Change of Coefficients for Drinfeld Modules, Shtuka, and Abelian Sheaves

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Abstract

We study the passage from Drinfeld-A'-modules to Drinfeld-A-modules for a given finite flat inclusion $A \subset A'$. We show that this defines a morphism from the moduli space of Drinfeld-A'-modules to the moduli space of Drinfeld-A-modules which is proper but in general not representable. For Drinfeld-Anderson shtuka and abelian sheaves instead of Drinfeld modules we obtain the same results.

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Introduction

Throughout this article let $\mathbb{F}_q$ be a finite field with q elements and characteristic p and let C and C' be two smooth projective geometrically irreducible curves over $\mathbb{F}_q$. Let $\pi : C' \to C$ be a fixed finite morphism of degree n. Let $\infty \in C$ be a closed point which does not split in C', that is, there is exactly one point $\infty' \in C'$ above ∞. Set $A := \Gamma(C \smallsetminus \{\infty\}, \mathcal{O}_C)$ and $A' := \Gamma(C' \smallsetminus \{\infty'\}, \mathcal{O}_{C'})$, then A' is a flat A-algebra via $\pi^* : A \to A'$.

In this situation π defines a *restriction of coefficients functor* from Drinfeld-A'-modules over S to Drinfeld-A-modules over S. This functor induces a morphism between the moduli spaces (moduli functors, or more sophisticated, moduli stacks) classifying Drinfeld-A'-modules, respectively Drinfeld-A-modules. We show in this article that this morphism is proper but not necessarily representable. Likewise we study the effect of π on Drinfeld-Anderson shtuka, see Definition 1.7, and on abelian sheaves, a notion introduced by the first author [9] as a higher dimensional generalization of Drinfeld modules, see Definition 1.5. For the case of Drinfeld-Anderson shtuka we may even relax the condition on π and drop the assumption on the ramification of ∞. The pushforward of sheaves along $\pi \times \mathrm{id}_S : C'_S \to C_S$ defines a *restriction of coefficients functor* from Drinfeld-Anderson shtuka on C' over S to Drinfeld-Anderson shtuka on C over S, respectively from abelian sheaves on C' over S to abelian sheaves on C over S. Again this yields proper but in general not representable morphisms between the moduli spaces classifying Drinfeld-Anderson shtuka on C', respectively Drinfeld-Anderson shtuka on C and similarly for abelian sheaves.

Of course the results for Drinfeld modules, Drinfeld-Anderson shtuka, and abelian sheaves are strongly related by the fact that the category of Drinfeld-A-modules over S is anti-equivalent to a full subcategory of the category of Drinfeld-Anderson shtuka on C over S

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and anti-equivalent to a full subcategory of the category of abelian sheaves on C over S. Nevertheless we give proofs also for the case of Drinfeld modules since these are particularly simple. After recalling the definitions and some basic properties in Section 1 we prove in Sections 2, 3, and 4 the properness and non-representability results for Drinfeld modules, abelian sheaves, respectively Drinfeld-Anderson shtuka.

This article has its origin in a conversation with F. Breuer who mentioned to us a special case of the proof for properness in the case of Drinfeld modules. Our proof of Proposition 2.3 below is a generalization of his. We like to express our gratitude to him.

1 Drinfeld Modules, Shtuka, and Abelian Sheaves

We retain the notation from the introduction. In addition, we set $\deg(\infty) := [\kappa(\infty) : \mathbb{F}_q]$ and we denote by ord_∞ the normalized valuation on the fraction field of A associated with the place ∞. For an $\mathbb{F}_q$-scheme S we set $C_S := C \times_{\mathbb{F}_q} S$. Unless mentioned explicitly we make no noetherian assumption on S.

For an $\mathbb{F}_q$-algebra B we denote by $B\{\tau\}$ the non-commutative polynomial ring in the variable τ over B with the commutation rule $\tau\, b = b^q\, \tau$ for all $b \in B$. As in [14, §1] one sees

Proposition 1.1. *There is an isomorphism of rings between* $B\{\tau\}$ *and* $\mathrm{End}_{B,\mathbb{F}_q}(\mathbb{G}_{a,B})$ *the ring of* $\mathbb{F}_q$*-linear endomorphisms of the additive group scheme over* $\operatorname{Spec} B$ *given by mapping* τ *to the* q*-th power Frobenius of* $\mathbb{G}_{a,B}$*.* □

Definition 1.2. (Drinfeld [5, §5.B])
Let S be an $\mathbb{F}_q$-scheme and assume there is a morphism $c : S \to \operatorname{Spec} A$. Let r be a positive integer. A *Drinfeld-A-module of rank r and characteristic c* over S is a pair (E, φ) where E is a commutative group scheme over S and

$$\varphi : \quad A \longrightarrow \mathrm{End}_S(E)$$

is a ring homomorphism from A to the ring $\mathrm{End}_S(E)$ of endomorphisms of the S-group scheme E such that

