^{geo})^\circ$ which is standard in the sense of Definition 7.2. This means that there is a formal group law F in N variables over $\mathbb{Z}_p$ and an isomorphism

$$\psi \colon G_F \xrightarrow{\sim} C_0 \tag{7.5}$$

of Lie groups over $\mathbb{Q}_p$, where $G_F = \mathbb{Z}_p^N$ is equipped with the group law (7.1) given by F. Since C^{geo} is compact the index $[C^{geo} : C_0]$ is finite.

Lemma 7.11. *Under the assumptions of Theorem 7.10 every translate of C_0 in Y^{geo} is Zariski-dense in the connected component of Y^{geo} which contains it.*

Proof. The Zariski-closure of a translate of C_0 is the translate of the Zariski-closure of C_0, so it will be enough to see that C_0 is Zariski-dense in $(Y^{geo})^\circ$. Since C^{geo} is Zariski-dense in Y^{geo} by assumption (a) of Theorem 7.10, also $C^{geo} \cap (Y^{geo})^\circ$ is Zariski-dense in $(Y^{geo})^\circ$. Let $\widetilde{C}_0$ be the largest subgroup of C_0 which is normal in $C^{geo} \cap (Y^{geo})^\circ$. Since C_0 has finite index in $C^{geo} \cap (Y^{geo})^\circ$, the same holds for $\widetilde{C}_0$. Let i be this index. Then the i-power map maps $C^{geo} \cap (Y^{geo})^\circ$ into $\widetilde{C}_0$, and therefore $\widetilde{C}_0$, and hence also C_0, is Zariski-dense in $(Y^{geo})^\circ$ by Lemma 7.3. $\square$

To establish Theorem 7.10 we will choose closed embeddings $Y \subset \mathbb{A}^a_L$ and $T \subset \mathbb{A}^b_L$, and consider the induced embedding $X \subset Y \times_L T \subset \mathbb{A}^{a+b}_L$. Recall that by [Ser92, Part II, § IV.9, Theorem 1] for every $\nu \in \mathbb{N}_{>0}$ the subset $p^\nu \mathbb{Z}_p^N \subset G_F = \mathbb{Z}_p^N$ is actually an open normal subgroup under the group law (7.1) given by F, and two elements x, y of G_F are congruent to each other modulo the subgroup $p^\nu \mathbb{Z}_p^N$ if and only if $x \equiv y \bmod p^\nu$.

Lemma 7.12. *Let T^c be a connected component of T and let $H \subset pr_2^{-1}(T^c) \subset \mathbb{A}^{a+b}_L$ be a Zariski-closed subset which does not contain any irreducible component of $pr_2^{-1}(T^c)$. Then there is a positive integer $d_{c,H}$ such that for all but finitely many $n \in \mathbb{Z}$ and for every $\nu \in \mathbb{N}_{>0}$ the image of $pr_1(H \cap pr_2^{-1}(ng)) \cap C$ under the quotient map $C \twoheadrightarrow C/\psi(p^\nu \mathbb{Z}_p^N)$ has cardinality at most $d_{c,H} \cdot p^{\nu(N-1)}$.*

Proof. We can write $H \subset \mathbb{A}^{a+b}_L$ as an intersection of finitely many hyper-surfaces defined by polynomials $h_\ell = h_\ell(\underline{y}, \underline{t}) \in L[\underline{y}, \underline{t}]$, where $\underline{y}$ and $\underline{t}$ denote the coordinates on $\mathbb{A}^a_L$ and $\mathbb{A}^b_L$, respectively. Let d_H be the maximum of the degrees of these polynomials. Note that for $ng \notin G \cap T^c$ the lemma holds trivially, because then $H \cap pr_2^{-1}(ng)$ is empty. On the other hand, Corollary 7.6 implies that for all but finitely many $ng \in G \cap T^c$ the intersection $H \cap pr_2^{-1}(ng)$ does not contain an entire connected component of $pr_2^{-1}(ng)$. We now fix an $ng \in G \cap T^c$ for which this holds.

We claim that $X \cap (C \times \{ng\})$ is either a $C^{geo} \times \{1\}$-coset in $C \times \{ng\}$ or empty (in which case the assertion of the lemma again holds trivially). Indeed, if this set is non empty, let $x_n = (y_n, ng)$ be a point in it with $y_n = pr_1(x_n) \in C$. Then $\varphi_1(y_n) = \varphi_1 pr_1(x_n) = \varphi_2 pr_2(x_n) = \varphi_2(ng)$ and so every other point $\tilde{x}_n = (\tilde{y}_n, ng) \in X \cap (C \times \{ng\})$ satisfies $x_n \cdot (y_n^{-1}\tilde{y}_n, 1) = \tilde{x}_n$ with $\varphi_1(y_n^{-1}\tilde{y}_n) = \varphi_1(y_n^{-1}) \cdot \varphi_1(\tilde{y}_n) = \varphi_2(ng)^{-1} \cdot \varphi_2(ng) = 1$, that is $y_n^{-1}\tilde{y}_n \in C \cap Y^{geo} = C^{geo}$. This implies $X \cap (C \times \{ng\}) \subset x_n \cdot (C^{geo} \times \{1\})$. For the converse inclusion note that for every $c \in C^{geo}$ the point $x_n \cdot (c, 1) = (y_n c, ng) \in (Y \times_L T)(L)$ lies in X because $\varphi_1(y_n c) = \varphi_1(y_n) = \varphi_2(ng)$. This proves that $X \cap (C \times \{ng\}) = x_n \cdot (C^{geo} \times \{1\})$. So $X \cap (C \times \{ng\})$ is the pairwise disjoint union of m cosets $x_{n,1} \cdot (C_0 \times \{1\}), \ldots, x_{n,m} \cdot (C_0 \times \{1\})$ of $C_0 \times \{1\}$, where we let $m := [C^{geo} : C_0]$ be the index.

By Lemma 7.11 this means that for every such coset $C' = x_{n,i} \cdot (C_0 \times \{1\})$ the intersection $H \cap C'$ is a proper subset of C'. Namely, the lemma says that $pr_1(x_{n,i}^{-1} \cdot C') = C_0$ is Zariski-dense in $(Y^{geo})^\circ$. Under the isomorphism from (7.4) this implies that $x_{n,i}^{-1} \cdot C'$ is Zariski-dense in the unity component $\ker(pr_2)^\circ$, and so C' is Zariski-dense in $x_{n,i} \cdot \ker(pr_2)^\circ$. The latter is a connected component of $pr_2^{-1}(ng)$ and not contained in H by our assumption on ng. Therefore, $H \cap C'$ is a proper closed subset of C' cut out by the finitely many polynomials $h_\ell(\underline{y}, ng) \in L[\underline{y}]$ of degree $\leq d_H$ obtained from $h_\ell(\underline{y}, \underline{t})$ by plugging in (the coordinates of) the point ng. Under the projection pr_1, which induces like in (7.4) an isomorphism $pr_1 \colon pr_2^{-1}(ng) \xrightarrow{\sim} pr_1(x_{n,i}) \cdot Y^{geo}$ of varieties, $pr_1(H \cap C')$ is a proper closed subset of $pr_1(C')$ cut out by the same polynomials $h_\ell(\underline{y}, ng) \in L[\underline{y}]$. We consider the images $\bar{h}_\ell(\underline{y}, ng)$ of these $h_\ell(\underline{y}, ng)$ in the coordinate ring $L[Y]$ of Y.

For every such coset $C' = x_{n,i} \cdot (C_0 \times \{1\})$, we consider the subset $H'_{n,i} = pr_1(x_{n,i})^{-1} \cdot pr_1(H \cap C') \subset C_0 \subset Y^{geo}$, which is a proper subset of C_0 cut out by finitely many hyper-surfaces of Y^{geo}. Namely, if we set $y_{n,i} := pr_1(x_{n,i}) \in C$, then $H'_{n,i}$ is cut out by the pullbacks $\bar{h}'_{\ell,n,i} := t^*_{y_{n,i}}(\bar{h}_\ell(\underline{y}, ng)) \in L[Y]$ of the polynomials $\bar{h}_\ell(\underline{y}, ng)$ under the translation $t_{y_{n,i}}$ by $y_{n,i}$. Let $\overline{W} \subset L[Y]$ be the L-linear vector subspace spanned by $t^*_y(\bar{h})$ for all $y \in Y(L)$ and all $\bar{h} \in L[Y]$, which are images of polynomials $h \in L[\underline{y}]$ of degree $\leq d_H$. Then $\overline{W}$ has finite dimension by [Bor91, I.1.9 Proposition] which only depends on d_H, and the polynomials $\bar{h}'_{\ell,n,i}$ cutting out $H'_{n,i}$ in C_0 belong to $\overline{W}$. Let $W \subset L[\underline{y}]$ be a finite dimensional L-linear subspace that surjects onto $\overline{W}$ and choose preimages $h'_{\ell,n,i} \in W$ of all $\bar{h}'_{\ell,n,i}$. Then $H'_{n,i}$ is cut out in C_0 by the polynomials $h'_{\ell,n,i}$. We note that $pr_1(H \cap pr_2^{-1}(ng)) \cap C$ equals the disjoint union $\coprod_{i=1}^m y_{n,i} \cdot H'_{n,i}$ by using again the isomorphism $pr_1 \colon pr_2^{-1}(ng) \xrightarrow{\sim} pr_1(x_{n,i}) \cdot Y^{geo}$. So it remains to count the elements in the finite set $H'_{n,i}/\psi(p^\nu \mathbb{Z}_p^N)$ or equivalently its preimage $\psi^{-1}(H'_{n,i})/(p^\nu \mathbb{Z}_p)^N$ under the isomorphism ψ from (7.5).

We claim that this preimage is contained in a proper analytic hyper-surface Z in $G_F = \mathbb{Z}_p^N$ whose degree is bounded independently of n and $H'_{n,i}$ by a constant depending only on the degree d_H and the isomorphism ψ. Indeed, we choose a $\mathbb{Q}_p$-basis $(\alpha_1, \ldots, \alpha_s)$ of L, where we set $s = [L : \mathbb{Q}_p]$. The map $\psi \colon G_F \xrightarrow{\sim} C_0 \subset Y(L) \subset (\mathbb{A}^a_L)(L)$ is given with respect to coordinates on $\mathbb{A}^a_L$ by power series $\psi_1, \ldots, \psi_a \in L\langle z_1, \ldots, z_N \rangle$ which converge for every $x = (x_1, \ldots, x_N)$ in $G_F = \mathbb{Z}_p^N$. Writing

$h'_{\ell,n,i} \in W$ for the polynomial equations cutting out $H'_{n,i} \subset C_0$ we see that $\psi^{-1}(H'_{n,i})$ is the zero locus of the $h'_{\ell,n,i}(\psi_1,\ldots,\psi_a) \in L\langle z_1,\ldots,z_N\rangle$. With respect to the $\mathbb{Q}_p$-basis $(\alpha_j)_j$ of L we can write $h'_{\ell,n,i}(\psi_1,\ldots,\psi_a) = \sum_j \alpha_j \cdot f_{\ell,n,i,j}$ with $f_{\ell,n,i,j} \in \mathbb{Q}_p\langle z_1,\ldots,z_N\rangle$. Then $\psi^{-1}(H'_{n,i})$ is the simultaneous zero locus of all $f_{\ell,n,i,j}$. More precisely, we view $h'_{\ell,n,i}$ as a morphism $Y \to \mathbb{A}_L$ and consider its Weil restriction $\mathrm{Res}_{L/\mathbb{Q}_p} h'_{\ell,n,i}\colon \mathrm{Res}_{L/\mathbb{Q}_p} Y \to \mathrm{Res}_{L/\mathbb{Q}_p} \mathbb{A}_L$. Here $\mathrm{Res}_{L/\mathbb{Q}_p} \mathbb{A}_L$ is the Weil restriction of $\mathbb{A}_L$, which is isomorphic to $\mathbb{A}^s_{\mathbb{Q}_p}$ under the identification $(\mathrm{Res}_{L/\mathbb{Q}_p} \mathbb{A}_L)(\mathbb{Q}_p) = L = \bigoplus_j \alpha_j \mathbb{Q}_p$. Then $\psi^{-1}(H'_{n,i})$ is the simultaneous zero locus of all the morphisms (for all ℓ)

$$\begin{array}{ccccccccc} G_F & \xrightarrow[\sim]{\psi} & C_0 & \subset & (\mathrm{Res}_{L/\mathbb{Q}_p} Y)(\mathbb{Q}_p) & \xrightarrow{\mathrm{Res}_{L/\mathbb{Q}_p} h'_{\ell,n,i}} & (\mathrm{Res}_{L/\mathbb{Q}_p} \mathbb{A}_L)(\mathbb{Q}_p) & \cong & \mathbb{A}^s_{\mathbb{Q}_p}(\mathbb{Q}_p)\,. \\ x & \longmapsto & & & & & \big(f_{\ell,n,i,j}(x_1,\ldots,x_N)\big)_{j=1,\ldots,s} & & \end{array}$$

Since $H'_{n,i} \neq C_0$, at least one $f_{\ell,n,i,j}$ is non-zero. Let $V \subset \mathbb{Q}_p\langle z_1,\ldots,z_N\rangle$ be the $\mathbb{Q}_p$-vector space generated by all f_j where h' runs through a $\mathbb{Q}_p$-basis of W and $h'(\psi_1,\ldots,\psi_a) = \sum_j \alpha_j \cdot f_j$. Then V is a finite dimensional $\mathbb{Q}_p$-vector space which only depends on d_H and ψ and not on ng and $x_{n,i}$. By Proposition 7.8 there is a constant c_V such that the Oesterlé degree $\deg(f) \le c_V$ for all $0 \neq f \in V$. It now follows from Oesterlé's result (7.3) that the cardinality of $H'_{n,i}/\psi(p^\nu \mathbb{Z}_p^N)$ is at most $c_V\, p^{\nu(N-1)}$. Thus the image of $pr_1(H \cap pr_2^{-1}(ng)) \cap C$ under the quotient map $C \twoheadrightarrow C/\psi(p^\nu\mathbb{Z}_p^N)$ has cardinality at most $m\, c_V\, p^{\nu(N-1)}$. $\square$

After these preparations it is easy to finish the

Proof of Theorem 7.10. To establish Theorem 7.10 we assume to the contrary that the Zariski-closure of F does not contain any connected component of X. Fix a connected component T^c of T and let $H \subset pr_2^{-1}(T^c)$ be the Zariski-closure of $F \cap pr_2^{-1}(T^c)$. Then the assumption implies that H does not contain any irreducible component of $pr_2^{-1}(T^c)$. Let $d_{c,H}$ be the positive integer from Lemma 7.12. Then for all but finitely many $n \in \mathbb{N}$ and for every $\nu \in \mathbb{N}_{>0}$ the image of the set $pr_1(F \cap pr_2^{-1}(ng))$ under the quotient map $C \twoheadrightarrow C/\psi(p^\nu\mathbb{Z}_p^N)$ has cardinality at most $d_{c,H} \cdot p^{\nu(N-1)}$. Let d be the sum $\sum_{T^c} d_{c,H}$ of the $d_{c,H}$ over all connected components T^c of T. Taking the union over all T^c we see that for all but finitely many $n \in \mathbb{N}$ and for every $\nu \in \mathbb{N}_{>0}$ the image of the set $pr_1(F \cap pr_2^{-1}(ng)) \subset C$ under the quotient map $C \twoheadrightarrow C/\psi(p^\nu\mathbb{Z}_p^N)$ has cardinality at most $d\, p^{\nu(N-1)}$.

Now let $\varepsilon > 0$ be the constant from assumption (b) of Theorem 7.10 and choose ν so large that $\varepsilon\, p^{\nu N} > d\, p^{\nu(N-1)}$. We consider the subgroup $D_1^{geo} := \psi(p^\nu\mathbb{Z}_p^N) \subset C^{geo}$ which is open in C^{geo}. Then $\#(C^{geo}/D_1^{geo}) \ge \#(C_0/D_1^{geo}) = \#(\mathbb{Z}_p/p^\nu\mathbb{Z}_p)^N = p^{\nu N}$. Since C^{geo} carries the subspace topology induced from C and the topology on C is the topology of C as a profinite group by [Ser92, Part II, § IV.8, Corollary 2], there is an open normal subgroup $D \subset C$ such that $D^{geo} := C^{geo} \cap D \subset D_1^{geo}$. By assumption (b) of Theorem 7.10 there is an infinite subset $S_D \subset \mathbb{N}$ such that for every $n \in S_D$ the image of $pr_1(F \cap pr_2^{-1}(ng)) \subset C$ under the quotient map $C \to C/D$ has cardinality at least $\varepsilon \cdot \#(C^{geo}/D^{geo})$. Since the map $C \to C/D$ factors through $C \to C/D^{geo} \to C/D$, the image of $pr_1(F \cap pr_2^{-1}(ng)) \subset C$ under the quotient map $C \to C/D^{geo}$ has also cardinality at least $\varepsilon \cdot \#(C^{geo}/D^{geo})$. The fibers of the map $C/D^{geo} \twoheadrightarrow C/D_1^{geo}$ are principal homogeneous spaces under the group D_1^{geo}/D^{geo}. This implies that the number of points in the image of $pr_1(F \cap pr_2^{-1}(ng))$ in C/D^{geo} which are mapped to the same point in C/D_1^{geo} is at most $\#(D_1^{geo}/D^{geo})$. In particular, the image of $pr_1(F \cap pr_2^{-1}(ng))$ in C/D_1^{geo} has cardinality at least $\varepsilon \cdot \#(C^{geo}/D^{geo})/\#(D_1^{geo}/D^{geo}) = \varepsilon \cdot \#(C^{geo}/D_1^{geo}) \ge \varepsilon\, p^{\nu N}$. But this contradicts the estimate from the previous paragraph. This completes the proof of Theorem 7.10. $\square$

We are now ready to give the

Proof of Theorem 1.8. Let $\mathcal{F} = \bigoplus_i \mathcal{F}_i$ be a direct sum of isoclinic convergent F-isocrystals $\mathcal{F}_i$ on U. Let $\frac{m_i}{n}$ with $m_i \in \mathbb{Z}$ and $n \in \mathbb{N}_{>0}$ be the slope of $\mathcal{F}_i$. Then the image $(\mathcal{F}_i)^{(n)}$ of $\mathcal{F}_i$ under the functor (3.1) is isoclinic of slope m_i and $\mathcal{U}_i := (\mathcal{F}_i)^{(n)} \otimes \mathcal{C}_i^\vee$ is unit-root where $\mathcal{C}_i$ is the pullback to U_n of the constant F^n-isocrystal on $\operatorname{Spec} \mathbb{F}_{q^n}$ given by $(K_n, F^n = \pi^{m_i})$. Then $\mathcal{U} := \bigoplus_i \mathcal{U}_i$ is unit-root and $\mathcal{C} := \bigoplus_i \mathcal{C}_i$ is a direct sum of constant F^n-isocrystals of rank one. Moreover, $\mathcal{F}^{(n)}$ lies in the Tannakian category $\langle\!\langle \mathcal{U} \oplus \mathcal{C} \rangle\!\rangle$, and so the monodromy group $\mathrm{Gr}(\mathcal{U} \oplus \mathcal{C}/U_n, u)$ surjects onto $\mathrm{Gr}(\mathcal{F}^{(n)}/U_n, u)$. Here we assume that e is divisible by n and that $u \in U_n(\mathbb{F}_{q^e})$. Note that every connected component of $\mathrm{Gr}(\mathcal{F}^{(n)}/U_n, u) \times_{K_e} \overline{K}$ maps isomorphically onto a connected component of $\mathrm{Gr}(\mathcal{F}/U, u) \times_{K_e} \overline{K}$ by Lemma 6.3(c). By Lemma 3.4

it will be enough to see that Conjecture 1.4 holds for $\mathcal{U} \oplus \mathcal{C}$. Let $\langle\langle \mathcal{U} \rangle\rangle_{const}$ be the full Tannakian subcategory of constant F-isocrystals in $\langle\langle \mathcal{U} \rangle\rangle$. It is generated by some constant unit-root F-isocrystal $\widetilde{\mathcal{C}}$. Then $\langle\langle \widetilde{\mathcal{C}} \rangle\rangle \subset \langle\langle \mathcal{U} \rangle\rangle \cap \langle\langle \mathcal{C} \oplus \widetilde{\mathcal{C}} \rangle\rangle \subset \langle\langle \mathcal{U} \rangle\rangle_{const} = \langle\langle \widetilde{\mathcal{C}} \rangle\rangle$. We now replace $\mathcal{C}$ by $\mathcal{C} \oplus \widetilde{\mathcal{C}}$ and thus may assume that $\langle\langle \mathcal{U} \rangle\rangle \cap \langle\langle \mathcal{C} \rangle\rangle = \langle\langle \mathcal{U} \rangle\rangle_{const}$. We can assume that $\mathcal{C}$ is itself not unit-root, by adding a constant F^n-isocrystal with non-zero slope to it if necessary. To ease notation we will pretend that $n = 1$ and say again "F-isocrystal" instead of "F^n-isocrystal".

To prove Conjecture 1.4 let $S \subset |U|$ be a subset of positive upper Dirichlet density. Choose a geometric base point $\bar{u}$ above u and let $\pi_1^{\text{ét}}(U, \bar{u})$ be the étale fundamental group of U. Let $\rho\colon \pi_1^{\text{ét}}(U, \bar{u}) \to \mathrm{GL}_r(K)$ be the representation corresponding to the unit-root F-isocrystal $\mathcal{U}$ under the canonically tensor equivalence between $F\text{-UR}_K(U) \subset F\text{-Isoc}_K(U)$ and $\mathrm{Rep}^{\mathrm{c}}_K\, \pi_1^{\text{ét}}(U, \bar{u})$ of Proposition 5.2. By Corollary 5.4 the restriction of this tensor equivalence onto $\langle\langle \mathcal{U} \rangle\rangle$ induces a tensor equivalence between $\langle\langle \mathcal{U} \rangle\rangle$ and $\langle\langle \rho \rangle\rangle$, and over a finite field extension L of K_e there is an isomorphism $\beta\colon \omega_f|_{\langle\langle \rho \rangle\rangle} \otimes_K L \xrightarrow{\sim} \omega_u|_{\langle\langle \mathcal{U} \rangle\rangle} \otimes_{K_e} L$ between the forgetful fiber functor ω_f on $\langle\langle \rho \rangle\rangle$ and the fiber functor ω_u on $\langle\langle \mathcal{U} \rangle\rangle$.

Let $Y = \mathrm{Gr}(\mathcal{U}/U, u)$ and $T = \mathrm{Gr}(\mathcal{C}/U, u)$, and let $X = \mathrm{Gr}(\mathcal{U} \oplus \mathcal{C}/U, u)$ and $Z = \mathrm{Gr}(\langle\langle \mathcal{U} \rangle\rangle \cap \langle\langle \mathcal{C} \rangle\rangle/U, u) = \mathrm{Gr}(\langle\langle \mathcal{U} \rangle\rangle_{const}/U, u)$. Then $Y^{geo} := \ker(Y \to Z)$ is the geometric monodromy group $\mathrm{Gr}(\mathcal{U}/U, u)^{geo}$ of $\mathcal{U}$ from Definition 4.9, and X is the fiber product $Y \times_Z T$ by Proposition 3.6(c). After enlarging L if necessary, the groups Z and T are each the product of a split torus with a finite group and possibly an additional factor $\mathbb{G}_{a,K_e}$ by Theorem 4.8(b) and [Bor91, III.8.11 Proposition]. The isomorphism β induces an isomorphism $\beta_*\colon\ \mathrm{Aut}^{\otimes}(\omega_f|_{\langle\langle \rho \rangle\rangle}) \times_K L \xrightarrow{\sim} \mathrm{Gr}(\mathcal{U}/U, u) \times_{K_e} L$ and the image $C := \beta_* \circ \rho\big(\pi_1^{\text{ét}}(U, \bar{u})\big)$ of the induced representation $\beta_* \circ \rho\colon \pi_1^{\text{ét}}(U, \bar{u}) \to \mathrm{Gr}(\mathcal{U}/U, u)(L)$ is dense in $\mathrm{Gr}(\mathcal{U}/U, u) \times_{K_e} L$ by Corollary 5.4. This image is a compact group, because $\pi_1^{\text{ét}}(U, \bar{u})$ is pro-finite. By adding a constant unit-root F-isocrystal to $\mathcal{U}$ as in Corollary 5.10, we can assume that $C^{geo} := Y^{geo}(L) \cap C$ equals the image $\beta_* \circ \rho\big(\pi_1^{\text{ét}}(U, \bar{u})^{geo}\big)$.

For every $x \in |U|$ the Frobenius conjugacy class $\mathrm{Frob}_x(\mathcal{U}) \subset \mathrm{Gr}(\mathcal{U}/U, u)(\overline{K})$ is by Corollary 5.4 generated by the image with respect to $\beta_* \circ \rho$ of the conjugacy class in $\pi_1^{\text{ét}}(U, \bar{u})$ of the geometric Frobenius Frob_x^{-1} at x. If $g \in T(K_e) \subset T(L)$ is the image of the Frobenius of $\mathcal{C}$ then $\mathrm{Frob}_x(\mathcal{C})$ consists of the single element $\deg(x) \cdot g$ for every $x \in |U|$ by Theorem 4.8(c). Since $\mathcal{C}$ was assumed to be not unit-root, the group $G \subset T(L)$ generated by g is infinite cyclic and Zariski-dense in T, see Theorem 4.8(b). Consider the Frobenius conjugacy class $\mathrm{Frob}_x(\mathcal{U} \oplus \mathcal{C}) \subset \mathrm{Gr}(\mathcal{U} \oplus \mathcal{C}/U, u)(\overline{K})$. By Lemma 3.3 it is mapped to the conjugacy classes of $\mathrm{Frob}_x(\mathcal{U})$ in $Y(\overline{K})$, respectively of $\mathrm{Frob}_x(\mathcal{C})$ in $T(\overline{K})$. Since these have representatives over L, also $\mathrm{Frob}_x(\mathcal{U} \oplus \mathcal{C})$ has a representative in $X(L)$. Thus we set

$$F = \bigcup_{x \in S} \mathrm{Frob}_x(\mathcal{U} \oplus \mathcal{C}) \cap (C \times G) \subset (C \times G) \cap X(L)\,.$$

Theorem 1.8 is therefore a consequence of the following

Claim. The octuple (T, Y, Z, X, C, G, g, F) satisfies the hypothesis of Theorem 7.10.

Denoting the morphism $X \to Y$ by pr_1 and the morphism $X \to T$ by pr_2, we see in particular, that for $n \in \mathbb{N}_{>0}$

$$F \cap pr_2^{-1}(ng) \;=\; \bigcup_{x \in S\colon\ \deg(x) = n} \mathrm{Frob}_x(\mathcal{U} \oplus \mathcal{C}) \cap (C \times G)$$

and

$$\begin{aligned} pr_1(F \cap pr_2^{-1}(ng)) \;&=\; \bigcup_{x \in S\colon\ \deg(x) = n} \mathrm{Frob}_x(\mathcal{U}) \cap C \\ &= \beta_* \circ \rho\left(\bigcup_{x \in S\colon\ \deg(x) = n} \text{conjugacy class of } \mathrm{Frob}_x^{-1} \text{ in } \pi_1^{\text{ét}}(U, \bar{u}) \right). \end{aligned}$$

Therefore, assumption (b) of Theorem 7.10 follows from Theorem 3.16. Moreover, Condition (a), which requires that $C^{geo} := Y^{geo}(L) \cap C = \beta_* \circ \rho\big(\pi_1^{\text{ét}}(U, \bar{u})^{geo}\big)$ is Zariski-dense in Y^{geo}, follows from Corollary 5.8. $\square$

8. The Theory of Maximal Quasi-Tori

We will need some results in the theory of algebraic groups in the following situation. In this entire section L is an algebraically closed field of characteristic 0.

Definition 8.1. For a not necessarily connected linear algebraic group G over L we let G° be its identity component and we identify G with the group $G(L)$ of its L-valued points. As usual we say that G is *reductive* if G° is. For every $h \in G$ and closed subgroup $H \subset G$ let $H^h := Z_{H^\circ}(h) := \{g \in H^\circ \colon gh = hg\}$ denote the centralizer of h in H° and let $H^{h\circ}$ denote the connected component of H^h containing 1. These are closed subgroups of H°.

We will need the following mild generalization of a classical result of Steinberg. It was announced in [KS99, Theorem 1.1A] with a brief sketch of proof. We include a full proof for the convenience of the reader.

Theorem 8.2. *Assume that G is reductive, let $h \in G$ be a semi-simple element which normalizes a Borel subgroup $B \subset G^\circ$ and a maximal torus $T \subset B$. Then G^h is reductive, $T^{h\circ}$ is a maximal torus in $G^{h\circ}$, and $B^{h\circ}$ is a Borel subgroup in $G^{h\circ}$.*

Note that T^h and B^h are not connected in general as can be seen from the following

Example 8.3. Let p be a prime number and let G be the $p-1$-dimensional torus:

$$G = \{(x_1, x_2, \dots, x_p) \in \mathbb{G}_m^p : x_1 \cdot x_2 \cdots x_p = 1\}.$$

Then the cyclic permutation

$$(x_1, x_2, \dots, x_p) \to (x_2, x_3, \dots, x_p, x_1)$$

is an automorphism of G of order p whose fixed points are

$$(\zeta, \zeta, \dots, \zeta),$$

where ζ is any p-th root of unity. The semi-direct product $G \rtimes \mathbb{Z}/p\mathbb{Z}$ where the generator h of $\mathbb{Z}/p\mathbb{Z}$ acts by the automorphism above is a counter-example to the connectivity of both the h-fixed points of a maximal torus and a Borel subgroup of G, since the latter are both equal to G.

Proof of Theorem 8.2. Steinberg [Ste68, Theorem 7.5 on page 51] proved the claim when G° is simply connected. We are going to reduce the general case to this one via two reduction steps. First assume that G° is semi-simple. In this case its étale fundamental group is finite, and hence the same holds for every connected component of G, too. Let $\varphi : \widetilde{G} \to G$ be the universal cover of G, which exits by the remarks above. We may equip $\widetilde{G}$ uniquely with the structure of a linear algebraic group such that φ is a group homomorphism. Let $\widetilde{K} \subset \widetilde{G}$ be the kernel of φ. It is a finite normal subgroup and $\widetilde{K} \cap \widetilde{G}^\circ$ lies in the center of $\widetilde{G}^\circ$ by [CGP10, Corollary A.4.11] and [Bor91, IV.11.21 Proposition].

Pick an element $\widetilde{h} \in \widetilde{G}$ in the pre-image of h, and let $\widetilde{T}, \widetilde{B}$ be the pre-image of T, B in $\widetilde{G}$, respectively. Note that $\widetilde{h}$ is semi-simple, because when we write $\widetilde{h} = \widetilde{h}_s \widetilde{h}_u$ as a product of its semi-simple part $\widetilde{h}_s$ and its unipotent part $\widetilde{h}_u$, then $\widetilde{h}_u$ lies in $\widetilde{K}$ by [Bor91, I.4.4 Theorem], because $\varphi(\widetilde{h}) = h$ is semi-simple. If we denote by n the order of $\widetilde{K}$, then $\widetilde{h}^n = \widetilde{h}_s^n \widetilde{h}_u^n = \widetilde{h}_s^n$ is semi-simple. Therefore, already $\widetilde{h}$ is semi-simple by Lemma 3.7(a).

Since T is connected the restriction $\varphi|_{\widetilde{T}^\circ} : \widetilde{T}^\circ \to T$ is surjective with finite kernel $\widetilde{K} \cap \widetilde{G}^\circ$, therefore the connected group $\widetilde{T}^\circ$ must be a torus. It has the same dimension as T, so it must be a maximal torus in $\widetilde{G}^\circ$, as the ranks of G° and $\widetilde{G}^\circ$ are the same. Likewise $\widetilde{B}^\circ$ is an extension of the solvable group B by the commutative group $\widetilde{K} \cap \widetilde{G}^\circ$. Therefore, $\widetilde{B}^\circ$ is connected solvable and of the same dimension as B, and hence a Borel subgroup of $\widetilde{G}^\circ$. Since

$$\varphi(\widetilde{h}^{-1} \widetilde{T} \widetilde{h}) = \varphi(\widetilde{h}^{-1}) \varphi(\widetilde{T}) \varphi(\widetilde{h}) = h^{-1} T h = T,$$

we get that $\widetilde{h}$ normalizes $\widetilde{T}$. A similar computation shows that $\widetilde{h}$ normalizes $\widetilde{B}$, too. Therefore, by Steinberg's theorem quoted above the subgroup $\widetilde{G}^{\widetilde{h}}$ is reductive, $\widetilde{T}^{\widetilde{h}\circ}$ is a maximal torus in $\widetilde{G}^{\widetilde{h}}$, and $\widetilde{B}^{\widetilde{h}\circ}$ is a Borel subgroup in $\widetilde{G}^{\widetilde{h}}$.

The regular map

$$\varphi^{-1}(G^h) \to \widetilde{K}, \quad \widetilde{x} \mapsto \widetilde{h}^{-1} \widetilde{x} \widetilde{h} \widetilde{x}^{-1}$$

has finite image, so it is constant, hence 1 on $\varphi^{-1}(G^h)^\circ$. Therefore, $\varphi^{-1}(G^h)^\circ \subset \widetilde{G}^{\widetilde{h}}$. Clearly $\widetilde{G}^{\widetilde{h}} \subset \varphi^{-1}(G^h)$, and hence $\widetilde{G}^{\widetilde{h}\circ} = \varphi^{-1}(G^h)^\circ$. A similar argument shows that $\widetilde{T}^{\widetilde{h}\circ} = \varphi^{-1}(T^h)^\circ$ and $\widetilde{B}^{\widetilde{h}\circ} = \varphi^{-1}(B^h)^\circ$. Since φ is finite to one and the connected subgroup $\widetilde{G}^{\widetilde{h}\circ}$ surjects onto $G^{h\circ}$ by [Bor91, I.1.4 Corollary], we get that $G^{h\circ}$ is reductive by [Bor91, IV.14.11 Corollary]. Similarly $\widetilde{T}^{\widetilde{h}\circ}$ surjects onto $T^{h\circ}$, so the latter is a maximal torus in $G^{h\circ}$ by [Bor91, IV.11.14 Proposition]. The same reasoning shows that $B^{h\circ}$ is a Borel subgroup in $G^{h\circ}$.

Consider now the general case and let $Z \subset G^\circ$ be the connected component of the center of G°. It equals the radical of G° and is a torus contained in T by [Bor91, IV.11.21 Proposition]. Set $\overline{G} = G/Z$ and let $\psi : G \to \overline{G}$ be the quotient map. Then $\overline{G}$ is semi-simple. Let $\overline{T}, \overline{B}$ be the image of T, B in $\overline{G}$, respectively. By [Bor91, IV.11.14 Proposition] the subgroup $\overline{T}$ is a maximal torus in $\overline{G}$ and $\overline{B}$ is a Borel subgroup in $\overline{G}$. Clearly $\overline{T} \subset \overline{B}$ and $\psi(h)$ normalizes this pair, so by the case which we have just proven the subgroup $\overline{G}^{\psi(h)}$ is reductive, $\overline{T}^{\psi(h)\circ}$ is a maximal torus in $\overline{G}^{\psi(h)\circ}$, and $\overline{B}^{\psi(h)\circ}$ is a Borel subgroup in $\overline{G}^{\psi(h)\circ}$. Now let $\widetilde{G}, \widetilde{T}, \widetilde{B}$ denote the pre-image of $\overline{G}^{\psi(h)}, \overline{T}^{\psi(h)\circ}$ and $\overline{B}^{\psi(h)\circ}$ with respect to ψ, respectively. Clearly $G^h \subset \widetilde{G}$ and $T^{h\circ} \subset \widetilde{T}^\circ \subset T$ and $B^{h\circ} \subset \widetilde{B}^\circ \subset B$, because $Z \subset T \subset B$.

Proposition 8.4. *The groups $G^{h\circ}, T^{h\circ}, B^{h\circ}$ surject onto $\overline{G}^{\psi(h)\circ}, \overline{T}^{\psi(h)\circ}$ and $\overline{B}^{\psi(h)\circ}$ with respect to ψ, respectively.*

Proof. Since Z is a characteristic subgroup of G, we have $h^{-1}Zh \subset Z$. Therefore, the map

$$Z \to Z, \quad z \mapsto h^{-1}zhz^{-1}$$

is a homomorphism of groups. Let J be the image of Z under this homomorphism. It is a closed subgroup of Z invariant under conjugation by h as the computation $h(h^{-1}zhz^{-1})h^{-1} = h^{-1}(hzh^{-1})h(hz^{-1}h^{-1})$ shows. Note that it will be enough to show that the regular map

$$\kappa : \widetilde{G}^\circ \to \widetilde{G}^\circ, \quad x \mapsto h^{-1}xhx^{-1}$$

has image in J. Indeed, if this is the case then for every $x \in \widetilde{G}^\circ$ there is a $z \in Z$ such that

$$h^{-1}xhx^{-1} = h^{-1}zhz^{-1},$$

so

$$h^{-1}xz^{-1}h(xz^{-1})^{-1} = h^{-1}xhh^{-1}z^{-1}hzx^{-1} = h^{-1}xhx^{-1}zh^{-1}z^{-1}h = h^{-1}xhx^{-1}(h^{-1}zhz^{-1})^{-1} = 1$$

using that both z and $h^{-1}z^{-1}h$ are in the center of G°. Therefore, $xz^{-1} \in G^{h\circ}$, but $\psi(x) = \psi(xz^{-1})$ as $z \in Z$. So $G^{h\circ}$ surjects onto $\psi(\widetilde{G}^\circ) = \overline{G}^{\psi(h)\circ}$ by [Bor91, I.1.4 Corollary]. Using $Z \subset T \subset B$, a similar argument shows that $T^{h\circ}$ and $B^{h\circ}$ surject onto $\overline{T}^{\psi(h)\circ}$ and $\overline{B}^{\psi(h)\circ}$ with respect to ψ, respectively.

The group $\widetilde{G}^\circ$ is the extension of the reductive group $\overline{G}^{\psi(h)\circ}$ by the torus Z, so it is reductive. Therefore, semi-simple elements are dense in $\widetilde{G}^\circ$. So it will be enough to show that κ maps every maximal torus $V \subset \widetilde{G}^\circ$ into J, since the latter is closed. Since Z is a central torus, it is contained in V. Therefore, $\psi^{-1}(\psi(V)) = V$. The image $\psi(V)$ is a torus on which the action of $\psi(h)$ is trivial, in particular $\psi(V)$ is normalized by $\psi(h)$. Therefore, h normalizes V. Since $h \in G^h \subset \widetilde{G}$, for some positive integer m we have $h^m \in \widetilde{G}^\circ$, so h^m is in the normalizer of V in $\widetilde{G}^\circ$. Since the normalizer of V in $\widetilde{G}^\circ$ is a finite extension of V, the conjugation action of h on V has finite order.

Conjugation by h leaves J invariant, as we have already remarked, so there is an induced action on $\overline{V} = V/J$. This action is trivial on the subgroup $\overline{Z} = Z/J$ by definition. We also noted that the induced action on the quotient $\overline{V}/\overline{Z} = V/Z$ is also trivial, because under the morphism ψ the latter is isomorphic to $\psi(V)$ on which $\psi(h)$ acts trivially. By Lemma 8.5 below this implies that this action on $\overline{V}$ is trivial. This is equivalent to $\kappa|_V$ taking values in J. □

Lemma 8.5. *Let T be a torus over a field of arbitrary characteristic, and let α be an automorphism of T of finite order. Assume that there is a sub-torus $T' \subset T$, such that α fixes every point of T', and the automorphism of the quotient T/T' induced by α is also the identity. Then α is trivial.*

Proof. For every torus T let $X_*(T)$ denote the group of its cocharacters. Then the rule $T \mapsto X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a fully faithful exact functor. In particular we have a short exact sequence:

$$0 \longrightarrow X_*(T') \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow X_*(T/T') \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow 0.$$

Let $\mathbf{b}'$ be a $\mathbb{Q}$-basis of $X_*(T') \otimes_{\mathbb{Z}} \mathbb{Q} \subset X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, and extend this to a $\mathbb{Q}$-basis $\mathbf{b}$ of $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since the induced action is trivial both on $X_*(T') \otimes_{\mathbb{Z}} \mathbb{Q}$ and $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}/X_*(T') \otimes_{\mathbb{Z}} \mathbb{Q} = X_*(T/T') \otimes_{\mathbb{Z}} \mathbb{Q}$, the matrix of the action on $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ in the basis $\mathbf{b}$ is upper triangular with ones on the diagonal. In particular it is unipotent. But this is also a matrix of finite order, so it is semi-simple, too. Therefore, this matrix is the identity by Lemma 3.7. □

The *Proof of Theorem 8.2* is now easy. By the proposition above $G^{h\circ}$ is the extension of $\overline{G}^{\psi(h)\circ}$ by a subgroup of Z^h. The group $\overline{G}^{\psi(h)}$ is reductive by the above, while the above-mentioned subgroup of Z^h is a subgroup of the torus Z, so it is also reductive. Therefore, $G^{h\circ}$ is also reductive. Since Z lies in T, the subgroup $Z^{h\circ}$ lies in $T^{h\circ}$. Therefore, $T^{h\circ}$ is the extension of a maximal torus in $\overline{G}^{\psi(h)}$ by a group containing the connected component of the kernel of the restriction of ψ onto $G^{h\circ}$, so it is a maximal torus in $G^{h\circ}$. A similar argument shows that $B^{h\circ}$ is a Borel subgroup in $G^{h\circ}$. ☐

Let $\alpha\colon G \twoheadrightarrow G^{\mathrm{red}}$ denote the maximal reductive quotient of G, i.e. the quotient of G by its unipotent radical R_uG. Note that $G/G^\circ \xrightarrow{\sim} G^{\mathrm{red}}/(G^{\mathrm{red}})^\circ$ because R_uG is connected and hence contained in G°. If $\gamma\colon G \twoheadrightarrow H$ is a surjection of algebraic groups, then there is an induced surjection $\gamma^{\mathrm{red}}\colon G^{\mathrm{red}} \twoheadrightarrow H^{\mathrm{red}}$ between the reductions, because the image of R_uG is a closed connected unipotent normal subgroup, and hence contained in R_uH.

Definition 8.6. (a) If G is reductive (that is G° is reductive) a closed subgroup $T \subset G$ is called a *maximal quasi-torus* if T equals the intersection $N_G(B^\circ) \cap N_G(T^\circ)$ of the normalizers in G of a Borel subgroup $B^\circ \subset G^\circ$ and a maximal Torus $T^\circ \subset B^\circ$.
(b) For general G, a closed subgroup $T \subset G$ is called a *maximal quasi-torus* if the quotient morphism $G \twoheadrightarrow G^{\mathrm{red}}$ maps T isomorphically onto a maximal quasi-torus in the reductive group G^{red}.

Remark 8.7. If G is reductive and $T \subset G$ is a maximal quasi-torus of the form $T = N_G(B^\circ)\cap N_G(T^\circ)$ then $T\cap G^\circ = N_{G^\circ}(B^\circ)\cap N_{G^\circ}(T^\circ) = N_{B^\circ}(T^\circ) = Z_{G^\circ}(T^\circ) = T^\circ$ by [Bor91, IV.11.16 Theorem, IV.10.6 Theorem and IV.13.17 Corollary 2], where $Z_{G^\circ}(T^\circ)$ denotes the centralizer. In particular, the identity component of T is the maximal torus T° (and the notation is consistent).