1. E is Zariski locally on S isomorphic to the additive group scheme $\mathbb{G}_{a,S}$,
2. if $U = \operatorname{Spec} B$ is an affine open subset of S and $\psi : E_U \xrightarrow{\sim} \mathbb{G}_{a,U}$ is an isomorphism of S-group schemes then for each $a \in A \smallsetminus \{0\}$

$$\psi \circ \varphi(a) \circ \psi^{-1} \;=\; \sum_{i=0}^{<\infty} \delta_i(a)\tau^i \;\in\; B\{\tau\}$$

   with $\delta_0(a) = c^*(a)$, $\delta_i(a) \in B^\times$ for $i = d(a) := -r\,\mathrm{ord}_\infty(a)\deg(\infty)$, and $\delta_i(a)$ nilpotent for $i > d(a)$.

A *morphism* of Drinfeld-A-modules $\varepsilon : (E, \varphi) \to (\widetilde{E}, \widetilde{\varphi})$ is a morphism of S-group schemes $\varepsilon : E \to \widetilde{E}$ which satisfies $\widetilde{\varphi}(a) \circ \varepsilon = \varepsilon \circ \varphi(a)$ for all $a \in A$.

If $f : S' \to S$ is a morphism of $\mathbb{F}_q$-schemes we can pull back Drinfeld-A-modules (E, φ) over S to Drinfeld-A-modules $(f^*E, f^*\varphi)$ over S'.

The following proposition is due to Drinfeld [5, Propositions 5.1 and 5.2]

Proposition 1.3. *Let (E,φ) be a Drinfeld-A-module of rank r over S. Then Zariski locally on S there exists an isomorphism $\varepsilon : (E,\varphi) \xrightarrow{\sim} (\mathbb{G}_{a,S},\psi)$ of Drinfeld-A-modules where ψ is of the* standard form

$$\psi:\ A \longrightarrow \mathcal{O}_S\{\tau\}\,, \qquad \psi(a) \;=\; \sum_{i=0}^{d(a)} \delta_i(a)\tau^i$$

with $d(a) := -r\operatorname{ord}_\infty(a)\deg(\infty)$ and $\delta_{d(a)} \in \mathcal{O}_S^\times$. Moreover if $\psi(a)$ is of the described form for one $a \in A \smallsetminus \mathbb{F}_q$ then it already is for any $a \in A$. □

Proposition 1.4. *The morphism $\pi : C' \to C$ defines a* restriction of coefficients functor *$\pi_* : (E',\varphi') \mapsto (E',\varphi'\circ\pi^*)$ from Drinfeld-A'-modules of rank r' over S to Drinfeld-A-modules of rank nr' over S, where n is the degree of π.*

Proof. The change of rank results from the fact that $n\operatorname{ord}_\infty(a)\deg(\infty) = \operatorname{ord}_{\infty'}(a)\deg(\infty')$ for all $a \in A$ since $\pi^{-1}(\infty) = \{\infty'\}$. The rest is clear from the definition. □

Remark. Consider the moduli problem, that is, the contravariant functor

$$\begin{aligned} \underline{\text{Dr-}A\text{-Mod}}^r :\ \mathcal{S}ch_{/\operatorname{Spec} A} &\longrightarrow \mathcal{S}ets \\ (c : S \to \operatorname{Spec} A) &\mapsto \Big\{\text{Isomorphism classes of Drinfeld-}A\text{-modules} \\ &\qquad\quad \text{of rank } r \text{ and characteristic } c \text{ over } S\Big\} \end{aligned}$$

from the category of schemes over $\operatorname{Spec} A$ to the category of sets. This functor is not representable (without adding level structures). Nevertheless the restriction of coefficients functor defines a *restriction of coefficients morphism*

$$\pi_* :\ \underline{\text{Dr-}A'\text{-Mod}}^{r'} \longrightarrow \underline{\text{Dr-}A\text{-Mod}}^{nr'}\,, \quad (E',\varphi') \;\mapsto\; \pi_*(E',\varphi')\,.$$

Remark. If we let S vary, the category of Drinfeld-A-modules of rank r becomes a stack $\mathcal{D}r\text{-}A\text{-}\mathcal{M}od^{\,r}$ for the fppf topology on the category of $\mathbb{F}_q$-schemes. It is an algebraic stack in the sense of Deligne-Mumford [4], see Laumon [12, Corollary 1.4.3]. The restriction of coefficients functor defines a *restriction of coefficients 1-morphism* $\pi_* : \mathcal{D}r\text{-}A'\text{-}\mathcal{M}od^{\,r'} \to \mathcal{D}r\text{-}A\text{-}\mathcal{M}od^{\,nr'}$.