Lemma 8.8. *Let G be arbitrary and let T be a maximal quasi-torus in G. Then the identity component T° of T is a maximal torus in G and all elements of T are semi-simple.*

Proof. Let $R_uG \subset G$ be the unipotent radical and let $\alpha\colon G \twoheadrightarrow G/R_uG = G^{\mathrm{red}} =: \widetilde{G}$ be the quotient morphism. Then $\widetilde{T} := \alpha(T)$ is a maximal quasi-torus in $\widetilde{G}$ and its connected component $\widetilde{T}^\circ = \alpha(T^\circ)$ is a maximal torus in $\widetilde{G}$ by the above. In particular, T° is a torus and contained in a maximal torus T' of G. It follows that $\widetilde{T}^\circ$ is contained in and hence equal to the torus $\alpha(T')$. Since $T' \cap \ker\alpha = \{1\}$, it follows that $\alpha\colon T' \twoheadrightarrow \alpha(T') = \alpha(T^\circ)$ is an isomorphism and so $T^\circ = T'$ is a maximal torus in G.

If g lies in T and n is the order of the finite group T/T° then $g^n \in T^\circ$. That is, g^n is semi-simple, and so g is semi-simple by Lemma 3.7(a). ☐

We need to establish a few more facts about maximal quasi-tori. We are grateful to Friedrich Knop for providing a proof of part (c) in the following theorem. Since we were not able to find a correct proof of this statement in the literature we include Knop's argument on https://mathoverflow.net/questions/280874 for the reader's convenience.

Theorem 8.9. *Assume that G is reductive.*

(a) Let T be a maximal quasi-torus in G. Then $T/T^\circ = G/G^\circ$. Every other maximal quasi-torus is conjugate to T under G°.
(b) An element $g \in G(L)$ is semi-simple if and only if it is contained in a maximal quasi-torus, if and only if its G-conjugacy class is closed.
(c) Every connected component of G contains a dense open subset consisting of semi-simple elements.

Proof. (b) If g is semi-simple then it normalizes a maximal torus T° and a Borel subgroup B° of G° containing T° by [Ste68, Theorem 7.5], and hence lies in the maximal quasi-torus $T = N_G(B^\circ) \cap N_G(T^\circ)$. The converse was proven in Lemma 8.8. The characterization in terms of the G-conjugacy class of g is given in [Spa82, Corollaire II.2.22].

(a) The conjugacy statement follows from the fact that all pairs $T^\circ \subset B^\circ$ of a maximal torus and a Borel subgroup in G° are conjugate under G° by [Hum75, § 21.3, Corollary A]. To show that T surjects onto G/G° fix a connected component of G and let $g \in G$ be an element in this connected component. In its multiplicative Jordan decomposition $g = g_sg_u$ the unipotent part g_u lies in G° by Lemma 3.7(b). Therefore, $g_s \in gG^\circ$. Let T' be a maximal quasi-torus containing g_s, which exists by (b). In particular, T' intersects the connected component gG° of G. Since T is conjugate to T' under G° which acts trivially on G/G° we see that also T intersects gG°. It follows that $T \twoheadrightarrow G/G^\circ$ is surjective. Since $T \cap G^\circ = T^\circ$ by Remark 8.7, we conclude that $T/T^\circ \xrightarrow{\sim} G/G^\circ$.

(c) Let $T \subset G$ be a maximal quasi-torus. By (a) every connected component of G is of the form hG° for an $h \in T$. To show that hG° contains a dense open subset consisting of semi-simple elements, we consider the conjugation action

$$\Phi\colon G^\circ \times_L hT^{h\circ} \to hG^\circ, \quad (g, ht) \mapsto ghtg^{-1},$$

where $T^h := \{g \in T^\circ \colon gh = hg\} \subset T^\circ$ as in Definition 8.1. All elements in the image of Φ are conjugate to elements in $hT^{h\circ} \subset T$, and so are semi-simple by (b). Since the image of Φ is constructible by Chevalley's theorem [EGA, IV$_1$, Corollaire 1.8.5], this image contains an open subset of its closure by [EGA, 0$_3$, Proposition 9.2.2]. It thus suffices to show that Φ is dominant. Under the isomorphism of varieties $hG^\circ \xrightarrow{\sim} G^\circ,\ x \mapsto h^{-1}x$ the morphism Φ corresponds to the morphism

$$\Phi'\colon G^\circ \times_L T^{h\circ} \to G^\circ, \quad (g,t) \mapsto h^{-1}ghtg^{-1}.$$

To prove that Φ' is dominant, we use Theorem 8.2 which says that $T^{h\circ}$ is a maximal torus in G^h. Therefore, the conjugation action $G^h \times_L T^{h\circ} \to G^h,\ (g,t) \mapsto gtg^{-1}$ is dominant by [Bor91, IV.11.10 Theorem and IV.13.17, Corollary 2]. Note that $gtg^{-1} = h^{-1}ghtg^{-1}$, because $g \in G^h$. Thus it suffices to show that the morphism

$$\Phi''\colon G^\circ \times_L G^h \to G^\circ, \quad (g,\tilde g) \mapsto h^{-1}gh\tilde g g^{-1}$$

is dominant. In the point $(g,\tilde g) = (1,1)$ the morphism Φ'' has differential

$$\begin{aligned}(\mathrm{Ad}(h^{-1})-1)\oplus \mathrm{id}\colon\ \mathrm{Lie}\,G^\circ \oplus \mathrm{Lie}\,G^h &\longrightarrow \mathrm{Lie}\,G^\circ,\\ (X\,,\,\widetilde X) &\longmapsto (\mathrm{Ad}(h^{-1})-1)(X) + \widetilde X \;=\; h^{-1}Xh - X + \widetilde X\,,\end{aligned} \tag{8.1}$$

where $\mathrm{Lie}\,G^\circ$ denotes the Lie algebra of G° and $\mathrm{Ad}\colon G \to \mathrm{Aut}_L(\mathrm{Lie}\,G^\circ)$ denotes the adjoint representation. Since $G^h = \{g \in G^\circ \colon h^{-1}ghg^{-1} = 1\}$ we obtain $\mathrm{Lie}\,G^h = \ker(\mathrm{Ad}(h^{-1})-1)$ and since h is semi-simple, also $\mathrm{Ad}(h^{-1})$ is semi-simple, and hence $\ker(\mathrm{Ad}(h^{-1})-1) + \mathrm{im}(\mathrm{Ad}(h^{-1})-1) = \mathrm{Lie}\,G^\circ$. This shows that the differential (8.1) is surjective in $(1,1)$, and therefore Φ'' is dominant (for example by [BLR90, § 2.2, Proposition 8]) and the theorem is proven. □

If G is not assumed to be reductive this implies the following

Theorem 8.10. *Let G be a not necessarily connected, linear algebraic group over L.*

- *(a) Let T° be a maximal torus of G°. Then there exists a maximal quasi-torus $T \subset G$ with $T \cap G^\circ = T^\circ$.*
- *(b) Any two maximal quasi-tori in G are conjugate under G°.*
- *(c) An element $g \in G(L)$ is semi-simple if and only if it is contained in a maximal quasi-torus.*
- *(d) Every maximal quasi-torus T in G satisfies $T/T^\circ = G/G^\circ$ and normalizes a Borel subgroup of G°, which contains the maximal torus $T^\circ \subset G^\circ$. In particular, $G^\circ \cap T = T^\circ$.*
- *(e) Conversely, a closed subgroup $T \subset G$ is a maximal quasi-torus if $T \twoheadrightarrow G/G^\circ$ is surjective, the connected component T° is a maximal torus of G°, and T normalizes a Borel subgroup $B^\circ \subset G^\circ$ containing T°.*

Proof. (a) Let $U := R_uG \subset G^\circ$ be the unipotent radical, set $\widetilde G := G^{\mathrm{red}} = G/U$ and let $\alpha\colon G \to \widetilde G$ be the quotient map. Then $\widetilde G$ is reductive with $G/G^\circ \cong \widetilde G/\widetilde G^\circ$ and $\widetilde T^\circ := \alpha(T^\circ) \subset G^\circ$ is a maximal torus by [Bor91, IV.11.20 Proposition]. Choose a Borel subgroup $\widetilde B^\circ \subset \widetilde G^\circ$ containing $\widetilde T^\circ$ and let $\widetilde T := N_{\widetilde G}(\widetilde B^\circ) \cap N_{\widetilde G}(\widetilde T^\circ)$ be the associated maximal quasi-torus in $\widetilde G$. By [Bor91, IV.11.19 Proposition] all such Borel subgroups $\widetilde B^\circ$ are conjugate under the Weyl group W of G°, and therefore also all maximal quasi-tori $\widetilde T$ containing $\widetilde T^\circ$ are conjugate under W. We shall lift $\widetilde T$ to a maximal quasi-torus T in G with $T \cap G^\circ = T^\circ$.

(i) We first consider the case that $U = G^\circ$ and $T^\circ = (1)$, in which $\widetilde G^\circ = (1)$ and $\widetilde G = G/G^\circ$ is a finite group. In this case $\widetilde T = \widetilde G$ is the unique maximal quasi-torus in $\widetilde G$ and the theorem asserts that the sequence

$$1 \longrightarrow U \longrightarrow G \longrightarrow G/G^\circ \longrightarrow 1 \tag{8.2}$$

splits and any two splittings are conjugate. Using the nilpotent filtration of U we reduce to the case that $U \cong \mathbb{G}_a^r$ is commutative. Then the class of the extension G is an element of $\mathrm{H}^2\big(G/G^\circ, U(L)\big)$. If n is the order of the finite group G/G° then multiplication with n is an isomorphism on $U \cong \mathbb{G}_a^r$, hence also on all cohomology groups $\mathrm{H}^i\big(G/G^\circ, U(L)\big)$ for $i \geq 1$. On the other hand n kills $\mathrm{H}^i\big(G/G^\circ, U(L)\big)$ by [Rot09, Proposition 9.40] and so $\mathrm{H}^i\big(G/G^\circ, U(L)\big) = (0)$ for $i \geq 1$. So the sequence (8.2) splits and the image of a splitting is a maximal quasi-torus T in G. Moreover, $\mathrm{H}^1\big(G/G^\circ, U(L)\big) = (0)$ implies that any two splittings are conjugate under U by [Rot09, Proposition 9.21].

(ii) To treat the general case we set $H := \alpha^{-1}(\widetilde{T}) \subset G$. Then $U \subset H$ is normal and $\alpha\colon H/U \xrightarrow{\sim} \widetilde{T}$. Its identity component is $H^\circ = \alpha^{-1}(\widetilde{T}^\circ) = U \rtimes T^\circ$ with $\alpha\colon H/H^\circ \xrightarrow{\sim} \widetilde{T}/\widetilde{T}^\circ$. We will spell out in terms of algebraic groups the following philosophy. The fact that any two maximal tori of $U \rtimes T^\circ$ are conjugate could be interpreted by saying that every stabilizing automorphism of $U \rtimes T^\circ$ is inner, and hence $\mathrm{H}^1(\widetilde{T}^\circ, U) = \mathrm{Stab}(\widetilde{T}^\circ, U)/\mathrm{Inn}(\widetilde{T}^\circ, U) = (0)$; compare [Rot09, § 9.1.3]. As a consequence there should be an exact sequence

$$0 \longrightarrow \mathrm{H}^2(G/G^\circ, U^{T^\circ}) \xrightarrow{\mathrm{Inf}^2} \mathrm{H}^2(\widetilde{T}, U) \xrightarrow{\mathrm{Res}^2} \mathrm{H}^2(\widetilde{T}^\circ, U)^{G/G^\circ}$$

as in [Rot09, Theorem 9.84], where $U^{T^\circ} = Z_{H^\circ}(T^\circ) \cap U$ is the group of fixed points in U under the conjugation action of T° and $Z_{H^\circ}(T^\circ)$ denotes the centralizer. We interpret H as a class in $\mathrm{H}^2(\widetilde{T}, U)$ which via pullback under $\widetilde{T}^\circ \to \widetilde{T}$ maps to its identity component $H^\circ = H \times_{\widetilde{T}} \widetilde{T}^\circ$, and hence to the trivial class in $\mathrm{H}^2(\widetilde{T}^\circ, U)$. From (8.2) interpreted as $\mathrm{H}^2(G/G^\circ, U^{T^\circ}) = (0)$ we should obtain that the class of H is trivial, that is $H \cong U \rtimes \widetilde{T}$, which provides a lift of $\widetilde{T}$.

Note that we will not use this philosophy, but construct a lift of $\widetilde{T}$ as follows. Let $N := N_H(T^\circ)$ be the normalizer and consider the sequence of algebraic groups

$$1 \longrightarrow U^{T^\circ} \longrightarrow N/T^\circ \longrightarrow G/G^\circ \longrightarrow 1\,, \tag{8.3}$$

which we claim is exact. Using the isomorphisms $H/H^\circ = \widetilde{T}/\widetilde{T}^\circ = \widetilde{G}/\widetilde{G}^\circ = G/G^\circ$ every element of G/G° has a representative $g \in H$. The conjugate $gT^\circ g^{-1}$ is a maximal torus in H°, and hence of the form $hT^\circ h^{-1}$ for some $h \in H^\circ$ by [Bor91, IV.11.3 Corollary]. Then $h^{-1}g \in N$ maps onto g in G/G° and this proves exactness on the right. The group $N \cap G^\circ = N_{H^\circ}(T^\circ) = Z_{H^\circ}(T^\circ)$ is connected by [Bor91, IV.10.6 Theorem], because H° is connected and solvable. It follows that $N^\circ = N \cap G^\circ = T^\circ \times N_u^\circ$, where $N_u^\circ = N^\circ \cap U = U^{T^\circ}$ is the group of unipotent elements in N°, use [Bor91, IV.12.1 Theorem]. This proves the exactness in the middle and on the left.

Since the identity component U^{T° of N/T° is a unipotent group, there is a section $s\colon G/G^\circ \to N/T^\circ$ by the the special case treated in (i) above. We now define $T := \beta^{-1}\big(s(G/G^\circ)\big)$ where $\beta\colon N \to N/T^\circ$ is the quotient map. Then $T/T^\circ \cong G/G^\circ$ and this implies that T is a maximal quasi-torus with identity component T° and $\alpha\colon T \xrightarrow{\sim} \widetilde{T}$.

(b) We must show that any two maximal quasi-tori T_1 and T_2 in G are conjugate under G° also in the general case. Let $\widetilde{T}_1$ and $\widetilde{T}_2$ be their isomorphic images in $\widetilde{G}$ under α. By Theorem 8.9 we can conjugate $\widetilde{T}_2$ into $\widetilde{T}_1$ under $\widetilde{G}^\circ = \alpha(G^\circ)$ and thus assume that they are equal $\widetilde{T} := \widetilde{T}_1 = \widetilde{T}_2$. In particular $T_1, T_2 \subset H := \alpha^{-1}(\widetilde{T})$ and the two maximal tori T_1° and T_2° of the connected group $\alpha^{-1}(\widetilde{T}^\circ)$, which is isomorphic to $U \rtimes T_1^\circ$ by the above, are conjugate by [Bor91, IV.11.3 Corollary]. So we may assume that they are equal $T^\circ := T_1^\circ = T_2^\circ$. It follows that both T_1 and T_2 are contained in the normalizer $N := N_H(T^\circ)$ and we may consider the subsets T_1/T° and T_2/T° of the group N/T° from (8.3). Since they both map isomorphically onto $\widetilde{T}/\widetilde{T}^\circ$ under the map $N/T^\circ \twoheadrightarrow G/G^\circ = \widetilde{G}/\widetilde{G}^\circ$ whose kernel is the unipotent radical U^{T° of N/T°, we see that T_1/T° and T_2/T° are maximal quasi-tori in N/T°. By what we proved in (a)(i), they are conjugate under U^{T° and this shows that T_1 and T_2 are conjugate under G°.

(c) By Lemma 8.8 every element of a maximal quasi-torus is semi-simple. Conversely, to show that every semi-simple element $g \in G$ lies in a maximal quasi-torus we use that g normalizes a Borel subgroup $B^\circ \subset G^\circ$ and a maximal torus $T^\circ \subset B^\circ$ by [Ste68, Theorem 7.5]. Then $\widetilde{T}^\circ := \alpha(T^\circ)$ and $\widetilde{B}^\circ := \alpha(B^\circ)$ are a maximal torus and a Borel subgroup of $\widetilde{G}^\circ$ by [Bor91, IV.11.14 Proposition], which are normalized by $\tilde{g} := \alpha(g) \in \widetilde{G}(L)$. In particular $\tilde{g} \in \widetilde{T} := N_{\widetilde{G}}(\widetilde{B}^\circ) \cap N_{\widetilde{G}}(\widetilde{T}^\circ)$ and by (a)(ii) we can choose a maximal quasi-torus $T \subset G$ containing T° mapping isomorphically onto $\widetilde{T}$. Let $g' \in T(L)$ be the preimage of $\tilde{g}$ under this isomorphism. Then g and g' both lie in $H := \alpha^{-1}(\widetilde{T})$ and even in $N := N_H(T^\circ)$. Moreover, they map to the same element in $\widetilde{T}/\widetilde{T}^\circ = G/G^\circ = (N/T^\circ)/U^{T^\circ} = N/N^\circ$; see (8.3). Considering the subgroup of G generated by g, g' and G°, which is automatically closed, we can assume that G/G° is cyclic. Then both g and g' define sections s and s' of (8.3) and the argument of (a)(i) above shows that s and s' are conjugate into each other by an element of N/T°. Since T was constructed in (a)(ii) as the preimage of $s(G/G^\circ)$ under $N \twoheadrightarrow N/T^\circ$ this shows that g can be conjugate into T by an element of N as desired.

(d) If $T \subset G$ is a maximal quasi-torus and $\widetilde{T} := \alpha(T)$, then $T/T^\circ = \widetilde{T}/\widetilde{T}^\circ = \widetilde{G}/\widetilde{G}^\circ = G/G^\circ$ by Theorem 8.9 and T° is a maximal torus in G° by Lemma 8.8. To prove that T normalizes a Borel subgroup of G, let $\widetilde{B}^\circ$ be a Borel subgroup of $\widetilde{G}$ containing $\widetilde{T}^\circ$ with $\widetilde{T} = N_{\widetilde{G}}(\widetilde{B}^\circ) \cap N_{\widetilde{G}}(\widetilde{T}^\circ)$; see Remark 8.7. Then $\alpha^{-1}(\widetilde{B}^\circ)$ is normalized by T and is an extension of $\widetilde{B}^\circ$ by R_uG, hence connected solvable. Moreover, $\alpha^{-1}(\widetilde{B}^\circ)$ is contained in a Borel subgroup B° of G°. But $\alpha(B^\circ)$ is connected solvable and contains $\widetilde{B}^\circ$, hence equals $\widetilde{B}^\circ$ by maximality of the Borel subgroup $\widetilde{B}^\circ$. This shows that $B^\circ = \alpha^{-1}(\widetilde{B}^\circ)$ is a Borel subgroup of G° normalized by T.

(e) Let $n := \#(G/G^\circ)$. For every element of T its n-th power lies in the torus T° and hence is semi-simple. Therefore, all elements of T are semi-simple by Lemma 3.7 and $U \cap T = (1)$. So the map α restricted to T is injective, and maps T isomorphically onto $\alpha(T) \subset \widetilde{G}$. The connected component $\alpha(T)^\circ = \alpha(T^\circ)$ is a maximal torus in $\widetilde{G}^\circ$ and $\alpha(B^\circ)$ is a Borel subgroup in $\widetilde{G}^\circ$ by [Bor91, IV.11.14 Proposition]. Since $\alpha(T)$ normalizes the pair $\alpha(T^\circ) \subset \alpha(B^\circ)$, it is contained in the maximal torus $\widetilde{T} = N_{\widetilde{G}}(\alpha(B^\circ)) \cap N_{\widetilde{G}}(\alpha(T^\circ))$, which satisfies $\widetilde{T}^\circ = \alpha(T^\circ)$ by Remark 8.7. Since $T \twoheadrightarrow G/G^\circ \xrightarrow{\sim} \widetilde{G}/\widetilde{G}^\circ = \widetilde{T}/\widetilde{T}^\circ$, we conclude that $\alpha\colon T \xrightarrow{\sim} \alpha(T) = \widetilde{T}$, and hence T is a maximal quasi-torus in G. □

Corollary 8.11. *Let $f\colon G \twoheadrightarrow H$ be a surjection of algebraic groups. Then the image of a maximal quasi-torus (resp. a maximal torus, resp. a Borel subgroup) in G is again a maximal quasi-torus (resp. a maximal torus, resp. a Borel subgroup) in H. Moreover, every maximal quasi-torus (resp. maximal torus, resp. Borel subgroup) in H arises in this way.*

Proof. For maximal tori and Borel subgroups this is just [Bor91, IV.11.14 Proposition]. So let $T \subset G$ be a maximal quasi-torus. By Theorem 8.10(d) there is a Borel subgroup $B^\circ \subset G^\circ$ which contains the maximal torus $T^\circ \subset G^\circ$ and is normalized by T. Then $f(T)$ normalizes the Borel subgroup $f(B^\circ)$ and the maximal torus $f(T^\circ) \subset f(B^\circ)$ of H. Since the surjection $T \twoheadrightarrow G/G^\circ \twoheadrightarrow H/H^\circ$ factors through $f(T)$, we see that $f(T) \twoheadrightarrow H/H^\circ$ is surjective, and hence $f(T)$ is a maximal quasi-torus in H by Theorem 8.10(e).

Conversely, if $T \subset G$ and $T' \subset H$ are any maximal quasi-tori (resp. maximal tori, resp. Borel subgroups), then by Theorem 8.10(b) (resp. [Bor91, IV.11.1 Theorem and IV.11.3 Corollary]) there is an element $h \in H$ with $T' = h^{-1}f(T)h$. For any preimage $g \in G$ the maximal quasi-torus $g^{-1}Tg$ in G surjects onto T'. This proves the corollary. □

The corollary leads to a very handy remark which will be used at least twice.

Remark 8.12. Let $G = G_1 \times_{G_3} G_2$ be the fiber product of two linear algebraic groups G_1 and G_2 over a third G_3 for epimorphisms $G_1 \twoheadrightarrow G_3$ and $G_2 \twoheadrightarrow G_3$. This means that G is a closed subgroup of $G_1 \times_L G_2$ and the restrictions of the projections $\pi_1 : G_1 \times_L G_2 \to G_1$ and $\pi_2 : G_1 \times_L G_2 \to G_2$ are surjective. Let $T \subset G$ be a maximal quasi-torus. Then for $i = 1, 2, 3$ the images $T_i \subset G_i$ of T are maximal quasi-tori by Corollary 8.11. Since T is a subgroup of $G_1 \times_L G_2$ we get that the product $\pi_1|_T \times \pi_2|_T$ is a closed immersion of T into $T_1 \times_L T_2$. Therefore, T is a fiber product of the maximal quasi-tori T_1 and T_2. In particular, if $G^\circ = G_1^\circ \times_L G_2^\circ$ then its intersection with T is the maximal torus $T^\circ = T_1^\circ \times_L T_2^\circ$, and if $G = G_1 \times_L G_2$ then $T = T_1 \times_L T_2$.

A useful condition for being a maximal quasi-torus is given in the following

Theorem 8.13. *Let G be reductive, and let $H \subset G$ be a closed subgroup with the following properties:*

(a) the connected component H° of H is a maximal torus in G°,
(b) the natural map $H/H^\circ \to G/G^\circ$ is surjective,
(c) the group H is commutative.

Then H is a maximal quasi-torus in G, and it is the only maximal quasi-torus in G containing H°.

To prove the theorem we will need the following

Lemma 8.14. *In the situation of the theorem, two elements of H are conjugate under G° if and only if they are conjugate under the normalizer $N_{G^\circ}(H^\circ)$ of H° in G°. In particular, the intersection of every G°-conjugacy class with H is finite. (Note however, that there is no action of $N_{G^\circ}(H^\circ)$ on H in general, because $N_{G^\circ}(H^\circ)$ only normalizes H° and not necessarily H.)*

Proof. In order to prove the first claim, note that one direction follows from the inclusion $N_{G^\circ}(H^\circ) \subset G^\circ$. To prove the converse, let $h, h' \in H$ be conjugate under G°, say $h = x^{-1}h'x$ for an $x \in G^\circ$. Since H is commutative by condition (c), h' centralizes H°, and hence the conjugate $h = x^{-1}h'x$ centralizes $x^{-1}H^\circ x$. But h also centralizes H°, so we get that H° and $x^{-1}H^\circ x$ lie in the centralizer G^h of h in G°. Since H° and $x^{-1}H^\circ x$ are maximal tori in G° by condition (a), they are maximal tori in G^h, too. So by [Bor91, IV.11.3 Corollary] there is a $y \in G^h$ such that $y^{-1}x^{-1}H^\circ xy = H^\circ$. Set $w = xy$. Clearly $w \in N_{G^\circ}(H^\circ)$, but also $w^{-1}h'w = y^{-1}x^{-1}h'xy = y^{-1}hy = h$, as $y \in G^h$. Therefore, h is conjugate to h' under $N_{G^\circ}(H^\circ)$.

To prove the second claim, note that the identity component of $N_{G^\circ}(H^\circ)$ is H° by assumption (a) and [Bor91, III.8.10, Corollary 2 and IV.13.17, Corollary 2]. By assumption (c) the latter acts trivially by conjugation on H, therefore the action of $N_{G^\circ}(H^\circ)$ factors through the Weyl group $W = N_{G^\circ}(H^\circ)/H^\circ$, which is finite. □

Proof of Theorem 8.13. By condition (a) and Remark 8.7 there is a maximal quasi-torus $T \subset G$ such that $T^\circ = H^\circ$. We have to show that $T = H$. To this end fix a $t \in T$. By condition (b) there is an $h \in H$ such that $h^{-1}t \in G^\circ$, and hence $tG^\circ = hG^\circ$.

Proposition 8.15. *We have $h^{-1}t \in T^\circ$.*

Proof. Since T° is commutative, every element $tx \in tT^\circ$ centralizes $T^t \subset T^\circ$. Therefore, the quotient group $Q = T^\circ/T^t$ acts faithfully on tT° by conjugation. By [MFK94, Chapter 1, § 2, Theorem 1.1] the categorical quotient Y of tT° by this action of Q exists as an affine scheme. Let Y^{red} be the reduced scheme underlying Y, and let $\pi\colon tT^\circ \to Y^{\mathrm{red}}$ be the quotient map. Consider the set

$$C = \{(x, y) \in hT^\circ \times Y^{\mathrm{red}} \colon \exists\, a \in tT^\circ\ \exists\, b \in G^\circ \text{ such that } \pi(a) = y \text{ and } b^{-1}ab = x)\}.$$

We claim that C is a constructible set. Namely consider the morphism

$$\varphi\colon\ tT^\circ \times G^\circ \longrightarrow hG^\circ \times Y^{\mathrm{red}}, \quad (a, b) \longmapsto (b^{-1}ab, \pi(a)) .$$

The preimage $\varphi^{-1}(hT^\circ \times Y^{\mathrm{red}}) \subset tT^\circ \times G^\circ$ is a closed subset, and $C = \varphi\big(\varphi^{-1}(hT^\circ \times Y^{\mathrm{red}})\big)$. Therefore, C is a constructible set by Chevalley's theorem [EGA, IV_1, Corollaire 1.8.5]. Let $\pi_1\colon C \to hT^\circ$ and $\pi_2\colon C \to Y^{\mathrm{red}}$ be the projections onto the first and the second factor, respectively. Every element of $hT^\circ = hH^\circ \subset H$ is semi-simple by Lemma 3.7, because some power of it lies in the torus H°. Therefore, this element is conjugate under G° to an element of tT° by Theorem 8.9(a),(b). Thus the map π_1 is surjective, and hence the dimension of C is at least $\dim(T^\circ) = \dim(hT^\circ)$ by [GW10, Proposition 14.107]. For every $y \in Y^{\mathrm{red}}$, the points in the fiber $\pi^{-1}(y)$ of $\pi\colon tT^\circ \to Y^{\mathrm{red}}$ are conjugate under T°, so the fiber of $\pi_2\colon C \to Y^{\mathrm{red}}$ above $y \in Y^{\mathrm{red}}$, which equals

$$\{\, x \in hT^\circ \colon \exists\, a \in \pi^{-1}(y), \exists\, b \in G^\circ \text{ with } x = b^{-1}ab \,\} \;=\; H \cap \{\, b^{-1}ab \colon a \in \pi^{-1}(y), b \in G^\circ \,\} ,$$

is the intersection of a G°-conjugacy class with H. This is a finite set by Lemma 8.14. The image $\pi_2(C) \subset Y^{\mathrm{red}}$ is constructible by Chevalley's theorem, and the fibers of the surjective map $\pi_2\colon C \to \pi_2(C)$ are finite by the above, so $\dim(\pi_2(C)) = \dim(C)$ by [GW10, Proposition 14.107], and this is at least $\dim(T^\circ) = \dim(tT^\circ)$. Thus $\dim(Y^{\mathrm{red}}) \geq \dim(\pi_2(C)) \geq \dim(tT^\circ) \geq \dim(Y^{\mathrm{red}})$. This means that Q is zero-dimensional and connected as a quotient of T°, and hence $T^t = T^\circ$. Therefore, both h and t centralize T°, and so $h^{-1}t$ centralizes T°, too. But $h^{-1}t \in G^\circ$ and the centralizer $Z_{G^\circ}(T^\circ)$ of T° in G° is T° itself by [Bor91, IV.13.17, Corollary 2]. □

Proof of Theorem 8.13 continued. By the proposition above $tT^\circ = hh^{-1}tT^\circ = hT^\circ = hH^\circ$, so we get that H contains T. Now we only need to show the reverse inclusion. Let $h \in H$ be again arbitrary. Since the natural map $T/T^\circ \to G/G^\circ$ is surjective by Theorem 8.9(a), there is a $t \in T$ such that $t^{-1}h \in G^\circ$. Since T is in H, we get that $t^{-1}h$ is in H, too. But H centralizes H°, so $t^{-1}h$ is in $Z_{G^\circ}(H^\circ)$. However, the latter is H° itself, therefore $hH^\circ = tt^{-1}hH^\circ = tH^\circ = tT^\circ$, and hence T contains H, and so $T = H$. This finishes the proof of Theorem 8.13. □

We end this section by proving the following

Theorem 8.16. *Let $\varphi\colon G \hookrightarrow H$ be an injective homomorphism of algebraic groups, and assume that there is a closed normal subgroup $N \triangleleft G$ such that $\varphi(N) \triangleleft H$ is also normal. Let $T \subset G$ be a maximal quasi-torus. If its image in $H/\varphi(N)$ is a maximal quasi-torus, then also its image in H is a maximal quasi-torus.*

To prove it we start with a

Lemma 8.17. *Let $\gamma\colon G \twoheadrightarrow H$ be a surjective homomorphism of linear algebraic groups whose kernel is a unipotent group, and let $T \subset G$ be a closed subgroup such that the restriction of γ to T is injective. If $\gamma(T)$ is a maximal quasi-torus in H, then T is a maximal quasi-torus in G.*

Proof. Let $\pi\colon H \twoheadrightarrow H^{\mathrm{red}}$ be the maximal reductive quotient. Then the composition $\pi \circ \gamma\colon G \twoheadrightarrow H^{\mathrm{red}}$ is a surjective map onto a reductive group. Its kernel is an extension of unipotent groups, so it is unipotent and hence connected by Lemma 3.7(b). Therefore, $\pi \circ \gamma\colon G \twoheadrightarrow H^{\mathrm{red}}$ is the maximal reductive quotient for G. Since $T \xrightarrow{\sim} \gamma(T) \xrightarrow{\sim} \pi \circ \gamma(T)$ is an isomorphism onto the maximal quasi-torus $\pi \circ \gamma(T)$ in H^{red} we conclude that T is a maximal quasi-torus in G. □

Definition 8.18. We say that a closed subgroup B in a linear algebraic group G is a *quasi-Borel subgroup* if its identity component B° is a Borel subgroup, and there is a maximal quasi-torus $T \subset G$ such that T lies in B and B is generated by B° and T.

Remark 8.19. (a) Every maximal quasi-torus is contained in a quasi-Borel subgroup. Namely, by Theorem 8.10(d) there is a Borel subgroup $B^\circ \subset G^\circ$ normalized by T with $T^\circ \subset B^\circ$. Let B be the group generated by B° and T. Since B° is normalized by T, the semi-direct product $B^\circ \rtimes T \twoheadrightarrow B$ surjects onto B with kernel $B^\circ \cap T = T^\circ$. We conclude that B is an extension $1 \to B^\circ \to B \to T/T^\circ \to 1$. In particular, the connected component of B is B°, and hence B is a quasi-Borel subgroup.

(b) For every quasi-Borel subgroup $B \subset G$ the map $B/B^\circ \to G/G^\circ$ induced by the inclusion of B into G is an isomorphism. Indeed, in the situation of the definition, T normalizes B° and $T^\circ \subset B^\circ$. Therefore, B arises as described in (a), and this shows that $G^\circ \cap B = B^\circ$, whence $B/B^\circ \to G/G^\circ$ is injective. The surjectivity follows from the surjectivity of $T \twoheadrightarrow G/G^\circ$.

(c) If $\gamma\colon G \twoheadrightarrow H$ is a surjection of algebraic groups, then the image $\gamma(B)$ of every quasi-Borel subgroup B in G is again a quasi-Borel subgroup. Indeed, in the situation of the definition $\gamma(T)$ is a maximal quasi-torus in H by Corollary 8.11, which is contained in $\gamma(B)$. Moreover, $\gamma(T)$ and $\gamma(B^\circ)$ generate $\gamma(B)$ and the identity component $\gamma(B)^\circ = \gamma(B^\circ)$ is a Borel subgroup of H by [Bor91, I.1.4 Corollary and IV.11.14 Proposition].

Lemma 8.20. *Let $\gamma\colon G \twoheadrightarrow G'$ be a surjection of linear algebraic groups, let $T \subset G$ be a maximal quasi-torus in G, and let $B \subset G$ be a quasi-Borel subgroup which contains T. Then T is a maximal quasi-torus in $H = \gamma^{-1}(\gamma(B))$.*

Proof. We will use Theorem 8.10(e) for the pair $T \subset H$. The identity component T° is a maximal torus in G, so it is a maximal torus in the smaller group H, too. Since B° is a Borel subgroup of G, and clearly $B^\circ \subset B \subset H$, we get that B° is a Borel subgroup in the smaller group H. By choice B° is normalized by T and B° contains T°. So by Theorem 8.10(e) we only need to show that the map $H/H^\circ \to G/G^\circ$ induced by the inclusion $H \hookrightarrow G$ is injective, because then the surjection $T \twoheadrightarrow G/G^\circ$ will factor through a surjection $T \twoheadrightarrow H/H^\circ$.

So let $h \in G^\circ \cap H$. Then $\gamma(h)$ lies in $\gamma(G^\circ) = G'^\circ$ and in $\gamma(H) = \gamma(B)$, which is a quasi-Borel subgroup of G' by Remark 8.19(c). Therefore, $\gamma(h) \in G'^\circ \cap \gamma(B) = \gamma(B)^\circ = \gamma(B^\circ)$ by Remark 8.19(b), and hence $\gamma(h) = \gamma(b)$ for an element $b \in B^\circ \subset H^\circ \subset G^\circ$. Thus we have to show that the element $\tilde h := hb^{-1} \in G^\circ \cap \ker\gamma$ actually lies in H°. For this purpose note, that $(\ker\gamma)^\circ$ is a characteristic subgroup of $\ker\gamma$, which in turn is normal in G. Therefore, $(\ker\gamma)^\circ$ is normal in G. Let $\pi\colon G \twoheadrightarrow G/(\ker\gamma)^\circ =: \overline{G}$ be the quotient morphism and set $\overline{H} := \pi(H)$ and $\overline{B}^\circ := \pi(B)^\circ = \pi(B^\circ)$. The latter is a Borel subgroup in $\overline{G}$ by [Bor91, IV.11.14 Proposition]. Now $\pi(\tilde h)$ lies in the finite group $C := \overline{G}^\circ \cap \ker\gamma/(\ker\gamma)^\circ$, which is normal in $\overline{G}^\circ$. The operation of $\overline{G}^\circ$ by conjugation on C factors through the finite automorphism group of C, and hence is trivial because $\overline{G}^\circ$ is connected. It follows that C is contained in the center of $\overline{G}^\circ$, and hence in $\overline{B}^\circ = \pi(B^\circ)$ by [Bor91, IV.11.11 Corollary]. Thus that $\pi(\tilde h) = \pi(\tilde b)$ for an element $\tilde b \in B^\circ$, and so $hb^{-1}\tilde b^{-1} = \tilde h \tilde b^{-1} \in \ker\pi = (\ker\gamma)^\circ \subset H^\circ$. We conclude that $h \in H^\circ$, and hence $G^\circ \cap H = H^\circ$ and $H/H^\circ \to G/G^\circ$ is injective as desired. This proves the lemma. □

Proposition 8.21. *Let $\gamma\colon G \twoheadrightarrow G'$ be a surjection of linear algebraic groups, and let $T \subset G$ be a maximal quasi-torus in G. Then T is a maximal quasi-torus in $H = \gamma^{-1}(\gamma(T))$.*

Proof. Let B be a quasi-Borel subgroup which contains T. By the lemma above T is a maximal quasi-torus in $\gamma^{-1}(\gamma(B))$. Therefore, we may replace G by $\gamma^{-1}(\gamma(B))$ and G' by $\gamma(B)$ without loss of generality. In other words we may assume that G'° is solvable. The idea is to show that the inclusion $H \hookrightarrow G$ induces an isomorphism $H^{\mathrm{red}} \xrightarrow{\sim} G^{\mathrm{red}}$. Let U be the unipotent radical of $K := \ker(\gamma)$. Since U is a characteristic subgroup in K, which is a normal subgroup both in G and in H, the group U is also normal both in G and in H. Therefore, $\gamma(U)$ is also normal in G'. Set $\overline{G} = G/U$ and $\overline{G}' = G'/\gamma(U)$ and let $\overline{\gamma}\colon \overline{G} \to \overline{G}'$ be the map induced by γ. By Corollary 8.11 the image $\overline{T}$ of T in $\overline{G}$ is a maximal quasi-torus. Set $\overline{H} = H/(H\cap U)$; then the kernel $H \cap U$ of the quotient map $H \to \overline{H}$ is unipotent, and intersects T trivially by Theorem 8.10(c). So by Lemma 8.17 it will be enough to see that $\overline{T}$ is a maximal quasi-torus in $\overline{H}$. We obtain the two upper exact rows in the following diagram.

$$\begin{tikzcd}
1 \arrow[r] & K/U \arrow[r] \arrow[d, equal] & \overline{H} \arrow[r] \arrow[d, hook] & \overline{\gamma}(\overline{T}) \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & K/U \arrow[r] \arrow[d, equal] & \overline{G} \arrow[r, "\overline{\gamma}"] \arrow[d, two heads] & \overline{G}' \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & K/U \arrow[r] & \overline{G}^{\mathrm{red}} \arrow[r] & \overline{G}'^{\mathrm{red}} \arrow[r] & 1\,.
\end{tikzcd}$$

Since $\overline{H}$ is the extension of a reductive group by another reductive group, it is reductive, i.e. $\overline{H} = H^{\mathrm{red}}$. The intersection $(K/U) \cap R_u\overline{G}^\circ$ of K/U with the unipotent radical $R_u\overline{G}^\circ$ of $\overline{G}^\circ$ is a closed unipotent normal subgroup of K/U, hence connected by Lemma 3.7(b). Since K/U is reductive, we obtain $(K/U)\cap R_u\overline{G}^\circ = (1)$. Moreover, $R_u\overline{G}'^\circ = \overline{\gamma}(R_u\overline{G}^\circ)$ by [Bor91, IV.14.11 Corollary]. This shows that the bottom row in the diagram is also exact. Since $\overline{\gamma}(\overline{T})^\circ$ is a maximal torus in the connected solvable group $\overline{G}'^\circ$, the composition of the morphisms in the right column is an isomorphism on the identity components $\overline{\gamma}(\overline{T})^\circ \xrightarrow{\sim} (\overline{G}'^{\mathrm{red}})^\circ$ by [Bor91, III.10.6 Theorem]. Since $\overline{\gamma}(\overline{T})$ is a maximal quasi-torus in $\overline{G}'$ by Corollary 8.11, the composition of the morphisms in the right column is also an isomorphism on the group of connected components by Theorem 8.9. This proves that $\overline{H} \xrightarrow{\sim} \overline{G}^{\mathrm{red}}$ is an isomorphism. In particular, $\overline{T}$ is a maximal quasi-torus in $\overline{H}$ and the proposition follows. □

Proof of Theorem 8.16. Let $\pi_1\colon G \to G/N$ and $\pi_2\colon H \to H/\varphi(N)$ be the quotient maps. Then $\pi_1(T) \subset G/N$ is a maximal quasi-torus by Corollary 8.11 and its image $\pi_2\varphi(T)$ in $H/\varphi(N)$ is a maximal quasi-torus by assumption. Note that φ maps the pre-image $F_1 = \pi_1^{-1}(\pi_1(T)) \subset G$ isomorphically onto the pre-image $F_2 = \pi_2^{-1}(\pi_2\varphi(T)) \subset H$. By Corollary 8.11 there is a maximal quasi-torus T' in H with $\pi_2(T') = \pi_2\varphi(T)$. By Proposition 8.21 the subgroup T is a maximal quasi-torus in F_1 and the subgroup T' is a maximal quasi-torus in F_2. Since $\varphi|_{F_1}\colon F_1 \to F_2$ is an isomorphism, $\varphi(T)$ is a maximal quasi-torus in F_2. Therefore, $\varphi(T)$ is conjugate to T' under F_2 by Theorem 8.10(b). Since T' is a maximal quasi-torus in H, we get that $\varphi(T)$ is conjugate to a maximal quasi-torus in H, so itself is a maximal quasi-torus in H. □

9. Intersecting Conjugacy Classes with Maximal Quasi-Tori

In this section we continue to consider linear algebraic groups G over an algebraically closed field L of characteristic 0. We collect several results which we will need in the following sections.

Notation 9.1. Let G be a linear algebraic group, let $T \subset G$ be a maximal quasi-torus. Note that T° commutes with $T^h := \{g \in T^\circ\colon gh = hg\}$ for every $h \in T$. So we have $T^{ht} = T^h$ for every $t \in T^\circ$. The conjugation action of T° on hT° is given for $t \in T^\circ$ and $hx \in hT^\circ$ by $thxt^{-1} = hh^{-1}thxt^{-1} = hxh^{-1}tht^{-1}$, because $h^{-1}th \in T^\circ$ commutes with $x \in T^\circ$. Therefore, the map

$$\varphi\colon T^\circ \longrightarrow T^\circ, \quad t \longmapsto h^{-1}tht^{-1} \tag{9.1}$$

satisfies $thxt^{-1} = hx\varphi(t)$ for every $t \in T^\circ$ and $hx \in hT^\circ$. Moreover, φ is a homomorphism of algebraic groups, namely the product of the endomorphisms $t \mapsto h^{-1}th$ and $t \mapsto t^{-1}$ of the commutative group T°. The kernel of φ is T^h. Let Q_h be the quotient T°/T^h. Then φ induces a closed immersion of tori

$$\overline{\varphi}\colon Q_h \hookrightarrow T^\circ, \quad \bar{t} = t \bmod T^h \longmapsto h^{-1}tht^{-1}.$$

Proposition 9.2. *In the situation of Notation 9.1 the following holds:*

(a) The natural group homomorphisms $T^{h\circ}\times_L Q_h \to T^\circ$, $(t_0,\bar t\,)\mapsto t_0\cdot\overline{\varphi}(\bar t)$ and $T^{h\circ}\to T^\circ/\overline{\varphi}(Q_h)$ are surjective with finite kernels.

(b) Every element of hT° is conjugate under T° to an element of $hT^{h\circ}$,

(c) The intersection of $hT^{h\circ}$ with any G°-conjugacy class (respectively G-conjugacy class) is finite.

Proof. (a) The kernel of the first homomorphism is the set $\{(t_0,\bar t\,)\colon t_0=\overline{\varphi}(\bar t)^{-1}=th^{-1}t^{-1}h\}$. This condition is equivalent to $h^{-1}th=t_0^{-1}t$. Since $t_0\in T^h$ we obtain $h^{-n}th^n=t_0^{-n}t$ for every positive integer n. If n equals the order of h in the group T/T° of connected components, then $h^n\in T^\circ$ and $h^{-n}th^n=t$. This shows that $t_0^n=1$ and $\overline{\varphi}(\bar t^{\,n})=\overline{\varphi}(\bar t)^n=t_0^{-n}=1$. In particular, the kernel of the first homomorphism is contained in the n-torsion subgroup of the torus $T^{h\circ}\times_L Q_h$ and the kernel of the second homomorphism is contained in the n-torsion subgroup of the torus $T^{h\circ}$. Both are finite groups. By [Bor91, I.1.4 Corollary] the surjectivity now follows from this, from the irreducibility of the targets, and from the comparison of dimensions $\dim Q_h=\dim T^\circ-\dim T^h$ and $\dim T^\circ/\overline{\varphi}(Q_h)=\dim T^\circ-\dim Q_h=\dim T^h=\dim T^{h\circ}$.

(b) By (a) every $hx\in hT^\circ$ is of the form $hx=ht_0\overline{\varphi}(\bar t)=ht_0\varphi(t)=t(ht_0)t^{-1}$ for $ht_0\in hT^{h\circ}$ and $t\in T^\circ$.

(c) Since the projection onto the maximal reductive quotient $G\to G^{\rm red}$ maps T isomorphically to its image, and it maps conjugate elements to conjugate elements, we may assume that G is reductive without loss of generality. (Note that there may be elements in T which are not conjugate under G, but whose images are conjugate under $G^{\rm red}$. So the cardinality of the intersection in question may grow by passing to $G^{\rm red}$.) The proposition is now a consequence of the following more precise statement. □

Proposition 9.3. *In the situation of Notation 9.1 let G be reductive, let $N_G(T^{h\circ})$ be the normalizer and let $W\subset\{\,w\in N_G(T^{h\circ})\colon whw^{-1}\in hT^{h\circ}\}$ be a finite subset which is maximal (under the inclusion of subsets) such that $W\hookrightarrow N_G(T^{h\circ})/N_G(T^{h\circ})^\circ$ is injective. Moreover, let $m>0$ be the smallest positive integer with $h^m\in T^{h\circ}$ and consider $Z:=\{z\in T^{h\circ}\colon z^m=1\}$, which is a finite group. If two elements $u,v\in hT^{h\circ}$ are conjugate under G, then there are elements $w\in W$ and $z\in Z$ with $u=zwvw^{-1}$.*

Remark 9.4. (a) The set $\{\,w\in N_G(T^{h\circ})\colon whw^{-1}\in hT^{h\circ}\}$ is a subgroup. But in general it does not contain $N_G(T^{h\circ})^\circ$.

(b) Note that we do *not* claim that for every $w\in W$ and $z\in Z$ the element $zwvw^{-1}$ is conjugate to v.

(c) When G is connected, $T=T^\circ$ and so $h\in T^\circ$ and $T^h=T^{h\circ}=T^\circ$ is a maximal torus in G. Thus we can take W as (a system of representatives of) the Weyl group of T°. Also $m=1$ and $Z=\{1\}$. In this way we recover the result of Steinberg [Ste74, § III.3.4, Corollary 2]: Two elements of T are conjugate under G if and only if they are conjugate under W.

Proof of Proposition 9.3. First note that some power of h lies in T°, and since this power commutes with h, it also lies in T^h. Multiplying the exponent further by the order of $T^h/T^{h\circ}$ produces an integer $m>0$ such that $h^m\in T^{h\circ}$. Also note that Z is a finite group because $T^{h\circ}$ is commutative.

Let $u,v\in hT^{h\circ}$ be conjugate under G, and pick an $x\in G$ such that $u=xvx^{-1}$. Since v centralizes $T^{v\circ}=T^{h\circ}=T^{u\circ}$, the conjugate $u=xvx^{-1}$ centralizes $xT^{u\circ}x^{-1}$. But u also centralizes $T^{u\circ}$, so we get that $T^{u\circ},xT^{u\circ}x^{-1}\subset G^{u\circ}$. By Theorem 8.2 the subgroups $T^{u\circ}\subset G^{u\circ}$ and $T^{v\circ}\subset G^{v\circ}$ are maximal tori, and hence $xT^{v\circ}x^{-1}=xT^{u\circ}x^{-1}$ is also a maximal torus in $xG^{v\circ}x^{-1}=G^{u\circ}$. So there is a $y\in G^{u\circ}$ such that $yxT^{u\circ}x^{-1}y^{-1}=T^{u\circ}$. Then $w:=yx\in N_G(T^{h\circ})$, but also $u=yuy^{-1}=yxvx^{-1}y^{-1}=wvw^{-1}$, as $y\in G^u$.

Writing $u=h\tilde u=\tilde uh$ and $v=h\tilde v=\tilde vh$ with $\tilde v,\tilde u\in T^{h\circ}$, we see that $h\tilde u=u=wvw^{-1}=whw^{-1}w\tilde vw^{-1}$ and thus $whw^{-1}=h\tilde u(w\tilde v^{-1}w^{-1})\in hT^{h\circ}$, because $\tilde v\in T^{h\circ}$ which is normalized by w. So there is an element $\tilde w\in W$ such that $w=\tilde wn$ for an $n\in N_G(T^{h\circ})^\circ$. Let $t\in T^{h\circ}$ with $\tilde wh\tilde w^{-1}=ht=th$. Then $u^{-1}(\tilde wv\tilde w^{-1})=\tilde u^{-1}h^{-1}(\tilde wh\tilde v\tilde w^{-1})=\tilde u^{-1}t(\tilde w\tilde v\tilde w^{-1})=(\tilde w\tilde v\tilde w^{-1})t\tilde u^{-1}=(\tilde w\tilde vh\tilde w^{-1})h^{-1}\tilde u^{-1}=(\tilde wv\tilde w^{-1})u^{-1}$, because $T^{h\circ}$ is commutative. We set $z:=u(\tilde wv\tilde w^{-1})^{-1}$ and compute $z^m=u^m(\tilde wv^m\tilde w^{-1})^{-1}$. By [Bor91, III.8.10, Corollary 2], $N_{G^\circ}(T^{h\circ})^\circ=Z_{G^\circ}(T^{h\circ})^\circ$, that is $T^{h\circ}$ centralizes $N_{G^\circ}(T^{h\circ})^\circ$. Now, $v^m=h^m\tilde v^m$ lies in $T^{h\circ}$, and hence commutes with n. We conclude that $u^m=wv^mw^{-1}=\tilde wnv^mn^{-1}\tilde w^{-1}=\tilde wv^m\tilde w^{-1}$. Therefore, $z^m=1$ and $z\in Z$, whence $u=z\tilde wv\tilde w^{-1}$ as claimed. □

Corollary 9.5. *In the situation of Notation 9.1 let C be the intersection of hT° with a G°-conjugacy class (respectively a G-conjugacy class) in G. Then C is a finite union of T°-conjugacy classes on hT°.*

Proof. Write C as a disjoint union:
$$C = \coprod_{i \in I} C_i$$
such that each $\emptyset \neq C_i \subset hT^\circ$ is a T°-conjugacy class. By Proposition 9.2(b) for each $i \in I$ the intersection $C_i \cap hT^{h\circ}$ is non-empty. Therefore, $\#(C \cap hT^{h\circ}) \geq \#I$. But the set $C \cap hT^{h\circ}$ is finite by Proposition 9.2(c). $\square$

Definition 9.6. For a closed subgroup $H \subset G$ and a set $C \subset G$ let ${}^H C = \bigcup\limits_{g \in H} gCg^{-1}$ be the union of the H-conjugacy classes of elements of C. Clearly the map $C \mapsto {}^H C$ on subsets of G preserves inclusions and satisfies ${}^H({}^H C) = {}^H C$.

Proposition 9.7. *In the situation of Notation 9.1 let $C \subset hT^\circ$ be a subset and let $H \subset G$ be a closed subgroup containing T°. Then ${}^H C \cap hT^{h\circ}$ is Zariski-dense in $hT^{h\circ}$ if and only if ${}^H C \cap hT^\circ$ is Zariski-dense in hT°.*

Proof. We set $C_1 := {}^H C \cap hT^{h\circ}$ and observe that ${}^H C_1 = {}^H C$, because the inclusion ${}^H C_1 \subset {}^H C$ is obvious and the opposite inclusion follows from Proposition 9.2(b), because $T^\circ \subset H$.

First assume that $hT^{h\circ}$ equals the Zariski-closure $\overline{C_1}$ of C_1, and let $x \in hT^\circ$. By Proposition 9.2(b) there is a $t \in T^\circ \subset H$ with $txt^{-1} \in hT^{h\circ} = \overline{C_1}$. Then $x \in t^{-1}\overline{C_1}t = \overline{t^{-1}C_1 t}$. Since $t^{-1}C_1 t$ is contained in ${}^H C_1 \cap hT^\circ = {}^H C \cap hT^\circ$ we conclude that $x \in \overline{t^{-1}C_1 t} \subset \overline{{}^H C \cap hT^\circ}$. Therefore, ${}^H C \cap hT^\circ$ is Zariski-dense in hT°.

For the converse implication we assume that $\overline{C_1} \neq hT^{h\circ}$ and consider the subset
$$D \;:=\; \{\, (c_1, b, g) \in C_1 \times hT^\circ \times H \text{ such that } b = gc_1g^{-1} \,\} \;\subset\; C_1 \times hT^\circ \times H$$
and the projections $\pi_1 \colon D \to C_1$ and $\pi_2 \colon D \to hT^\circ$. Then $\pi_2(D) = {}^H C_1 \cap hT^\circ = {}^H C \cap hT^\circ$. We consider the following diagram which is *not* commutative
$$\begin{tikzcd} D \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & hT^\circ \arrow[d, "\gamma"] \\ C_1 \subset hT^{h\circ} \arrow[r, "\beta"] & hT^\circ/\overline{\varphi}(Q_h) \end{tikzcd}$$
where β is induced from the homomorphism from Proposition 9.2(a). Although the diagram is not commutative, we claim that $\gamma\pi_2(D) \subset \beta\pi_1(D)$. Indeed, let $x \in \gamma\pi_2(D)$ and let $(c_1, b, g) \in D$ be a preimage of x, that is $b = gc_1g^{-1}$. By Proposition 9.2(b) there is an element $t \in T^\circ \subset H$ such that $c := tbt^{-1} \in hT^{h\circ}$. Then $c = (tg)c_1(tg)^{-1} \in {}^H C_1 \cap hT^{h\circ} = C_1$. Moreover, $b = t^{-1}ct = c\varphi(t)$, and hence $(c, b, t^{-1}) \in D$. This shows that $x = \gamma\pi_2(c_1, b, g) = b \cdot \overline{\varphi}(Q_h) = c \cdot \overline{\varphi}(Q_h) = \beta\pi_1(c, b, t^{-1}) \in \beta\pi_1(D)$ and proves the claim.

Since $\overline{C_1} \neq hT^{h\circ}$ and $hT^{h\circ}$ is irreducible, we get the inequality $\dim \overline{C_1} < \dim hT^{h\circ}$ for the dimensions. By [EGA, IV$_2$, Théorème 4.1.2] we have $\dim \overline{\beta(\overline{C_1})} \leq \dim \overline{C_1} < \dim hT^{h\circ} = \dim hT^\circ/\overline{\varphi}(Q_h)$, and therefore $\overline{\beta(\overline{C_1})}$ is a proper closed subset of $hT^\circ/\overline{\varphi}(Q_h)$ which contains $\gamma\pi_2(D)$. Since γ is surjective, the preimage of $\overline{\beta(\overline{C_1})}$ under γ is a proper closed subset of hT° which contains $\pi_2(D) = {}^H C \cap hT^\circ$. This shows that the Zariski-closure of ${}^H C \cap hT^\circ$ is strictly contained in hT° and finishes the proof. $\square$

Proposition 9.8. *Let G be a linear algebraic group, let $T \subset G$ be a maximal quasi-torus, let $h \in T$, and let $C \subset hT^\circ$ be a subset. Let $\alpha \colon G \twoheadrightarrow \widetilde{G} := G/R_uG$ be the projection onto the maximal reductive quotient of G. Let $H \subset G$ and $\widetilde{H} \subset \widetilde{G}$ be closed subgroups with $T^\circ \subset H$ and $\alpha(H) \subset \widetilde{H}$. Then ${}^H C \cap hT^\circ$ is Zariski-dense in hT° if and only if ${}^{\widetilde{H}}\alpha(C) \cap \alpha(hT^\circ)$ is Zariski-dense in $\alpha(hT^\circ)$.*

Remark. Note that the "if"-direction is not obvious, because there may be elements in hT° which are not conjugate under H, but whose images are conjugate under $\alpha(H)$.

Proof. Note that $\alpha({}^H C) \subset {}^{\widetilde{H}}\alpha(C)$, and hence $\alpha({}^H C \cap hT^\circ) \subset {}^{\widetilde{H}}\alpha(C) \cap \alpha(hT^\circ)$. Since $\alpha \colon hT^\circ \xrightarrow{\sim} \alpha(hT^\circ)$ is an isomorphism, the "only if"-direction is clear.

To prove the converse, we use Notation 9.1 and let $C_1 := {}^H C \cap hT^{h\circ}$. Then ${}^H C_1 = {}^H C$ as in the proof of Proposition 9.7. Moreover, ${}^{\widetilde{H}}\alpha(C_1) = {}^{\widetilde{H}}\alpha(C)$, because the inclusion ${}^{\widetilde{H}}\alpha(C_1) \subset {}^{\widetilde{H}}\alpha(C)$ follows from $\alpha(C_1) \subset \alpha({}^H C) \subset {}^{\widetilde{H}}\alpha(C)$, and the opposite inclusion follows from $\alpha(C) \subset \alpha({}^H C_1) \subset {}^{\widetilde{H}}\alpha(C_1)$. So by Proposition 9.7 it suffices to show that C_1 is Zariski-dense in $hT^{h\circ}$ provided that ${}^{\widetilde{H}}\alpha(C_1)\cap\alpha(hT^{h\circ})$ is Zariski-dense in $\alpha(hT^{h\circ})$.

Let $u\in {}^{\widetilde{H}}\alpha(C_1)\cap\alpha(hT^{h\circ})$, that is $u=\tilde g v\tilde g^{-1}$ for some $\tilde g\in\widetilde H$ and $v\in\alpha(C_1)\subset\alpha(hT^{h\circ})$. Then in the notation of Proposition 9.3 applied to $\alpha(T^{h\circ})\subset\widetilde G$, there are elements $z\in Z$ and $w\in W$ with $u=zwvw^{-1}$. We conclude that

$$ {}^{\widetilde{H}}\alpha(C_1)\cap\alpha(hT^{h\circ})\ \subset\ \bigcup_{z\in Z,w\in W} zw\alpha(C_1)w^{-1}\ \subset\ \alpha(hT^{h\circ})\,. $$

If the Zariski-closure $\overline{C_1}$ of C_1 is strictly contained in $hT^{h\circ}$, then $\overline{\alpha(C_1)}=\alpha(\overline{C_1})\neq\alpha(hT^{h\circ})$ and we obtain an inequality of dimensions $\dim\overline{\alpha(C_1)}<\dim\alpha(hT^{h\circ})$, because $\alpha(hT^{h\circ})$ is irreducible. On the other hand the Zariski-closure

$$ \overline{{}^{\widetilde{H}}\alpha(C_1)\cap\alpha(hT^{h\circ})}\ \subset\ \bigcup_{z\in Z,w\in W} zw\,\overline{\alpha(C_1)}\,w^{-1}\ \subset\ \alpha(hT^{h\circ})\,, $$

because the union is closed as a finite union of closed subsets. Since $\dim zw\,\overline{\alpha(C_1)}\,w^{-1}=\dim\overline{\alpha(C_1)}<\dim\alpha(hT^{h\circ})$ and $\alpha(hT^{h\circ})$ is irreducible, it cannot be the finite union of proper closed subsets. This implies that ${}^{\widetilde{H}}\alpha(C_1)\cap\alpha(hT^{h\circ})$ is not Zariski-dense in $\alpha(hT^{h\circ})$ and proves the proposition. □

Lemma 9.9. *Let G be reductive (but not necessarily connected) and let $T\subset G$ be a maximal quasi-torus.*

- *(a) For every $g\in G(L)$ we have $\overline{{}^G\{g\}}\cap T={}^G\{g_s\}\cap T$, where g_s is the semi-simple part in the multiplicative Jordan decomposition of g.*
- *(b) Let $\{C_x : x\in S\}$ be a collection of conjugacy classes in $G(L)$ and let $h\in T(L)$. Then $hG^\circ\cap\bigcup_{x\in S}C_x$ is Zariski-dense in hG° if and only if $hT^\circ\cap\bigcup_{x\in S}\overline{C_x}$ is Zariski-dense in hT°.*

Proof. Let $c\colon G\times G\to G$ be the map given by the rule $(h,g)\mapsto hgh^{-1}$.

(a) Theorem 8.9(a), (b) implies that ${}^G\{g\}$ is Zariski-closed in G if and only ${}^G\{g\}\cap T\neq\emptyset$. So if ${}^G\{g\}\cap T=\emptyset$ there is a $g'\in\overline{{}^G\{g\}}\smallsetminus{}^G\{g\}$. By Chevalley's theorem [EGA, IV$_1$, Théorème 1.8.4] ${}^G\{g\}=c(G\times\{g\})$ is constructible, so there is a non-empty Zariski-open subset of its closure $O\subset\overline{{}^G\{g\}}$ with $O\subset{}^G\{g\}$ by [EGA, 0$_{\rm III}$, Proposition 9.2.2]. In particular ${}^G\{g'\}\subset\overline{{}^G\{g\}}\smallsetminus O$ and $\overline{{}^G\{g'\}}\subsetneq\overline{{}^G\{g\}}$. Proceeding in this way will eventually produce a $g'\in\overline{{}^G\{g\}}\cap T$. Now we use that the map $s\colon h\mapsto h_s$ on $\overline{{}^G\{g\}}$ sending $h\in G$ to its semi-simple part h_s is actually a morphism of schemes by Lemma 9.10 below. Since ${}^G\{g\}$ is mapped into ${}^G\{g_s\}$ under s, its Zariski-closure $\overline{{}^G\{g\}}$ is mapped into $\overline{{}^G\{g_s\}}$. But g_s is semi-simple, so ${}^G\{g_s\}$ is closed, and hence the image of $\overline{{}^G\{g\}}$ under s lies in ${}^G\{g_s\}$. Therefore, $g'=g'_s$ also lies in ${}^G\{g_s\}$, and this shows that ${}^G\{g_s\}={}^G\{g'\}\subset\overline{{}^G\{g\}}$. It moreover shows that all semi-simple elements $h=h_s\in\overline{{}^G\{g\}}$ are mapped under s to ${}^G\{g_s\}$. Since $h=s(h)$, we conclude that $\overline{{}^G\{g\}}\cap T\subset{}^G\{g_s\}\cap T$ proving (a).

(b) Let $A:=\overline{hG^\circ\cap\bigcup_{x\in S}C_x}$ and $B:=\overline{hT^\circ\cap\bigcup_{x\in S}\overline{C_x}}$ denote the Zariski-closures. Note that A contains B, and since A is invariant under conjugation by G°, it also contains $\overline{c(G^\circ\times B)}$. From Theorem 8.9(c) we conclude that the set of semi-simple elements in hG°, which equals $c(G^\circ\times hT^\circ)$, is dense in hG°, that is $\overline{c(G^\circ\times hT^\circ)}=hG^\circ$.

Now, if $B=hT^\circ$ then this implies that A contains $\overline{c(G^\circ\times hT^\circ)}=hG^\circ$.

So we only have to show that $A=hG^\circ$ implies $B=hT^\circ$. Assume that this is not the case and let $V\subset hT^\circ$ be the open complement of B. We claim that the Zariski-closure of $c(G^\circ\times V)$ equals hG°. Indeed, $(G^\circ\times hT^\circ)\smallsetminus c^{-1}(\overline{c(G^\circ\times V)})$ is open in $G^\circ\times hT^\circ$ and its image in hT° with respect to the projection does not meet V. Since the projection $G^\circ\times hT^\circ\to hT^\circ$ is flat of finite presentation, this image is open by [EGA, IV$_2$, Théorème 2.4.6], and hence empty because V is open and dense in the irreducible variety hT°. Thus $(G^\circ\times hT^\circ)\subset c^{-1}(\overline{c(G^\circ\times V)})$ and $c(G^\circ\times hT^\circ)\subset\overline{c(G^\circ\times V)}$. Since $\overline{c(G^\circ\times hT^\circ)}=hG^\circ$, we have $\overline{c(G^\circ\times V)}=hG^\circ$, which is our claim. By Chevalley's theorem [EGA, IV$_1$, Théorème 1.8.4] $c(G^\circ\times V)$ is constructible, so it must contain a non-empty Zariski-open subset $O\subset hG^\circ$ by [EGA, 0$_{\rm III}$, Proposition 9.2.2]. Because $hG^\circ\cap\bigcup_{x\in S}C_x$ is Zariski-dense in hG°, we get that there is an $x\in S$ such that $C_x\cap O\neq\emptyset$. Hence there is a $h\in G^\circ$ such that $h^{-1}Vh\cap C_x\neq\emptyset$, and since C_x is a conjugacy class, we get $V\cap C_x\neq\emptyset$. This is a contradiction. □

Lemma 9.10. *Let G be a linear algebraic group, and consider the maps of sets $s\colon G \to G$, $h \mapsto h_s$ and $u\colon G \to G$, $h \mapsto h_u$, which send every element h to its semi-simple part h_s, respectively unipotent part h_u in the multiplicative Jordan decomposition. Let $g \in G$. Then the restriction of these maps to the reduced closed subscheme $\overline{{}^G\{g\}}$, where ${}^G\{g\}$ is the conjugacy class of g, are morphisms of schemes.*

Proof. Let $\rho\colon G \hookrightarrow \mathrm{GL}_n =: H$ be a faithful linear representation. Since $\rho(\overline{{}^G\{g\}}) \subset \overline{{}^H\{\rho(g)\}}$ and the multiplicative Jordan decomposition is compatible with ρ by [Bor91, I.4.4 Theorem], it will be enough to show the claim for $\overline{{}^H\{\rho(g)\}}$, or in other words we may assume that $G = \mathrm{GL}_n$ without loss of generality. Note that the characteristic polynomial, as a conjugation-invariant regular function, is constant on $\overline{{}^G\{g\}}$. Let $\chi(t) \in L[t]$ be this value, and write

$$\chi(t) = \prod_{j=1}^{r} (t - \lambda_j)^{r_j},$$

where $\lambda_j \in L$ are pairwise different, and the r_j are positive integers. Since the polynomials

$$P_j(t) = \chi(t)/(t - \lambda_j)^{r_j}, \quad (j = 1, 2, \ldots, r)$$

have no common divisor, and $L[t]$ is a principal ideal domain, there are $Q_1, Q_2, \ldots, Q_r \in L[t]$ such that

$$1 = Q_1 P_1 + Q_2 P_2 + \cdots + Q_r P_r.$$

In particular $Q_j(t)P_j(t) \equiv 1 \bmod (t - \lambda_j)^{r_j}$ for all j, because $(t - \lambda_j)^{r_j}$ divides $P_i(t)$ for every $i \neq j$, and hence $(Q_j P_j)^2 \equiv Q_j P_j \bmod \chi$. Let $h \in \overline{{}^G\{g\}}$. Then the characteristic polynomial of h equals χ and the Cayley-Hamilton theorem implies that the endomorphism $Q_j(h)P_j(h) \in \mathrm{End}_L(L^n)$ is the projection onto the generalized eigenspace $\ker(h - \lambda_j)^{r_j}$ of h for the eigenvalue λ_j. Thus the polynomial

$$P(t) := \lambda_1 Q_1(t)P_1(t) + \lambda_2 Q_2(t)P_2(t) + \cdots + \lambda_r Q_r(t)P_r(t) \ \in\ L[t]$$

satisfies

$$s(h) = P(h) = \lambda_1 Q_1(h)P_1(h) + \lambda_2 Q_2(h)P_2(h) + \cdots + \lambda_r Q_r(h)P_r(h) \in M_{n\times n}(L)$$

for every $h \in \overline{{}^G\{g\}}$. Clearly this is a polynomial map, hence a morphism of schemes. Thus $u(h) = s(h)^{-1} \cdot h$ is also a morphism. □

Corollary 9.11. *Let G be a reductive group (which is not necessarily connected). Let $F \subset G$ be a union of conjugacy classes and let $F^{ss} = \{g_s\colon g \in F\}$ be the set consisting of the semi-simple parts g_s of the elements g of F. Then F is Zariski-dense in a connected component of G if and only if F^{ss} is Zariski-dense in that connected component.*

Proof. For every $g \in G$ the Zariski-closure $\overline{{}^G\{g\}}$ of the conjugacy class ${}^G\{g\}$ of g contains the conjugacy class ${}^G\{g_s\}$ of the semi-simple part g_s of g by Lemma 9.9(a). So the Zariski-closure of F^{ss} lies in the Zariski-closure of F, and hence one implication holds. On the other hand let $hG^\circ \subset G$ be a connected component in which $F \cap hG^\circ$ is Zariski-dense. By Theorem 8.9(c) there is a dense open subset $O \subset hG^\circ$ such that O only contains semi-simple elements. Then $F \cap O$ is also Zariski-dense in hG°, but $F \cap O \subset F^{ss} \cap hG^\circ$. Therefore, $F^{ss} \cap hG^\circ$ is also Zariski-dense in hG°. □

10. The Weakly Pink Hypothesis and its Consequences

Conjecture 10.1. *Consider an open sub-curve $f\colon V \hookrightarrow U$ and a base point $u \in V(\mathbb{F}_{q^e})$. Let $\mathcal{F} \in F\text{-}\mathrm{Isoc}_K(U)$ be a convergent F-isocrystal on U. Then $\mathrm{Gr}(f^*\mathcal{F}/V, u) \subset \mathrm{Gr}(\mathcal{F}/U, u)$ is a parabolic subgroup.*

In private conversation with one of us this was formulated by Richard Pink as a question in the special case when $\mathcal{F}$ comes from a p-divisible group on U. Note that by [Bor91, IV.11.2 Corollary] and the following lemma, the conjecture is equivalent to the assertion that the natural injective morphism $\mathrm{Gr}(f^*\mathcal{F}/V, u) \to \mathrm{Gr}(\mathcal{F}/U, u)$ maps every Borel subgroup of $\mathrm{Gr}(f^*\mathcal{F}/V, u) \times_{K_e} \overline{K}$ onto a Borel subgroup of $\mathrm{Gr}(\mathcal{F}/U, u) \times_{K_e} \overline{K}$.

Lemma 10.2. *The natural morphism $\mathrm{Gr}(f^*\mathcal{F}/V, u) \to \mathrm{Gr}(\mathcal{F}/U, u)$ always is a closed immersion.*

Proof. Every object of $\langle\!\langle f^*\mathcal{F} \rangle\!\rangle$ is a subquotient of an object of the form $\bigoplus_i f^*\mathcal{F}^{\otimes m_i} \otimes (f^*\mathcal{F}^\vee)^{\otimes n_i} = f^*\big(\bigoplus_i \mathcal{F}^{\otimes m_i} \otimes (\mathcal{F}^\vee)^{\otimes n_i}\big)$. Now the statement follows from Proposition A.14(b). □

Definition 10.3. Let $\mathcal{F} \in F\text{-Isoc}_K(U)$ be a convergent F-isocrystal on U and let $f\colon V \hookrightarrow U$ be an open sub-curve with a base point $u \in V(\mathbb{F}_{q^e})$.

(a) We will call $\mathcal{F}$ *pink* [1] with respect to $f\colon V \hookrightarrow U$ if under the inclusion $\mathrm{Gr}(f^*\mathcal{F}/V,u) \subset \mathrm{Gr}(\mathcal{F}/U,u)$ a Borel subgroup of $\mathrm{Gr}(f^*\mathcal{F}/V,u) \times_{K_e} \overline{K}$ is also a Borel subgroup of $\mathrm{Gr}(\mathcal{F}/U,u) \times_{K_e} \overline{K}$.

(b) We will call $\mathcal{F}$ *weakly pink* with respect to $f\colon V \hookrightarrow U$ if under the inclusion $\mathrm{Gr}(f^*\mathcal{F}/V,u) \subset \mathrm{Gr}(\mathcal{F}/U,u)$ a maximal quasi-torus of $\mathrm{Gr}(f^*\mathcal{F}/V,u) \times_{K_e} \overline{K}$ is also a maximal quasi-torus of the group $\mathrm{Gr}(\mathcal{F}/U,u) \times_{K_e} \overline{K}$.

(c) We will call $\mathcal{F}$ *conservative* with respect to f if $\mathrm{Gr}(f^*\mathcal{F}/V,u) \to \mathrm{Gr}(\mathcal{F}/U,u)$ is an isomorphism.

Note that $\mathcal{F}$ is (weakly) pink with respect to $f\colon V \hookrightarrow U$ if and only if *every* Borel subgroup (resp. *every* maximal quasi-torus) of $\mathrm{Gr}(f^*\mathcal{F}/V,u) \times_{K_e} \overline{K}$ is also a Borel subgroup (resp. maximal quasi-torus) of $\mathrm{Gr}(\mathcal{F}/U,u) \times_{K_e} \overline{K}$, because they are all conjugate by [Bor91, IV.11.1 Theorem] (resp. Theorem 8.10).

The reason why we are interested in this concept is the following theorem, which reformulates Theorem 1.11 and which we will prove at the end of this section.

Theorem 10.4. *Let $\mathcal{F} \in F\text{-Isoc}_K(U)$ be a convergent semi-simple F-isocrystal on U which is weakly pink with respect to an open sub-curve $f\colon V \hookrightarrow U$ for which $f^*\mathcal{F}$ has a slope filtration on V (with isoclinic subquotients). Then $\mathcal{F}$ satisfies Conjectures 1.2, 1.3 and 1.4.*

Note that by the specialization theorem of Grothendieck and Katz [Kat79, Corollary 2.3.2] there always is an open sub-curve $f\colon V \hookrightarrow U$ on which the Newton polygon of $\mathcal{F}$ is constant, and by the slope filtration theorem [Kat79, Corollary 2.6.3] the restriction $f^*\mathcal{F}$ has a slope filtration with isoclinic subquotients. Our assumption is that $\mathcal{F}$ is weakly pink with respect to such an f.

Before we prove the theorem (after Definition 10.15) let us establish a few facts about (weakly) pink F-isocrystals.

Proposition 10.5. *Let $\mathcal{F} \in F\text{-Isoc}_K(U)$ be a convergent F-isocrystal on U and let $f\colon V \hookrightarrow U$ be an open sub-curve with a base point $u \in V(\mathbb{F}_{q^e})$.*

(a) If $\mathcal{F}$ is pink with respect to f, then it is weakly pink with respect to f.

(b) If $\mathcal{F}$ is weakly pink with respect to f then the natural inclusion $\beta\colon \mathrm{Gr}(f^*\mathcal{F}/V,u) \hookrightarrow \mathrm{Gr}(\mathcal{F}/U,u)$ induces an isomorphism on the groups of connected components and every maximal torus of $\mathrm{Gr}(f^*\mathcal{F}/V,u)$ is also a maximal torus of $\mathrm{Gr}(\mathcal{F}/U,u)$. (See Warning 10.6 for the converse.)

(c) Without assumption on $\mathcal{F}$, the inclusion $\mathrm{Gr}(f^*\mathcal{F}/V,u) \subset \mathrm{Gr}(\mathcal{F}/U,u)$ always induces a surjection $\big(\mathrm{Gr}(f^*\mathcal{F}/V,u)/\mathrm{Gr}(f^*\mathcal{F}/V,u)^\circ\big) \times_{K_e} \overline{K} \twoheadrightarrow \big(\mathrm{Gr}(\mathcal{F}/U,u)/\mathrm{Gr}(\mathcal{F}/U,u)^\circ\big) \times_{K_e} \overline{K}$ on the groups of geometrically connected components.

Proof. (b) Let T_1 be a maximal quasi-torus of $G_1 := \mathrm{Gr}(f^*\mathcal{F}/V,u) \times_{K_e} \overline{K}$ and set $G_2 := \mathrm{Gr}(\mathcal{F}/U,u) \times_{K_e} \overline{K}$. If $\mathcal{F}$ is weakly pink with respect to f, then $T_2 := \beta(T_1)$ is a maximal quasi-torus of G_2 and $\beta(T_1^\circ) = T_2^\circ$ is a maximal torus of G_2 by Lemma 8.8. Furthermore, Theorem 8.10 implies that $G_1/G_1^\circ \cong T_1/T_1^\circ \cong T_2/T_2^\circ \cong G_2/G_2^\circ$ is an isomorphism on the groups of connected components over $\overline{K}$. Since this isomorphism is already defined over K_e, this proves (b).

(c) Let $\mathcal{U}$ be an object of $\langle\!\langle \mathcal{F} \rangle\!\rangle$ such that the surjective homomorphism $\mathrm{Gr}(\mathcal{F}/U,u) \twoheadrightarrow \mathrm{Gr}(\mathcal{U}/U,u)$ has kernel equal to the characteristic subgroup $G_2^\circ \subset G_2$; see Remark A.17 and Corollary A.16(b). Then $\mathcal{U}$ is unit-root by Lemma 5.1, and hence the inclusion map $\mathrm{Gr}(f^*\mathcal{U}/V,u) \subset \mathrm{Gr}(\mathcal{U}/U,u)$ is an isomorphism by Corollary 5.11. Since $f^*\mathcal{U}$ is an object of $\langle\!\langle f^*\mathcal{F} \rangle\!\rangle$ the corresponding homomorphism $\mathrm{Gr}(f^*\mathcal{F}/V,u) \times_{K_e} \overline{K} \twoheadrightarrow \mathrm{Gr}(f^*\mathcal{U}/V,u) \times_{K_e} \overline{K} \xrightarrow{\sim} \mathrm{Gr}(\mathcal{U}/U,u) \times_{K_e} \overline{K} = G_2/G_2^\circ$ is surjective by Lemma 3.3. It follows that $G_1/G_1^\circ \twoheadrightarrow G_2/G_2^\circ$ is surjective.

(a) Let T_1 be a maximal quasi-torus in G_1 and let $\alpha\colon G_2 \twoheadrightarrow G_2/R_uG_2 =: G_3$ be the maximal reductive quotient of G_2, where R_uG_2 is the unipotent radical of G_2. We have to show that α maps $\beta(T_1)$ isomorphically onto a maximal quasi-torus of G_3. By Theorem 8.10 there is a Borel subgroup B_1° of G_1° which is normalized by T_1. Since $\beta(G_1) \subset G_2$ is parabolic, $\beta(T_1^\circ)$ is a maximal torus and $\beta(B_1^\circ)$ is a Borel subgroup of G_2° by [Bor91, IV.11.2 Corollary and IV.11.3 Corollary]. Now [Bor91, IV.11.14 Proposition] shows that $T_3^\circ := \alpha\beta(T_1^\circ)$ is a maximal torus and $B_3^\circ := \alpha\beta(B_1^\circ)$ is a Borel subgroup of G_3°. The latter is normalized

[1] as an abbreviation for "die Monodromie-Gruppe wird parabolisch unter der Inklusion von Kurven" (the monodromy group becomes parabolic under the inclusion of curves)

by $\alpha\beta(T_1)$. Therefore, $\alpha\beta(T_1)$ is contained in the maximal quasi-torus $T_3 := N_{G_3}(B_3^\circ) \cap N_{G_3}(T_3^\circ)$. Since all elements of $\beta(T_1)$ are semi-simple by Theorem 8.10 and the kernel of α is unipotent, the restriction of α to $\beta(T_1)$ is injective, and therefore the map $\alpha\beta$ induces an injection $T_1/T_1^\circ \hookrightarrow T_3/T_3^\circ$. On the other hand, this injection equals the surjection $T_1/T_1 = G_1/G_1^\circ \twoheadrightarrow G_2/G_2^\circ = G_3/G_3^\circ = T_3/T_3^\circ$ from (c). This shows that α maps $\beta(T_1)$ isomorphically onto T_3 as desired. □

Warning 10.6. If we consider a closed subgroup G_1 of a non-connected linear algebraic group G_2 and a maximal quasi-torus T_1 in G_1 one can ask whether there is a maximal quasi-torus of G_2 containing T_1. We believe that this is not true in general, even under the assumption that $G_1/G_1^\circ = G_2/G_2^\circ$ and that T_1° is a maximal torus in G_2°, but see Theorem 11.6 below. That is, we believe that the converse of Proposition 10.5(b) does not hold in general. In order to prove this converse one would have to show that (in the notation of the proof of Proposition 10.5) every maximal quasi-torus T_1 of G_1 is mapped isomorphically to a maximal quasi-torus in the maximal reductive quotient G_3 of G_2. By our hypothesis we obtain an isomorphism $G_1/G_1^\circ \xrightarrow{\sim} G_2/G_2^\circ \xrightarrow{\sim} G_3/G_3^\circ$. The proof now reduces to the following group theoretic statement. By hypothesis $T_3^\circ = \alpha\beta(T_1^\circ)$ is a maximal torus in G_3° and we choose a Borel subgroup $B_3^\circ \subset G_3^\circ$ containing T_3°. Then $T_3 := N_{G_3}(B_3^\circ) \cap N_{G_3}(T_3^\circ)$ is a maximal quasi-torus in G_3. The isomorphism $G_3/G_3^\circ \xrightarrow{\sim} T_3/T_3^\circ \subset N_{G_3}(T_3^\circ)/T_3^\circ$ from Theorem 8.9 yields a split exact sequence of groups

$$1 \longrightarrow N_{G_3^\circ}(T_3^\circ)/T_3^\circ \longrightarrow N_{G_3}(T_3^\circ)/T_3^\circ \longrightarrow G_3/G_3^\circ \longrightarrow 1\,. \tag{10.1}$$

Here $W := N_{G_3^\circ}(T_3^\circ)/T_3^\circ$ is the Weyl group. The morphism $G_3/G_3^\circ \xrightarrow{\sim} G_1/G_1^\circ \xrightarrow{\sim} T_1/T_1^\circ \xrightarrow{\alpha\beta} N_{G_3}(T_3^\circ)/T_3^\circ$ yields another splitting of (10.1). One has to show that the two splittings are conjugate. Every conjugacy class of splittings $s\colon G_3/G_3^\circ \to N_{G_3}(T_3^\circ)/T_3^\circ$ defines a cohomology class $\varphi\colon G_3/G_3^\circ \to W$ in $\mathrm{H}^1(G_3/G_3^\circ, W)$ as follows. Let $g \in G_3/G_3^\circ$. Since $s(g)$ normalizes T_3° it conjugates B_3° to another Borel subgroup containing T_3°. The latter is of the form $s(g)^{-1}B_3^\circ s(g) = \varphi(g)B_3^\circ\varphi(g)^{-1}$ for a uniquely determined element $\varphi(g) \in W$ by [Bor91, II.11.19 Proposition]. The cohomology class φ is trivial if and only if the splitting comes from a maximal quasi-torus, because if $s(g) \in T_3 \subset N_{G_3}(B_3^\circ)$ then $\varphi(g) = 1$. So $\alpha\beta(T_1)$ is a maximal quasi-torus if and only if the corresponding cohomology class is trivial. Now it is not difficult to construct a group G_3 with $\mathrm{H}^1(G_3/G_3^\circ, W) \neq 0$ and to choose a splitting s with non-zero cohomology class. Unfortunately we were not able to show that this situation cannot arise from a group homomorphism $G_1 \to G_2$ or from a convergent F-isocrystal $\mathcal{F} \in F\text{-Isoc}_K(U)$.

Proposition 10.7. *Let $\mathcal{F} \in F\text{-Isoc}_K(U)$ be a convergent F-isocrystal on U which is weakly pink with respect to an open immersion $f\colon V \hookrightarrow U$, and let $\mathcal{G}$ be a convergent F-isocrystal on U which is conservative with respect to f. Then $\mathcal{F} \oplus \mathcal{G}$ is weakly pink with respect to f.*

Proof. By Proposition 3.6(c) there are two Cartesian diagrams

$$\begin{tikzcd}[column sep=tiny] & \mathrm{Gr}(f^*(\mathcal{F}) \oplus f^*(\mathcal{G})/V, u) \arrow[dl, two heads, "\pi_1"'] \arrow[dr, two heads, "\rho_1"] & \\ \mathrm{Gr}(f^*\mathcal{F}/V, u) \arrow[dr, two heads] & \square & \mathrm{Gr}(f^*\mathcal{G}/V, u) \arrow[dl, two heads, "\sigma_1"] \\ & \mathrm{Gr}(\langle\!\langle f^*(\mathcal{F})\rangle\!\rangle \cap \langle\!\langle f^*(\mathcal{G})\rangle\!\rangle/V, u) & \end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=tiny] & \mathrm{Gr}(\mathcal{F} \oplus \mathcal{G}/U, u) \arrow[dl, two heads, "\pi_2"'] \arrow[dr, two heads, "\rho_2"] & \\ \mathrm{Gr}(\mathcal{F}/U, u) \arrow[dr, two heads] & \square & \mathrm{Gr}(\mathcal{G}/U, u) \arrow[dl, two heads, "\sigma_2"] \\ & \mathrm{Gr}(\langle\!\langle \mathcal{F}\rangle\!\rangle \cap \langle\!\langle \mathcal{G}\rangle\!\rangle/U, u), & \end{tikzcd}$$

of algebraic groups, where the maps are all induced by the inclusion functors on the respective Tannakian categories. Let N_1 and N_2 denote the kernel of π_1 and π_2, respectively. Then N_1 and N_2 are closed normal subgroups in $\mathrm{Gr}(f^*(\mathcal{F}) \oplus f^*(\mathcal{G})/V, u)$ and $\mathrm{Gr}(\mathcal{F} \oplus \mathcal{G}/U, u)$, respectively. We claim that N_1 maps isomorphically onto N_2 with respect to the map:

$$\varphi\colon\ \mathrm{Gr}(f^*(\mathcal{F}) \oplus f^*(\mathcal{G})/V, u) \to \mathrm{Gr}(\mathcal{F} \oplus \mathcal{G}/U, u)$$

induced by pull-back with respect to f. The map φ extends to a map of the Cartesian diagrams above which is also induced by pull-back with respect to f. Therefore, $\varphi(N_1) \subset N_2$. Now N_1 and N_2 can be described as the fiber products

$$\begin{tikzcd} & N_1 \arrow[dl] \arrow[dr, "\rho_1"] & \\ \{1\} \arrow[dr] & \square & \mathrm{Gr}(f^*\mathcal{G}/V,u) \arrow[dl, "\sigma_1"] \\ & \mathrm{Gr}(\langle\langle f^*\mathcal{F}\rangle\rangle \cap \langle\langle f^*\mathcal{G}\rangle\rangle/V,u) & \end{tikzcd}$$

and

$$\begin{tikzcd} & N_2 \arrow[dl] \arrow[dr, "\rho_2"] & \\ \{1\} \arrow[dr] & \square & \mathrm{Gr}(\mathcal{G}/U,u) \arrow[dl, "\sigma_2"] \\ & \mathrm{Gr}(\langle\langle \mathcal{F}\rangle\rangle \cap \langle\langle \mathcal{G}\rangle\rangle/U,u), & \end{tikzcd}$$

of algebraic groups, respectively. Since the map

$$\varphi\colon\ \mathrm{Gr}(f^*\mathcal{G}/V,u) \to \mathrm{Gr}(\mathcal{G}/U,u)$$

is an isomorphism by the assumption that $\mathcal{G}$ is conservative, and since

$$\varphi\colon\ \mathrm{Gr}(\langle\langle f^*\mathcal{F}\rangle\rangle \cap \langle\langle f^*\mathcal{G}\rangle\rangle/V,u) \to \mathrm{Gr}(\langle\langle \mathcal{F}\rangle\rangle \cap \langle\langle \mathcal{G}\rangle\rangle/U,u)$$

is a closed immersion by Lemma 10.2, we conclude that $\varphi|_{N_1}\colon N_1 \to N_2$ is an isomorphism. Note that

$$\mathrm{Gr}(f^*(\mathcal{F}) \oplus f^*(\mathcal{G})/V,u)/N_1 = \mathrm{Gr}(f^*\mathcal{F}/V,u) \text{ and } \mathrm{Gr}(\mathcal{F} \oplus \mathcal{G}/U,u)/N_2 = \mathrm{Gr}(\mathcal{F}/U,u),$$

and the group homomorphism

$$\widetilde{\varphi}\colon\ \mathrm{Gr}(f^*(\mathcal{F}) \oplus f^*(\mathcal{G})/V,u)/N_1 \to \mathrm{Gr}(\mathcal{F} \oplus \mathcal{G}/U,u)/N_2$$

induced by φ is the map:

$$\mathrm{Gr}(f^*\mathcal{F}/V,u) \to \mathrm{Gr}(\mathcal{F}/U,u)$$

induced by the pull-back with respect to f. If $T \subset \mathrm{Gr}(f^*(\mathcal{F}) \oplus f^*(\mathcal{G})/V,u) \times_{K_e} \overline{K}$ is a maximal quasi-torus, its images in $\mathrm{Gr}(f^*\mathcal{F}/V,u) \times_{K_e} \overline{K}$ and in $\mathrm{Gr}(\mathcal{F}/U,u) \times_{K_e} \overline{K}$ are maximal quasi-tori by Corollary 8.11 and by the assumption that $\mathcal{F}$ is weakly pink with respect to f. Therefore, the image of T in $\mathrm{Gr}(\mathcal{F} \oplus \mathcal{G}/U,u) \times_{K_e} \overline{K}$ is a maximal quasi-torus by Theorem 8.16, and hence $\mathcal{F} \oplus \mathcal{G}$ is also weakly pink with respect to f. $\square$

Definition 10.8. Let $\gamma\colon G_1 \to G_2$ be a homomorphism of linear algebraic groups over an algebraically closed field L of characteristic 0. We say that γ is *pink*, if it maps a Borel subgroup of G_1 onto a Borel subgroup of G_2. Similarly, we say that γ is *weakly pink*, if it maps a maximal quasi-torus of G_1 onto a maximal quasi-torus of G_2.