Next we study the analogous situation for abelian sheaves. This notion was introduced in [9]. While Drinfeld modules are analogues for elliptic curves in the arithmetic of function fields, abelian sheaves are the appropriate analogues for abelian varieties as the results of [9, 2] amply demonstrate.

Let r and d be positive integers and write $\frac{d}{r\deg(\infty)} = \frac{k}{\ell}$ with relatively prime positive integers k and ℓ. Let S be an $\mathbb{F}_q$-scheme and fix a morphism $c : S \to C$. Let $\mathcal{J}$ be the ideal sheaf on C_S of the graph of c. We let $\sigma := \operatorname{id}_C \times \operatorname{Frob}_q$ be the endomorphism of C_S that acts as the identity on the underlying topological space and on the coordinates of C and as $b \mapsto b^q$ on the elements $b \in \mathcal{O}_S$. Let $pr : C_S \to S$ be the projection onto the second factor. For an integer m denote by $\mathcal{O}_{C_S}(m\cdot\infty)$ the invertible sheaf on C_S associated with the divisor $m\cdot\infty$ and set $\mathcal{F}(m\cdot\infty) := \mathcal{F}\otimes_{\mathcal{O}_{C_S}}\mathcal{O}_{C_S}(m\cdot\infty)$ for any sheaf of $\mathcal{O}_{C_S}$-modules on C_S.

Definition 1.5. An *abelian sheaf* $\underline{\mathcal{F}} = (\mathcal{F}_i, \Pi_i, \tau_i)$ *on* C *of rank* r*, dimension* d*, and characteristic* c *over* S is a ladder of locally free sheaves $\mathcal{F}_i$ on C_S of rank r and injective homomorphisms Π_i, τ_i of $\mathcal{O}_{C_S}$-modules ($i \in \mathbb{Z}$) of the form

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathcal{F}_{i-1} & \xrightarrow{\Pi_{i-1}} & \mathcal{F}_i & \xrightarrow{\Pi_i} & \mathcal{F}_{i+1} & \xrightarrow{\Pi_{i+1}} & \cdots \\
 & & \uparrow{\scriptstyle \tau_{i-2}} & & \uparrow{\scriptstyle \tau_{i-1}} & & \uparrow{\scriptstyle \tau_i} & & \\
\cdots & \longrightarrow & \sigma^*\mathcal{F}_{i-2} & \xrightarrow{\sigma^*\Pi_{i-2}} & \sigma^*\mathcal{F}_{i-1} & \xrightarrow{\sigma^*\Pi_{i-1}} & \sigma^*\mathcal{F}_i & \xrightarrow{\sigma^*\Pi_i} & \cdots
\end{array}$$

subject to the following conditions (for all $i \in \mathbb{Z}$):

1. the above diagram is commutative,
2. the morphism $\Pi_{i+\ell-1} \circ \ldots \circ \Pi_i$ identifies $\mathcal{F}_i$ with the subsheaf $\mathcal{F}_{i+\ell}(-k \cdot \infty)$ of $\mathcal{F}_{i+\ell}$,
3. $pr_* \operatorname{coker} \Pi_i$ is a locally free $\mathcal{O}_S$-module of rank d,
4. $\operatorname{coker} \tau_i$ is annihilated by $\mathcal{J}^d$ and $pr_* \operatorname{coker} \tau_i$ is a locally free $\mathcal{O}_S$-module of rank d.

A *morphism* between two abelian sheaves $(\mathcal{F}_i, \Pi_i, \tau_i)$ and $(\mathcal{F}'_i, \Pi'_i, \tau'_i)$ is a collection of morphisms $\mathcal{F}_i \to \mathcal{F}'_i$ which commute with the Π's and the τ's.

Remark. Abelian sheaves of dimension $d = 1$ are called *elliptic sheaves* and were studied by Drinfeld [6] and Blum-Stuhler [1]. The category of Drinfeld-A-modules of rank r over S is anti-equivalent to the category of elliptic sheaves of rank r over S which satisfy $\deg \mathcal{F}_0 = 1 - r$, see [1, Theorem 3.2.1].