Note that γ is (weakly) pink if and only if it maps *every* Borel subgroup (resp. *every* maximal quasi-torus) of G_1 onto a Borel subgroup (resp. maximal quasi-torus) of G_2, because they are all conjugate under G_1° by [Bor91, IV.11.1 Theorem] (resp. Theorem 8.10).

Lemma 10.9. *Let*

$$\begin{tikzcd} G_1 \arrow[r, "\gamma"] \arrow[d, "\pi_1"'] & G_2 \arrow[d, "\pi_2"] \\ H_1 \arrow[r, "\chi"] & H_2 \end{tikzcd}$$

be a commutative diagram in the category of linear algebraic groups over an algebraically closed field L of characteristic 0, such that π_1 and π_2 are surjective. If γ is pink (resp. weakly pink), then χ is also pink (resp. weakly pink).

Proof. Let Γ be a Borel subgroup (resp. maximal quasi-torus) in G_1. Since π_1 is surjective, its image $\pi_1(\Gamma)$ is also a Borel subgroup (resp. maximal quasi-torus) in H_1 by Corollary 8.11. By assumption the image $\gamma(\Gamma)$ is also a Borel subgroup (resp. a maximal quasi-torus) in G_2. Since π_2 is surjective, the image $\pi_2(\gamma(\Gamma))$ of $\gamma(\Gamma)$ is also a Borel subgroup (resp. a maximal quasi-torus) in H_2 by Corollary 8.11. But $\pi_2(\gamma(\Gamma)) = \chi(\pi_1(\Gamma))$, so χ is pink (resp. weakly pink). $\square$

Proposition 10.10. *Let $\mathcal{F} \in F\text{-}\mathrm{Isoc}_K(U)$ be a convergent F-isocrystal on U which is pink (resp. weakly pink, resp. conservative) with respect to an open sub-curve $f\colon V \hookrightarrow U$, and let $\mathcal{G}$ be an object of $\langle\langle \mathcal{F} \rangle\rangle$. Then $\mathcal{G}$ is pink (resp. weakly pink, resp. conservative) with respect to f.*

Proof. We have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Gr}(f^*\mathcal{F}/V,u) & \hookrightarrow & \mathrm{Gr}(\mathcal{F}/U,u) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ \mathrm{Gr}(f^*\mathcal{G}/V,u) & \hookrightarrow & \mathrm{Gr}(\mathcal{G}/U,u) \end{array}$$

such that the vertical maps are surjective and the horizontal maps are induced by the inclusion $f\colon V \hookrightarrow U$. If the top horizontal group homomorphism is pink (resp. weakly pink), the lower horizontal group homomorphism is also pink (resp. weakly pink) by Lemma 10.9. On the other hand, if the top horizontal group homomorphism is an isomorphism, the lower horizontal group homomorphism is also an isomorphism, because it is a surjective closed immersion. $\square$

Next we will establish the following

Proposition 10.11. *Let $\mathcal{F}$ be a direct sum of isoclinic convergent F-isocrystals on U and let $f\colon V \hookrightarrow U$ be an open sub-curve with a base point $u \in V(\mathbb{F}_{q^e})$. Then $\mathcal{F}$ is conservative with respect to f, i.e. the map $\mathrm{Gr}(f^*\mathcal{F}/V,u) \hookrightarrow \mathrm{Gr}(\mathcal{F}/U,u)$ induced by the inclusion $f\colon V \hookrightarrow U$ is an isomorphism. In particular $\mathcal{F}$ is pink with respect to f.*

Before proving the proposition, we note the following

Corollary 10.12. *Let $\mathcal{F}$ be a convergent F-isocrystal on U such that the identity component $\mathrm{Gr}(\mathcal{F}/U,u)^\circ$ of its monodromy group is a torus. Let $f\colon V \hookrightarrow U$ be an open sub-curve with a base point $u \in V(\mathbb{F}_{q^e})$. Then $\mathcal{F}$ is conservative with respect to f.*

Proof. There is a finite field extension K_n' of K_e over which the torus $\mathrm{Gr}(\mathcal{F}/U,u)^\circ$ splits. Let n be the inertia degree of K_n' over K. Then K_n' is a totally ramified field extension of K_n. By Corollary 6.5 we may find a finite étale covering $g\colon \widetilde{U} \to U$ and take K_n' and n large enough, such that u lifts to a point $\tilde{u} \in \widetilde{V}(\mathbb{F}_{q^n})$ for $\widetilde{V} := \widetilde{U} \times_U V$, and such that $\mathrm{Gr}(g^*\mathcal{F}/\widetilde{U},\tilde{u}) \xrightarrow{\sim} \mathrm{Gr}(\mathcal{F}/U,u)^\circ$ is an isomorphism. We consider the functor $F\text{-}\mathrm{Isoc}_K(\widetilde{U}) \to F^n\text{-}\mathrm{Isoc}_{K_n'}(\widetilde{U}_n)$, $g^*\mathcal{F} \mapsto (g^*\mathcal{F})^{(n)} \otimes_{K_n} K_n'$ which is the composition of the functor $(\,.\,)^{(n)}$ from (3.1) on $\widetilde{U}$ and the extension functor $\otimes_{K_n} K_n'$ of the coefficients from K_n to K_n'. We get by Lemma 6.3(c) an open and closed immersion

$$\mathrm{Gr}(g^*\mathcal{F}^{(n)} \otimes_{K_n} K_n'/\widetilde{U}_n,\tilde{u}) \hookrightarrow \mathrm{Gr}(g^*\mathcal{F}/\widetilde{U},\tilde{u}) \times_{K_e} K_n'. \tag{10.2}$$

Since $\mathrm{Gr}(g^*\mathcal{F}/\widetilde{U},\tilde{u})$ is geometrically connected, the map (10.2) is an isomorphism and $\mathrm{Gr}(g^*\mathcal{F}^{(n)} \otimes_{K_n} K_n'/\widetilde{U}_n,\tilde{u})$ is a split torus. Therefore, its one-dimensional K_n'-rational representations generate the Tannakian category of all its representations. So $g^*\mathcal{F}^{(n)} \otimes_{K_n} K_n'$ belongs to the Tannakian sub-category $\langle\langle \widetilde{\mathcal{F}} \rangle\rangle \subset F^n\text{-}\mathrm{Isoc}_{K_n'}(\widetilde{U}_n)$ generated by an F^n-isocrystal $\widetilde{\mathcal{F}}$ on $\widetilde{U}_n$ which is a sum of one-dimensional F^n-isocrystals. For each one-dimensional F^n-isocrystal the slope filtration is constant, so $\widetilde{\mathcal{F}}$ is a direct sum of isoclinic convergent F^n-isocrystals. By Propositions 10.10 and 10.11 the F^n-isocrystals $\widetilde{\mathcal{F}}$ and $g^*\mathcal{F}^{(n)} \otimes_{K_n} K_n'$ are conservative with respect to the open sub-curve $pr_{\widetilde{U}}\colon \widetilde{V} \hookrightarrow \widetilde{U}$. By (10.2) this implies that the closed immersion $\mathrm{Gr}(\tilde{pr}_{\widetilde{U}}^* g^*\mathcal{F}/\widetilde{V},\tilde{u}) \hookrightarrow \mathrm{Gr}(g^*\mathcal{F}/\widetilde{U},\tilde{u})$ is an isomorphism after the faithfully flat base-change from K_e to K_n'. Thus it is an isomorphism already over K_e. Since $g \circ pr_{\widetilde{U}} = f \circ pr_V\colon \widetilde{V} \to U$, Lemmas 6.2 and 10.2 provide closed immersions of group schemes $\mathrm{Gr}(\tilde{pr}_{\widetilde{U}}^* g^*\mathcal{F}/\widetilde{V},\tilde{u}) \hookrightarrow \mathrm{Gr}(f^*\mathcal{F}/V,u) \hookrightarrow \mathrm{Gr}(\mathcal{F}/U,u)$ whose composition is an isomorphism onto the identity component by the above. Now Proposition 10.5(c) implies that $\mathrm{Gr}(f^*\mathcal{F}/V,u) \hookrightarrow \mathrm{Gr}(\mathcal{F}/U,u)$ is an isomorphism and $\mathcal{F}$ is conservative. $\square$

Next we prove Proposition 10.11, which by Proposition A.14(a) follows immediately from the following two lemmas:

Lemma 10.13. *The full sub-category $\mathcal{S}(U)$ of direct sums of isoclinic convergent F-isocrystals is a full Tannakian sub-category.*

Proof. Clearly $\mathcal{S}(U)$ is closed under direct sums. Since the tensor product and duals of isoclinic F-isocrystals are isoclinic, the category $\mathcal{S}(U)$ is closed under tensor products and taking duals, too. Therefore, it will be sufficient to show that if $\mathcal{F}$ is a direct sum of isoclinic F-isocrystals and $\mathcal{G} \subset \mathcal{F}$ is a sub F-isocrystal, then $\mathcal{G}$ and $\mathcal{F}/\mathcal{G}$ are also direct sums of isoclinic F-isocrystals with the same slopes than $\mathcal{F}$. By looking at the dual $(\mathcal{F}/\mathcal{G})^\vee \subset \mathcal{F}^\vee$ and observing that the dual of an isoclinic F-isocrystal is again isoclinic with the negative slope, it suffices to prove the statement for $\mathcal{G}$. Write $\mathcal{F}$ as

$$\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_n,$$

where each $\mathcal{F}_i$ is an isoclinic F-isocrystal of slope λ_i and the slopes λ_i are pair-wise different. We are going to show the claim by induction on n. The case $n = 1$ is trivial.

Now assume that $n \geq 2$ and the claim is true for $n-1$. Let $\pi_i : \mathcal{F} \to \mathcal{F}_i$ be the projection onto the i-th factor for each $i = 1, 2, \ldots, n$. Set $\mathcal{G}_1 = \ker(\pi_1) \cap \mathcal{G}$; it is a sub F-isocrystal of $\mathcal{G}$. It is also isomorphic to a sub F-isocrystal of $\ker(\pi_1) = \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_n$, so it is a direct sum of isoclinic F-isocrystals by the induction hypothesis. Set $\mathcal{G}_2 = \ker(\pi_2 \oplus \cdots \oplus \pi_n) \cap \mathcal{G}$; it is a sub F-isocrystal of $\mathcal{G}$. It is also isomorphic to a sub F-isocrystal of $\mathcal{F}_1$, so it is isoclinic. Since $\mathcal{G}_1 \cap \mathcal{G}_2 \subset \ker(\pi_1) \cap \ker(\pi_2)$, the intersection $\mathcal{G}_1 \cap \mathcal{G}_2$ is the trivial crystal, and therefore we have an injection

$$\mathcal{G}_1 \oplus \mathcal{G}_2 \hookrightarrow \mathcal{F}$$

which is an isomorphism onto $\mathcal{G}_1 + \mathcal{G}_2$, the sub-isocrystal generated by $\mathcal{G}_1$ and $\mathcal{G}_2$. Since $\mathcal{G}$ contains both $\mathcal{G}_1$ and $\mathcal{G}_2$, it contains $\mathcal{G}_1 + \mathcal{G}_2$, too. The quotient $\mathcal{G}/(\mathcal{G}_1 + \mathcal{G}_2)$ is isomorphic both to a subquotient of $\mathcal{F}_1$ and $\mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_n$. By our induction hypothesis, every subquotient of $\mathcal{F}_1$ is isoclinic with slope λ_1 and every subquotient of $\mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_n$ is a direct sum of isoclinics with slopes in $\{\lambda_2, \ldots, \lambda_n\}$. Since λ_1 does not lie in $\{\lambda_2, \ldots, \lambda_n\}$, this can only be the case if the quotient $\mathcal{G}/(\mathcal{G}_1 + \mathcal{G}_2)$ is trivial, and hence $\mathcal{G} \cong \mathcal{G}_1 \oplus \mathcal{G}_2$. The claim follows. □

Lemma 10.14. *Let $f: V \hookrightarrow U$ be the inclusion of a non-empty open sub-curve and let $\mathcal{F}$ be an object of $\mathcal{S}(U)$. Then the pull-back functor $\mathcal{G} \mapsto f^*\mathcal{G}$ from $\langle\langle \mathcal{F} \rangle\rangle$ to $\langle\langle f^*\mathcal{F} \rangle\rangle$ is a tensor equivalence of Tannakian categories.*

Proof. The pull-back functor is obviously a tensor functor. It is fully faithful by Kedlaya's full faithfulness theorem [Ked04, Theorem 1.1]. So it remains to show the following: if $\mathcal{F}$ is a direct sum of isoclinic F-isocrystals on U and $\mathcal{G} \subset f^*\mathcal{F}$ is a sub F-isocrystal (over V), then $\mathcal{G}$ is of the form $f^*\mathcal{H}$ for some sub F-isocrystal $\mathcal{H} \subset \mathcal{F}$. By the proof of the lemma above $\mathcal{G}$ is the direct sum of sub F-isocrystals of pull-backs of the isoclinic direct summands of $\mathcal{F}$ via f. So we may assume without loss of generality that $\mathcal{F}$ is isoclinic of slope λ. Then there are two constant F-isocrystals $\mathcal{D}_1, \mathcal{D}_2$ of slopes $\lambda_1 = -\lambda$ and $\lambda_2 = \lambda$, respectively, such that $\mathcal{F} \otimes \mathcal{D}_1$ is unit-root and $\mathcal{D}_1 \otimes \mathcal{D}_2$ is trivial of rank n^2 for some positive integer n. Then $\mathcal{G} \otimes f^*\mathcal{D}_1 \subset f^*(\mathcal{F} \otimes \mathcal{D}_1)$ is of the form $f^*\mathcal{H}_1$ for a unique F-isocrystal $\mathcal{H}_1 \subset \mathcal{F} \otimes \mathcal{D}_1$ on U by Corollary 5.11. Therefore

$$\mathcal{G}^{\oplus n^2} \cong \mathcal{G} \otimes f^*\mathcal{D}_1 \otimes f^*\mathcal{D}_2 \subset f^*(\mathcal{F}) \otimes f^*(\mathcal{D}_1 \otimes \mathcal{D}_2) \cong f^*(\mathcal{F}^{\oplus n^2})$$

is of the form $f^*(\mathcal{H}_1 \otimes \mathcal{D}_2)$ for the F-isocrystal $\mathcal{H}_1 \otimes \mathcal{D}_2 \subset \mathcal{F}^{\oplus n^2}$ on U. By projecting onto a direct summand of $\mathcal{F}^{\oplus n^2}$ we get the claim. □

After these general results we now turn towards the proof of Theorem 10.4. We make use of the following

Definition 10.15. Let $\mathcal{F}$ be a convergent F-isocrystal and fix a maximal quasi-torus $T \subset \mathrm{Gr}(\mathcal{F}, u)$ with connected component T° and an element $t \in T(\overline{K})$. For every closed point x of U let

$$\mathrm{Frob}_x^{ss}(\mathcal{F}, tT^\circ) := \overline{\mathrm{Frob}_x(\mathcal{F})} \cap tT^\circ.$$

We will frequently use the following useful fact: let $\mathcal{G}$ be an object of $\langle\langle\mathcal{F}\rangle\rangle$ and let $h\colon \mathrm{Gr}(\mathcal{F},u) \twoheadrightarrow \mathrm{Gr}(\mathcal{G},u)$ be the corresponding surjective homomorphism. Choose two maximal quasi-tori $T_1 \subset \mathrm{Gr}(\mathcal{F},u)$ and $T_2 \subset \mathrm{Gr}(\mathcal{G},u)$ such that h maps T_1 into T_2. Let $t_1 \in T_1(\overline{K})$ and $t_2 := h(t_1)$. Then for every closed point x of U the morphism h maps $\mathrm{Frob}_x^{ss}(\mathcal{F}, t_1T_1^\circ)$ into $\mathrm{Frob}_x^{ss}(\mathcal{G}, t_2T_2^\circ)$. (This is clear since h maps $\mathrm{Frob}_x(\mathcal{F})$ into $\mathrm{Frob}_x(\mathcal{G})$ by Lemma 3.3.)

Proof of Theorem 10.4. Let $\mathcal{F}$ be a convergent semi-simple F-isocrystal on U and let $S \subset |U|$ be a set of positive upper Dirichlet density. By assumption there is an open sub-curve $f\colon V \hookrightarrow U$ which respect to which $\mathcal{F}$ is weakly pink, such that $f^*\mathcal{F}$ has a filtration

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m$$

with isoclinic factors $\mathcal{F}_i/\mathcal{F}_{i-1}$. We replace S by $S \cap |V|$ which has the same Dirichlet density as S because $|U| \smallsetminus |V|$ is finite. Moreover, set

$$\mathcal{G} = \bigoplus_{i=1}^{m} (\mathcal{F}_i/\mathcal{F}_{i-1})^{ss}.$$

Then $\alpha\colon G := \mathrm{Gr}(f^*\mathcal{F}/V,u) \twoheadrightarrow \widetilde{G} := \mathrm{Gr}(\mathcal{G}/V,u)$ is the maximal reductive quotient and α induces an isomorphism on the groups of connected components $G/G^\circ \xrightarrow{\sim} \widetilde{G}/\widetilde{G}^\circ$ by Lemma 3.8. By Theorem 8.10 there exists a maximal quasi-torus $T \subset G$. Then T° is a maximal torus in G by Lemma 8.8 and $\widetilde{T} := \alpha(T)$ is a maximal quasi-torus in $\widetilde{G}$ with identity component $\widetilde{T}^\circ = \alpha(T^\circ)$ and $\alpha|_T\colon T \xrightarrow{\sim} \widetilde{T}$ is an isomorphism.

By Theorem 1.8, Conjecture 1.4 holds for $\mathcal{G}/V$. Let $\tilde{t} \in \widetilde{T}(\overline{K})$ be an element such that the connected component $\tilde{t}\widetilde{G}^\circ$ of $\mathrm{Gr}(\mathcal{G}/V,u)$ is contained in the Zariski-closure of the set $\bigcup_{x\in S} \mathrm{Frob}_x(\mathcal{G})$. We claim that this is not changed if we remove from S all points x for which $\mathrm{Frob}_x(\mathcal{G})$ does not consist of semi-simple elements or does not meet $\tilde{t}\widetilde{G}^\circ$. Namely, by Theorem 8.9(c) there is an open set O in $\tilde{t}\widetilde{G}^\circ$ consisting of semi-simple elements. Since the Zariski-closure $\overline{X} = \tilde{t}\widetilde{G}^\circ(\overline{K})$ of $X := \tilde{t}\widetilde{G}^\circ(\overline{K}) \cap \bigcup_{x\in S} \mathrm{Frob}_x(\mathcal{G})$ is irreducible and contained in the union of $\overline{X \cap O(\overline{K})}$ and $\overline{X \smallsetminus O(\overline{K})} \subset (\tilde{t}\widetilde{G}^\circ \smallsetminus O)(\overline{K})$, we conclude that $\tilde{t}\widetilde{G}^\circ$ equals the Zariski-closure of $X \cap O(\overline{K})$ which consists of semi-simple elements only.

Since $\widetilde{G}$ is reductive, the semi-simple conjugacy class $\mathrm{Frob}_x(\mathcal{G})$ is Zariski-closed in $\widetilde{G}$ for every $x \in S$ by Theorem 8.9(b). Therefore, $\mathrm{Frob}_x^{ss}(\mathcal{G}, \tilde{t}\widetilde{T}^\circ) = \tilde{t}\widetilde{T}^\circ \cap \mathrm{Frob}_x(\mathcal{G})$ and Lemma 9.9(b) implies that $\tilde{t}\widetilde{T}^\circ$ is the Zariski-closure of $\widetilde{C} := \tilde{t}\widetilde{T}^\circ \cap \bigcup_{x\in S} \mathrm{Frob}_x(\mathcal{G}) = \bigcup_{x\in S} \mathrm{Frob}_x^{ss}(\mathcal{G}, \tilde{t}\widetilde{T}^\circ)$. We now lift the situation to $G = \mathrm{Gr}(f^*\mathcal{F}/V,u)$. Let $t := (\alpha|_T)^{-1}(\tilde{t}) \in T(\overline{K})$ and view it as an element of $\mathrm{Gr}(\mathcal{F}/U,u)$ by Lemma 10.2. Since $\mathcal{F}$ is weakly pink with respect to f, T is also a maximal quasi-torus in $\mathrm{Gr}(\mathcal{F}/U,u)$. By Lemma 9.9(b) and Proposition 6.7 the theorem is now a consequence of the following

Claim. The set $C := \bigcup_{x\in S} \mathrm{Frob}_x^{ss}(\mathcal{F}, tT^\circ)$ is Zariski-dense in tT°.

To prove the claim, let $x \in S$ be arbitrary and pick a $g \in \mathrm{Frob}_x(f^*\mathcal{F}) \cap tG^\circ(\overline{K})$. Write $g = g_s \cdot g_u$ for the multiplicative Jordan decomposition of g, where $g_s, g_u \in G$ are the semi-simple and unipotent parts of g, respectively. Since g_s is semi-simple, Theorem 8.10 shows that g_s lies in a maximal quasi-torus of G and can be conjugate by an element $h \in G^\circ(\overline{K})$ such that $h^{-1}g_sh$ lies in $T \cap tG^\circ = tT^\circ$. Then $h^{-1}gh$ is also an element of $\mathrm{Frob}_x(f^*\mathcal{F})(\overline{K})$, and its multiplicative Jordan decomposition is $h^{-1}gh = (h^{-1}g_sh)\cdot(h^{-1}g_uh)$, where $h^{-1}g_sh, h^{-1}g_uh \in \mathrm{Gr}(f^*\mathcal{F}/V,u)$ are the semi-simple and unipotent parts of $h^{-1}gh$, respectively. So we may assume without loss of generality that $g_s \in tT^\circ(\overline{K})$. Since g_s is also the semi-simple part of g in the larger group $G_1 := \mathrm{Gr}(\mathcal{F}/U,u)$ (see [Bor91, I.4.4 Theorem]) which is reductive, and $\mathrm{Frob}_x(f^*\mathcal{F}) \subset \mathrm{Frob}_x(\mathcal{F})$ by definition, we get that $g_s \in \mathrm{Frob}_x^{ss}(\mathcal{F}, tT^\circ)$ using Lemma 9.9(a) and that $\mathrm{Frob}_x^{ss}(\mathcal{F}, tT^\circ) = \overline{\mathrm{Frob}_x(\mathcal{F})} \cap tT^\circ = {}^{G_1}\{g_s\} \cap tT^\circ$. Therefore, also ${}^{G_1}\mathrm{Frob}_x^{ss}(\mathcal{F}, tT^\circ) \cap tT^\circ = \mathrm{Frob}_x^{ss}(\mathcal{F}, tT^\circ)$.

Since by [Bor91, I.4.4 Theorem] the homomorphism $\alpha\colon \mathrm{Gr}(f^*\mathcal{F}/V,u) \to \mathrm{Gr}(\mathcal{G}/V,u)$ preserves Jordan decompositions we get that $\alpha(g) = \alpha(g_s)\cdot\alpha(g_u)$ is the multiplicative Jordan decomposition of $\alpha(g)$, where $\alpha(g_s), \alpha(g_u) \in \mathrm{Gr}(\mathcal{G}/V,u)$ are the semi-simple and unipotent parts of $\alpha(g)$, respectively. Moreover, as $\alpha(g) \in \mathrm{Frob}_x(\mathcal{G})$ by Lemma 3.3 and $\mathrm{Frob}_x(\mathcal{G})$ is semi-simple by our assumption on S we get $\alpha(g_s) = \alpha(g) \in \mathrm{Frob}_x(\mathcal{G})$. Therefore, $\mathrm{Frob}_x(\mathcal{G}) = {}^{\widetilde{G}}\{\alpha(g_s)\}$ in terms of Definition 9.6, and this means that ${}^{\widetilde{G}}\alpha(\mathrm{Frob}_x^{ss}(\mathcal{F}, tT^\circ))$ contains $\mathrm{Frob}_x(\mathcal{G})$. Therefore, also ${}^{\widetilde{G}}\alpha(\mathrm{Frob}_x^{ss}(\mathcal{F}, tT^\circ)) \cap \tilde{t}\widetilde{T}^\circ$ contains $\mathrm{Frob}_x(\mathcal{G}) \cap \tilde{t}\widetilde{T}^\circ = \mathrm{Frob}_x^{ss}(\mathcal{G}, \tilde{t}\widetilde{T}^\circ)$, and hence ${}^{\widetilde{G}}\alpha(C) \cap \tilde{t}\widetilde{T}^\circ = \bigcup_{x\in S} {}^{\widetilde{G}}\alpha(\mathrm{Frob}_x^{ss}(\mathcal{F}, tT^\circ)) \cap \tilde{t}\widetilde{T}^\circ$ contains $\bigcup_{x\in S} \mathrm{Frob}_x^{ss}(\mathcal{G}, \tilde{t}\widetilde{T}^\circ) = \widetilde{C}$. Since $\widetilde{C}$ is Zariski-dense in $\tilde{t}\widetilde{T}^\circ$, also ${}^{\widetilde{G}}\alpha(C) \cap \tilde{t}\widetilde{T}^\circ$ is Zariski-dense in $\tilde{t}\widetilde{T}^\circ$, and by Proposition 9.8 we conclude that ${}^G C \cap tT^\circ$ is Zariski-dense in tT°. Finally ${}^G C \cap tT^\circ \subset {}^{G_1} C \cap tT^\circ = \bigcup_{x\in S} {}^{G_1}\mathrm{Frob}_x^{ss}(\mathcal{F}, tT^\circ) \cap tT^\circ = C$. So $C = {}^G C \cap tT^\circ$ is Zariski-dense in tT° as claimed. ☐

11. The Weakly Pink Conjecture for Weakly Firm F-Isocrystals

In this section we will show that Chebotarëv density is equivalent to the weakly pink property for the locally weakly firm F-isocrystals from Definition 1.9. The following claim provides examples of locally firm and locally weakly firm F-isocrystals.

Proposition 11.1. *Let $\mathcal{F}$ be a convergent F-isocrystal on U. Consider the following list of properties:*

- *(a) the rank of $\mathcal{F}$ is two and $\mathcal{F}$ is not isoclinic.*
- *(b) each slope of the generic Newton polygon of $\mathcal{F}$ has multiplicity one,*
- *(c) each isoclinic component of the generic slope filtration of $\mathcal{F}$ has an abelian monodromy group,*
- *(d) the convergent F-isocrystal $\mathcal{F}$ is locally firm.*
- *(e) the convergent F-isocrystal $\mathcal{F}$ is locally weakly firm.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e).

Proof. If $\mathcal{F}$ is not isoclinic, then every isoclinic component of the generic slope filtration of $\mathcal{F}$ has rank strictly less than the rank of $\mathcal{F}$. Therefore, if the rank of $\mathcal{F}$ is two then the former must have rank one. So (a) implies (b). If (b) holds let $V \subset U$ be a dense open subset over which the Newton polygon of $\mathcal{F}$ is constant. By [Kat79, Corollary 2.6.2] the restriction of $\mathcal{F}$ to V has a slope filtration whose factors are isoclinic of rank one. Thus the monodromy group of each factor is a closed subgroup of the multiplicative group $\mathbb{G}_{m,K_e}$. In particular it is abelian, so (c) holds. Now assume that (c) is true for $\mathcal{F}$. Let $V \subset U$ be the dense open subset over which the Newton polygon of $\mathcal{F}$ is constant and let $\mathcal{F}_i$ be the isoclinic components of the slope filtration of $\mathcal{F}$ on V. Then $\mathcal{F}|_V^{ss}$ is an object of $\langle\langle \mathcal{F}_1, \mathcal{F}_2, \dots \rangle\rangle$, so $\mathrm{Gr}(\mathcal{F}|_V^{ss}/V, u)$ is the quotient of a subgroup of the product of the $\mathrm{Gr}(\mathcal{F}_i/V, u)$. The latter are abelian, so the same holds for their product, and hence for their subgroups, the quotients of the latter, and so for $\mathrm{Gr}(\mathcal{F}|_V^{ss}/V, u)$, too. So (d) holds. We already explained in Definition 1.9 why (d) implies (e). ☐

Proposition 11.2. *The sub-categories consisting of convergent F-isocrystals which are successive extensions of isoclinic convergent F-isocrystals on U is a full Tannakian sub-categories of F-$\mathrm{Isoc}_K(U)$.*

Proof. We will call an F-isocrystal which is a successive extension of isoclinic convergent F-isocrystals on U *isofiltered*. Obviously the trivial F-isocrystal is isofiltered, so we need to check that the category of isofiltered convergent F-isocrystals on U is closed under taking directs sums, tensor products, duals, and subobjects. Let $\mathcal{F}$ and $\mathcal{G}$ be two isofiltered F-isocrystals on U, and let

$$\mathcal{F}_{-n} \subset \mathcal{F}_{-n+1} \subset \cdots \subset \mathcal{F}_n \qquad \text{and} \qquad \mathcal{G}_{-n} \subset \mathcal{G}_{-n+1} \subset \cdots \subset \mathcal{G}_n$$

be filtrations on $\mathcal{F}$ and $\mathcal{G}$, respectively, such that the subquotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ and $\mathcal{G}_i/\mathcal{G}_{i-1}$ are isoclinic of slopes of valuation $-i$ (assuming that the valuation is suitably normalized to take only integer values). Note that the definitions

$$(\mathcal{F} \oplus \mathcal{G})_i = \mathcal{F}_i \oplus \mathcal{G}_i \quad \text{and} \quad (\mathcal{F} \otimes \mathcal{G})_i = \sum_{j+k \le i} \mathcal{F}_j \otimes \mathcal{G}_k$$

furnish filtrations

$$(\mathcal{F} \oplus \mathcal{G})_{-n} \subset (\mathcal{F} \oplus \mathcal{G})_{-n+1} \subset \cdots \subset (\mathcal{F} \oplus \mathcal{G})_n \text{ and } (\mathcal{F} \otimes \mathcal{G})_{-2n} \subset (\mathcal{F} \otimes \mathcal{G})_{-2n+1} \subset \cdots \subset (\mathcal{F} \otimes \mathcal{G})_{2n}$$

on $\mathcal{F} \oplus \mathcal{G}$ and $\mathcal{F} \otimes \mathcal{G}$, respectively, such that the subquotients $(\mathcal{F} \oplus \mathcal{G})_i/(\mathcal{F} \oplus \mathcal{G})_{i-1}$ and $(\mathcal{F} \otimes \mathcal{G})_i/(\mathcal{F} \otimes \mathcal{G})_{i-1}$ are quotients of

$$\mathcal{F}_i/\mathcal{F}_{i-1} \oplus \mathcal{G}_i/\mathcal{G}_{i-1} \quad \text{and} \quad \bigoplus_{j+k=i} \mathcal{F}_j/\mathcal{F}_{j-1} \otimes \mathcal{G}_k/\mathcal{G}_{k-1},$$

respectively. Any quotient of these is isoclinic. We get that $\mathcal{F} \oplus \mathcal{G}$ and $\mathcal{F} \otimes \mathcal{G}$ are isofiltered. The dual $\mathcal{F}^\vee$ has a filtration:

$$\mathcal{F}_n^\vee \subset \mathcal{F}_{n-1}^\vee \subset \cdots \subset \mathcal{F}_{-n}^\vee$$

such that $\mathcal{F}_i^\vee/\mathcal{F}_{i+1}^\vee$ is isomorphic to the dual of $\mathcal{F}_{i+1}/\mathcal{F}_i$, and in particular is isoclinic, too. Thus $\mathcal{F}^\vee$ is isofiltered, too. Finally let $\mathcal{H} \subset \mathcal{F}$ be a sub F-isocrystal. Then

$$\mathcal{F}_{-n} \cap \mathcal{H} \subset \mathcal{F}_{-n+1} \cap \mathcal{H} \subset \cdots \subset \mathcal{F}_n \cap \mathcal{H}$$

is a filtration on $\mathcal{H}$ such that $\mathcal{F}_i \cap \mathcal{H}/\mathcal{F}_{i-1} \cap \mathcal{H}$ is a subquotient of $\mathcal{F}_i/\mathcal{F}_{i-1}$. In particular, it is likewise isoclinic. Therefore $\mathcal{H}$ is isofiltered, too. □

Proposition 11.3. *The sub-categories consisting of firm, locally firm, weakly firm, and locally weakly firm convergent F-isocrystals on U are full Tannakian sub-categories of F-$\mathrm{Isoc}_K(U)$.*

Proof. Obviously the trivial F-isocrystal is firm, so we need to check that the categories of (locally) firm convergent F-isocrystals on U are closed under taking direct sums, tensor products, duals, and subobjects. Since these operations commute with shrinking U, we only need to treat the firm case. Let $\mathcal{F}$ and $\mathcal{G}$ be two firm convergent F-isocrystals on U, and let $\mathcal{H} \in \langle\!\langle \mathcal{F}, \mathcal{G} \rangle\!\rangle$. By Proposition 11.2 above $\mathcal{H}$ is isofiltered. Moreover $\mathcal{H}^{ss} \in \langle\!\langle \mathcal{F}^{ss}, \mathcal{G}^{ss} \rangle\!\rangle$, so $\mathrm{Gr}(\mathcal{H}^{ss}, u)$ is a quotient of a fiber product of $\mathrm{Gr}(\mathcal{F}^{ss}, u)$ and $\mathrm{Gr}(\mathcal{G}^{ss}, u)$ by Proposition 3.6(c). Since the fiber products and quotients of abelian groups are abelian, we get that $\mathrm{Gr}(\mathcal{H}^{ss}, u)$ is also abelian, and hence $\mathcal{H}$ is firm, too.

The proof for weakly firm and locally weakly firm F-isocrystals is similar. By shrinking U, we only need to treat the weakly firm case. Let $\mathcal{F}$ and $\mathcal{G}$ be two weakly firm convergent F-isocrystals on U, and let $\mathcal{H} \in \langle\!\langle \mathcal{F}, \mathcal{G} \rangle\!\rangle$. By Proposition 11.2 above $\mathcal{H}$ is isofiltered. Moreover $\mathrm{Gr}(\mathcal{H}, u)$ is a quotient of a fiber product of $\mathrm{Gr}(\mathcal{F}, u)$ and $\mathrm{Gr}(\mathcal{G}, u)$ by Proposition 3.6(c), so we only need to show that the category of linear algebraic groups whose maximal quasi-torus is abelian is closed under quotients and fiber products.

The latter can be proven as follows. If F is a linear algebraic group and G is a quotient of F by a closed normal subgroup, and $T \subset F$ is a maximal quasi-torus then its image T' in G is a maximal quasi-torus in G by Corollary 8.11. If T is abelian, then so its quotient T', and hence the maximal quasi-torus of G is abelian. Now the maximal quasi-torus of a fiber product H of two linear algebraic groups F and G is isomorphic to a fiber product of the maximal quasi-tori of F and G by Remark 8.12. If the latter are abelian, so is this fiber product, so in other words the maximal quasi-torus of the fiber product H is abelian, too. □

Definition 11.4. Let $\mathcal{F}$ be a convergent F-isocrystal on U and let $f\colon V \hookrightarrow U$ be a non-empty open sub-curve. We say that $\mathcal{F}$ is *almost weakly pink* (with respect to the inclusion $f\colon V \hookrightarrow U$) if every maximal torus of the monodromy group $\mathrm{Gr}(f^*\mathcal{F}/V, u)$ of the crystal on the shrunken curve is a maximal torus in the monodromy group $\mathrm{Gr}(\mathcal{F}/U, u)$ of the crystal, too.

Theorem 11.5. *Let $\mathcal{F}$ be a semi-simple convergent F-isocrystal on U and let $f\colon V \hookrightarrow U$ be a non-empty open sub-curve. Assume that $\bigcup_{x \in |V|} \mathrm{Frob}_x(\mathcal{F}) \subset \mathrm{Gr}(\mathcal{F}/U, u)$ is Zariski dense. (This is a weak form of the Chebotarëv density Conjecture 1.2.) Then $\mathcal{F}$ is almost weakly pink with respect to $f\colon V \hookrightarrow U$.*

Remark. We believe that the conclusion of the theorem does not imply that $\mathcal{F}$ is weakly pink; compare Warning 10.6, but see Theorem 11.6 below.

Proof. Let $G = \mathrm{Gr}(\mathcal{F}/U, u)(\overline{K})$ be the $\overline{K}$-valued points of the monodromy group of $\mathcal{F}$ and let $H := \mathrm{Gr}(f^*\mathcal{F}/V, u)(\overline{K}) \subset G$ be the $\overline{K}$-valued points of the monodromy group of $f^*\mathcal{F}$ on the shrunken curve. We view both groups as linear algebraic groups over $\overline{K}$. Let $F = \bigcup_{x \in |V|} \mathrm{Frob}_x(f^*\mathcal{F}) \subset H$ be the union of the Frobenius conjugacy classes (conjugacy under H), and let $F^{ss} = \{g_s \colon g \in F\} \subset H$ be the set of the semi-simple parts g_s of the elements g of F. For a subset X of G let ${}^G X$ be the union of the conjugacy classes under G of the elements of X. Then ${}^G F = \bigcup_{x \in |V|} \mathrm{Frob}_x(\mathcal{F}) \subset G$ is the union of the Frobenius conjugacy classes (conjugacy under G), and ${}^G(F^{ss}) = ({}^G F)^{ss} := \{g_s \colon g \in {}^G F\} \subset G$. By our assumption ${}^G F$ is dense in G. By Corollary 9.11 we get that $({}^G F)^{ss}$ is Zariski-dense in G, too. Let $T \subset H$ be a maximal quasi-torus. Since every element of F^{ss} is conjugate to an element of T by Theorem 8.9(a),(b), we get that ${}^G T$ is also Zariski-dense in G, and hence that ${}^G T \cap G^\circ$ is Zariski-dense in G°.

At this point the proof is now purely group-theoretical: We consider conjugation under G° and we write ${}^{G^\circ} T$, respectively ${}^{G^\circ} T^\circ$ for the union of the conjugacy classes under G° of the elements of T, respectively of T°. Then $T/T^\circ = H/H^\circ$ surjects onto G/G° by Theorem 8.9(a) and Proposition 10.5(c). Thus ${}^G T = {}^{G^\circ} T$, because every element $g \in G$ can be written as $g = g_0 h$ with $g_0 \in G^\circ$ and $h \in T$, and so $gTg^{-1} = g_0 T g_0^{-1}$. Therefore, ${}^{G^\circ} T \cap G^\circ$ is Zariski-dense in G°. We observe that ${}^{G^\circ} T \cap G^\circ = {}^{G^\circ}(T \cap G^\circ)$. Now let $A \subset G^\circ$ be a maximal torus in G° containing T°. Recall that there is a finite group Γ acting on A such that any two elements of A in the same conjugacy class under the action of G° actually lie in the same orbit under the action of Γ by Steinberg's result [Ste74, § III.3.4, Corollary 2], see also Remark 9.4(c). If T° were a proper closed subscheme of A then the same would hold for the union of its images under the action of Γ, because the latter is finite and A is irreducible. Therefore, ${}^{G^\circ}(T\cap G^\circ)\cap A$ is a proper closed subscheme of A. By Lemma 9.9(b) this contradicts that ${}^{G^\circ}(T\cap G^\circ)$ is Zariski-dense in G°. We conclude that $T^\circ = A$ is a maximal torus in G°. $\square$

Our main result of this section is the following equivalence:

Theorem 11.6. *Let $\mathcal{F}$ be a semi-simple convergent F-isocrystal on U and let $f\colon V \hookrightarrow U$ be an open sub-curve for which $f^*\mathcal{F}$ is locally weakly firm. Then the following properties are equivalent:*

(a) $\mathcal{F}$ is weakly pink with respect to f,
(b) $\mathcal{F}$ is almost weakly pink with respect to f,
(c) $\mathcal{F}$ satisfies Conjecture 1.4,
(d) $\mathcal{F}$ satisfies Conjecture 1.2.

Proof. The implications (a)⇒(c), (c)⇒(d), and (d)⇒(b) were already proven in Theorem 10.4, Proposition 6.7, and Theorem 11.5, respectively. Now we only need to concern ourselves with (b)⇒(a). By assumption $\mathcal{G} = f^*\mathcal{F}$ is the successive extension of isoclinic convergent F-isocrystals $\mathcal{G}_i$ on V and the maximal quasi-torus of the monodromy group $\mathrm{Gr}(\mathcal{G}^{ss}/V,u)$ is abelian, that is the direct product of a torus with a finite abelian group H; use [Bor91, III.10.6 Theorem (4)]. Let us fix the following notation. We write

$$\begin{aligned} G &:= \mathrm{Gr}(\mathcal{F}/U,u)\times_{K_e}\overline{K} \qquad \text{and}\\ \widetilde{G} &:= \mathrm{Gr}(\mathcal{G}/V,u)\times_{K_e}\overline{K} \qquad \text{and}\\ \widetilde{G}^{red} &:= \mathrm{Gr}(\mathcal{G}^{ss}/V,u)\times_{K_e}\overline{K} \end{aligned}$$

for the linear algebraic groups over $\overline{K}$ obtained by base-change. Let $T\subset\widetilde{G}$ be a maximal quasi-torus, and consider T as a closed subgroup of G via the embedding $\widetilde{G}\hookrightarrow G$ furnished by $f\colon V\hookrightarrow U$. By Theorem 8.13 we only have to show the following:

(i) the connected component T° of T is a maximal torus in G°,
(ii) the natural map $T/T^\circ \to G/G^\circ$ is surjective,
(iii) the group T is commutative.

First we establish (ii). By Theorem 8.10(d) the map $T/T^\circ \xrightarrow{\sim} \widetilde{G}/\widetilde{G}^\circ$ is an isomorphism, since T is a maximal quasi-torus in $\widetilde{G}$. By Proposition 10.5(c) the map $\widetilde{G}/\widetilde{G}^\circ \twoheadrightarrow G/G^\circ$ is surjective for every F-isocrystal $\mathcal{F}$ on U, so (ii) holds. Claim (iii) holds, because the maximal quasi-torus in $\widetilde{G}^{red}$ is abelian, since $\mathcal{G}$ is weakly firm, and the maximal quasi-torus T maps isomorphically onto its image under the map $\widetilde{G}\twoheadrightarrow\widetilde{G}^{red}$ which is a maximal quasi-torus in $\widetilde{G}^{red}$ by Corollary 8.11. Finally, by our assumption (b) we have condition (i). This finishes the proof of Theorem 11.6. $\square$

Proposition 11.7. *Let $\mathcal{F}$ be a semi-simple convergent F-isocrystal on U whose monodromy group $G = \mathrm{Gr}(\mathcal{F}/U,u)$ has semi-simple rank at most one. This means that the maximal tori in the semi-simple quotient G/RG of G by the radical RG have dimension at most one. Then $\mathcal{F}$ is weakly pink with respect to any open sub-curve $f\colon V\hookrightarrow U$ as above.*

Remark 11.8. Note that the monodromy group of every rank two semi-simple convergent F-isocrystal has semi-simple rank at most one, so the proposition above applies to this case.

Proof of Proposition 11.7. We will use the set-up and notation in the proof of Theorem 11.6 and write $G := \mathrm{Gr}(\mathcal{F}/U,u)\times_{K_e}\overline{K}$ and $\widetilde{G} := \mathrm{Gr}(f^*\mathcal{F}/V,u)\times_{K_e}\overline{K}$. We use Proposition 3.9 and let $\mathcal{S},\mathcal{T}\in\langle\langle\mathcal{F}\rangle\rangle$ be the convergent F-isocrystals whose monodromy groups are $\mathrm{Gr}(\mathcal{S}/U,u)\times_{K_e}\overline{K} = G/Z$ and $\mathrm{Gr}(\mathcal{T}/U,u)\times_{K_e}\overline{K} = G/[G^\circ,G^\circ]$, where $Z\subset G^\circ$ is the center and $[G^\circ,G^\circ]$ is the derived group of G°, which are both characteristic subgroups defined over K_e. Then $\mathcal{S}$ has semi-simple monodromy group by [Bor91, IV.11.21 Proposition] and the identity component $\mathrm{Gr}(\mathcal{T}/U,u)^\circ$ of the monodromy group of $\mathcal{T}$ is a torus, use [Bor91, IV.14.11 Corollary and III.10.6 Theorem]. We consider the diagram

$$\begin{tikzcd} G = \mathrm{Gr}(\mathcal{F}/U,u)\times_{K_e}\overline{K} \arrow[r, two heads, "\pi"] & \mathrm{Gr}(\mathcal{S}\oplus\mathcal{T}/U,u)\times_{K_e}\overline{K} =: H \\ \widetilde{G} = \mathrm{Gr}(f^*\mathcal{F}/V,u)\times_{K_e}\overline{K} \arrow[u, hook] \arrow[r, two heads, "\widetilde{\pi}"] & \mathrm{Gr}(f^*(\mathcal{S}\oplus\mathcal{T})/V,u)\times_{K_e}\overline{K} =: \widetilde{H} \arrow[u, hook] \end{tikzcd}$$

We claim that it suffices to show that $\mathcal{S}$ is weakly pink with respect to f. Indeed, $\mathcal{T}$ is conservative with respect to f by Corollary 10.12 and so $\mathcal{S} \oplus \mathcal{T}$ will be weakly pink with respect to f by Proposition 10.7. Let $\widetilde{T} \subset \widetilde{G}$ be a maximal quasi-torus and let T be the image of $\widetilde{T}$ in G. Then the images $\widetilde{\pi}(\widetilde{T}) \subset \widetilde{H}$ and $\pi(T) \subset H$ of $\widetilde{T}$ are maximal quasi-tori by Corollary 8.11 and by the property that $\mathcal{S} \oplus \mathcal{T}$ is weakly pink. Let $C \subset G$ be the kernel of π, which is a finite group contained in the center of G° by Proposition 3.9. Let $\overline{B} \subset H^\circ$ be a Borel subgroup with $\pi(T^\circ) = \pi(T)^\circ \subset \overline{B}$ which is normalized by the maximal quasi-torus $\pi(T) \subset H$, and let $B \subset G^\circ$ be a Borel subgroup with $\overline{B} = \pi(B)$; use [Bor91, IV.11.14 Proposition]. Since $C \subset B$ by [Bor91, IV.11.11 Corollary] we must have $B = \pi^{-1}(\overline{B})$ and hence $T^\circ \subset B$. We now use Theorem 8.10(e) to show that $T \subset G$ is a maximal quasi-torus. Firstly, every $t \in T$ normalizes B, because $tBt^{-1} \subset \pi^{-1}\big(\pi(tBt^{-1})\big) = \pi^{-1}\big(\pi(t) \cdot \overline{B} \cdot \pi(t^{-1})\big) = \pi^{-1}(\overline{B}) = B$. Secondly, $T \twoheadrightarrow \pi(T) \twoheadrightarrow H/H^\circ \cong G/G^\circ$, because $\pi(T)$ is a maximal quasi-torus in H and the kernel of π is contained in G°. And finally, T° surjects onto the maximal torus $\pi(T)^\circ$ in H°. On the other hand T° is contained in a maximal torus T_1° of G° which also surjects onto $\pi(T)^\circ$. Since the kernel of π is finite, the tori T° and T_1° both have the same dimension as $\pi(T)^\circ$. So they coincide, and it follows from Theorem 8.10(e) that $T \subset G$ is a maximal quasi-torus. Thus $\mathcal{F}$ is weakly pink.

We now prove that $\mathcal{S}$ is weakly pink, and to this end, we replace $\mathcal{F}$ by $\mathcal{S}$. Then $\mathcal{F}$ is still semi-simple and locally firm with respect to f by Lemma 3.8 and Proposition 11.3, and moreover $G := \mathrm{Gr}(\mathcal{F}/U, u) \times_{K_e} \overline{K}$ is semi-simple of rank at most one. If this rank is zero, then $\{1\}$ is the maximal torus in G and G° is a unipotent group by [Bor91, IV.11.5 Corollary]. Being semi-simple, G° is trivial. By Lemma 10.2 and Proposition 10.5(c) the map $\widetilde{G} := \mathrm{Gr}(f^*\mathcal{F}/V, u) \times_{K_e} \overline{K} \hookrightarrow G$ is a closed immersion which is surjective on $\widetilde{G}/\widetilde{G}^\circ \twoheadrightarrow G/G^\circ = G$. This implies that $\widetilde{G} \xrightarrow{\sim} G$ is an isomorphism, and $\mathcal{F}$ is weakly pink (and even conservative).

If G has semi-simple rank 1, then the maximal tori in G are one-dimensional. If $\mathcal{F}$ were not weakly pink, then $\widetilde{T}^\circ = \{1\}$ would be the maximal torus in $\widetilde{G}^\circ$. Then $\widetilde{G}^\circ$ is actually a unipotent group by [Bor91, IV.11.5 Corollary]. Therefore, by [Hum75, 17.5 Theorem] there is a filtration $W_\bullet$ on the K_e-linear space $W := \omega_u(\mathcal{F})$ left invariant by $\widetilde{G}^\circ$ such that $\widetilde{G}^\circ$ acts trivially on the successive quotients. Let $(0) =: \widetilde{\mathcal{H}}_0 \subsetneq \widetilde{\mathcal{H}}_1 \subset f^*\mathcal{F}$ be the convergent F-sub-isocrystal on V corresponding to the fixed vectors of $\widetilde{G}^\circ$ on W by Proposition A.18. Iterating this we obtain a filtration $(0) = \widetilde{\mathcal{H}}_0 \subset \widetilde{\mathcal{H}}_1 \subset \widetilde{\mathcal{H}}_2 \subset \cdots \subset f^*\mathcal{F}$ of $f^*\mathcal{F}$ by convergent F-sub-isocrystals on V corresponding to the filtration $W_\bullet$ above by successively applying Proposition A.18 to the F-isocrystals $f^*\mathcal{F}/\widetilde{\mathcal{H}}_i$ for all i. Having finite monodromy group, all quotients $\widetilde{\mathcal{H}}_{i+1}/\widetilde{\mathcal{H}}_i$ are unit root by Lemma 5.1. Therefore, $f^*\mathcal{F}$ is also unit root. By the Katz-Grothendieck semi-continuity theorem [Kat79, Corollary 2.3.2] we get that $\mathcal{F}$ itself is unit root, and hence conservative by Corollary 10.12. This contradicts that $\mathcal{F}$ is not weakly pink. □

Combining Propositions 10.7, 10.11 and 11.7 we have the following immediate

Corollary 11.9. *Let $\mathcal{F}$ be a semi-simple convergent F-isocrystal on U whose monodromy group $G = \mathrm{Gr}(\mathcal{F}/U, u)$ has semi-simple rank at most one. Let $\mathcal{G}$ be a direct sum of isoclinic convergent F-isocrystals on U. Then $\mathcal{F} \oplus \mathcal{G}$ is weakly pink with respect to f, and hence Conjecture 1.4 holds for it.* □

12. Maximal compact subgroups of complex linear algebraic groups

In the rest of this article we want to look at the *overconvergent* analog of Conjectures 1.2, 1.3, 1.4, and prove Theorem 1.12. To this end we start with some facts about maximal compact subgroups of complex linear algebraic groups in the present section. If G is a linear algebraic group over $\mathbb{C}$, then $G(\mathbb{C})$ has the structure of a complex analytic group such that $G^\circ(\mathbb{C})$ is the connected component of the identity in the usual topological sense. In particular the group of connected components of the Lie group $G(\mathbb{C})$ is finite. Therefore, [Hoc65, Chapter XV, Theorem 3.1] applies to $G(\mathbb{C})$, and provides the following result:

Theorem 12.1. *There is a compact (and hence closed) Lie subgroup $\mathbb{K} \subset G(\mathbb{C})$ and a closed differentiable sub-manifold $\mathbb{E} \subset G^\circ(\mathbb{C})$ diffeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$ and containing the identity such that*

(a) the multiplication map $\mathbb{E} \times \mathbb{K} \to G(\mathbb{C})$ is a diffeomorphism,
(b) we have $x^{-1}\mathbb{E}x = \mathbb{E}$ for every $x \in \mathbb{K}$,
(c) for every compact subgroup $L \subset G(\mathbb{C})$ there is an $e \in \mathbb{E}$ such that $e^{-1}Le \subset \mathbb{K}$. □

Definition 12.2. We will call a subgroup $\mathbb{K} \subset G(\mathbb{C})$ as in Theorem 12.1 a *maximal compact subgroup.* It is easy to derive the following basic properties of these subgroups:

Remark 12.3. Recall that by Cartan's theorem, (that is, by the original $\mathbb{R}$-version of Theorem 7.1 above) every closed subgroup of $G(\mathbb{C})$ is a real Lie group and a real sub-manifold of $G(\mathbb{C})$, see[Ser92, Part II, § V.9, Corollary to Theorem 1 on page 155].

Proposition 12.4. *The following hold:*

(a) *For every maximal compact subgroup $\mathbb{K} \subset G(\mathbb{C})$ the inclusion $\mathbb{K} \subset G(\mathbb{C})$ induces an isomorphism $\mathbb{K}/\mathbb{K}^\circ \cong G(\mathbb{C})/G(\mathbb{C})^\circ \cong G/G^\circ$ of groups of connected components. In particular, $\mathbb{K} \cap G(\mathbb{C})^\circ = \mathbb{K}^\circ$, and this is a maximal compact subgroup of $G(\mathbb{C})^\circ$.*
(b) *Any two maximal compact subgroups in $G(\mathbb{C})$ are conjugate by an element in $G^\circ(\mathbb{C})$.*
(c) *For every automorphism φ of the complex Lie group $G(\mathbb{C})$ and every maximal compact subgroup $\mathbb{K}$ in $G(\mathbb{C})$ the image $\varphi(\mathbb{K})$ is a maximal compact subgroup in $G(\mathbb{C})$.*
(d) *Every compact subgroup $L \subset G(\mathbb{C})$ is contained in a maximal compact subgroup of $G(\mathbb{C})$. In particular, a subgroup $L \subset G(\mathbb{C})$ is compact and maximal with respect to the inclusion of subgroups if and only if it is a maximal compact subgroup in the sense of Definition 12.2.*
(e) *Every element x of a compact subgroup $L \subset G(\mathbb{C})$ is semi-simple.*

Proof. (a) Let G and $\mathbb{K}$ be as in the claim, and let $\mathbb{E}$ be as in Theorem 12.1. To prove the surjectivity of $\mathbb{K} \to G(\mathbb{C})/G(\mathbb{C})^\circ$ let $g \in G(\mathbb{C})$ and write it as $g = ec$ with $e \in \mathbb{E}$ and $c \in \mathbb{K}$. Then $g \in \mathbb{E}c \subset G(\mathbb{C})^\circ c$ as desired. Since both $\mathbb{E}$ and $\mathbb{K}^\circ$ are connected, so is their product, and hence the multiplication map restricted onto $\mathbb{E} \times \mathbb{K}^\circ$ is a diffeomorphism $\mathbb{E} \times \mathbb{K}^\circ \to G(\mathbb{C})^\circ$, and $\mathbb{K} \cap G(\mathbb{C})^\circ = \mathbb{K}^\circ$. Since $\mathbb{K}^\circ$ is a subgroup of $\mathbb{K}$, part (b) of Theorem 12.1 also holds for $\mathbb{K}^\circ$ trivially. Now let $L \subset G(\mathbb{C})^\circ$ be a compact subgroup. Then by Theorem 12.1(c), there is an $e \in \mathbb{E}$ such that $e^{-1}Le \subset \mathbb{K}$. As $\mathbb{E}$ is connected and contains the identity, we have $e \in \mathbb{E} \subset G(\mathbb{C})^\circ$. Since also $L \subset G(\mathbb{C})^\circ$ we get $e^{-1}Le \in \mathbb{K} \cap G(\mathbb{C})^\circ = \mathbb{K}^\circ$. So $\mathbb{K}^\circ$ is a maximal compact subgroup of $G(\mathbb{C})^\circ$.

(b) Let $\mathbb{K}_1, \mathbb{K}_2 \subset G(\mathbb{C})$ be two maximal compact subgroups and view them as real Lie groups by Remark 12.3. Then there is an $e \in \mathbb{E}$ such that $e^{-1}\mathbb{K}_1 e \subset \mathbb{K}_2$ by Theorem 12.1(c), because $\mathbb{K}_1$ is a compact subgroup and $\mathbb{K}_2$ is a maximal compact subgroup. Therefore, $\dim(\mathbb{K}_1) \leq \dim(\mathbb{K}_2)$. By reversing the roles of $\mathbb{K}_1$ and $\mathbb{K}_2$ we get that $\dim(\mathbb{K}_2) \leq \dim(\mathbb{K}_1)$, too, and hence $\dim(e^{-1}\mathbb{K}_1 e) = \dim(\mathbb{K}_1) = \dim(\mathbb{K}_2)$. It follows that the connected components are equal: $e^{-1}\mathbb{K}_1^\circ e = \mathbb{K}_2^\circ$. By claim (a) above we get that

$$e^{-1}\mathbb{K}_1 e/(e^{-1}\mathbb{K}_1 e)^\circ \cong \mathbb{K}_1/\mathbb{K}_1^\circ \cong G(\mathbb{C})/G(\mathbb{C})^\circ \cong \mathbb{K}_2/\mathbb{K}_2^\circ\,.$$

So $e^{-1}\mathbb{K}_1 e$ and $\mathbb{K}_2$ must be equal and (b) is proven.

(c) Let $\mathbb{E}$ be as in Theorem 12.1 above. Then $\varphi(\mathbb{E})$ is diffeomorphic to $\mathbb{E}$, so it is also a closed differentiable sub-manifold in $G(\mathbb{C})$ diffeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$ and containing the identity. We get that the multiplication map $\varphi(\mathbb{E}) \times \varphi(\mathbb{K}) \to \varphi(G(\mathbb{C})) = G(\mathbb{C})$ is a diffeomorphism. If $x \in \varphi(\mathbb{K})$ then $x = \varphi(y)$ for some $y \in \mathbb{K}$, and hence

$$x^{-1}\varphi(\mathbb{E})x \;=\; \varphi(y)^{-1}\varphi(\mathbb{E})\varphi(y) \;=\; \varphi(y^{-1}\mathbb{E}y) \;=\; \varphi(\mathbb{E})\,,$$

so part (b) of Theorem 12.1 holds for $\varphi(\mathbb{K})$ and $\varphi(\mathbb{E})$, too. If $L \subset G(\mathbb{C})$ is any compact subgroup then also $\varphi^{-1}(L) \subset G(\mathbb{C})$ is a compact subgroup. So there is an $e \in \mathbb{E}$ such that $e^{-1}\varphi^{-1}(L)e \subset \mathbb{K}$ by Theorem 12.1(c). Then $\varphi(e) \in \varphi(\mathbb{E})$ satisfies $\varphi(e)^{-1}L\varphi(e) = \varphi(e^{-1}\varphi^{-1}(L)e) \subset \varphi(\mathbb{K})$, so part (c) of Theorem 12.1 holds for $\varphi(\mathbb{K})$. This proves claim (c).

(d) Now let $L \subset G(\mathbb{C})$ be a compact subgroup and let $\mathbb{K}$ be an arbitrary maximal compact subgroup of $G(\mathbb{C})$. By Theorem 12.1(c) there is an $e \in \mathbb{E}$ such that $e^{-1}Le \subset \mathbb{K}$. Then $L \subset e\mathbb{K}e^{-1}$. However, $e\mathbb{K}e^{-1}$ is a maximal compact subgroup of $G(\mathbb{C})$ by claim (c), and so claim (d) follows. To prove the second statement let $\mathbb{K}$ be a maximal compact subgroup in the sense of Definition 12.2 and let $L \subset G(\mathbb{C})$ be a compact subgroup which contains $\mathbb{K}$. Then there exists a maximal compact subgroup $\mathbb{K}'$ containing L, and hence also $\mathbb{K}$. Arguing as in the proof of (b) above shows that $\dim \mathbb{K} = \dim \mathbb{K}'$, and hence $\mathbb{K} = \mathbb{K}'$ and $\mathbb{K} = L$. It follows that $\mathbb{K}$ is maximal under inclusion. Conversely if L is any compact subgroup which is maximal under inclusion there exists a maximal compact subgroup $\mathbb{K}$ in the sense of Definition 12.2 containing L. Since L is maximal under inclusion, it is equal to $\mathbb{K}$.

(e) Let $H \subset G$ be the Zariski closure of the group $x^{\mathbb{Z}}$ generated by x. Then H is commutative and is isomorphic to the product $H_s \times H_u$ of the closed subgroups H_s and H_u consisting of all semi-simple, respectively unipotent elements of H; see [Bor91, I.4.7 Theorem]. The element x lies in $H(\mathbb{C}) \cap L$, which is a compact group, because L is compact and $H \subset G$ is a closed subgroup. The image of x under the projection homomorphism $\pi_u \colon H \twoheadrightarrow H_u$ lies in the compact subgroup $\pi_u(H(\mathbb{C}) \cap L)$. Since H_u is a successive extension of additive groups $\mathbb{G}_{a,\mathbb{C}}$ and $\mathbb{G}_{a,\mathbb{C}}$ contains no compact subgroups other than $\{0\}$, the image $\pi_u(H(\mathbb{C}) \cap L)$ must be $\{0\}$. This shows that $x \in H_s$, and proves the claim. □

Proposition 12.5. *Let G be a (not necessarily connected)* reductive *linear algebraic group over $\mathbb{C}$ and let $\mathbb{K} \subset G(\mathbb{C})$ be a maximal compact subgroup. Then $\mathbb{K}$ is Zariski dense in G. Moreover, G has in a unique way the structure of an algebraic group over $\mathbb{R}$ such that $\mathbb{K} = G(\mathbb{R})$.*

Remark. It is important to note that we need to assume that G is reductive. The claim is not true for $G = \mathbb{G}_a^n$, for example, where the maximal compact subgroup is just the identity.

Proof of Proposition 12.5. By [Ser93, § 5.2 and Théorème 1] the $\mathbb{R}$-linear algebraic envelope $G_{\mathbb{R}} := \mathbb{K}^{\mathbb{R}\text{-alg}}$ of $\mathbb{K}$, see Definition 4.1, is a linear algebraic group satisfying $G_{\mathbb{R}}(\mathbb{R}) = \mathbb{K}$. By [Ser93, Beginning of § 5.3 and Théorème 4 and Remarque] it satisfies $G_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} = G$, and is thus the desired unique real form. If $H \subset G_{\mathbb{R}}$ is the Zariski closure of $\mathbb{K}$ in $G_{\mathbb{R}}$ then the universal property of $G_{\mathbb{R}}$ implies that $H = G_{\mathbb{R}}$, see [Ser93, § 4.3 and Exemple b]. So $\mathbb{K}$ is Zariski-dense in $G_{\mathbb{R}}$ and G. □

Example 12.6. The group $G = \mathbb{G}_{m,\mathbb{C}}^n = \operatorname{Spec} \mathbb{C}[z_\nu^{\pm 1} \colon \nu = 1, \ldots, n]$ is commutative, and hence has a unique maximal compact subgroup $\mathbb{K}$ by Proposition 12.4(a),(b). A real structure on G is given by $G_{\mathbb{R}} := \operatorname{Spec} \mathbb{C}[a_\nu, b_\nu \colon \nu = 1, \ldots, n]/(a_\nu^2 + b_\nu^2 - 1)$ with isomorphism given by $a_\nu = \frac{1}{2}(z_\nu + z_\nu^{-1})$ and $b_\nu = \frac{1}{2i}(z_\nu - z_\nu^{-1})$, as well as $z_\nu = a_\nu + ib_\nu$ and $z_\nu^{-1} = a_\nu - ib_\nu$. Clearly $G_{\mathbb{R}}(\mathbb{R}) = (S^1)^n$ is compact, so by Proposition 12.4(d) it is contained in the maximal compact subgroup $\mathbb{K} \subset G(\mathbb{C})$. Under the map $t \mapsto \big(\log |z_\nu(\varphi(t))|\big)_{\nu=1\ldots n}$ the quotient Lie group $G(\mathbb{C})/G_{\mathbb{R}}(\mathbb{R})$ is isomorphic to $(\mathbb{R}^n, +)$, whose only compact subgroup is the identity. Since the image of $\mathbb{K}$ under the continuous quotient map $G(\mathbb{C}) \to G(\mathbb{C})/G_{\mathbb{R}}(\mathbb{R})$ is compact, it is the identity. Therefore, $G_{\mathbb{R}}(\mathbb{R}) = \mathbb{K}$ and hence $G_{\mathbb{R}}(\mathbb{R})$ is the maximal compact subgroup in G and $G_{\mathbb{R}}$ is the unique associated compact real form of G from Proposition 12.5.

Proposition 12.7. *Let $G \subset \mathrm{GL}_n$ be a closed subgroup over $\mathbb{C}$ and let $\mathbb{K} \subset G(\mathbb{C})$ be a maximal compact subgroup. Let $z \in G(\mathbb{C})$ be a semi-simple element such that every eigenvalue of z on the standard representation $\mathbb{C}^n$ of GL_n has complex norm 1. Then z is conjugate to an element of $\mathbb{K}$ under $G^\circ(\mathbb{C})$.*

Proof. It will be enough to find a compact subgroup $C \subset \mathrm{GL}_n(\mathbb{C})$ which contains z. In this case the subgroup $C \cap G(\mathbb{C}) \subset G(\mathbb{C})$, being the intersection of a compact and a closed subgroup, is compact, so there is an $x \in G^\circ(\mathbb{C})$ such that $x^{-1}(C \cap G(\mathbb{C}))x \subset \mathbb{K}$ by Theorem 12.1(c), and hence $x^{-1}zx \in \mathbb{K}$, too. Since z is semi-simple, the standard representation $\mathbb{C}^n$ of GL_n has a basis B in which the matrix of z is diagonal. Let $C \subset \mathrm{GL}_n(\mathbb{C})$ be the subgroup of all elements which in the basis B are diagonal such that all diagonal entries have norm 1. Then C is isomorphic to $(S^1)^n$ as a Lie group, so C is compact, and it clearly contains z. □

We will also need the following lemma due to Deligne:

Lemma 12.8. *Let G be a linear algebraic group over $\mathbb{C}$, let $\mathbb{K} \subset G(\mathbb{C})$ be a maximal compact subgroup, and let $x, y \in \mathbb{K}$ be two elements conjugate under $G(\mathbb{C})$, respectively under $G^\circ(\mathbb{C})$. Then they are already conjugate under $\mathbb{K}$, respectively under $\mathbb{K}^\circ$.*

Proof. Deligne's proof in [Del80, Lemma 2.2.2] lacks references, and is only really formulated in the connected semi-simple case, so here is a short proof using the results from [Hoc65]. Let $\mathbb{E}$ be as in Theorem 12.1 above. Let $a \in G(\mathbb{C})$ be such that $axa^{-1} = y$. Then we may write a uniquely as $a = ec$, where $e \in \mathbb{E}$ and $c \in \mathbb{K}$. If $a \in G^\circ(\mathbb{C})$ then $\mathbb{E} \subset G^\circ(\mathbb{C})$ implies that $c \in \mathbb{K} \cap G^\circ(\mathbb{C}) = \mathbb{K}^\circ$; use Proposition 12.4(a). Then $e(cxc^{-1})e^{-1} = y$. Since $c \in \mathbb{K}$, the element $z = cxc^{-1}$ is in $\mathbb{K}$, so it will be enough to show that $z = y$. Note that $eze^{-1} = y$ implies

$$ezy^{-1} = yey^{-1}.$$

By Theorem 12.1(b) we have $yey^{-1} \in \mathbb{E}$. Since $y, z \in \mathbb{K}$ we have $zy^{-1} \in \mathbb{K}$, therefore $yey^{-1} = e$ and $zy^{-1} = 1$ by Theorem 12.1(a). □

We next combine the above with the theory of maximal quasi-tori from Section 8.

Definition 12.9. A linear algebraic group T over $\mathbb{C}$ is a *quasi-torus* if its connected component T° is a torus, i.e. T° is isomorphic to $\mathbb{G}^n_{m,\mathbb{C}}$ for some n. A real Lie group $\mathbb{T}$ is a *compact quasi-torus* if it is compact and $\mathbb{T}^\circ$ is a compact real torus, i.e. $\mathbb{T}^\circ$ is isomorphic to $(S^1)^n$ for some natural number n. (Here $S^1 = \{z \in \mathbb{C}\colon |z| = 1\}$ as usual.)

Definition 12.10. A subgroup $\mathbb{T} \subset G(\mathbb{C})$ is a *maximal compact quasi-torus* in $G(\mathbb{C})$ if there is an (algebraic) maximal quasi-torus $T \subset G$ in the sense of Definition 8.6, such that $\mathbb{T}$ is a maximal compact subgroup in $T(\mathbb{C})$. By Example 12.6 every such $\mathbb{T}$ is a compact quasi-torus, so the terminology is at least partially justified.

Proposition 12.11. *The following holds:*

(a) any two maximal compact quasi-tori in $G(\mathbb{C})$ are conjugate under $G^\circ(\mathbb{C})$,
(b) every subgroup in $G(\mathbb{C})$ conjugate to a maximal compact quasi-torus under $G(\mathbb{C})$ is a maximal compact quasi-torus,
(c) every maximal compact subgroup $\mathbb{K} \subset G(\mathbb{C})$ contains a maximal compact quasi-torus $\mathbb{T}$,
(d) for every $\mathbb{K}$ and $\mathbb{T}$ as in (c) the following holds: every $x \in \mathbb{K}$ is conjugate under $\mathbb{K}^\circ$ to an element of $\mathbb{T}$.

Proof. (a) Let $\mathbb{T}_1, \mathbb{T}_2 \subset G(\mathbb{C})$ be two maximal compact quasi-tori. Let $T_1, T_2 \subset G$ be two maximal (algebraic) quasi-tori in G such that $\mathbb{T}_1, \mathbb{T}_2$ is a maximal compact subgroup in $T_1(\mathbb{C}), T_2(\mathbb{C})$, respectively. Then there is an $a \in G^\circ(\mathbb{C})$ such that $a^{-1}T_1a = T_2$ by Theorem 8.10(b). Since the map $x \mapsto a^{-1}xa$ is an isomorphism from T_1 onto T_2, the image $a^{-1}\mathbb{T}_1a$ of $\mathbb{T}_1$ under this map is a maximal compact subgroup in T_2. So $a^{-1}\mathbb{T}_1a$ is conjugate to $\mathbb{T}_2$ under an element of $T_2^\circ(\mathbb{C})$ by Proposition 12.4(b), and hence $\mathbb{T}_1$ is conjugate to $\mathbb{T}_2$ by an element of $G^\circ(\mathbb{C})$.

(b) Let $\mathbb{T} \subset G(\mathbb{C})$ be a maximal compact quasi-torus and let $a \in G(\mathbb{C})$ be arbitrary. Let $T \subset G$ be a maximal quasi-torus in G such that $\mathbb{T}$ is a maximal compact subgroup in $T(\mathbb{C})$. Since the map $x \mapsto a^{-1}xa$ is an isomorphism from T onto $a^{-1}Ta$, the image $a^{-1}\mathbb{T}a$ of $\mathbb{T}$ under this map is a maximal compact subgroup in the algebraic maximal quasi-torus $a^{-1}Ta$.

(c) Let $\mathbb{T} \subset G(\mathbb{C})$ be again a maximal compact quasi-torus and let $\mathbb{K} \subset G(\mathbb{C})$ be a maximal compact subgroup. Note that $\mathbb{T}$ exists by Theorems 8.10(a) and 12.1. By Theorem 12.1(c) there is an $x \in G(\mathbb{C})^\circ$ such that $x^{-1}\mathbb{T}x \subset \mathbb{K}$. By claim (b) the group $x^{-1}\mathbb{T}x$ is a maximal compact quasi-torus.

(d) Finally let $\mathbb{K}, \mathbb{T}$ and $x \in \mathbb{K}$ be as in claim (d). Then x is semi-simple by Proposition 12.4(e). Let $T \subset G$ be an algebraic maximal quasi-torus in G such that $\mathbb{T}$ is a maximal compact subgroup in $T(\mathbb{C})$. Then there is an $a \in G^\circ(\mathbb{C})$ such that $a^{-1}xa \subset T(\mathbb{C})$ by Theorem 8.10. Since $L = a^{-1}\mathbb{K}a \cap T(\mathbb{C})$ is the intersection of a compact and a closed subgroup, it is a compact subgroup in $T(\mathbb{C})$. Therefore, there is a $b \in T^\circ(\mathbb{C})$ such that $b^{-1}Lb \subset \mathbb{T}$ by Theorem 12.1(c). Then $(ab)^{-1}xab = b^{-1}a^{-1}xab \in b^{-1}Lb \subset \mathbb{T}$. Claim (d) now follows from Deligne's Lemma 12.8 applied to the two elements $x \in \mathbb{K}$ and $(ab)^{-1}xab \in \mathbb{T} \subset \mathbb{K}$. □

Next let us prove the analog of Proposition 9.2 for maximal compact quasi-tori. We consider the following situation: Let $G \subset \mathrm{GL}_n$ be a closed algebraic subgroup over $\mathbb{C}$, let $\mathbb{K} \subset G(\mathbb{C})$ be a maximal compact subgroup, and let $\mathbb{T} \subset G(\mathbb{C})$ be a maximal compact quasi-torus contained in $\mathbb{K}$, such that $\mathbb{T}$ is a maximal compact subgroup of a maximal quasi-torus T of G. Let $h \in \mathbb{T}$ and recall that we defined $T^h := \{g \in T^\circ\colon gh = hg\}$ in Notation 9.1. Set $\mathbb{T}^h = \{g \in \mathbb{T}^\circ\colon gh = hg\} = T^h(\mathbb{C}) \cap \mathbb{T}^\circ$.

Lemma 12.12. *The group $\mathbb{T}^h$ is a maximal compact subgroup in T^h. In particular, it is a compact quasi-torus.*

Proof. Since $\mathbb{T}^h = T^h \cap \mathbb{T}^\circ$ is the intersection of a closed algebraic subgroup with a compact Lie group, it is a compact subgroup. Let $\mathbb{S}$ be a maximal compact subgroup of T^h which contains $\mathbb{T}^h$. Then $\mathbb{S}$ is contained in a maximal compact subgroup of T°. Since T° is abelian, it has a unique maximal compact subgroup by Proposition 12.4(b), which is $\mathbb{T}^\circ = \mathbb{T} \cap T^\circ$ by Proposition 12.4(a). We get that $\mathbb{S} \subset T^h \cap \mathbb{T}^\circ = \mathbb{T}^h$, so $\mathbb{S} = \mathbb{T}^h$, and hence the latter is a maximal compact subgroup in T^h. □

Let $\mathbb{T}^{h\circ}$ denote the connected component of the identity of $\mathbb{T}^h$, as usual.

Proposition 12.13. *The following hold:*

(a) *every element of* $h\mathbb{T}^\circ$ *is conjugate under* $\mathbb{T}^\circ$ *to an element of* $h\mathbb{T}^{h\circ}$,
(b) *there is a positive integer* M *such that the intersection of* $h\mathbb{T}^{h\circ}$ *with any* $G(\mathbb{C})$*-conjugacy class has at most* M *elements.*

Before we start the proof of this proposition, we will need a lemma. Every torus T over $\mathbb{C}$ is abelian, so it has a unique maximal compact subgroup by Proposition 12.4(b), which we denote by $c(T)$.

Lemma 12.14. *Let* P, Q *be two tori over* $\mathbb{C}$*. Then the following holds:*
(a) *we have* $c(P\times_{\mathbb{C}} Q) = c(P)\times c(Q)$,
(b) *every surjective homomorphism* $\varphi : P \to Q$ *maps* $c(P)$ *onto* $c(Q)$.

Proof. By Tychonoff's theorem $c(P)\times c(Q)$ is a compact subgroup of $P\times_{\mathbb{C}} Q$. Therefore, $c(P\times_{\mathbb{C}} Q)$ contains $c(P)\times c(Q)$. On the other hand the image of $c(P\times_{\mathbb{C}} Q)$ with respect to the projection $P\times_{\mathbb{C}} Q\to P$ is a compact subgroup of P, so it lies in $c(P)$. Similarly the image of $c(P\times_{\mathbb{C}} Q)$ with respect to the projection $P\times_{\mathbb{C}} Q\to Q$ lies in $c(Q)$. Therefore, $c(P\times_{\mathbb{C}} Q)$ lies in $c(P)\times c(Q)$, so claim (a) holds.

Now we are going to prove claim (b). First assume that φ is an isogeny, that is it has finite kernel. Recall that the maximal compact subgroup of a torus R is a real torus whose real dimension is the same as the dimension of R by Example 12.6. Since φ induces an isomorphism on tangent spaces we get that $\varphi(c(P))$ is a compact Lie group whose dimension is the same as the dimension of $c(Q)$, which contains it. Since both groups are connected, they are equal.

Next assume that $P = R\times Q$ and φ is the projection $R\times Q\to Q$ to the second factor. Then $c(P) = c(R)\times C(Q)$ by part (a), so claim (b) holds in this case, too. Finally let φ be arbitrary. The connected component of the kernel of φ is a torus R. Since over the algebraically closed field $\mathbb{C}$ any extension of tori splits, we have $P = R\times_{\mathbb{C}} S$ for $S := P/R$. Then $\varphi : P = R\times_{\mathbb{C}} S\to Q$ is the composition of the projection $R\times_{\mathbb{C}} S\to S$ followed by the isogeny $S\to Q$. By the above the claim holds for the composition of a projection and an isogeny, so it holds for φ, too. □

Proof of Proposition 12.13. Let $T\subset G$ be maximal quasi-torus such that $\mathbb{T}$ is a maximal compact subgroup in $T(\mathbb{C})$. Let $\varphi : T^\circ\to T^\circ$ be the map $\varphi(t) = h^{-1}tht^{-1}$. As we saw in Notation 9.1 this map is a group homomorphism. The restriction of this homomorphism maps $\mathbb{T}^\circ$ into itself, because $\varphi(\mathbb{T}^\circ)$ is contained in the maximal compact subgroup $\mathbb{T}^\circ$. We denote the resulting homomorphism $\mathbb{T}^\circ\to\mathbb{T}^\circ$ by φ as well. In order to prove (a) it will be sufficient to show that for every $hx\in h\mathbb{T}^\circ$ there are elements $t_0\in\mathbb{T}^{h\circ}$ and $t\in\mathbb{T}^\circ$ with

$$hx = ht_0\varphi(t) = h\varphi(t)t_0 = hh^{-1}tht^{-1}t_0 = tht^{-1}t_0 = t(ht_0)t^{-1}.$$

So it will be sufficient to show that the map $\mathbb{T}^{h\circ}\times\mathbb{T}^\circ\to\mathbb{T}^\circ$ given by $(t_0, t)\mapsto t_0\varphi(t)$ is surjective.

Let $\psi : T^{h\circ}\times T^\circ\to T^\circ$ be the homomorphism defined similarly, i.e. given by the formula $(t_0, t)\mapsto t_o\varphi(t)$. By Proposition 9.2(a) we know that ψ is surjective. By Lemma 12.12 and Lemma 12.14(a) the group $\mathbb{T}^{h\circ}\times\mathbb{T}^\circ$ is a maximal compact subgroup of $T^{h\circ}\times T^\circ$. Therefore, claim (a) holds by Lemma 12.14(b). Claim (b) is an immediate consequence of Proposition 9.3. □

Lemma 12.14 also implies the following

Corollary 12.15. *Let* $\varphi\colon G\twoheadrightarrow\widetilde{G}$ *be an epimorphism of linear algebraic groups over* $\mathbb{C}$ *and let* $T\subset G$ *be a maximal quasi-torus and* $\mathbb{T}\subset T(\mathbb{C})\subset G(\mathbb{C})$ *a maximal compact quasi-torus. Then* $\varphi(\mathbb{T})$ *is a maximal compact quasi-torus in* $\varphi(T)$ *and* $\widetilde{G}$.

Proof. By Corollary 8.11 the image $\varphi(T)$ is a maximal quasi-torus in $\widetilde{G}$. Since $\varphi(\mathbb{T})\subset\varphi(T)$ is compact, it is contained in a maximal compact subgroup $\widetilde{\mathbb{T}}$ of $\varphi(T)$. We must show that $\varphi(\mathbb{T}) = \widetilde{\mathbb{T}}$. In the connected component of unity we have $\varphi(\mathbb{T}^\circ) = \varphi(c(T^\circ)) = c(\varphi(T^\circ)) = \widetilde{\mathbb{T}}^\circ$ by Lemma 12.14(b). The claim now follows from the surjectivity $\mathbb{T}/\mathbb{T}^\circ = T/T^\circ = G/G^\circ\twoheadrightarrow\widetilde{G}/\widetilde{G}^\circ = \widetilde{\mathbb{T}}/\widetilde{\mathbb{T}}^\circ$ for which we use Proposition 12.4(a) and Theorem 8.10(d). □

13. Chebotarëv for Overconvergent F-Isocrystals

Finally in this section we shall prove Theorem 1.12 about Chebotarëv density for overconvergent F-isocrystals. For every field K and curve U as above let $F\text{-Isoc}^\dagger_K(U)$ denote the K-linear rigid tensor category of K-linear overconvergent F-isocrystals on U. There is a forgetful functor $f_U\colon F\text{-Isoc}^\dagger_K(U)\to F\text{-Isoc}_K(U)$ which is fully faithful by a fundamental theorem of Kedlaya [Ked04, Theorem 1.1]. Fix a point $u \in U(\mathbb{F}_{q^e})$. Then the composition of f_U and the fiber functor ω_u on $F\text{-Isoc}_K(U)$ corresponding to u makes $F\text{-Isoc}_K^\dagger(U)$ a Tannakian category. By abuse of notation we denote $\omega_u \circ f_U$ again by ω_u. For every object $\mathcal{F}$ of $F\text{-Isoc}_K^\dagger(U)$ let $\langle\!\langle \mathcal{F} \rangle\!\rangle$ be the Tannakian sub-category of $F\text{-Isoc}_K^\dagger(U)$ generated by $\mathcal{F}$, see Definition A.3, and let $\mathrm{Gr}^\dagger(\mathcal{F}/U,u)$ or $\mathrm{Gr}^\dagger(\mathcal{F},u)$ denote the automorphism group of the fiber functor $\omega_u \colon \langle\!\langle \mathcal{F} \rangle\!\rangle \to K_e$-vector spaces. The category $\langle\!\langle \mathcal{F} \rangle\!\rangle$ is tensor equivalent to the K_e-rational representations of a K_e/K-groupoid $\mathfrak{Gr}^\dagger(\mathcal{F}/U,u)$, see Definition A.5 and Theorem A.11, and $\mathrm{Gr}^\dagger(\mathcal{F},u) = \mathfrak{Gr}^\dagger(\mathcal{F}/U,u)^\Delta$ is its kernel group. It is a smooth linear algebraic group over K_e by Proposition A.13. With every closed point $x \in U$ one can associate a (stable) Frobenius conjugacy class $\mathrm{Frob}_x^\dagger(\mathcal{F}) \subset \mathrm{Gr}^\dagger(\mathcal{F},u)$ in the same way as for $\mathrm{Gr}(\mathcal{F},u)$; see Definition 3.1. Note that the functor f_U induces a homomorphism $\mathrm{Gr}(f_U(\mathcal{F}),u) \to \mathrm{Gr}^\dagger(\mathcal{F},u)$ which is not an isomorphism in general.

Lemma 13.1. *The natural morphism $\mathrm{Gr}(f_U(\mathcal{F}),u) \to \mathrm{Gr}^\dagger(\mathcal{F},u)$ always is a closed immersion and induces an epimorphism $\mathrm{Gr}(f_U(\mathcal{F}),u)/\mathrm{Gr}(f_U(\mathcal{F}),u)^\circ \twoheadrightarrow \mathrm{Gr}^\dagger(\mathcal{F},u)/\mathrm{Gr}^\dagger(\mathcal{F},u)^\circ$ on the groups of connected components.*

Proof. Every object of $\langle\!\langle f_U(\mathcal{F}) \rangle\!\rangle$ is a subquotient of an object of the form $\bigoplus_i f_U(\mathcal{F})^{\otimes m_i} \otimes (f_U(\mathcal{F})^\vee)^{\otimes n_i} = f_U\big(\bigoplus_i \mathcal{F}^{\otimes m_i} \otimes (\mathcal{F}^\vee)^{\otimes n_i}\big)$. So the first statement follows from Proposition A.14(b).

To prove the second let $\mathcal{U}$ be an object of $\langle\!\langle \mathcal{F} \rangle\!\rangle$ such that the surjective homomorphism $\mathrm{Gr}^\dagger(\mathcal{F},u) \twoheadrightarrow \mathrm{Gr}^\dagger(\mathcal{U},u)$ has kernel equal to the characteristic subgroup $\mathrm{Gr}^\dagger(\mathcal{F},u)^\circ \subset \mathrm{Gr}^\dagger(\mathcal{F},u)$; see Remark A.17 and Corollary A.16(b). Then $\mathcal{U}$ has finite monodromy group and is unit-root by Lemma 5.1. Moreover, every convergent F-isocrystal $\mathcal{G} \in \langle\!\langle f_U(\mathcal{U}) \rangle\!\rangle$ has finite monodromy, and in particular finite local monodromy at the points in the complement of U. By Tsuzuki's monodromy theorem [Tsu98, Theorem 7.2.3 on page 1165] this implies that $\mathcal{G}$ has an overconvergent extension on U. Therefore, $f_U \colon \langle\!\langle \mathcal{U} \rangle\!\rangle \xrightarrow{\sim} \langle\!\langle f_U(\mathcal{U}) \rangle\!\rangle$ is a tensor equivalence of Tannakian categories and the inclusion map $\mathrm{Gr}(f_U(\mathcal{U}),u) \subset \mathrm{Gr}^\dagger(\mathcal{U},u)$ is an isomorphism. Since $f_U(\mathcal{U})$ is an object of $\langle\!\langle f_U(\mathcal{F}) \rangle\!\rangle$ the corresponding homomorphism $\mathrm{Gr}(f_U(\mathcal{F}),u) \twoheadrightarrow \mathrm{Gr}(f_U(\mathcal{U}),u) \xrightarrow{\sim} \mathrm{Gr}^\dagger(\mathcal{U},u) = \mathrm{Gr}^\dagger(\mathcal{F},u)/\mathrm{Gr}^\dagger(\mathcal{F},u)^\circ$ is surjective by Lemma 3.3. This proves the claim. □

Using the p-adic version of Deligne's equidistribution theorem due to Crew and Kedlaya [Ked06] and arguments inspired by the proof of Theorem 1.8 we will now prove Theorem 1.12, that is we prove the following

Theorem 13.2. *For every semi-simple overconvergent F-isocrystal $\mathcal{F}$ and for every subset $S \subset |U|$ of positive upper Dirichlet density the Zariski-closure of the set $\bigcup_{x\in S} \mathrm{Frob}_x^\dagger(\mathcal{F})$ contains a connected component of the group $\mathrm{Gr}^\dagger(\mathcal{F}/U,u) \times_{K_e} \overline{K}$.*

In other words, the overconvergent analog of Conjectures 1.4 and 1.2 hold for every semi-simple overconvergent F-isocrystal $\mathcal{F}$. This implies as in Proposition 6.7 that also the overconvergent analog of Conjecture 1.3 hold:

Corollary 13.3. *For every semi-simple overconvergent F-isocrystal $\mathcal{F}$ and for every subset $S \subset |U|$ of Dirichlet density one the set $\bigcup_{x\in S} \mathrm{Frob}_x^\dagger(\mathcal{F})$ is Zariski-dense in $\mathrm{Gr}^\dagger(\mathcal{F}/U,u)$.* □

We immediately have the following further

Corollary 13.4. *Let $S \subset |U|$ be a subset of Dirichlet density one and let $\mathcal{F},\mathcal{G}$ be two overconvergent F-isocrystals such that for every $x \in S$ we have $\mathrm{Tr}(\mathrm{Frob}_x^\dagger(\mathcal{F})) = \mathrm{Tr}(\mathrm{Frob}_x^\dagger(\mathcal{G}))$. Then the semi-simplifications of $\mathcal{F}$ and $\mathcal{G}$ are isomorphic.*

Proof. Note that $\mathcal{F}^{ss} \oplus \mathcal{G}^{ss}$ is also an overconvergent F-isocrystal, so by Theorem 13.2 we know that the set $\bigcup_{x\in S} \mathrm{Frob}_x^\dagger(\mathcal{F}^{ss} \oplus \mathcal{G}^{ss})$ is Zariski-dense in $\mathrm{Gr}^\dagger(\mathcal{F}^{ss} \oplus \mathcal{G}^{ss},u)$. Now we may argue exactly as we did in the proof of Corollary 1.5. □

Remark 13.5. This corollary was proven by Nobuo Tsuzuki when $\mathcal{F}$ and $\mathcal{G}$ are ι-mixed (see Definition 13.9 below) via a simpler direct method, see [Abe18b, Proposition A.4.1], but his argument also uses the p-adic analogue of Weil II, like ours. (Note that the proof of this claim in [Abe18a, Proposition in Section 3] is incorrect.) Note that the natural p-adic analogue of the Langlands correspondence was proven by Tomoyuki Abe in [Abe18b], so the condition of being ι-mixed is not very restrictive. (See Remark 13.11 below for an explanation.)

We will continue by showing how Corollary 13.4 yields a

p-adic proof of Theorem 1.1. We only need to show that when $V_p(A) \cong V_p(B)$ then there is an isogeny $f\colon A \to B$. As in the l-adic proof above we may assume that for every $x \in |U|$ the abelian varieties A and B have good ordinary reduction at x. For every such x let A_x and B_x denote the reductions of A and B over x, respectively. Arguing as in the l-adic proof in Section 2 we can conclude that the L-functions $L(A_x, t)$ and $L(B_x, t)$ are equal for every $x \in |U|$. By slight abuse of notation let A, B also denote the unique abelian schemes over U whose generic fiber is A, B, respectively.

Let $\mathbb{Z}_q$ and $\mathbb{Q}_q$ denote the ring of Witt vectors of $\mathbb{F}_q$ of infinite length and its fraction field, respectively. For every abelian scheme C over U let $\mathbf{D}^\dagger(C)$ denote the overconvergent (rational) crystalline Dieudonné module of A over U (for a construction see [KT03], sections 4.3–4.8). It is a $\mathbb{Q}_q$-linear overconvergent F-isocrystal equipped with the q-Frobenius which is semi-simple by [Pál15, Theorem 1.2]. Moreover by [Pál15, Theorem 1.1] we also know the following:

Theorem 13.6. *The map:*

$$\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}_q \xrightarrow{\alpha} \mathrm{Hom}(\mathbf{D}^\dagger(A), \mathbf{D}^\dagger(B))$$

induced by the functoriality of overconvergent Dieudonné modules is an isomorphism.

Note that for every abelian scheme C over U the convergent F-isocrystal underlying $\mathbf{D}^\dagger(C)$ is the convergent Dieudonné module of the p-divisible group $C[p^\infty]$. In particular for every $x \in |U|$ the factor of the L-function $L(U, \mathbf{D}^\dagger(C), t)$ of $\mathbf{D}^\dagger(C)$, defined by Étesse–Le Stum (see [ES93]), at x is equal to the L-function $L(C_x, t)$ of the reduction of C over x. In particular $\mathbf{D}^\dagger(C)$ is pure of weight 1. We get that the trace of the Frobenius elements $\mathrm{Frob}_x^\dagger(\mathbf{D}^\dagger(A))$ and $\mathrm{Frob}_x^\dagger(\mathbf{D}^\dagger(B))$ are the same for every $x \in |U|$. Therefore, by Corollary 13.4 the overconvergent F-isocrystals $\mathbf{D}^\dagger(A)$ and $\mathbf{D}^\dagger(B)$ must be isomorphic, so by Theorem 13.6 above the abelian varieties A and B are isogenous and the p-adic proof of Theorem 1.1 is complete. □

In order to prove Theorem 13.2 we next describe the overconvergent analog of Definition 4.9, Remark 4.10 and Proposition 4.11.

Definition 13.7. Let $\mathrm{Isoc}_K^\dagger(U)$ denote the category of K-linear overconvergent isocrystals on U. Fix a point $u \in U(\mathbb{F}_{q^e})$. The pull-back with respect to u furnishes a functor from $\mathrm{Isoc}_K^\dagger(U)$ into the category of finite dimensional K_e-vector spaces which makes $\mathrm{Isoc}_K^\dagger(U)$ into a Tannakian category, see [Cre92, § 2.1]. By slight abuse of notation let ω_u be the corresponding fiber functor on $\mathrm{Isoc}_K^\dagger(U)$. Let $(\cdot)^\sim\colon F\text{-}\mathrm{Isoc}_K^\dagger(U) \to \mathrm{Isoc}_K^\dagger(U)$ denote the functor furnished by forgetting the Frobenius structure. For every object $\mathcal{F}$ of $F\text{-}\mathrm{Isoc}_K^\dagger(U)$ let $\mathrm{DGal}^\dagger(\mathcal{F}, u)$ denote the Tannakian fundamental group of the Tannakian sub-category $\langle\!\langle \mathcal{F}^\sim \rangle\!\rangle$ of $\mathrm{Isoc}_K^\dagger(U)$ generated by $\mathcal{F}^\sim$ with respect to the fiber functor ω_u. Moreover for every such $\mathcal{F}$ let $\langle\!\langle \mathcal{F} \rangle\!\rangle_{const}$ and $\mathbf{W}(\mathcal{F}, u)$ denote the Tannakian sub-category of constant objects of $\langle\!\langle \mathcal{F} \rangle\!\rangle$ and the Tannakian fundamental group of $\langle\!\langle \mathcal{F} \rangle\!\rangle_{const}$ with respect to the fiber functor ω_u, respectively. Then $\mathbf{W}(\mathcal{F}, u)$ is commutative by Theorem 4.8(b).

The monodromy group $\mathrm{DGal}^\dagger(\mathcal{F}, u)$ was introduced by Crew [Cre92]. Next we describe its relationship to $\mathrm{Gr}^\dagger(\mathcal{F}, u)$. Let $\alpha\colon \mathrm{DGal}^\dagger(\mathcal{F}, u) \to \mathrm{Gr}^\dagger(\mathcal{F}, u)$ be the homomorphism induced by the forgetful functor $(\cdot)^\sim\colon \langle\!\langle \mathcal{F} \rangle\!\rangle \to \langle\!\langle \mathcal{F}^\sim \rangle\!\rangle$, and let $\beta\colon \mathrm{Gr}^\dagger(\mathcal{F}, u) \to \mathbf{W}(\mathcal{F}, u)$ be the homomorphism induced by the inclusion $\langle\!\langle \mathcal{F} \rangle\!\rangle_{const} \subset \langle\!\langle \mathcal{F} \rangle\!\rangle$.

Proposition 13.8. *If $\mathcal{F}^\sim$ is semi-simple, then the sequence:*

$$0 \longrightarrow \mathrm{DGal}^\dagger(\mathcal{F}, u) \xrightarrow{\ \alpha\ } \mathrm{Gr}^\dagger(\mathcal{F}, u) \xrightarrow{\ \beta\ } \mathbf{W}(\mathcal{F}, u) \longrightarrow 0.$$

is exact. This is the case for example if the overconvergent F-isocrystal $\mathcal{F}$ is semi-simple.

Proof. This follows by the same proof as in Proposition 4.11, or alternatively like in [Pál15, Proposition 3.8]. The last statement follows from [Cre92, Corollary 4.10], which states that for a semi-simple $\mathcal{F}$ the group $\mathrm{DGal}^\dagger(\mathcal{F}, u)^\circ$ is a semi-simple group. Since it is the monodromy group of $\mathcal{F}^\sim$, the latter is semi-simple by Lemma 3.8. □

In the rest of the section we shall prove Theorem 13.2 by reducing to the case when $\mathcal{F}$ is ι-mixed as in the following

Definition 13.9. Fix an isomorphism of fields $\iota\colon \overline{K} \to \mathbb{C}$ and let $|\cdot|\colon \mathbb{C} \to \mathbb{R}_{\geq 0}$ be the usual archimedean absolute value on $\mathbb{C}$. We say that an overconvergent F-isocrystal $\mathcal{F}$ on U is (point-wise) *ι-pure* of weight w, where $w \in \mathbb{Z}$, if for every $x \in |U|$ and for every eigenvalue $\alpha \in \overline{K}$ of $\mathrm{Frob}_x^\dagger(\mathcal{F})$ acting on $\omega_u(\mathcal{F}) \otimes_{K_e,\iota} \mathbb{C}$ we have $|\iota(\alpha)| = q^{w \deg(x)/2}$. We say that an overconvergent F-isocrystal $\mathcal{F}$ is *ι-mixed* if it is a successive extension of ι-pure overconvergent F-isocrystals. Let $F\text{-Isoc}_K^{\dagger,\iota}(U)$ denote the the full sub-category of $F\text{-Isoc}_K^\dagger(U)$ whose objects are ι-mixed overconvergent F-isocrystals on U.

Proposition 13.10. *The category $F\text{-Isoc}_K^{\dagger,\iota}(U)$ is a full Tannakian sub-category of $F\text{-Isoc}_K^\dagger(U)$.*

Proof. The category $F\text{-Isoc}_K^{\dagger,\iota}(U)$ contains the trivial (constant, hence overconvergent) F-isocrystal $\underline{\mathbf{1}}_U$ and is closed under forming direct sums by definition. Because tensor products, duals and subquotients of ι-pure overconvergent F-isocrystals are also ι-pure, we get that $F\text{-Isoc}_K^{\dagger,\iota}(U)$ is also closed under tensor products, duals and subquotients. □

Remark 13.11. We would like to clarify what we mean by the natural p-adic analogue of the Langlands correspondence in Remark 13.5, and why such a claim would imply that a large class of overconvergent F-isocrystals on U is ι-mixed. Let $\mathbb{A}$ denote the ring of adèles of $k = \mathbb{F}_q(U)$. Let $\mathcal{F}$ be an absolutely irreducible K-linear overconvergent F-isocrystal on U of rank n, by which we mean that for every finite field extension L of K the L-linear overconvergent F-isocrystal $\mathcal{F} \otimes_K L$ on U we get from $\mathcal{F}$ by extension of scalars from K to L is irreducible. Also assume that the determinant $\det(\mathcal{F}) = \bigwedge^n(\mathcal{F})$ of $\mathcal{F}$ is a unit-root F-isocrystal with finite monodromy and let ρ be the corresponding p-adic representation $\rho\colon \pi_1^{\text{ét}}(U, \bar{u}) \to K^\times$; see Proposition 5.2. Then the aforementioned p-adic Langlands correspondence of Abe [Abe18a, Theorem 4.2.2] claims that there is a cuspidal automorphic representation Π of $GL_n(\mathbb{A})$ such that the central character of Π is ρ under the identification furnished by class field theory, and if $\Pi = \otimes_{x \in |X|} \Pi_x$ is the factorization of Π into the tensor product of local representations then Π_x is unramified for every $x \in |U|$ and its Hecke parameters are equal to the eigenvalues of $\mathrm{Frob}_x^\dagger(\mathcal{F})$. Because by [Laf02, Théorème VI.10(i)] the Ramanujan-Petersson conjecture holds for Π we get that $\mathcal{F}$ is ι-pure of weight zero.

Next we will state a convenient form of the p-adic version of Deligne's equidistribution theorem. For the rest of the section we keep the following

Notation 13.12. We fix a ι-pure overconvergent F-isocrystal $\mathcal{U} \in F\text{-Isoc}_K^{\dagger,\iota}(U)$ on U of weight zero. We call $G^{\text{arith}} := \mathrm{Gr}^\dagger(\mathcal{U}, u)$ and $G^{\text{geom}} := \mathrm{DGal}^\dagger(\mathcal{U}, u)$ its arithmetic and geometric monodromy groups, respectively. Assume that the connected component $(G^{\text{arith}})^\circ$ of G^{arith} is a semi-simple group. By (the analog of) Lemma 3.8 this implies that $\mathcal{U}$ is a semi-simple overconvergent F-isocrystal. Therefore, the connected component $(G^{\text{geom}})^\circ$ of G^{geom} is semi-simple by [Cre92, Corollary 4.10] and equals the derived group of the connected component $(G^{\text{arith}})^\circ$ of G^{arith} by [Pál15, Proposition 4.12]. By [Bor91, IV.14.2 Corollary] we have $(G^{\text{arith}})^\circ = (G^{\text{geom}})^\circ$, and it follows from Proposition 13.8 that $\mathbf{W}(\mathcal{U}, u)$ is finite. It follows further that $\mathbf{W}(\mathcal{U}, u)$ is cyclic, because it is generated by the Frobenius f of a tensor generator (W, f) of $\langle\langle\mathcal{U}\rangle\rangle_{const}$ by Theorem 4.8(b).

Using the embedding ι we may extend scalars and define the semi-simple algebraic groups $G_{\mathbb{C}}^{\text{arith}}$ and $G_{\mathbb{C}}^{\text{geom}}$ over $\mathbb{C}$. Let $G^{\text{arith}}(\mathbb{C})$ and $G^{\text{geom}}(\mathbb{C})$ denote their complex points, which we regard as complex semi-simple Lie groups. We will denote by $\mathbb{K}^{\text{arith}}$ and $\mathbb{K}^{\text{geom}}$ maximal compact subgroups of $G^{\text{arith}}(\mathbb{C})$ and $G^{\text{geom}}(\mathbb{C})$ such that $\mathbb{K}^{\text{geom}} \subset \mathbb{K}^{\text{arith}}$. Clearly $(\mathbb{K}^{\text{geom}})^\circ \subset (\mathbb{K}^{\text{arith}})^\circ$. Since $(G^{\text{arith}})^\circ = (G^{\text{geom}})^\circ$, both $(\mathbb{K}^{\text{geom}})^\circ$ and $(\mathbb{K}^{\text{arith}})^\circ$ are maximal compact subgroups in $(G^{\text{arith}})^\circ = (G^{\text{geom}})^\circ$ by Proposition 12.4(a). Therefore, they are conjugate by Proposition 12.4(b), and hence have the same dimension. Since they are also connected we get that $(\mathbb{K}^{\text{arith}})^\circ = (\mathbb{K}^{\text{geom}})^\circ$, and therefore we deduce from Proposition 12.4(a) that

$$\begin{aligned}
\mathbb{K}^{\text{arith}}/\mathbb{K}^{\text{geom}} &\cong \big(\mathbb{K}^{\text{arith}}/(\mathbb{K}^{\text{arith}})^\circ\big)/\big(\mathbb{K}^{\text{geom}}/(\mathbb{K}^{\text{geom}})^\circ\big) \\
&\cong \big(G_{\mathbb{C}}^{\text{arith}}/(G_{\mathbb{C}}^{\text{arith}})^\circ\big)/\big(G_{\mathbb{C}}^{\text{geom}}/(G_{\mathbb{C}}^{\text{geom}})^\circ\big) \\
&\cong G_{\mathbb{C}}^{\text{arith}}/G_{\mathbb{C}}^{\text{geom}} \\
&\cong \mathbf{W}(\mathcal{U}, u)(\mathbb{C}) \;=:\; \Gamma
\end{aligned}$$

is a finite cyclic group with canonical generator being the Frobenius f from the previous paragraph. In what follows we consider the group homomorphism

$$\mathbb{Z} \twoheadrightarrow \Gamma, \quad m \longmapsto [m] := f^m. \tag{13.1}$$

Definition 13.13. Fix an element $\gamma \in \Gamma \cong \mathbb{K}^{\mathrm{arith}}/\mathbb{K}^{\mathrm{geom}}$ and let $\mathbb{K}^{\mathrm{arith}}_\gamma$ denote the inverse image of γ in $\mathbb{K}^{\mathrm{arith}}$. We denote the set of $\mathbb{K}^{\mathrm{arith}}$-conjugacy classes in $\mathbb{K}^{\mathrm{arith}}$ by $\mathbb{K}^{\mathrm{arith},\#}$ and the ones which meet $\mathbb{K}^{\mathrm{arith}}_\gamma$ by $\mathbb{K}^{\mathrm{arith},\#}_\gamma$. We equip these sets with the quotient topology. $\mathbb{K}^{\mathrm{arith},\#}_\gamma$ is a union of connected components of $\mathbb{K}^{\mathrm{arith},\#}$, because it equals the preimage of γ under the induced map $\mathbb{K}^{\mathrm{arith},\#} \to \Gamma$. Let $\mu_{\mathrm{Haar},\gamma}$ be the $\mathbb{K}^{\mathrm{geom}}$-translation invariant measure on $\mathbb{K}^{\mathrm{arith}}_\gamma$ of total mass 1. (We may take the left or right invariant measure as either is bi-invariant by [Bou04, Chapitre VII, § 1.3, Corollaire].) Let $\mu^\#_{\mathrm{Haar},\gamma}$ be its push-forward onto $\mathbb{K}^{\mathrm{arith},\#}_\gamma$, see Definition B.1. The main equidistribution statement will be that a suitably normalized sum of point masses corresponding to Frobenius elements converges to the measure $\mu^\#_{\mathrm{Haar},\gamma}$.

Lemma 13.14. *The space $\mathbb{K}^{arith,\#}$ is a compact topological Hausdorff space.*

Proof. By [Bou95, Chapter I, § 8.3, Proposition 8] it suffices to show that the quotient map $\mathbb{K}^{\mathrm{arith}} \to \mathbb{K}^{\mathrm{arith},\#}$ is open and the graph of the conjugation action $\mathbb{K}^{\mathrm{arith}} \times \mathbb{K}^{\mathrm{arith}} \to \mathbb{K}^{\mathrm{arith}} \times \mathbb{K}^{\mathrm{arith}}$, $(x, g) \mapsto (x, gxg^{-1})$ is closed. The openness of the quotient map follows from [Bou95, Chapter III, § 2.4, Lemma 2], and the graph is closed because $\mathbb{K}^{\mathrm{arith}} \times \mathbb{K}^{\mathrm{arith}}$ is compact and Hausdorff. □

We will need more results on the structure of the quotient, which are due to Brumfiel by viewing $\mathbb{K}^{\mathrm{arith}}$ as a closed, bounded affine semi-algebraic group, see [DK81, § 7, Definition 3].

Theorem 13.15 ([Bru87, Corollary 1.6])**.** *If H is a closed, bounded affine semi-algebraic group which acts continuously and semi-algebraically on an affine semi-algebraic space X then the quotient space X/H exits as an affine semi-algebraic space and the quotient map $X \to X/H$ is continuous and semi-algebraic.*

Remark 13.16. Theorem 13.15 applies to all real closed fields, but we will only use it for the real number field $\mathbb{R}$. It is pointed out in [Bru87, Remark 1.3] that in the latter case the quotient map $X \to X/H$ is *topological*, which implies that the topology on X/H is the usual quotient topology. We conclude that $\mathbb{K}^{\mathrm{arith},\#}$ is an affine semi-algebraic space and the quotient map $\mathbb{K}^{\mathrm{arith}} \to \mathbb{K}^{\mathrm{arith},\#}$ is semi-algebraic.

Definition 13.17. We consider the semi-simplification $\mathrm{Frob}^\dagger_x(\mathcal{U})^{ss}$ of the Frobenius conjugacy class $\mathrm{Frob}^\dagger_x(\mathcal{U})$. The eigenvalues $\alpha \in \overline{K}$ of $\mathrm{Frob}^\dagger_x(\mathcal{U})^{ss}$ and of $\mathrm{Frob}^\dagger_x(\mathcal{U})$ acting on $\omega_u(\mathcal{U})$ are the same. Since $\mathcal{U}$ is ι-pure of weight zero, all these eigenvalues have complex norm 1 in the action on $\omega_u(\mathcal{U}) \otimes_{K_e,\iota} \mathbb{C}$. So $\mathrm{Frob}^\dagger_x(\mathcal{U})^{ss}$ is conjugate under $G^{\mathrm{arith}}(\mathbb{C})^\circ$ to an element of $\mathbb{K}^{\mathrm{arith}}$ by Proposition 12.7. The $\mathbb{K}^{\mathrm{arith}}$-conjugacy class of this element is well-defined, because by Lemma 12.8 two elements of $\mathbb{K}^{\mathrm{arith}}$ which are conjugate under $G^{\mathrm{arith}}(\mathbb{C})$ are already conjugate under $\mathbb{K}^{\mathrm{arith}}$. We denote this $\mathbb{K}^{\mathrm{arith}}$-conjugacy class by $\theta(x)$. Note that by definition of $\mathbf{W}(\mathcal{U}, u)$, the image of $\mathrm{Frob}^\dagger_x(\mathcal{U})$ and $\mathrm{Frob}^\dagger_x(\mathcal{U})^{ss}$ in $\Gamma = \mathbf{W}(\mathcal{U}, u)(\mathbb{C})$ is just the image $\gamma := [\deg(x)]$ of $\deg(x)$ under the map (13.1). Thus the conjugacy class $\theta(x)$ is an element of $\mathbb{K}^{\mathrm{arith},\#}_\gamma$.

Definition 13.18. For each integer m let U_m denote the set of points in U of degree m. For every $y \in \mathbb{K}^{\mathrm{arith},\#}$ let δ_y denote the Dirac delta of y on $\mathbb{K}^{\mathrm{arith},\#}$. We define a measure μ_m on the set of conjugacy classes $\mathbb{K}^{\mathrm{arith},\#}$ to be the discrete measure:

$$\mu_m := \frac{1}{\#U_m} \sum_{x \in U_m} \delta_{\theta(x)}.$$

Note that this measure is supported on $\mathbb{K}^{\mathrm{arith},\#}_\gamma$ where γ is equal to the image $[m]$ of m in Γ from (13.1).

Theorem 13.19 (p-adic version of Deligne's equidistribution theorem)**.** *For every $\gamma \in \Gamma$ the measures μ_m, for which $[m] = \gamma \in \Gamma$ in* (13.1)*, converge weakly to $\mu^\#_{Haar,\gamma}$ on $\mathbb{K}^{arith,\#}_\gamma$ as $m \to \infty$.*

Proof. In the l-adic case Ulmer [Ulm04, § 9.4] attributes this theorem to Katz–Sarnak [KS99, Theorem 9.7.10] and points out that this is a mild generalization of Deligne's original result [Del80, Theorem 3.5.3]. The p-adic case can be handled in the same way, based on the work of Crew, Kedlaya, Abe and Caro [AC18, Cre98, Ked06]. □

Finally we are ready to give the

Proof of Theorem 13.2. The proof is very similar to the proof of Theorem 1.8 at the end of Section 7. We start with similar reduction steps. Let $\mathcal{F}$ be a semi-simple overconvergent F-isocrystal with simple summands $\mathcal{F}_i$. Fix an i and look at the element $\alpha_i := \mathrm{Frob}_u^\dagger(\det \mathcal{F}_i) := u^*F^e$ in the (abelian) monodromy group $\mathrm{Gr}^\dagger(\det \mathcal{F}_i/U,u)(K_e) \subset K_e^\times$. We take an (r_ie)-th root $\widetilde{\alpha}_i$ of α_i where r_i is the rank of $\mathcal{F}_i$, and at the expense of replacing K by the finite extension $K(\widetilde{\alpha}_i)$ we let $\mathcal{C}_i$ be the constant, hence overconvergent F-isocrystal on U induced from the F-isocrystal $(K, F = \widetilde{\alpha}_i^{-1}) \in F\text{-}\mathrm{Isoc}_K(\operatorname{Spec}\mathbb{F}_q)$ on $\mathbb{F}_q$. Set $\mathcal{U}_i := \mathcal{F}_i \otimes \mathcal{C}_i$. Then the conjugacy class $\mathrm{Frob}_u^\dagger(\det\mathcal{U}_i)$ of $\det\mathcal{U}_i = \det(\mathcal{F}_i \otimes \mathcal{C}_i) = (\det \mathcal{F}_i) \otimes \mathcal{C}_i^{\otimes r_i}$ is equal to $\{1\} \subset \mathrm{Gr}^\dagger(\det\mathcal{U}_i/U,u)(K_e) \subset K_e^\times$. In particular, $\det\mathcal{U}_i$ is unit root at u. Since it has rank one, it is unit root on the entire curve U. By Lemma 3.3 also the tensor generator $\mathcal{E}_i$ of $\langle\!\langle \det\mathcal{U}_i\rangle\!\rangle_{const}$ has $\mathrm{Frob}_u^\dagger(\mathcal{E}_i) = \{1\}$ and so $\mathrm{Gr}^\dagger(\mathcal{E}_i/U,u)$ is a finite group by Theorem 4.8(b) and (c). By Proposition 13.8 this implies that the monodromy group of $\det\mathcal{U}_i$ is finite, since its geometric monodromy group is also finite by [Cre87, 4.13 Corollary]. Moreover, $\mathcal{U}_i$ is irreducible, because $\mathcal{F}_i$ is. So $\mathcal{U}_i$ is ι-pure of weight zero by Abe's result, see Remark 13.11. Now we let $\mathcal{U} := \bigoplus_i \mathcal{U}_i$ and $\mathcal{C} := \bigoplus_i \mathcal{C}_i$. Then $\mathcal{F}$ belongs to $\langle\!\langle \mathcal{U} \oplus \mathcal{C}\rangle\!\rangle$ and by Lemma 3.3 it is enough to show the theorem for $\mathcal{U} \oplus \mathcal{C}$. So we may assume that $\mathcal{F} = \mathcal{U} \oplus \mathcal{C}$, where $\mathcal{U}$ is a semi-simple and ι-pure overconvergent F-isocrystals of weight zero and $\mathcal{C}$ is the direct sum of irreducible constant F-isocrystals of varying weights.

The group $\mathrm{DGal}^\dagger(\mathcal{U},u)^\circ$ is semi-simple by [Cre92, Corollary 4.10] and equals the derived group of the connected component $\mathrm{Gr}^\dagger(\mathcal{U},u)^\circ$ by [Pál15, Proposition 4.12]. Let Z_1 be the center of $\mathrm{Gr}^\dagger(\mathcal{U},u)^\circ$. We use (the overconvergent analog of) Proposition 3.9 and let $\mathcal{S}, \mathcal{T} \in \langle\!\langle \mathcal{U}\rangle\!\rangle$ be the overconvergent F-isocrystals whose monodromy groups are $\mathrm{Gr}^\dagger(\mathcal{S},u) = \mathrm{Gr}^\dagger(\mathcal{U},u)/Z_1$ and $\mathrm{Gr}^\dagger(\mathcal{T},u) = \mathrm{Gr}^\dagger(\mathcal{U},u)/\,\mathrm{DGal}^\dagger(\mathcal{U},u)^\circ$. There is a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Gr}^\dagger(\mathcal{U}\oplus\mathcal{C},u) & \hookrightarrow & \mathrm{Gr}^\dagger(\mathcal{U},u) \times_{K_e} \mathrm{Gr}^\dagger(\mathcal{C},u) \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
\mathrm{Gr}^\dagger(\mathcal{S}\oplus\mathcal{T}\oplus\mathcal{C},u) & \hookrightarrow & \mathrm{Gr}^\dagger(\mathcal{S}\oplus\mathcal{T},u) \times_{K_e} \mathrm{Gr}^\dagger(\mathcal{C},u).
\end{array}$$

By Proposition 3.9 the vertical map on the right has finite kernel. Since the horizontal maps are injective by Proposition 3.6(c), we get that the vertical map on the left has finite kernel, too. By Lemma 3.5 it is enough to prove Theorem 13.2 for $\mathcal{S}\oplus\mathcal{T}\oplus\mathcal{C}$. Since ι-pure overconvergent F-isocrystals of weight zero form a full Tannakian sub-category, the first claims in the two following lemmas are clear.

Lemma 13.20. *The overconvergent F-isocrystal $\mathcal{S}$ is ι-pure of weight zero such that $\mathrm{Gr}^\dagger(\mathcal{S},u)^\circ$ is semi-simple and has trivial center.*

Proof. The group $\mathrm{Gr}^\dagger(\mathcal{S},u)^\circ$ is semi-simple by [Bor91, IV.11.21 Proposition and IV.14.11 Corollary]. Let $\overline{Z}$ be its center. For an element $\bar{z} \in \overline{Z}(\overline{K})$ and a preimage $z \in \mathrm{Gr}^\dagger(\mathcal{U},u)^\circ(\overline{K})$ of $\bar{z}$ the map $\mathrm{Gr}^\dagger(\mathcal{U},u)^\circ \to Z_1 \cap [\mathrm{Gr}^\dagger(\mathcal{U},u)^\circ, \mathrm{Gr}^\dagger(\mathcal{U},u)^\circ]$, $g \mapsto gzg^{-1}z^{-1}$ factors through the connected component of $Z_1 \cap [\mathrm{Gr}^\dagger(\mathcal{U},u)^\circ, \mathrm{Gr}^\dagger(\mathcal{U},u)^\circ]$ which is trivial by [Bor91, IV.14.2 Proposition]. Thus $z \in Z_1$ and $\bar{z} = 1$. □

Lemma 13.21. *The overconvergent F-isocrystal $\mathcal{T}$ is ι-pure of weight zero such that $\mathrm{Gr}^\dagger(\mathcal{T},u)^\circ$ is a torus and $\mathrm{DGal}^\dagger(\mathcal{T},u)$ is finite.*

Proof. The second claim follows from the fact that the connected commutative reductive group Z_1° surjects onto $\mathrm{Gr}^\dagger(\mathcal{T},u)^\circ$ by [Bor91, I.1.4 Corollary, IV.14.2 Proposition, III.8.4 Corollary and III.8.5 Proposition]. The last claim follows from the fact that we divided out $\mathrm{Gr}^\dagger(\mathcal{U},u)$ by $\mathrm{DGal}^\dagger(\mathcal{U},u)^\circ$ and that $\mathrm{DGal}^\dagger(\mathcal{U},u)^\circ$ surjects onto $\mathrm{DGal}^\dagger(\mathcal{T},u)^\circ$, because $\mathcal{T} \in \langle\!\langle \mathcal{U}\rangle\!\rangle$. □

Let $\mathcal{E}$ be a tensor generator of $\langle\!\langle \mathcal{T}\rangle\!\rangle_{const}$. It is a constant F-isocrystal whose monodromy group is $\mathrm{Gr}^\dagger(\mathcal{E},u) = \mathbf{W}(\mathcal{T},u) = \mathrm{Gr}^\dagger(\mathcal{T},u)/\,\mathrm{DGal}^\dagger(\mathcal{T},u)$ by Proposition 13.8. There is a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Gr}^\dagger(\mathcal{S}\oplus\mathcal{T}\oplus\mathcal{C},u) & \hookrightarrow & \mathrm{Gr}^\dagger(\mathcal{S},u) \times_{K_e} \mathrm{Gr}^\dagger(\mathcal{T},u) \times_{K_e} \mathrm{Gr}^\dagger(\mathcal{C},u) \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
\mathrm{Gr}^\dagger(\mathcal{S}\oplus\mathcal{E}\oplus\mathcal{C},u) & \hookrightarrow & \mathrm{Gr}^\dagger(\mathcal{S},u) \times_{K_e} \mathrm{Gr}^\dagger(\mathcal{E},u) \times_{K_e} \mathrm{Gr}^\dagger(\mathcal{C},u).
\end{array}$$

By Lemma 13.21 the map on the right has finite kernel. Since the horizontal maps are injective by Proposition 3.6(c), we get that the vertical map on the left has finite kernel, too. By Lemma 3.5 it is enough to prove Theorem 13.2 for $\mathcal{S} \oplus \mathcal{E} \oplus \mathcal{C}$.

Note that $\mathcal{E} \oplus \mathcal{C}$ is the direct sum of constant ι-pure F-isocrystals of varying weights. By slight abuse of notation writing $\mathcal{U}$ for $\mathcal{S}$ and $\mathcal{C}$ for $\mathcal{E} \oplus \mathcal{C}$ we may assume without the loss of generality that $\mathcal{F} = \mathcal{U} \oplus \mathcal{C}$, where $\mathcal{U}$ is a semi-simple and ι-pure overconvergent F-isocrystals of weight zero such that $\mathrm{Gr}^\dagger(\mathcal{U}, u)^\circ$ is semi-simple with trivial center, and $\mathcal{C}$ is the direct sum of constant ι-pure F-isocrystals of varying weights. In this case $\mathrm{Gr}^\dagger(\mathcal{F}, u)$ is the fiber product of $\mathrm{Gr}^\dagger(\mathcal{U}, u)$ and $\mathrm{Gr}^\dagger(\mathcal{C}, u)$ over $\mathrm{Gr}^\dagger(\langle\langle \mathcal{U} \rangle\rangle \cap \langle\langle \mathcal{C} \rangle\rangle, u)$ by Proposition 3.6(c). Let $f \in \mathrm{Gr}^\dagger(\mathcal{C}, u)(K_e)$ be the Frobenius of the constant F-isocrystal $\mathcal{C}$. Then $\mathrm{Gr}^\dagger(\mathcal{C}, u)$ is the Zariski closure of $f^{\mathbb{Z}}$ and is commutative by Theorem 4.8(b). Since $\mathrm{Gr}^\dagger(\langle\langle \mathcal{U} \rangle\rangle \cap \langle\langle \mathcal{C} \rangle\rangle, u)^\circ$ is a quotient of the commutative group $\mathrm{Gr}^\dagger(\mathcal{C}, u)^\circ$, it is commutative. Since $\mathrm{Gr}^\dagger(\langle\langle \mathcal{U} \rangle\rangle \cap \langle\langle \mathcal{C} \rangle\rangle, u)^\circ$ is also a quotient of $\mathrm{Gr}^\dagger(\mathcal{U}, u)^\circ$, which has no commutative quotients by [Bor91, IV.14.2 Proposition], we conclude that $\mathrm{Gr}^\dagger(\langle\langle \mathcal{U} \rangle\rangle \cap \langle\langle \mathcal{C} \rangle\rangle, u)^\circ$ is trivial. So $\mathrm{Gr}^\dagger(\mathcal{F}, u)^\circ$ is actually the direct product of $\mathrm{Gr}^\dagger(\mathcal{U}, u)^\circ$ and $\mathrm{Gr}^\dagger(\mathcal{C}, u)^\circ$.

We now consider the base change of these groups to $\mathbb{C}$ via ι

$$G_1 := \mathrm{Gr}^\dagger(\mathcal{F}, u) \times_{K_e, \iota} \mathbb{C}, \qquad G_2 := \mathrm{Gr}^\dagger(\mathcal{U}, u) \times_{K_e, \iota} \mathbb{C}, \qquad G_3 := \mathrm{Gr}^\dagger(\mathcal{C}, u) \times_{K_e, \iota} \mathbb{C}.$$

Let $T_1 \subset G_1$ be a maximal quasi-torus. It is the fiber product of two maximal quasi-tori $T_2 \subset G_2$ and $T_3 \subset G_3$ with $T_1^\circ = T_2^\circ \times_{\mathbb{C}} T_3^\circ$ by Remark 8.12. Note that actually $T_3 = G_3$, because G_3° is a torus. Let $\mathbb{T}_1$ be a maximal compact quasi-torus in T_1, see Definition 12.10 and let $\mathbb{T}_j$ for $j = 2, 3$ be the image of $\mathbb{T}_1$ under the projections $T_1 \twoheadrightarrow T_j$. Then $\mathbb{T}_j$ is a maximal compact quasi-torus in T_j and G_j by Corollary 12.15.

Let $S \subset |U|$ be a subset of positive upper Dirichlet density $\overline{\delta}(S) > 0$. For every connected component $h_1 G_1^\circ$ of G_1 we consider the subset $S(h_1)$ of those $x \in S$ for which $h_1 G_1^\circ$ contains a point of $\mathrm{Frob}_x^\dagger(\mathcal{F})$. Then S is the finite union of the subsets $S(h_1)$. By Lemma 3.12 at least one of them has positive upper Dirichlet density. We replace S by this subset and then consider the connected component $h_1 G_1^\circ$ of G_1 which meets $\mathrm{Frob}_x^\dagger(\mathcal{F})$ for every $x \in S$. Since $T_1 \subset G_1$ is a maximal quasi-torus, we may assume that $h_1 \in \mathbb{T}_1$ by Theorem 8.9 and Proposition 12.4(a). For $j = 2, 3$ let $h_j \in \mathbb{T}_j$ be the image of h_1 under the projection $T_1 \twoheadrightarrow T_j$. Then $T_3^{h_3} := \{t_3 \in T_3^\circ \colon t_3 h_3 = h_3 t_3\} = T_3^\circ$ using Notation 9.1, because $T_3 = G_3$ is commutative. Therefore, $T_1^{h_1} := \{t_1 = (t_2, t_3) \in T_1^\circ = T_2^\circ \times_{\mathbb{C}} T_3^\circ \colon t_1 h_1 = h_1 t_1\} = T_2^{h_2} \times_{\mathbb{C}} T_3^\circ$ and $h_1 T_1^{h_1 \circ} = h_2 T_2^{h_2 \circ} \times_{\mathbb{C}} h_3 T_3^\circ$. Moreover, $\mathbb{T}_j^{h_j} = T_j^{h_j} \cap \mathbb{T}_j^\circ$ is a maximal compact subgroup in $T_j^{h_j}$ for $j = 1, 2, 3$ by Lemma 12.12.

We consider the semi-simplification $\mathrm{Frob}_x^\dagger(\mathcal{F})^{ss}$ of the Frobenius conjugacy class $\mathrm{Frob}_x^\dagger(\mathcal{F})$ and similarly for the F-isocrystal $\mathcal{U}$. In order to show that $h_1 G_1^\circ \cap \bigcup_{x \in S} \mathrm{Frob}_x^\dagger(\mathcal{F})$ is Zariski-dense in $h_1 G_1^\circ$, it is by Lemma 9.9(b) and Proposition 9.7 enough to show that $h_1 T_1^{h_1 \circ} \cap \bigcup_{x \in S} \mathrm{Frob}_x^\dagger(\mathcal{F})^{ss}$ is Zariski-dense in $h_1 T_1^{h_1 \circ}$. Also note that $h_1 T_1^{h_1 \circ} \cap \mathrm{Frob}_x^\dagger(\mathcal{F})^{ss} \neq \emptyset$ for every $x \in S$ by our choice of h_1 and by Theorem 8.9 and Proposition 9.2(b).

As in Notation 13.12 we now consider maximal compact subgroups $\widetilde{\mathbb{K}}_2^{\mathrm{arith}}$ of G_2 and $\widetilde{\mathbb{K}}_2^{\mathrm{geom}}$ of $G_2^{\mathrm{geom}} := \mathrm{DGal}^\dagger(\mathcal{U}, u) \times_{K_e, \iota} \mathbb{C}$ such that $\widetilde{\mathbb{K}}_2^{\mathrm{arith}}$ contains $\widetilde{\mathbb{K}}_2^{\mathrm{geom}}$. By Theorem 12.1(c) and Proposition 12.4(c) there is an element $e \in G_2^{\mathrm{arith}}(\mathbb{C})^\circ$ such that $\mathbb{K}_2^{\mathrm{arith}} := e\, \widetilde{\mathbb{K}}_2^{\mathrm{arith}} e^{-1}$ contains $\mathbb{T}_2$ and is a maximal compact subgroup of G_2. Since G_2^{geom} is normal in G_2 conjugation by e is an automorphism of G_2^{geom}, and hence $\mathbb{K}_2^{\mathrm{geom}} := e\, \widetilde{\mathbb{K}}_2^{\mathrm{geom}} e^{-1}$ is a maximal compact subgroup of G_2^{geom} contained in $\mathbb{K}_2^{\mathrm{arith}}$ by Proposition 12.4(c). To lighten the notation we drop the superscript "arith" and just write $\mathbb{K}_2 := \mathbb{K}_2^{\mathrm{arith}}$. We let $\gamma \in \Gamma$ be the image of the element $h_2 \in \mathbb{T}_2$, and we denote by $\mathbb{K}_{2,\gamma}$ the preimage of γ in $\mathbb{K}_2$. It is a union of connected components containing $h_2 \mathbb{K}_2^\circ$, where we write $\mathbb{K}_2^\circ$ for the connected component of $\mathbb{K}_2$. Note that by Lemma 12.8 two elements of $\mathbb{K}_2$ which are conjugate under $G_2(\mathbb{C})$ are already conjugate under $\mathbb{K}_2$. We denote the set of conjugacy classes of $\mathbb{K}_2$ by $\mathbb{K}_2^\#$, the ones which meet $\mathbb{K}_{2,\gamma}$ by $\mathbb{K}_{2,\gamma}^\#$, and the ones which meet $h_2 \mathbb{K}_2^\circ$ by $(h_2 \mathbb{K}_2^\circ)^\#$. We equip these sets with the quotient topology. Then $(h_2 \mathbb{K}_2^\circ)^\#$ is a connected component of $\mathbb{K}_{2,\gamma}^\#$. We consider the following diagram

(13.2)
$$\begin{tikzcd} & h_2 \mathbb{K}_2^\circ \arrow[r, hook] \arrow[d, two heads, "\psi"] & \mathbb{K}_{2,\gamma} \arrow[d, two heads, "\psi"] \\ h_2 \mathbb{T}_2^{h_2 \circ} \arrow[r, two heads, "\varphi"] & (h_2 \mathbb{K}_2^\circ)^\# \arrow[r, hook] & \mathbb{K}_{2,\gamma}^\# . \end{tikzcd}$$

In this diagram the map ψ is surjective by construction, and φ is surjective by Propositions 12.11(d) and 12.13(a). In particular, it follows from this and Definition 13.17 that $\theta(x) \in (h_2\mathbb{K}_2^\circ)^\#$, and hence

$$h_2\mathbb{T}_2^{h_2\circ} \cap \mathrm{Frob}_x^\dagger(\mathcal{U})^{ss} \;=\; \varphi^{-1}(\theta(x)) \;\neq\; \emptyset \qquad \text{for every } x \in S\,. \tag{13.3}$$

Lemma 13.22. *(a) The continuous map φ between compact Hausdorff spaces is nice in the sense of Definition B.3.*

(b) There is a positive integer M such that all fibers of φ have cardinality at most M.

(c) For every semi-algebraic subset $X \subset (h_2\mathbb{T}_2^{h_2\circ})$ we have $\dim X = \dim \varphi(X)$.

(d) Every closed semi-algebraic subset $Y \subset (h_2\mathbb{K}_2^\circ)^\#$ with $\dim Y < \dim(h_2\mathbb{K}_2^\circ)^\#$ has volume $\mu^\#_{Haar,\gamma}(Y) = 0$.

(e) There is a closed semi-algebraic subset $Z \subset (h_2\mathbb{K}_2^\circ)^\#$ with $\dim Z < \dim(h_2\mathbb{K}_2^\circ)^\#$ whose complement is a finite disjoint union $(h_2\mathbb{K}_2^\circ)^\# \smallsetminus Z = \coprod_{i=1}^n Y_i$ of open subsets Y_i such that φ is trivial over Y_i in the sense that there is a finite discrete set F_i and a semi-algebraic isomorphism $\varphi^{-1}(Y_i) \xrightarrow{\sim} F_i \times Y_i$ compatible with the projections onto Y_i.

Proof. (b) follows from Proposition 12.13(b).

(a) By Lemma 13.14 the quotient $(h_2\mathbb{K}_2^\circ)^\#$ is compact. By Remark 13.16 the quotient map $h_2\mathbb{K}_2^\circ \to (h_2\mathbb{K}_2^\circ)^\#$ is continuous and semi-algebraic as the restriction of $\mathbb{K}_2 \to \mathbb{K}_2^\#$ to the connected component $h_2\mathbb{K}_2^\circ$, and $(h_2\mathbb{K}_2^\circ)^\#$ is an affine semi-algebraic space. So also φ is continuous and semi-algebraic as the composition of the inclusion $h_2\mathbb{T}_2^{h_2\circ} \to h_2\mathbb{K}_2^\circ$ and the quotient map $h_2\mathbb{K}_2^\circ \to (h_2\mathbb{K}_2^\circ)^\#$. Moreover, $h_2\mathbb{T}_2^{h_2\circ}$ is semi-algebraic and compact, hence complete by [DK81, Theorem 9.4]. Then [DK85, Chapter I, Remark 5.5(v) and § 6, Definition 4] implies that φ is a finite semi-algebraic map. Therefore, [DK85, Chapter II, Theorem 6.13] implies that there are semi-algebraic triangulations τ_1, τ_2 of $h_2\mathbb{T}_2^{h_2\circ}$ and $(h_2\mathbb{K}_2^\circ)^\#$, respectively, such that the restriction of φ onto any simplex of τ_1 is a simplicial map to a simplex of τ_2, up to continuous semi-algebraic isomorphism. Since every finite to one simplicial map between simplices is trivially injective, we get that each such restriction is injective. The claim now follows from Remark B.4.

(e) By Hardt's Local-Triviality-Theorem [DK82, Theorem 6.4] for the semi-algebraic map $h_2\mathbb{T}_2^{h_2\circ} \to (h_2\mathbb{K}_2^\circ)^\#$ there is a decomposition $(h_2\mathbb{K}_2^\circ)^\# = \coprod_{i=1}^{\tilde n} \widetilde Y_i$ into finitely many semi-algebraic subsets $Y_i \subset (h_2\mathbb{K}_2^\circ)^\#$, such that φ is trivial over $\widetilde Y_i$ in the above sense. By [DK81, Proposition 8.2(b)] there is an $n \le \tilde n$ with $\dim(h_2\mathbb{K}_2^\circ)^\# = \dim \widetilde Y_1 = \ldots = \dim \widetilde Y_n > \dim \widetilde Y_i$ for all $i > n$. For $i \le n$ let Y_i be the open interior of $\widetilde Y_i$, which is a semi-algebraic subset of $(h_2\mathbb{K}_2^\circ)^\#$. Let further $Z := \bigcup_{i\le n}(\widetilde Y_i \smallsetminus Y_i) \cup \bigcup_{i>n} \widetilde Y_i$. Then $\dim Z < \dim(h_2\mathbb{K}_2^\circ)^\# = \dim Y_i$ for every $i \le n$ by [DK81, Theorem 8.10]. Indeed, if $\dim(\widetilde Y_i \smallsetminus Y_i)$ was equal to $\dim(h_2\mathbb{K}_2^\circ)^\#$ then $\widetilde Y_i \smallsetminus Y_i$ would contain a non-empty open subset by loc. cit. in contradiction to Y_i being the largest open subset of $\widetilde Y_i$. This proves (e).

(c) The inequality $\dim X \ge \dim \varphi(X)$ follows from [DK81, Proposition 8.3]. We next consider the decomposition $(h_2\mathbb{K}_2^\circ)^\# = \coprod_{i=1}^{\tilde n} \widetilde Y_i$ from the previous paragraph and the semi-algebraic subsets $\varphi(X) \cap \widetilde Y_i$ of $(h_2\mathbb{K}_2^\circ)^\#$. Then $X \cap \varphi^{-1}(\widetilde Y_i) \subset \varphi^{-1}(\varphi(X) \cap \widetilde Y_i) \cong F_i \times (\varphi(X) \cap \widetilde Y_i)$. Therefore,

$$\dim X \;=\; \max\{\dim X \cap \varphi^{-1}(\widetilde Y_i) \colon 1 \le i \le \tilde n\} \;\le\; \max\{\dim \varphi(X) \cap \widetilde Y_i \colon 1 \le i \le \tilde n\} \;=\; \dim \varphi(X)$$

by [DK81, Proposition 8.2(b)].

(d) By (c) the semi-algebraic set $\varphi^{-1}(Y)$ satisfies $\dim \varphi^{-1}(Y) < \dim h_2\mathbb{T}_2^{h_2\circ}$. By definition this means that the Zariski closure $\overline{\varphi^{-1}(Y)}$ in $h_2T_2^{h_2\circ}$ is strictly contained in $h_2T_2^{h_2\circ}$. Let ${}^{G_2}\varphi^{-1}(Y)$ be the union of the G_2-conjugacy classes of the elements of $\varphi^{-1}(Y) \subset h_2G_2^\circ$. The preimage $\psi^{-1}(Y)$ equals the intersection of ${}^{G_2}\varphi^{-1}(Y)$ with $h_2\mathbb{K}_2^\circ$. If $\psi^{-1}(Y)$ was Zariski-dense in $h_2G_2^\circ$, then ${}^{G_2}\varphi^{-1}(Y)$ would be Zariski-dense in $h_2G_2^\circ$, too. This would imply by Lemma 9.9(b) and Proposition 9.7 that ${}^{G_2}\varphi^{-1}(Y) \cap h_2T_2^{h_2\circ}$ is Zariski-dense in $h_2T_2^{h_2\circ}$. In the notation of Proposition 9.3 we have ${}^{G_2}\varphi^{-1}(Y) \cap h_2T_2^{h_2\circ} \subset \bigcup_{w\in W, z\in Z} zw\varphi^{-1}(Y)w^{-1} \subset h_2T_2^{h_2\circ}$. This is a finite union. Since $h_2T_2^{h_2\circ}$ is irreducible, already one component $zw\varphi^{-1}(Y)w^{-1}$ must be Zariski-dense in $h_2T_2^{h_2\circ}$ for certain w and z. But then $\varphi^{-1}(Y)$ would be Zariski-dense in $w^{-1}z^{-1}(h_2T_2^{h_2\circ})w = h_2T_2^{h_2\circ}$ which yields a contradiction. Therefore, $\overline{\psi^{-1}(Y)}$ must be contained in a proper hyperplane $H \subsetneq h_2G_2^\circ$ and then $\psi^{-1}(Y) \subset H \cap h_2\mathbb{K}_2^\circ \subsetneq h_2\mathbb{K}_2^\circ$, the latter being a strict inclusion by Proposition 12.5. Since H is defined by a polynomial equation in the coordinates of $h_2G_2^\circ$ and $\mu_{\mathrm{Haar},\gamma}$ is absolutely continuous with respect to the Lebesgue measure on the charts of the differentiable manifold $h_2\mathbb{K}_2^\circ$, we conclude $0=\mu_{\mathrm{Haar},\gamma}\big(\psi^{-1}(Y)\big)=\mu^{\#}_{\mathrm{Haar},\gamma}(Y)$. □

We continue with the proof of Theorem 13.2. On the sets in Diagram (13.2) we consider various measures: On $\mathbb{K}_{2,\gamma}$ and $h_2\mathbb{K}_2^\circ$ the (restriction of the) Haar measure $\mu_{\mathrm{Haar},\gamma}$ from Definition 13.13 and on $\mathbb{K}_{2,\gamma}^{\#}$ and $(h_2\mathbb{K}_2^\circ)^{\#}$ its push-forward $\mu^{\#}_{\mathrm{Haar},\gamma}$. Moreover, on $\mathbb{K}_{2,\gamma}^{\#}$ and $(h_2\mathbb{K}_2^\circ)^{\#}$ we consider the (restriction of the) measures μ_m from Definition 13.18. By the equidistribution Theorem 13.19, when $m\to\infty$ the measures μ_m, for which $[m]=\gamma\in\Gamma$ in (13.1), converge weakly to $\mu^{\#}_{\mathrm{Haar},\gamma}$ on $\mathbb{K}_{2,\gamma}^{\#}$ and on $(h_2\mathbb{K}_2^\circ)^{\#}$. Since φ is nice by the previous lemma, we can pull back measures along φ, see Definition B.7. By Lemma 13.22 and Proposition B.8 the pullback measures $\varphi^*\mu_m$, for which the class $[m]$ of m in Γ equals γ, converge weakly to the measure $\lambda:=\varphi^*\mu^{\#}_{\mathrm{Haar},\gamma}$ on $h_2\mathbb{T}_2^{h_2\circ}$ when $m\to\infty$.

Lemma 13.23. *The pull-back measure $\lambda:=\varphi^*\mu^{\#}_{Haar,\gamma}$ satisfies $\lambda(h_2\mathbb{T}_2^{h_2\circ})<\infty$ and $\lambda(H\cap h_2\mathbb{T}_2^{h_2\circ})=0$ for every proper hypersurface $H\subsetneq h_2T_2^{h_2\circ}$.*

Proof. By Lemma 13.22(b) the cardinality of every fiber $\varphi^{-1}(y)$ is at most M. Therefore,

$$\lambda(h_2\mathbb{T}_2^{h_2\circ})\ :=\ \int_{(h_2\mathbb{K}_2^\circ)^{\#}}\#\varphi^{-1}(y)\,d\mu^{\#}_{\mathrm{Haar},\gamma}(y)\ \le\ M\cdot\mu^{\#}_{\mathrm{Haar},\gamma}\big((h_2\mathbb{K}_2^\circ)^{\#}\big)\ :=\ M\cdot\mu_{\mathrm{Haar},\gamma}(h_2\mathbb{K}_2^\circ)\ \le\ M$$

by Definition 13.13 of the measure $\mu_{\mathrm{Haar},\gamma}$.

Next let H be as in the second statement. By [DK81, § 8, Definitions 1 and 2] the dimension of $H\cap h_2\mathbb{T}_2^{h_2\circ}$ is the dimension of its Zariski closure in $h_2T_2^{h_2\circ}$. Therefore, $\dim(H\cap h_2\mathbb{T}_2^{h_2\circ})\le\dim H<\dim h_2T_2^{h_2\circ}=\dim h_2\mathbb{T}_2^{h_2\circ}=\dim(h_2\mathbb{K}_2^\circ)^{\#}$, because $h_2T_2^{h_2\circ}$ is irreducible. Here the second-to-last equality follows from [DK81, Proposition 8.6] by considering a real structure on $T_2^{h_2\circ}$ with $\mathbb{T}_2^{h_2\circ}=T_2^{h_2\circ}(\mathbb{R})$ as in Proposition 12.5. The last equality follows from Lemma 13.22(c). By [DK81, Proposition 8.3] we have

$$\dim\varphi(H\cap h_2\mathbb{T}_2^{h_2\circ})\ \le\ \dim(H\cap h_2\mathbb{T}_2^{h_2\circ})\ <\ \dim(h_2\mathbb{K}_2^\circ)^{\#}\,,$$

and hence Lemma 13.22(d) implies $\mu^{\#}_{\mathrm{Haar},\gamma}\big(\varphi(H\cap h_2\mathbb{T}_2^{h_2\circ})\big)=0$. Now for a point $y\in(h_2\mathbb{K}_2^\circ)^{\#}$ the cardinality of $\varphi^{-1}(y)\cap H\cap h_2\mathbb{T}_2^{h_2\circ}$ is zero if $y\notin\varphi(H\cap h_2\mathbb{T}_2^{h_2\circ})$ and otherwise at most M by Lemma 13.22(b). Thus we compute

$$\lambda(H\cap h_2\mathbb{T}_2^{h_2\circ})\ :=\ \int_{(h_2\mathbb{K}_2^\circ)^{\#}}\#\big(\varphi^{-1}(y)\cap H\cap h_2\mathbb{T}_2^{h_2\circ}\big)\,d\mu^{\#}_{\mathrm{Haar},\gamma}(y)\ \le\ M\cdot\mu^{\#}_{\mathrm{Haar},\gamma}\big(\varphi(H\cap h_2\mathbb{T}_2^{h_2\circ})\big)\ =\ 0$$

as desired. □

The pullback measure $\varphi^*\mu_m$ on $h_2\mathbb{T}_2^{h_2\circ}$ has the following description. Recall, that for every $m\in\mathbb{N}$ we set $U_m:=\{\,x\in U\colon\ \deg x=m\}$. Let $A\subset h_2\mathbb{T}_2^{h_2\circ}$ be a Borel-subset. Then by Definition B.7

$$\varphi^*\mu_m(A)\ :=\ \frac{1}{\#U_m}\sum_{x\in U_m}(\varphi^*\delta_{\theta(x)})(A)\ =\ \frac{1}{\#U_m}\sum_{x\in U_m}\#(A\cap\mathrm{Frob}_x(\mathcal{U})^{ss})\,.$$

Now let $S_m:=S\cap U_m$. Then Lemma 3.17 gives us an infinite subset $R\subset\mathbb{N}$ such that

$$\frac{\#S_m}{\#U_m}\ \ge\ \frac{\overline{\delta}(S)}{2}\quad\text{for every } m\in R\,.$$

In particular, for every $m\in R$ we have $S_m\neq\emptyset$, and $h_2\mathbb{T}_2^{h_2\circ}\cap\mathrm{Frob}_x^\dagger(\mathcal{U})^{ss}\neq\emptyset$ for every $x\in S_m$ by (13.3). Thus the image $[m]$ of m in $\Gamma=\mathbf{W}(\mathcal{U},u)(\mathbb{C})$ under the map (13.1) coincides with the image of h_2 which we called γ.

Consider closed immersions $h_2T_2^{h_2\circ}\hookrightarrow\mathbb{C}^{d_2}$ and $h_3T_3^\circ\hookrightarrow\mathbb{C}^{d_3}$. It will be sufficient to prove that no proper hyper-surface $\mathbf{H}\subset\mathbb{C}^{d_2+d_3}$ with $\mathbf{H}\cap h_1T_1^{h_1\circ}\subsetneq h_1T_1^{h_1\circ}=h_2T_2^{h_2\circ}\times_{\mathbb{C}}h_3T_3^\circ\subset\mathbb{C}^{d_2}\times\mathbb{C}^{d_3}$ contains $h_1T_1^{h_1\circ}\cap\bigcup_{x\in S}\mathrm{Frob}_x^\dagger(\mathcal{F})^{ss}$. Assume the contrary and let $\mathbf{H}$ be such a counterexample. Let D be the degree of $\mathbf{H}$ in the variables of the first factor $\mathbb{C}^{d_2}$ and for every $m\in R$ set

$$H_m=\{z\in\mathbb{C}^{d_2}:(z,f^m)\in\mathbf{H}\}\,.$$

Note that $f^m=\mathrm{Frob}_x^\dagger(\mathcal{C})=\mathrm{Frob}_x^\dagger(\mathcal{C})^{ss}\in h_3T_3^\circ$ for every $x\in S_m$. Then $\mathrm{Frob}_x^\dagger(\mathcal{F})^{ss}=\mathrm{Frob}_x^\dagger(\mathcal{U})^{ss}\times\{f^{\deg x}\}$ implies that $h_2T_2^{h_2\circ}\cap\mathrm{Frob}_x^\dagger(\mathcal{U})^{ss}\subset H_m$ for every $x\in S_m$. Each H_m is a hyper-surface in $\mathbb{C}^{d_2}$ of degree $\le D$ such that $H_m\cap h_2T_2^{h_2\circ}$ is properly contained in $h_2T_2^{h_2\circ}$ for all but finitely many m by

Corollary 7.6 (applied with $L=\mathbb{C}, T=T_3, G=f^{\mathbb{Z}}, X=G_1, T^c=h_3T_3^\circ$). So by shrinking R we may assume that $H_m\cap h_2T_2^{h_2\circ}$ is properly contained in $h_2T_2^{h_2\circ}$ for every $m\in R$. By considering a real structure on $T_2^{h_2}$ with $h_2\mathbb{T}_2^{h_2\circ}=h_2T_2^{h_2\circ}(\mathbb{R})$ as in Proposition 12.5, using Theorem B.10, and shrinking R further we may even assume that there is a proper hyper-surface $H\subsetneq h_2T_2^{h_2\circ}\subset\mathbb{C}^{d_2}$ of degree at most D such that the sequence $H_1\cap h_2\mathbb{T}_2^{h_2\circ}, H_2\cap h_2\mathbb{T}_2^{h_2\circ},\ldots,H_m\cap h_2\mathbb{T}_2^{h_2\circ},\ldots$ converges to $H\cap h_2\mathbb{T}_2^{h_2\circ}$ in the sense of Definition B.9. Then $H\cap h_2\mathbb{T}_2^{h_2\circ}\subsetneq h_2\mathbb{T}_2^{h_2\circ}$ by Proposition 12.5 and $\lambda(H\cap h_2\mathbb{T}_2^{h_2\circ})=0$ by Lemma 13.23.

By Lemma B.12 there is a small $\varepsilon>0$ such that $\lambda\big((H\cap h_2\mathbb{T}_2^{h_2\circ})(2\varepsilon)\big)<\frac{1}{2}\,\overline{\delta}(S)$. By the triangle inequality $(H\cap h_2\mathbb{T}_2^{h_2\circ})(2\varepsilon)$ contains the closure $\overline{(H\cap h_2\mathbb{T}_2^{h_2\circ})(\varepsilon)}$ of $(H\cap h_2\mathbb{T}_2^{h_2\circ})(\varepsilon)$ in the metric space $h_2\mathbb{T}_2^{h_2\circ}$, and hence $\lambda\big(\overline{(H\cap h_2\mathbb{T}_2^{h_2\circ})(\varepsilon)}\big)<\frac{1}{2}\,\overline{\delta}(S)$. Choose an $m_\varepsilon\in\mathbb{N}$ such that for every index $m\in R$ with $m\geq m_\varepsilon$ we have $H_m\cap h_2\mathbb{T}_2^{h_2\circ}\subseteq(H\cap h_2\mathbb{T}_2^{h_2\circ})(\varepsilon)$. For every $x\in S_m$ the intersection $h_2\mathbb{T}_2^{h_2\circ}\cap\mathrm{Frob}_x^\dagger(\mathcal{U})^{ss}$ is contained in $H_m\cap h_2\mathbb{T}_2^{h_2\circ}$ by assumption. Moreover, $h_2\mathbb{T}_2^{h_2\circ}\cap\mathrm{Frob}_x^\dagger(\mathcal{U})^{ss}$ is non-empty by (13.3). So for every $m\in R$ with $m\geq m_\varepsilon$ and for every $x\in S_m$ we have $\#\big(\overline{(H\cap h_2\mathbb{T}_2^{h_2\circ})(\varepsilon)}\cap\mathrm{Frob}_x(\mathcal{U})^{ss}\big)\geq 1$, and hence

$$\varphi^*\mu_m\big(\overline{(H\cap h_2\mathbb{T}_2^{h_2\circ})(\varepsilon)}\big)\;=\;\frac{1}{\#U_m}\sum_{x\in U_m}\#\big(\overline{(H\cap h_2\mathbb{T}_2^{h_2\circ})(\varepsilon)}\cap\mathrm{Frob}_x(\mathcal{U})^{ss}\big)\;\geq\;\frac{\#S_m}{\#U_m}\;\geq\;\frac{\overline{\delta}(S)}{2}\,.$$

But taking $\limsup_{m\to\infty}$ and by the weak convergence of $\varphi^*\mu_m$ to λ and the Portemanteau theorem [Kle14, Theorem 13.16] we have

$$\frac{\overline{\delta}(S)}{2}\;\leq\;\limsup_{m\to\infty}\varphi^*\mu_m\big(\overline{(H\cap h_2\mathbb{T}_2^{h_2\circ})(\varepsilon)}\big)\;\leq\;\lambda\big(\overline{(H\cap h_2\mathbb{T}_2^{h_2\circ})(\varepsilon)}\big)\;<\;\frac{\overline{\delta}(S)}{2}\,,$$

which is a contradiction. This rules out the existence of $\mathbf{H}$ and finishes the proof of Theorem 13.2. □

Theorem 13.2 has the following consequence for convergent F-isocrystals.

Theorem 13.24. *Let $\mathcal{F}$ be a semi-simple convergent F-isocrystal on U. Assume that $\mathcal{F}$ has an overconvergent extension whose monodromy group $\mathrm{Gr}^\dagger(\mathcal{F}/U,e)$ has an abelian maximal quasi-torus. Then Conjectures 1.2, 1.3 and 1.4 hold true for $\mathcal{F}$.*

Remark 13.25. Of course by the assumption in the theorem we mean that there is an overconvergent isocrystal $\mathcal{F}^\dagger$ such that the convergent isocrystal underlying $\mathcal{F}^\dagger$ is isomorphic to $\mathcal{F}$. Note that $\mathcal{F}^\dagger$ is necessarily semi-simple. Indeed if $\mathcal{G}^\dagger\subset\mathcal{F}^\dagger$ is an overconvergent sub-isocrystal, then it has a convergent complement $\mathcal{H}\subset\mathcal{F}$. This $\mathcal{H}$ is isomorphic to the convergent isocrystal underlying the quotient $\mathcal{F}^\dagger/\mathcal{G}^\dagger$, so by Kedlaya's extension theorem [Ked04, Theorem 1.1] the embedding $\mathcal{H}\hookrightarrow\mathcal{F}$ extends to an embedding $\mathcal{F}^\dagger/\mathcal{G}^\dagger\hookrightarrow\mathcal{F}^\dagger$, and hence $\mathcal{G}^\dagger$ has an overconvergent complement, too.

Proof of Theorem 13.24. Let $\mathrm{Gr}^\dagger(\mathcal{F}^\dagger,u)$ be the monodromy group of $\mathcal{F}^\dagger$ and let $\mathrm{Gr}(\mathcal{F},u)\subset\mathrm{Gr}^\dagger(\mathcal{F}^\dagger,u)$ be the monodromy group of $\mathcal{F}$, see Lemma 13.1. We view $G^\dagger:=\mathrm{Gr}^\dagger(\mathcal{F}^\dagger,u)(\overline{K})$ as an algebraic group over $\overline{K}$. Let $S\subset|U|$ be a subset of positive upper Dirichlet density. Let $F=\bigcup_{x\in S}\mathrm{Frob}_x(\mathcal{F})\subset\mathrm{Gr}(\mathcal{F},u)(\overline{K})$ be the union of the Frobenius conjugacy classes, and let $F^{ss}=\{g_s\colon g\in F\}$ be the set consisting of the semi-simple parts g_s of the elements g of F. Using the sub-additivity of upper Dirichlet density from Lemma 3.12, we may assume without the loss of generality that every element of F and hence F^{ss} lies in the same conjugacy class $\mathbf{C}$ of connected components of $\mathrm{Gr}(\mathcal{F},u)$ by shrinking S, if it is necessary. For every subset X of an algebraic group H let ${}^HX$ be the union of the conjugacy classes of elements of X. By our overconvergent density Theorem 13.2 and by Corollary 9.11 the set ${}^{G^\dagger}F^{ss}$ is Zariski-dense in a connected component $hG^{\dagger\circ}$ of $G^\dagger$. We want to deduce that F^{ss} is Zariski-dense in a connected component of $\mathrm{Gr}(\mathcal{F},u)$. By Corollary 9.11 this is enough to prove Conjecture 1.4 for $\mathcal{F}$. Choose any connected component $C\subset\mathrm{Gr}(\mathcal{F},u)$ lying in $\mathbf{C}$ and in $hG^{\dagger\circ}$. We will actually show that $F^{ss}\cap C$ is Zariski-dense in C.

Let T be a maximal quasi-torus in $\mathrm{Gr}(\mathcal{F},u)$ and let T' be the unique connected component of T contained in C. Let Z be the Zariski-closure of $F^{ss}\cap T'$. Since every element of F^{ss} lies in $\mathbf{C}$, it can be conjugate into C. By Theorem 8.9 it is thus conjugate under $\mathrm{Gr}(\mathcal{F},u)$ to an element of Z. Therefore, ${}^{G^\dagger}Z$ contains ${}^{G^\dagger}F^{ss}$, and hence ${}^{G^\dagger}Z$ is Zariski-dense in $hG^{\dagger\circ}$, too. Let $T^\dagger\subset G^\dagger$ be a maximal quasi-torus, whose connected component $T^{\dagger\circ}$ contains the maximal torus T° of $\mathrm{Gr}(\mathcal{F},u)^\circ$, use Theorem 8.10(a). By

Theorem 8.9(a) we may assume that $h \in T^\dagger$. Consider the set

$$D = \{(z,t) \in Z \times hT^{\dagger\circ} \colon \exists\, g \in G^\dagger \text{ such that } g^{-1}zg = t)\,\}.$$

We claim that D is a constructible subset of $Z \times hT^{\dagger\circ}$. Namely consider the morphism

$$\varphi\colon\; Z \times G^\dagger \;\longrightarrow\; Z \times G^\dagger, \quad (z,g) \;\longmapsto\; (z, g^{-1}zg)\,.$$

The preimage $\varphi^{-1}(Z \times hT^{\dagger\circ}) \subset Z \times G^\dagger$ is a closed subset, and $D = \varphi\big(\varphi^{-1}(Z \times hT^{\dagger\circ})\big)$. Therefore, D is a constructible set by Chevalley's theorem [EGA, IV_1, Corollaire 1.8.5]. Let $\pi_1\colon D \to Z$ and $\pi_2\colon D \to hT^{\dagger\circ}$ be the projections onto the first and the second factor, respectively. Since ${}^{G^\dagger}Z$ is Zariski-dense in $hG^{\dagger\circ}$ and is the union of semi-simple conjugacy classes, Lemma 9.9(b) tells us that $\pi_2(D) = {}^{G^\dagger}Z \cap hT^{\dagger\circ}$ is Zariski-dense in $hT^{\dagger\circ}$. This implies $\dim(D) \geq \dim(hT^{\dagger\circ}) = \dim(T^{\dagger\circ})$ by [EGA, IV_2, Théorème 4.1.2].

Fix an element $(z,t) \in D$. For every other point (z,t') in D with $\pi_1(z,t) = \pi_1(z,t')$ the two elements t and t' of $hT^{\dagger\circ}$ are conjugate under $G^\dagger$. Since $T^\dagger$ is assumed to be commutative, we have $(T^\dagger)^h := \{g \in T^{\dagger\circ} \colon gh = hg\} = T^{\dagger\circ}$. So Proposition 9.3 implies that there are only finitely many $t' \in hT^{\dagger\circ}$ with $\pi_1(z,t) = \pi_1(z,t')$. In other words, the fibers of the surjective map $\pi_1\colon D \to \pi_1(D)$ are finite, and so $\dim(D) = \dim(\pi_1(D)) \leq \dim(Z)$ by [GW10, Proposition 14.107]. On the other hand $\dim(T') = \dim(T^\circ) \leq \dim(T^{\dagger\circ}) \leq \dim(D)$, because T° is contained in $T^{\dagger\circ}$. Since $Z \subset T'$ with $\dim(Z) \geq \dim(D) \geq \dim(T')$, and T' is irreducible, we get that Z is T', so $F^{ss} \cap T'$ is Zariski-dense in T'. By Lemma 9.9(b) we get that $F^{ss} \cap C$ is Zariski-dense in C as desired. □

The proof of the following theorem will use most of the results which we prove in our paper up to this point. This theorem will imply Theorem 1.10 in the introduction, as we shall see shortly.

Theorem 13.26. *Let $\mathcal{F}$ be a semi-simple convergent F-isocrystal on U which has an overconvergent extension $\mathcal{F}^\dagger$ on U, and such that $\mathcal{F}$ is locally weakly firm with respect to an open sub-curve $f\colon V \hookrightarrow U$. Then $\mathcal{F}$ is weakly pink with respect to f.*

Proof. By Theorem 11.6 it is enough to show that $\mathcal{F}$ is almost weakly pink with respect to f. Let $G^\dagger := \mathrm{Gr}^\dagger(\mathcal{F}^\dagger/U,u)(\overline{K})$ be the $\overline{K}$-valued points of the monodromy group of $\mathcal{F}^\dagger$ and let $\mathrm{Gr}(\mathcal{F}/U,u) \subset \mathrm{Gr}^\dagger(\mathcal{F}^\dagger/U,u)$ be the monodromy group of $\mathcal{F}$; see Lemma 13.1. Let $H := \mathrm{Gr}(f^*\mathcal{F}/V,u)(\overline{K}) \subset \mathrm{Gr}(\mathcal{F}/U,u)(\overline{K}) \subset G^\dagger$ be the $\overline{K}$-valued points of the monodromy group of $f^*\mathcal{F}$ on the shrunken curve; see Lemma 10.2. We view all three groups as linear algebraic groups over $\overline{K}$. Now the proof proceeds exactly as the proof of Theorem 11.5. Namely, let $F = \bigcup_{x\in|V|} \mathrm{Frob}_x(f^*\mathcal{F}) \subset H$ be the union of the Frobenius conjugacy classes (conjugacy under H), and let $F^{ss} = \{g_s \colon g \in F\} \subset H$ be the set of the semi-simple parts g_s of the elements g of F. For a subset X of $G^\dagger$ let ${}^{G^\dagger}X$ be the union of the conjugacy classes under $G^\dagger$ of the elements of X. Then ${}^{G^\dagger}F = \bigcup_{x\in|V|} \mathrm{Frob}_x^\dagger(\mathcal{F}^\dagger) \subset G^\dagger$ is the union of the Frobenius conjugacy classes (conjugacy under $G^\dagger$), and ${}^{G^\dagger}(F^{ss}) = ({}^{G^\dagger}F)^{ss} := \{g_s \colon g \in {}^{G^\dagger}F\} \subset G^\dagger$. By Remark 13.25 and Corollary 13.3 the set ${}^{G^\dagger}F$ is dense in $G^\dagger$. By Corollary 9.11 we get that $({}^{G^\dagger}F)^{ss}$ is Zariski-dense in $G^\dagger$, too. Let $T \subset H$ be a maximal quasi-torus. Since every element of F^{ss} is conjugate to an element of T by Theorem 8.9(a),(b), we get that ${}^{G^\dagger}T$ is also Zariski-dense in $G^\dagger$, and hence that ${}^{G^\dagger}T \cap G^{\dagger\circ}$ is Zariski-dense in $G^{\dagger\circ}$. As in the second (the purely group theoretic) part of the proof of Theorem 11.5 we see that T° is a maximal torus in $G^{\dagger\circ} = \mathrm{Gr}^\dagger(\mathcal{F}^\dagger/U,u)(\overline{K})^\circ$, and hence T° must be a maximal torus in the subgroup $\mathrm{Gr}(\mathcal{F}/U,u)(\overline{K})^\circ$, too. □

Proof of Theorem 1.10. Let $f\colon U \hookrightarrow V$ be an open sub-curve on which $f^*\mathcal{F}$ is weakly firm. Then $\mathcal{F}$ and $\mathcal{G}$ are weakly pink with respect to f by Theorem 13.26 and Proposition 10.10. Since $\mathcal{J}$ is conservative by Proposition 10.11 we see that $\mathcal{G} \oplus \mathcal{J}$ is weakly pink with respect to f by Proposition 10.7. Moreover, $\mathcal{G} \oplus \mathcal{J}$ is semi-simple by Lemma 3.8 because $\mathcal{F}$ and $\mathcal{J}$ are by assumption. Therefore, Conjectures 1.2, 1.3 and 1.4 hold true for $\mathcal{G} \oplus \mathcal{J}$ by Theorem 10.4. □

Appendix A. Non-Neutral Tannakian Categories and Representations of Groupoids

In this appendix we briefly recall the basics about Tannakian categories, groupoids and how they relate to monodromy groups of F-isocrystals. We closely follow the articles of Deligne and Milne [DM82, Del89, Del90, Mil92].

Definition A.1 ([Mil92, (A.7.1) and (A.7.2), page 222])**.** Let K be a field. A K-linear abelian tensor category $\mathscr{C}$ with unit object $\mathbb{1}$ such that $K = \mathrm{End}(\mathbb{1})$ is a *Tannakian category over* K if

(a) for every object X of $\mathscr{C}$ there exists an object $X^\vee$ of $\mathscr{C}$, called the *dual* of X, and morphisms $\mathrm{ev}\colon X\otimes X^\vee \to \mathbb{1}$ and $\delta\colon \mathbb{1}\to X^\vee\otimes X$ such that

$$(\mathrm{ev}\otimes \mathrm{id}_X)\circ(\mathrm{id}_X\otimes\delta) \;=\; \mathrm{id}_X \;:\; X \xrightarrow{\mathrm{id}_X\otimes\delta} X\otimes X^\vee\otimes X \xrightarrow{\mathrm{ev}\otimes\mathrm{id}_X} X \qquad \text{and}$$
$$(\mathrm{id}_{X^\vee}\otimes\mathrm{ev})\circ(\delta\otimes\mathrm{id}_{X^\vee}) \;=\; \mathrm{id}_{X^\vee}\colon\; X^\vee \xrightarrow{\delta\otimes\mathrm{id}_{X^\vee}} X^\vee\otimes X\otimes X^\vee \xrightarrow{\mathrm{id}_{X^\vee}\otimes\mathrm{ev}} X^\vee,$$

(b) and for some non-zero K-algebra L there is an exact faithful K-linear tensor functor ω from $\mathscr{C}$ to the category of finitely generated L-modules. Any such functor ω is called an *L-rational fiber functor for $\mathscr{C}$*.

A K-rational fiber functor for $\mathscr{C}$ is called *neutral*. If $\mathscr{C}$ has a neutral fiber functor it is called a *neutral Tannakian category over K*.

Remark A.2. (a) According to [DM82, § 1] being a *tensor category* means that there is a "tensor product" functor $\mathscr{C}\times\mathscr{C}\to\mathscr{C}$, $(X,Y)\mapsto X\otimes Y$ which is associative and commutative, such that $\mathscr{C}$ has a *unit object*. The latter is an object $\mathbb{1}\in\mathscr{C}$ together with an isomorphism $\mathbb{1}\xrightarrow{\sim}\mathbb{1}\otimes\mathbb{1}$ such that $\mathscr{C}\to\mathscr{C}$, $X\mapsto \mathbb{1}\otimes X$ is an equivalence of categories. A unit object is unique up to unique isomorphism; see [DM82, Proposition 1.3]. One sets $X^{\otimes 0} := \mathbb{1}$ and $X^{\otimes n} := X\otimes X^{\otimes n-1}$ for $n\in\mathbb{N}_{>0}$.

(b) Being *K-linear* means that $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ is a K-vector space for all $X,Y\in\mathscr{C}$.

(c) Being *abelian* means that $\mathscr{C}$ is an abelian category. Then automatically $\otimes$ is a bi-additive functor and is exact in each factor; see [DM82, Proposition 1.16].

(d) By [Del90, §§ 2.1–2.5] the conditions of Definition A.1 imply that $\mathrm{End}_{\mathscr{C}}(\mathbb{1}) = K$ and that the tensor product is K-bilinear and exact in each variable. It further implies that $\mathcal{H}om(X,Y) := X^\vee\otimes Y$ is an *internal hom* in $\mathcal{C}$, that is an object which represents the functor $\mathscr{C}^\circ\to \mathtt{Vec}_K$, $T\mapsto \mathrm{Hom}_{\mathscr{C}}(T\otimes X,Y)$. This means that $\mathrm{Hom}_{\mathscr{C}}(T\otimes X,Y) = \mathrm{Hom}_{\mathscr{C}}(T,\mathcal{H}om(X,Y))$. Then $\mathscr{C}$ is a *rigid* abelian K-linear tensor category in the sense of [DM82, Definition 2.19]. This further means that the natural morphisms $X\to(X^\vee)^\vee$ are isomorphisms and that $\bigotimes_{i=1}^n \mathcal{H}om(X_i,Y_i) = \mathcal{H}om(\bigotimes_i X_i, \bigotimes_i Y_i)$ for all $X_i, Y_i\in\mathscr{C}$.

(e) A functor $F\colon\mathscr{C}\to\mathscr{C}'$ between rigid abelian K-linear tensor categories is a *tensor functor* if $F(\mathbb{1})$ is a unit object in $\mathscr{C}'$ and there are fixed isomorphisms $F(X\otimes Y)\cong F(X)\otimes F(Y)$ compatible with the associativity and commutativity laws. A tensor functor automatically satisfies $F(X^\vee) = F(X)^\vee$ and $F\big(\mathcal{H}om(X,Y)\big) = \mathcal{H}om\big(F(X),F(Y)\big)$; see [DM82, Proposition 1.9]. In particular, for an L-rational fiber functor ω this means $\omega(\mathbb{1})\cong L$.

Definition A.3. A sub-category $\mathscr{C}'$ of a category $\mathscr{C}$ is *strictly full* if it is full and contains with every $X\in\mathscr{C}'$ also all objects of $\mathscr{C}$ isomorphic to X.

A strictly full sub-category $\mathscr{C}'$ of a rigid tensor category $\mathscr{C}$ is a *rigid tensor sub-category* if $\mathbb{1}\in\mathscr{C}'$ and $X\otimes Y, X^\vee\in\mathscr{C}'$ for all $X,Y\in\mathscr{C}'$. If in addition $\mathcal{C}$ is abelian and $\mathcal{C}'$ is closed under forming direct sums and subquotients, we call $\mathcal{C}'$ a *rigid abelian tensor sub-category*.

If $\mathscr{C}$ is a Tannakian category over K and $X\in\mathscr{C}$, the rigid abelian tensor sub-category of $\mathscr{C}$ containing as objects all subquotients of all $\bigoplus_{i=1}^r X^{\otimes n_i}\otimes(X^\vee)^{\otimes m_i}$ for all $r, n_i, m_i\in\mathbb{N}_0$ is called the *Tannakian sub-category generated by* X and is denoted $\langle\!\langle X\rangle\!\rangle$. Any L-rational fiber functor ω on $\mathscr{C}$ restricts to an L-rational fiber functor $\omega|_{\langle\!\langle X\rangle\!\rangle}$ on $\langle\!\langle X\rangle\!\rangle$ and makes $\langle\!\langle X\rangle\!\rangle$ indeed into a Tannakian category over K.

To describe the Tannakian duality in the *non-neutral* case, we need the following

Definition A.4. A *groupoid in sets* is a category in which every morphism has an inverse. Thus to give a groupoid in sets is to give a set S (of objects), a set G (of arrows), two maps $t,s\colon G\to S$ (sending an arrow to its target and source respectively), and a law of composition

$$\circ\colon\; G\underset{s,S,t}{\times} G \longrightarrow G\,, \qquad \text{where} \quad G\underset{s,S,t}{\times} G \;=\; \{(h,g)\in G\times G\colon s(h)=t(g)\}\,,$$

such that $\circ$ is a map over $S\times S$, each object has an identity morphism, composition of arrows is associative, and each arrow has an inverse.

In the rest of this appendix let K be a field and let K_e be a (finite or infinite) field extension which is Galois with Galois group $\mathscr{G}$. Let $S_0 := \operatorname{Spec} K$ and $S := \operatorname{Spec} K_e$. If $\mathscr{G}$ is finite, then the isomorphism

$$K_e \otimes_K K_e \xrightarrow{\sim} \prod_{\mathscr{G}} K_e\,, \quad a \otimes b \longmapsto \big(\sigma(a)\cdot b\big)_{\sigma\in\mathscr{G}}$$

gives rise to a commutative diagram

(A.1)
$$\begin{tikzcd}
S \times_{S_0} S \arrow[d, "pr_2"] \arrow[r, phantom, "="] & \operatorname{Spec} K_e \otimes_K K_e \arrow[r, phantom, "\cong"] & \coprod_{\mathscr{G}} \operatorname{Spec} K_e \arrow[d] \arrow[r, phantom, "=:"] & \underline{\mathscr{G}}_S \\
S \arrow[rr, equal] & & \operatorname{Spec} K_e\,, &
\end{tikzcd}$$

in which we view $\underline{\mathscr{G}}_S$ as a finite étale group scheme over S. If $\mathscr{G}$ is infinite, it is the projective limit of its finite quotients. Therefore, we can view $\underline{\mathscr{G}}_S$ as a profinite affine group scheme over S and the projective limit of the diagrams (A.1) over the finite quotients of $\mathscr{G}$ induces the corresponding diagram (A.1) also in this case.

Definition A.5. (a) A *K_e/K-groupoid* (or a *K-groupoid in schemes acting on K_e*) is a scheme $\mathfrak{G}$ over K together with two morphisms $t, s\colon \mathfrak{G} \to \operatorname{Spec} K_e$, called *target* and *source*, and a *law of composition*

$$\circ\colon \mathfrak{G} \underset{s,S,t}{\times} \mathfrak{G} \longrightarrow \mathfrak{G}\,,$$

which is an $S \times_{S_0} S$-morphism such that for all K-schemes T the category with objects $S(T)$, morphisms $\mathfrak{G}(T)$, target and source maps t and s, and composition law $\circ$ is a groupoid in sets.

(b) The K-groupoid $\mathfrak{G}$ is *transitive* if the morphism $(t,s)\colon \mathfrak{G} \to S \times_{S_0} S$ is surjective, it is *affine* if $\mathfrak{G}$ is an affine scheme and it is *algebraic* if $\mathfrak{G} \to S \times_{S_0} S$ is of finite type.

(c) The *kernel of a K_e/K-groupoid* $\mathfrak{G}$ is the pullback

$$G := \mathfrak{G}^\Delta := \Delta^*\mathfrak{G}$$

of $\mathfrak{G}$ under the diagonal morphism $\Delta\colon S \to S \times_{S_0} S$. It is a group scheme over S which is affine if $\mathfrak{G}$ is affine.

(d) A *morphism* between K_e/K-groupoids $\mathfrak{G}$ and $\mathfrak{H}$ is a morphism of $S \times_{S_0} S$-schemes $\alpha\colon \mathfrak{G} \to \mathfrak{H}$ which is compatible with the composition laws and induces a homomorphism of group schemes $\alpha^\Delta\colon \mathfrak{G}^\Delta \to \mathfrak{H}^\Delta$. Equivalently, α induces a functor between the categories $\big(S(T), \mathfrak{G}(T)\big) \to \big(S(T), \mathfrak{H}(T)\big)$ which is the identity map $S(T) \to S(T)$ on objects.

Remark A.6. Let $\mathfrak{G}$ be a K_e/K-groupoid with kernel $G := \mathfrak{G}^\Delta$. Then pr_2^*G acts on $\mathfrak{G}$ over $S \times_{S_0} S$ and makes it into a right G-torsor. The groupoid $\mathfrak{G}$ acts on G by conjugation:

(A.2)
$$(g, x) \longmapsto g \circ x \circ g^{-1} \qquad \text{for} \quad g \in \mathfrak{G}(T) \quad \text{and} \quad x \in G(T)$$

for any K-scheme T.

Definition A.7. If G_0 is a group scheme over K then $\mathfrak{G} := G_0 \times_{S_0} (S \times_{S_0} S)$ together with the composition induced by the group law of G_0 is a K_e/K-groupoid. It is called the *neutral groupoid* defined by G_0. It is affine, respectively algebraic, if G_0 is.

Example A.8. Let V be a finite dimensional vector space over K_e. Let $\mathfrak{Gl}(V)$ be the scheme over $S \times_{S_0} S$ representing the functor that sends a scheme $(b,a)\colon T \to S \times_{S_0} S$ over $S \times_{S_0} S$ to the set $\operatorname{Isom}_{\mathcal{O}_T}(a^*V, b^*V)$ of $\mathcal{O}_T$-isomorphisms from a^*V to b^*V. Then the composition of isomorphisms makes $\mathfrak{Gl}(V)$ into a K_e/K-groupoid which is affine, algebraic and transitive.

Example A.9. Here is a generalization of the previous example. Let V be a finite dimensional vector space over K_e and let $V_\bullet = (0 = V_0 \subset V_1 \subset \ldots \subset V_n = V)$ be a flag of K_e-linear subspaces $V_i \subset V$. Let $\mathfrak{Gl}(V_\bullet)$ be the scheme over $S \times_{S_0} S$ representing the functor that sends a scheme $(b,a)\colon T \to S \times_{S_0} S$ over $S \times_{S_0} S$ to the set

$$\operatorname{Isom}_{\mathcal{O}_T}(a^*V_\bullet, b^*V_\bullet) := \big\{\, f \in \operatorname{Isom}_{\mathcal{O}_T}(a^*V, b^*V)\colon f(a^*V_i) = b^*V_i \text{ for all } i \,\big\}.$$

We explain that this subfunctor is representable by a closed subscheme of $\mathfrak{Gl}(V)$. Namely, by induction on i the condition $f(a^*V_i) = b^*V_i$ is equivalent to the condition that the composite $\mathcal{O}_T$-homomorphism

$a^*V_i \hookrightarrow a^*V_{i+1} \xrightarrow{f} b^*V_{i+1} \twoheadrightarrow b^*(V_{i+1}/V_i)$ is the zero homomorphism. The latter is represented by a closed subscheme by [EGA, $\mathrm{I}_{\mathrm{new}}$, Proposition 9.7.9.1]. Again the composition of isomorphisms makes $\mathfrak{Gl}(V_\bullet)$ into a K_e/K-groupoid which is affine, algebraic, transitive, and is a closed subgroupoid of $\mathfrak{Gl}(V)$. Obvioiusly, $\mathfrak{Gl}(0 \subset V) = \mathfrak{Gl}(V)$.

For every i the restriction to V_i defines a morphism $\mathfrak{Gl}(V_\bullet) \to \mathfrak{Gl}(V_i)$ of groupoids, which is an epimorphism onto the closed subgroupoid $\mathfrak{Gl}(0 = V_0 \subset \ldots \subset V_i) \subset \mathfrak{Gl}(V_i)$.

Definition A.10. A *representation* of a groupoid $\mathfrak{G}$ is a morphism $\rho\colon \mathfrak{G} \to \mathfrak{Gl}(V)$ of groupoids for a finite dimensional K_e-vector space V. We let $\mathrm{Rep}(K_e\colon \mathfrak{G})$ be the category of representations of $\mathfrak{G}$ for varying finite dimensional K_e-vector spaces V. It has a natural tensor structure relative to which it forms a Tannakian category. The forgetful functor $\omega_0\colon (\rho, V) \mapsto V$ is a fiber functor over K_e.

Tannakian duality says that every K-linear Tannakian category which has a fiber functor over K_e is of this form:

Theorem A.11 (Tannakian duality [Del90, Théorème 1.12], [Mil92, Theorem A.8])**.** *Let $\mathscr{C}$ be a K-linear Tannakian category with a fiber functor ω over K_e.*

(a) There is an affine transitive K_e/K-groupoid $\mathfrak{G}$ that represents the functor $\mathrm{Aut}_K^{\otimes}(\omega)$ sending an $S \times_{S_0} S$-scheme $(b,a)\colon T \to S \times_{S_0} S$ to the set of isomorphisms of tensor functors $a^*\omega \xrightarrow{\sim} b^*\omega$; see [DM82, p. 116].

(b) The fiber functor ω induces an equivalence of tensor categories $\mathscr{C} \xrightarrow{\sim} \mathrm{Rep}(K_e\colon \mathfrak{G})$.

Conversely, let $\mathfrak{G}$ be an affine transitive K_e/K-groupoid, and let ω_0 be the forgetful fiber functor of $\mathrm{Rep}(K_e\colon \mathfrak{G})$*. Then the natural map $\mathfrak{G} \to \mathrm{Aut}_K^{\otimes}(\omega_0)$ is an isomorphism of groupoids.*

Definition A.12. The groupoid $\mathfrak{G} = \mathrm{Aut}_K^{\otimes}(\omega)$ is called the *Tannakian fundamental groupoid* of $(\mathcal{C}, \omega)$.

Proposition A.13. *The K_e/K-groupoid $\mathfrak{G}$ is algebraic if and only if* $\mathrm{Rep}(K_e\colon \mathfrak{G})$ *has a tensor generator X; compare Definition A.3. In this case $\mathfrak{G} \hookrightarrow \mathfrak{Gl}(\omega_0(X))$ is a closed immersion.*

Proof. By construction [Del90, § 6.8], $\mathfrak{G} = \mathrm{Spec}\, L_K(\omega_0, \omega_0)$, where $L_K(\omega_0, \omega_0)$ is defined in [Del90, § 4.7 and § 4.10(iii)] as an inductive limit of quotients of $\omega_0(X)^\vee \otimes_K \omega_0(X)$ where X runs through all objects of $\mathrm{Rep}(K_e\colon \mathfrak{G})$. In particular, if $\mathrm{Rep}(K_e\colon \mathfrak{G})$ has a tensor generator X then $L_K(\omega_0, \omega_0)$ is a quotient of $\omega_0(X)^\vee \otimes_K \omega_0(X)$ and a finitely generated algebra over $\coprod_{\mathscr{G}} K_e$. Conversely, if $L_K(\omega_0, \omega_0)$ is a finitely generated algebra over $\coprod_{\mathscr{G}} K_e$, then it is a quotient of $\omega_0(X)^\vee \otimes_K \omega_0(X)$ for some object X of $\mathrm{Rep}(K_e\colon \mathfrak{G})$, which necessarily must be a tensor generator. Obviously $\mathfrak{G} \hookrightarrow \mathfrak{Gl}(\omega_0(X))$ is a closed immersion. □

Proposition A.14. *Let $\alpha\colon \mathfrak{G} \to \mathfrak{H}$ be a homomorphism of affine transitive K_e/K-groupoids, and let ω^α be the corresponding functor* $\mathrm{Rep}(K_e\colon \mathfrak{H}) \to \mathrm{Rep}(K_e\colon \mathfrak{G})$.

(a) Then α is faithfully flat if and only if ω^α is fully faithful and every subobject of $\omega^\alpha(Y)$, for $Y \in$ $\mathrm{Rep}(K_e\colon \mathfrak{H})$*, is isomorphic to the image of a subobject of Y.*

(b) α is a closed immersion if and only if every object of $\mathrm{Rep}(K_e\colon \mathfrak{G})$ *is isomorphic to a subquotient of an object of the form $\omega^\alpha(Y)$ for an object $Y \in$* $\mathrm{Rep}(K_e\colon \mathfrak{H})$.

Proof. This was proven in [DM82, Proposition 2.21] for neutral Tannakian categories and group schemes instead of groupoids. But the proof likewise works in the non-neutral case for groupoids. □

Proposition A.15 ([Del89, § 10.8])**.** *Let $\mathfrak{G}$ and $\mathfrak{H}$ be K_e/K-groupoids with kernels $G := \mathfrak{G}^\Delta$ and $H := \mathfrak{H}^\Delta$, and let $\varphi\colon G \to H$ be a homomorphism of group schemes. If there is given an action of $\mathfrak{G}$ on H compatible with its action on G from* (A.2)*, then the $pr_2 H$-torsor deduced from $\mathfrak{G}$ by pushing out by the morphism $pr_2^*\varphi\colon pr_2^*G \to pr_2^*H$ is endowed with the structure of a groupoid whose kernel is H. We denote it by $\varphi_*\mathfrak{G}$.*

We apply this proposition in the following form.

Corollary A.16. *Let $\mathscr{C}$ be a K-linear Tannakian category equiped with a K_e-linear fiber functor ω. Let $\mathcal{F}$ be an object of $\mathscr{C}$ and let $\mathfrak{G} := \mathrm{Aut}_K^{\otimes}(\omega|_{\langle\langle \mathcal{F} \rangle\rangle})$ be the Tannakian fundamental groupoid of $\mathcal{F}$.*

(a) Every Tannakian sub-category of $\langle\langle X \rangle\rangle$ has a tensor generator $Y \in \langle\langle X \rangle\rangle$ and the natural fully faithful embedding $\langle\langle Y \rangle\rangle \to \langle\langle X \rangle\rangle$ of Tannakian categories induces an epimorphism of K_e/K-groupoids $\mathfrak{G} = \mathrm{Aut}^{\otimes}(\omega|_{\langle\langle X \rangle\rangle}) \twoheadrightarrow \mathrm{Aut}^{\otimes}(\omega|_{\langle\langle Y \rangle\rangle}) =: \mathfrak{H}$ and an epimorphism $\mathfrak{G}^\Delta \twoheadrightarrow \mathfrak{H}^\Delta$ of the kernel groups.

(b) *Conversely, let* $\varphi\colon \mathfrak{G}^\Delta \twoheadrightarrow H$ *be an epimorphism of algebraic groups over* K_e *whose kernel is invariant under the conjugation action of* $\mathfrak{G}$ *on* $\mathfrak{G}^\Delta$ *from* (A.2). *Then there exists an object* $Y \in \langle\langle X\rangle\rangle$ *and an epimorphism of groupoids* $\alpha\colon \mathfrak{G} \twoheadrightarrow \mathfrak{H} := \mathrm{Aut}^\otimes(\omega|_{\langle\langle Y\rangle\rangle})$ *whose restriction to the kernel groups* $\alpha^\Delta\colon \mathfrak{G}^\Delta \twoheadrightarrow \mathfrak{H}^\Delta$ *is isomorphic to* $\varphi\colon \mathfrak{G}^\Delta \twoheadrightarrow H$.

Proof. (b) Since the kernel of φ is stabilized by the conjugation action of $\mathfrak{G}$ on $\mathfrak{G}^\Delta$ from (A.2), this induces a compatible action of $\mathfrak{G}$ on H, and by Proposition A.15 we can form the groupoid $\mathfrak{H} := \varphi_*\mathfrak{G}$ with kernel $\mathfrak{H}^\Delta = H$. Since $\varphi\colon \mathfrak{G}^\Delta \twoheadrightarrow H$ is faithfully flat, also its extension $\alpha\colon \mathfrak{G} \twoheadrightarrow \mathfrak{H}$ is faithfully flat by Remark A.6 and so the functor $\mathrm{Rep}(K_e\colon \mathfrak{H}) \to \mathrm{Rep}(K_e\colon \mathfrak{G}) \cong \langle\langle X\rangle\rangle$ is an isomorphism onto a full Tannakian subcategory by Proposition A.14(a). Since $\mathfrak{G}$ is of finite type over $S\times_{S_0} S$ by Proposition A.13, also $\mathfrak{H}$ is of finite type over $S\times_{S_0} S$, and $\mathrm{Rep}(K_e\colon \mathfrak{H})$ has a tensor generator Y again by Proposition A.13.

(a) follows directly from Proposition A.14(a). The existence of a tensor generator was just proven in (b). □

Remark A.17. Recall that a subgroup of an algebraic group G which is invariant under all automorphisms of G is a *characteristic* subgroup. In particular, if the kernel of $\varphi\colon \mathfrak{G}^\Delta \twoheadrightarrow H$ is a characteristic subgroup of $\mathfrak{G}^\Delta$, then it is stabilized by the conjugation action of $\mathfrak{G}$ on $\mathfrak{G}^\Delta$ from (A.2), and so the hypotheses of Corollary A.16(b) are satisfied.

Proposition A.18. *Assume that* K *has characteristic zero. Let* $\rho\colon \mathfrak{G} \to \mathfrak{Gl}(V)$ *be a representation of an affine transitive* K_e/K*-groupoid* $\mathfrak{G}$ *on a* K_e*-vector space* V *and let* $H \subset \mathfrak{G}^\Delta$ *be a closed algebraic subgroup over* K_e *which is stable under the conjugation action of* $\mathfrak{G}$ *on* $\mathfrak{G}^\Delta$ *from* (A.2). *Let* $W \subset V$ *be the* K_e*-linear subspace of fixed vectors of* H *and consider the flag* $V_\bullet = (0 \subset W \subset V)$. *Then the representation* ρ *factors through the* K_e/K*-groupoid* $\mathfrak{Gl}(V_\bullet) \subset \mathfrak{Gl}(V)$ *from Example A.9. In particular, the representation* (ρ_W, W) *of* $\mathfrak{G}$ *is a subrepresentation of* (ρ, V), *where* ρ_W *is the composition of* $\rho\colon \mathfrak{G} \to \mathfrak{Gl}(V_\bullet)$ *followed by the epimorphism* $\mathfrak{Gl}(V_\bullet) \twoheadrightarrow \mathfrak{Gl}(W)$ *from Example A.9.*

Remark A.19. The subspace W in the proposition is defined as follows. Let $\overline{K}$ be a separable closure of K_e and let

$$\overline{W} \;:=\; \{\, v \in \overline{V} := V\otimes_{K_e}\overline{K}\colon \rho(h)(v) = v \text{ for all } h \in H(\overline{K})\,\}\,.$$

Since H is defined over K_e the subspace $\overline{W} \subset \overline{V}$ descends to a K_e-linear subspace $W \subset V$ satisfying $\overline{W} = W\otimes_{K_e}\overline{K}$.

Proof of Proposition A.18. We use Remark A.19 and let $\overline{S} := \mathrm{Spec}\,\overline{K}$. By faithfully flat descent [EGA, IV$_2$, Proposition 2.2.1] it suffices to prove that the morphism $\rho\colon \mathfrak{G} \to \mathfrak{Gl}(V)$ factors through $\mathfrak{Gl}(V_\bullet)$ after base-change from $S\times_{S_0} S$ to $\overline{S}\times_{S_0}\overline{S}$. Since in characteristic zero $\mathfrak{G}^\Delta$ is smooth, and hence $\mathfrak{G}^\Delta$ and $\mathfrak{G}$ are reduced schemes, we have to show that for every $\overline{S}\times_{S_0}\overline{S}$-scheme of the form $(b,a)\colon T = \mathrm{Spec}\,\overline{K} \to \overline{S}\times_{S_0}\overline{S}$ the map $\mathfrak{G}(T) \to \mathfrak{Gl}(V)(T)$ factors through $\mathfrak{Gl}(V_\bullet)(T)$. In addition, we consider T via the morphism $\Delta\circ a\colon T \to \overline{S} \hookrightarrow \overline{S}\times_{S_0}\overline{S}$ as another $\overline{S}\times_{S_0}\overline{S}$-scheme, which we denote by T_a. Likewise, we define T_b. Let $g \in \mathfrak{G}(T)$ and let $h \in H(T_b) \subset \mathfrak{G}^\Delta(T_b) = \mathfrak{G}(T_b)$. Then $g^{-1}hg \in H(T_a) \subset \mathfrak{G}^\Delta(T_a) = \mathfrak{G}(T_a)$ by the assumption that H is stable under the conjugation action of $\mathfrak{G}$ on $\mathfrak{G}^\Delta$ from (A.2). We have to show that $\rho(g) \in \mathfrak{Gl}(V)(T) = \mathrm{Isom}_{\mathcal{O}_T}(a^*V, b^*V)$ satisfies $\rho(g)(w) \in b^*W$ for all $w \in a^*W$. Since $\rho(g^{-1}hg)(w) = w$ in a^*V, we compute in b^*V

$$\rho(h)\rho(g)(w) \;=\; \rho(g)\rho(g^{-1}hg)(w) \;=\; \rho(g)(w)\,.$$

As this holds for all $h \in H(T_b)$, we obtain that $\rho(g)(w) \in b^*W$, and hence ρ factors through $\mathfrak{Gl}(V_\bullet)$ as desired. The last assertion is clear. □

Remark A.20. Recall from [DM82, p. 155ff] that for a Tannakian category $\mathscr{C}$ over K there is an extension of scalars $\mathscr{C}\otimes_K K_e$ which is a Tannakian category over K_e. It is equipped with a tensor functor $\bullet\otimes_K K_e\colon \mathscr{C} \to \mathscr{C}\otimes_K K_e$, $X \mapsto X\otimes_K K_e$, which satisfies

$$\mathrm{Hom}_{\mathscr{C}}(X,Y)\otimes_K K_e \;=\; \mathrm{Hom}_{\mathscr{C}\otimes_K K_e}(X\otimes_K K_e, Y\otimes_K K_e)\,.$$

If ω is a K_e-rational fiber functor on $\mathscr{C}$ then ω extends canonically to a K_e-rational, hence neutral fiber functor on $\mathscr{C}\otimes_K K_e$. Then [Mil92, Proposition A.12 and Example A.13] implies that Theorem A.11

induces a commutative diagram of Tannakian categories

$$\begin{tikzcd} X \arrow[d, mapsto] & \mathscr{C} \arrow[r, "\sim", "\omega"'] \arrow[d] & \mathrm{Rep}(K_e : \mathfrak{G}) \arrow[d] & \rho\colon \mathfrak{G} \to \mathfrak{Gl}(V) \arrow[d, mapsto] \\ X \otimes_K K_e & \mathscr{C} \otimes_K K_e \arrow[r, "\sim", "\omega"'] & \mathrm{Rep}(K_e : \mathfrak{G}^\Delta) & \rho^\Delta\colon \mathfrak{G}^\Delta \to \mathfrak{Gl}(V)^\Delta \end{tikzcd} \tag{A.3}$$

where the right vertical map is the restriction to the kernel groups.

Proposition A.21. *Let X and Y be two objects of a Tannakian category $\mathscr{C}$ over K. If there is a field extension $L \supset K$ which is Galois and an isomorphism $f\colon X \otimes_K L \xrightarrow{\sim} Y \otimes_K L$ in the category $\mathscr{C} \otimes_K L$, then there is an isomorphism $g\colon X \xrightarrow{\sim} Y$ in $\mathscr{C}$.*

Proof. Consider the group scheme $\mathrm{Aut}_{\mathscr{C}} Y$ over K which is defined for any K-algebra R by its R-valued points

$$(\mathrm{Aut}_{\mathscr{C}} Y)(R) \;:=\; \big(\mathrm{End}_{\mathscr{C}}(Y) \otimes_K R\big)^{\times}.$$

For every $\sigma \in \mathrm{Gal}(L/K)$ let $h_\sigma := f \circ \sigma^* f^{-1} \in (\mathrm{Aut}_{\mathscr{C}} Y)(L) = \mathrm{Aut}_{\mathscr{C}\otimes_K L}(Y \otimes_K L)$. They satisfy $h_{\sigma\tau} = h_\sigma \circ \sigma(h_\tau)$ and form a cocycle in $\mathrm{H}^1(\mathrm{Gal}(L/K), \mathrm{Aut}_{\mathscr{C}} Y)$. However, this group is trivial by [Ser79, Chapter X, § 1, Exercise 2]. So there is an element $h \in (\mathrm{Aut}_{\mathscr{C}} Y)(L)$ with $h_\sigma = h^{-1} \circ \sigma^* h$. In particular, $g := h \circ f\colon X \otimes_K L \xrightarrow{\sim} Y \otimes_K L$ satisfies $g = \sigma^* g$ for every $\sigma \in \mathrm{Gal}(L/K)$. Therefore $g\colon X \xrightarrow{\sim} Y$ is the desired isomorphism in $\mathscr{C}$. □

Proposition A.22. *Assume that K has characteristic zero and let $L \supset K$ be a finite Galois extension. Let $\mathscr{C}$ be a Tannakian category over K and let X be an object of $\mathscr{C}$. Then X is semi-simple if and only if the object $X \otimes_K L$ of $\mathscr{C} \otimes_K L$ is semi-simple.*

Proof. If X is semi-simple then $X \otimes_K L$ is semi-simple by [Sta08, Proposition 1.5.1]. Conversely let $X \otimes_K L$ be semi-simple and let $f\colon X \twoheadrightarrow Y$ be an epimorphism in $\mathscr{C}$. Since the extension functor $\bullet \otimes_K L$ has a right adjoint, it is right exact and $f \otimes 1\colon X \otimes_K L \twoheadrightarrow Y \otimes_K L$ is an epimorphism. Since $X \otimes_K L$ is semi-simple, there exists a morphism $g \in \mathrm{Hom}_{\mathscr{C}\otimes_K L}(Y \otimes_K L, X \otimes_K L)$ with $(f \otimes 1) \circ g = \mathrm{id}_{Y \otimes_K L}$. Then $\tilde{g} := \frac{1}{[L:K]} \sum_{\sigma \in \mathrm{Gal}(L/K)} \sigma^*(g) \in \mathrm{Hom}_{\mathscr{C}\otimes_K L}(Y \otimes_K L, X \otimes_K L)$ satisfies $(f \otimes 1) \circ \tilde{g} = \mathrm{id}_{Y \otimes_K L}$ and $\sigma^*(\tilde{g}) = \tilde{g}$ for every $\sigma \in \mathrm{Gal}(L/K)$. Therefore $\tilde{g} \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$ splits f, and hence X is semi-simple. □

Appendix B. Results from measure theory and real algebraic geometry

Definition B.1. Let $f\colon X \to Y$ be a continuous map between two topological spaces, each equipped with the Borel σ-algebra. For a measure μ on X the *push-forward measure* $f_*\mu$ is defined as $(f_*\mu)(V) := \mu(f^{-1}V)$ for every Borel-measurable subset $V \subset Y$. It satisfies $\int_Y h(y)\, d\, f_*\mu(y) = \int_X h(f(x))\, d\mu(x)$ for every measurable function h on Y.

Our next aim is to define the notion of a pull-back of measures under certain nice maps.

Lemma B.2. *Let $f : X \to Y$ be an injective continuous map between compact Hausdorff spaces. Then f maps Borel-measurable sets in X to Borel-measurable sets in Y.*

Proof. It will be enough to show the following:

- the image of every closed subset of X is Borel-measurable,
- the collection of subsets of X:

  $$\mathcal{C} = \{Z \subset X \mid f(Z) \subset Y \text{ is Borel-measurable}\}$$

  is a σ-algebra.

We first show the first claim. Since X is compact and Y is Hausdorff, the image of every closed subset of X is compact, and hence closed, and these are Borel-measurable. Now we prove the second claim. Since $\emptyset$ and X are closed in X, we get that $\emptyset, X \in \mathcal{C}$ by the above. If $B \in \mathcal{C}$, then

$$f(X \smallsetminus B) = f(X) \smallsetminus f(B)$$

using that f is injective. Since Borel-sets form a σ-algebra, the right hand side is a Borel-set, and hence $X \smallsetminus B \in \mathcal{C}$. If $B_i \in \mathcal{C}$, where $i \in \mathbb{N}$, then

$$f(\bigcup_{i\in\mathbb{N}} B_i) = \bigcup_{i\in\mathbb{N}} f(B_i),$$

so using that Borel-sets form a σ-algebra, we get that $\bigcup_{i\in\mathbb{N}} B_i \in \mathcal{C}$. □

Definition B.3. Let $f : X \to Y$ be a continuous map between topological spaces. We say that the map f is *nice* if there is a countable pair-wise disjoint decomposition

$$X = \coprod_{i=1}^{\infty} X_i \tag{B.1}$$

of X into Borel-measurable subsets X_i such that the restriction of f onto the closure $\overline{X}_i$ of each X_i is injective.

Remark B.4. Note that a continuous map $f : X \to Y$ is nice if and only if there is a countable cover $\{Z_i\}_{i\in\mathbb{N}}$ of X by closed subsets such that $f|_{Z_i}$ is injective for every $i \in \mathbb{N}$. Indeed if f is nice and

$$X = \coprod_{i=1}^{\infty} X_i$$

is a decomposition of X as in (B.1) above, then $\{\overline{X}_i\}_{i\in\mathbb{N}}$ of X is such a cover. On the other hand if $\{Z_i\}_{i\in\mathbb{N}}$ is a countable cover of X by closed subsets such that $f|_{Z_i}$ is injective for every $i \in \mathbb{N}$ then

$$X_i = Z_i \smallsetminus \bigcup_{j<i} Z_j$$

is Borel-measurable, and

$$X = \coprod_{i=1}^{\infty} X_i$$

is a decomposition of X. Since $X_i \subset Z_i$, we have $\overline{X}_i \subset \overline{Z}_i = Z_i$, and hence $f|_{\overline{X}_i}$ is injective for every $i \in \mathbb{N}$.

Lemma B.5. *Let $f : X \to Y$ be a nice continuous map between compact Hausdorff spaces. Then f maps Borel-measurable sets in X to Borel-measurable sets in Y.*

Proof. Fix a decomposition of X as in (B.1) above. For every Borel-measurable $Z \subset X$ we have:

$$f(Z) = \bigcup_{i\in\mathbb{N}} f(Z \cap X_i),$$

so it will be sufficient to show that $f(Z \cap X_i)$ is Borel-measurable for every $i \in \mathbb{N}$. Since $\overline{X}_i$ is closed, it is a compact Hausdorff space. Also the restriction of f onto $\overline{X}_i$ is injective by assumption. As $Z \cap X_i$ is Borel-measurable in $\overline{X}_i$, the claim follows from Lemma B.2. □

Lemma B.6. *Let $f : X \to Y$ be a nice continuous map between compact Hausdorff spaces. For every Borel-measurable subset $Z \subset X$ the counting function $c_{Z/Y} : Y \to \mathbb{R} \cup \{\infty\}$ given by the rule:*

$$y \mapsto \#(f^{-1}(y) \cap Z)$$

is measurable.

Proof. Fix a decomposition of X as in (B.1) above. Clearly

$$c_{Z/Y}(y) = \sum_{i=1}^{\infty} c_i(y),$$

where for every $i \in \mathbb{N}$ the function $c_i : Y \to \mathbb{R} \cup \{\infty\}$ is given by the rule:

$$y \mapsto \#(f^{-1}(y) \cap Z \cap X_i).$$

Since the latter are non-negative, $c_{Z/Y}$ is the point-wise supremum of the sequence $\left(\sum_{i=1}^{j} c_i\right)_{j\in\mathbb{N}}$. So it will be enough to show that each c_i is measurable. However, the restriction of f onto X_i is injective, so

c_i is just the characteristic function of $f(Z \cap X_i)$. By Lemma B.5 the set $f(Z \cap X_i)$ is Borel-measurable, since $Z \cap X_i$ is, so c_i is also measurable. $\square$

Definition B.7. Let $f : X \to Y$ be a nice continuous map between compact Hausdorff spaces. Let μ be a Borel-measure on Y. We define the pull-back measure $f^*\mu$ on X by the formula:

$$(f^*\mu)(Z) = \int_Y c_{Z/Y}(y) d\mu(y)$$

for every Borel-measurable $Z \subset X$. By the lemma above the integrand is measurable, and it is also non-negative, so the integral is well-defined. When there is an upper bound on the cardinality of the fibers of f, we get that $f^*(\mu)$ is bounded, too, i.e. it only takes finite values if the same is true for μ.

Proposition B.8. *Let $f\colon X \to Y$ be a nice, continuous, surjective map between compact Hausdorff spaces and let μ_m for $m \in \mathbb{N}$ be a sequence of measures on Y which converge weakly to a measure ν on Y. Assume that there is a positive integer M such that all fibers of f have cardinality at most M. Assume further that there is a closed subset $Z \subset Y$ of measure $\nu(Z) = 0$ whose complement is a finite disjoint union $Y \smallsetminus Z = \coprod_{i=1}^n Y_i$ of open subsets Y_i such that f is trivial over Y_i in the sense that there is a finite discrete set F_i and a homeomorphism $g_i\colon F_i \times Y_i \xrightarrow{\sim} f^{-1}(Y_i)$ compatible with the projections onto Y_i. Then the pullback measures $f^*\mu_m$ converge weakly to $f^*\nu$.*

Proof. We first observe that

$$(f^*\nu)(f^{-1}Z) \;:=\; \int_Z \#f^{-1}(y)\; d\nu(y) \;\le\; M \cdot \nu(Z) \;=\; 0$$

and likewise

$$(f^*\mu_m)(f^{-1}Z) \;:=\; \int_Z \#f^{-1}(y)\; d\mu_m(y) \;\le\; M \cdot \mu_m(Z)\,.$$

Since $\limsup_{m\to\infty} \mu_m(Z) \le \nu(Z) = 0$ by the weak convergence and the Portemanteau theorem [Kle14, Theorem 13.16], we conclude that $\lim_{m\to\infty}(f^*\mu_m)(f^{-1}Z) = 0$.

Let $U \subset X$ be an open subset. Then

$$(f^*\nu)(U) \;=\; (f^*\nu)(U \cap f^{-1}Z) + \sum_{i=1}^n (f^*\nu)(U \cap f^{-1}Y_i) \;=\; \sum_{i=1}^n \sum_{x\in F_i} (f^*\nu)\big(U \cap g_i(x \times Y_i)\big)$$

and likewise

$$\liminf_{m\to\infty}(f^*\mu_m)(U) \;=\; \sum_{i=1}^n \sum_{x\in F_i} \liminf_{m\to\infty}(f^*\mu_m)\big(U \cap g_i(x \times Y_i)\big)\,,$$

and similarly for lim sup. Since the projection map $x \times Y_i \to Y_i$ is a homeomorphism, the set $f\big(U \cap g_i(x \times Y_i)\big) \subset Y_i$ is open and $(f^*\nu)\big(U \cap g_i(x\times Y_i)\big) = \nu\big(f(U\cap g_i(x\times Y_i))\big)$ and $(f^*\mu_m)\big(U\cap g_i(x\times Y_i)\big) = \mu_m\big(f(U\cap g_i(x\times Y_i))\big)$. From this and the weak convergence of μ_m to ν we obtain $\liminf_{m\to\infty}(f^*\mu_m)(U) \ge (f^*\nu)(U)$.

If $U = X$ we also must show that $\limsup_{m\to\infty}(f^*\mu_m)(X) \le (f^*\nu)(X)$. Note that

$$\limsup_{m\to\infty}(f^*\mu_m)\big(g_i(x \times Y_i)\big) \;=\; \limsup_{m\to\infty}\mu_m(Y_i) \;=\; \limsup_{m\to\infty}\mu_m(Z \cup Y_i) \;\le\; \nu(Z \cup Y_i)$$

because $Z \cup Y_i$ is closed in Y and μ_m converges weakly to ν. We compute

$$\nu(Z \cup Y_i) \;=\; \nu(Z) + \nu(Y_i) \;=\; \nu(Y_i) \;=\; (f^*\nu)\big(g_i(x \times Y_i)\big)\,.$$

This implies $\limsup_{m\to\infty}(f^*\mu_m)(X) \le (f^*\nu)(X)$ and shows that the sequence of measures $f^*\mu_m$ converges weakly to $f^*\nu$. $\square$

In the rest of this appendix we consider the following

Definition B.9. As usual let $|z|$ denote the absolute value of a complex number $z \in \mathbb{C}$. Let $Y \subset \mathbb{A}^d_{\mathbb{R}}$ be a smooth affine scheme over $\mathbb{R}$, and assume that its set of real points

$$C = Y(\mathbb{R}) \subset \mathbb{R}^d$$

is compact and Zariski-dense in the base-change $Y_{\mathbb{C}} \subset \mathbb{A}^d_{\mathbb{C}}$ of Y to $\mathbb{C}$. Let

$$\mu \tag{B.2}$$

be the Lebesgue measure furnished by the volume form of the Riemannian metric specified on C. Let $d(\cdot,\cdot):\mathbb{R}^d\to\mathbb{R}_{\geq 0}$ be the usual Euclidean metric. For every $\varepsilon>0$ and every subset $A\subset C$ let $A(\varepsilon)$ denote the open ε-neighborhood of A in C:

$$A(\varepsilon) \ := \ \{z\in C\colon \exists\ y\in A \text{ such that } d(z,y)<\varepsilon\} \ = \ \bigcup_{y\in A}\{z\in C\colon d(z,y)<\varepsilon\}\,.$$

Note that for a subset $A\subset C$ which is closed, and hence compact, the closure $\overline{A(\varepsilon)}$ with respect to the metric on C is contained in $A(2\varepsilon)$. We say that a sequence $A_1,A_2,\ldots,A_m,\ldots$ of subsets of C *converges* to a subset $A\subseteq C$ if for every $\varepsilon>0$ there is an index $m_\varepsilon\in\mathbb{N}$ such that $A_m\subseteq A(\varepsilon)$ for every $m\geq m_\varepsilon$.

Theorem B.10. *Let $D\in\mathbb{N}$, and let $H_1,H_2,\ldots,H_m,\ldots$ be a sequence of algebraic hyper-surfaces of $\mathbb{C}^d$ of degree at most D such that the intersection $H_m\cap Y_{\mathbb{C}}$ is a proper hyper-surface in $Y_{\mathbb{C}}$ for every m. Then there is a subsequence $H_{m_1},H_{m_2},\ldots,H_{m_n},\ldots$ and an algebraic hyper-surface $H\subset\mathbb{C}^d$ of degree at most D with $H\cap Y_{\mathbb{C}}\subsetneq Y_{\mathbb{C}}$, such that the sequence $H_{m_1}\cap C,H_{m_2}\cap C,\ldots,H_{m_n}\cap C,\ldots$ converges to $H\cap C$ in the sense of Definition B.9.*

To prove the theorem we let $\mathbf{P}_D\subset\mathbb{C}[x_1,x_2,\ldots,x_d]$ denote the complex vector space of all complex polynomials on $\mathbb{C}^d$ of total degree at most D and we let $\overline{\mathbf{P}}_D$ be its image in the coordinate ring $\mathbb{C}[x_1,x_2,\ldots,x_d]/I(Y_{\mathbb{C}})$ of $Y_{\mathbb{C}}$. Here $I(Y_{\mathbb{C}})$ is the ideal of functions vanishing on $Y_{\mathbb{C}}$. The images of the monomials $x_1^{i_1}x_2^{i_2}\cdots x_d^{i_d}$ for $\underline{i}=(i_1,i_2,\ldots,i_d)\in\mathbb{N}_0^d$ with $i_1+i_2+\cdots+i_d\leq D$ form a generating system of the $\mathbb{C}$-vector space $\overline{\mathbf{P}}_D$. We may shrink this generating system to a basis $\mathcal{B}$. For every

$$F(x_1,x_2,\ldots,x_d) \ = \ \sum_{\underline{i}\in\mathcal{B}} c_{\underline{i}}\cdot x_1^{i_1}x_2^{i_2}\cdots x_d^{i_d} \ \in \ \overline{\mathbf{P}}_D \qquad \text{with} \quad c_{\underline{i}}\in\mathbb{C}$$

let

$$\|F\| \ := \ \max_{\underline{i}\in\mathcal{B}}|c_{\underline{i}}|.$$

This is clearly a norm on $\overline{\mathbf{P}}_D$.

Lemma B.11. *Let $M\in\mathbb{R}$, $M\geq 1$ be such that $d(0,x)\leq M$ for all $x\in C$.*

(a) *For every $x,y\in C$ and for every $F\in\overline{\mathbf{P}}_D$ we have:*

$$|F(x)-F(y)|\leq\|F\|\cdot d(x,y)\cdot M^{D-1}\cdot D\cdot\#\mathcal{B}.$$

(b) *For every $x\in C$ and for every $F_1,F_2\in\overline{\mathbf{P}}_D$ we have:*

$$|F_1(x)-F_2(x)|\leq\|F_1-F_2\|\cdot M^D\cdot\#\mathcal{B}.$$

Proof. (a) Write $x=(x_1,x_2,\ldots,x_d)$ and $y=(y_1,y_2,\ldots,y_d)$. Then

$$|x_i-y_i|\leq d(x,y) \qquad \text{and} \qquad |x_i|,|y_i|\leq M \quad (\forall i=1,2,\ldots,d).$$

In particular for every multi-index $\underline{i}=(i_1,i_2,\ldots,i_d)\in\mathcal{B}$ by telescoping we get:

$$\begin{aligned}
|x_1^{i_1}x_2^{i_2}\cdots x_d^{i_d}-y_1^{i_1}y_2^{i_2}\cdots y_d^{i_d}| \ &= \ |(x_1^{i_1}x_2^{i_2}\cdots x_d^{i_d}-y_1^{i_1}x_2^{i_2}\cdots x_d^{i_d})+(y_1^{i_1}x_2^{i_2}\cdots x_d^{i_d}-y_1^{i_1}y_2^{i_2}x_3^{i_3}\cdots x_d^{i_d})+\cdots| \\
&\leq \ \sum_{j=1}^{d}|x_j^{i_j}-y_j^{i_j}|\cdot|y_1^{i_1}\cdots y_{j-1}^{i_{j-1}}\cdot x_{j+1}^{i_{j+1}}\cdots x_d^{i_d}| \\
&\leq \ \sum_{j=1}^{d}|x_j-y_j|\cdot|x_j^{i_j-1}+x_j^{i_j-2}y_j+\cdots+y_j^{i_j-1}|\cdot M^{i_1+\ldots+i_{j-1}+i_{j+1}+\ldots+i_d} \\
&\leq \ d(x,y)\cdot M^{D-1}\cdot\sum_{j=1}^{d}i_j \\
&\leq \ d(x,y)\cdot M^{D-1}\cdot D.
\end{aligned}$$

Write

$$F(t_1,t_2,\ldots,t_d)=\sum_{\underline{i}\in\mathcal{B}}c_{\underline{i}}\cdot t_1^{i_1}t_2^{i_2}\cdots t_d^{i_d}.$$

By the above:

$$\begin{aligned}
|F(x)-F(y)| &= |\sum_{\underline{i}\in\mathcal{B}} c_{\underline{i}}\cdot(x_1^{i_1}x_2^{i_2}\cdots x_d^{i_d}-y_1^{i_1}y_2^{i_2}\cdots y_d^{i_d})| \\
&\leq \sum_{\underline{i}\in\mathcal{B}} |c_{\underline{i}}|\cdot d(x,y)\cdot M^{D-1}\cdot D \\
&\leq \|F\|\cdot d(x,y)\cdot M^{D-1}\cdot D\cdot \#\mathcal{B}.
\end{aligned}$$

(b) Writing $F_1(t_1,\ldots,t_d)=\sum_{\underline{i}\in\mathcal{B}} b_{\underline{i}}\cdot t_1^{i_1}\cdots t_d^{i_d}$ and $F_2(t_1,\ldots,t_d)=\sum_{\underline{i}\in\mathcal{B}} c_{\underline{i}}\cdot t_1^{i_1}\cdots t_d^{i_d}$ we compute

$$\begin{aligned}
|F_1(x)-F_2(x)| &\leq \sum_{\underline{i}\in\mathcal{B}} |b_{\underline{i}}-c_{\underline{i}}|\cdot|x_1|^{i_1}|x_2|^{i_2}\cdots|x_d|^{i_d} \\
&\leq \sum_{\underline{i}\in\mathcal{B}} \|F_1-F_2\|\cdot M^D \\
&\leq \|F_1-F_2\|\cdot M^D\cdot\#\mathcal{B}.
\end{aligned}$$

□

Proof of Theorem B.10. For every $m\in\mathbb{N}$ let $F_m\in\overline{\mathbf{P}}_D$ be a polynomial such that $\|F_m\|=1$ and the zero set of F_m in $Y_{\mathbb{C}}$ is $H_m\cap Y_{\mathbb{C}}$. Because the unit sphere $\{F\in\overline{\mathbf{P}}_D\colon \|F\|=1\}$ is compact, there is a subsequence $F_{m_1},F_{m_2},\ldots,F_{m_n},\ldots$ and an element $F\in\overline{\mathbf{P}}_D$ such that $\|F\|=1$ and the sequence $F_{m_1},F_{m_2},\ldots,F_{m_n},\ldots$ converges to F with respect to the norm $\|\,.\,\|$. We may even assume this subsequence is the full sequence after re-indexing.

We claim that in this case $H_1\cap C, H_2\cap C,\ldots,H_n\cap C,\ldots$ converges to $H\cap C$, where $H\cap Y_{\mathbb{C}}$ is the zero set of F. Assume that this is false. Then there is a small $\varepsilon>0$ such that, after taking a suitable subsequence, $H_m\cap C$ does not lie in $(H\cap C)(\varepsilon)$ for every $m\in\mathbb{N}$. Choose an $x_m\in(H_m\cap C)\smallsetminus(H\cap C)(\varepsilon)$ for every m. Since the set $C\smallsetminus(H\cap C)(\varepsilon)$ is closed in C, it is compact, so we may assume, after taking a suitable subsequence, that x_m converges to a point $x\in C\smallsetminus(H\cap C)(\varepsilon)$. Note that

$$|F_m(x)|=|F_m(x)-F_m(x_m)|\leq M^{D-1}\cdot D\cdot\#\mathcal{B}\cdot d(x,x_m)$$

and

$$|F(x)-F_m(x)|\leq M^D\cdot\#\mathcal{B}\cdot\|F-F_m\|$$

by Lemma B.11. Therefore

$$F(x)=\lim_{m\to\infty}F_m(x)+\lim_{m\to\infty}(F(x)-F_m(x))=0,$$

and $x\in H\cap C$. But this is a contradiction and so our claim is proven. To finish the proof of Theorem B.10 we note that $F\neq 0$ in $\overline{\mathbf{P}}_D$, because $\|F\|=1$, and so $H\cap Y_{\mathbb{C}}$ is a proper hyper-surface in $Y_{\mathbb{C}}$. □

To prove Theorem 13.2 we will also need the following

Lemma B.12. *Keep the notation of Definition B.9. Let λ be a measure on C satisfying $\lambda(C)<\infty$. Then for every algebraic hyper-surface $H\subset\mathbb{C}^d$ with $H\cap C\subsetneq C$ and $\lambda(H\cap C)=0$ we have:*

$$\lim_{\varepsilon\to 0}\lambda\big((H\cap C)(\varepsilon)\big)=0.$$

Proof. Assume that the claim is false. Then there is a strictly decreasing sequence $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n,\ldots$ such that $\varepsilon_n\to 0$, and

$$\lambda\big((H\cap C)(\varepsilon_n)\big)\geq\delta,$$

for some positive δ. Set $A_n=(H\cap C)(\varepsilon_n)-(H\cap C)(\varepsilon_{n+1})$. Then by σ-additivity applied to $(H\cap C)(\varepsilon_n)=(H\cap C)\cup\coprod_{m\geq n}A_m$ we have

$$\lambda\big((H\cap C)(\varepsilon_n)\big) = \lambda(H\cap C)+\sum_{m\geq n}\lambda(A_m) = \sum_{m\geq n}\lambda(A_m)\,. \tag{B.3}$$

Since $\lambda\big((H\cap C)(\varepsilon_n)\big)\leq\lambda(C)<\infty$ the sum $\sum_{m=1}^{\infty}\lambda(A_m)$ is convergent. Therefore, we have:

$$\lim_{n\to\infty}\sum_{m\geq n}\lambda(A_m)=0.$$

So by taking the limit in (B.3) we get $\lambda(H \cap C) \geq \delta$, which is a contradiction. $\square$

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