Proposition 1.6. *The push forward along* $\pi : C'_S \to C_S$ *defines a* restriction of coefficients functor

$$\pi_* : \ \underline{\mathcal{F}}' \ = \ (\mathcal{F}'_i, \Pi'_i, \tau'_i) \ \longmapsto \ \pi_*\underline{\mathcal{F}}' \ := \ (\pi_*\mathcal{F}'_i, \pi_*\Pi'_i, \pi_*\tau'_i)$$

from abelian sheaves on C' *of rank* r'*, dimension* d' *and characteristic* $c' : S \to C'$ *over* S *to abelian sheaves on* C *of rank* nr'*, dimension* d' *and characteristic* $\pi \circ c' : S \to C$ *over* S*. Here* n *is the degree of* π*.*

Proof. Since π is finite and flat the sheaves $\pi_*\mathcal{F}_i$ are locally free of rank nr' by [3, Corollary 2 to Proposition II.3.2.5]. Let k and ℓ be relatively prime positive integers with $\frac{k}{\ell} = \frac{d'}{nr' \deg(\infty)}$. Let e be the ramification index of π at ∞'. Then $n = e \deg(\infty')/\deg(\infty)$ and hence $k = k'/\gcd(k', e)$ and $\ell = \ell' e/\gcd(k', e)$. From axiom 2 of Definition 1.5 we obtain an isomorphism

$$\Pi'_{i+\ell-1} \circ \ldots \circ \Pi'_i \ : \ \mathcal{F}'_i \ \xrightarrow{\ \sim\ } \ \mathcal{F}'_{i+\ell} \otimes_{\mathcal{O}_{C'_S}} \mathcal{O}_{C'_S}(-ke \cdot \infty') \, .$$

Since $\pi^*\mathcal{O}_{C_S}(\infty) = \mathcal{O}_{C'_S}(e \cdot \infty')$ the projection formula

$$\pi_*\big(\mathcal{F}'_{i+\ell} \otimes_{\mathcal{O}_{C'_S}} \mathcal{O}_{C'_S}(-ke \cdot \infty')\big) \ = \ (\pi_*\mathcal{F}'_{i+\ell}) \otimes_{\mathcal{O}_{C_S}} \mathcal{O}_{C_S}(-k \cdot \infty)$$

yields

$$\pi_*\Pi'_{i+\ell-1} \circ \ldots \circ \pi_*\Pi'_i \ : \ \pi_*\mathcal{F}'_i \ \xrightarrow{\ \sim\ } \ (\pi_*\mathcal{F}'_{i+\ell}) \otimes_{\mathcal{O}_{C_S}} \mathcal{O}_{C_S}(-k \cdot \infty)$$

from which the proposition is evident. □

Remark. Consider the contravariant moduli functor

$$\begin{aligned} \underline{C\text{-Ab-Sh}}^{r,d}: \ \mathcal{S}ch_{/C} &\longrightarrow \mathcal{S}ets \\ (c: S \to C) &\mapsto \left\{ \begin{array}{l} \text{Isomorphism classes of abelian sheaves on } C \text{ of} \\ \quad \text{rank } r, \text{ dimension } d, \text{ and characteristic } c \text{ over } S \end{array} \right\} \end{aligned}$$

Also this functor is not representable (not even after adding level structures, see [9, Remark 4.2]). Again the restriction of coefficients functor defines a *restriction of coefficients morphism*

$$\pi_*: \ \underline{C'\text{-Ab-Sh}}^{r',d'} \longrightarrow \underline{C\text{-Ab-Sh}}^{nr',d'}, \quad \underline{\mathcal{F}}' \mapsto \pi_* \underline{\mathcal{F}}'.$$

Remark. If we let S vary, the category of abelian sheaves on C of rank r and dimension d becomes a stack $C\text{-}\mathcal{A}b\text{-}\mathcal{S}h^{r,d}$ for the fppf topology on the category of $\mathbb{F}_q$-schemes. It is an algebraic stack in the sense of Deligne-Mumford [4] by [9, Theorem 3.1]. The restriction of coefficients functor defines a *restriction of coefficients 1-morphism* $\pi_*: C'\text{-}\mathcal{A}b\text{-}\mathcal{S}h^{r',d'} \to C\text{-}\mathcal{A}b\text{-}\mathcal{S}h^{nr',d'}$.

The construction of [1, Theorem 3.2.1] yields a 1-isomorphism of $\mathcal{D}r\text{-}A\text{-}\mathcal{M}od^r$ with an open and closed substack of $C\text{-}\mathcal{A}b\text{-}\mathcal{S}h^{r,1}$, see [9, Example 1.8] such that the following diagram is 2-commutative

$$\begin{tikzcd} \mathcal{D}r\text{-}A'\text{-}\mathcal{M}od^{r'} \arrow[r, hook] \arrow[d, "\pi_*"] & C'\text{-}\mathcal{A}b\text{-}\mathcal{S}h^{r',1} \arrow[d, "\pi_*"] \\ \mathcal{D}r\text{-}A\text{-}\mathcal{M}od^{nr'} \arrow[r, hook] & C\text{-}\mathcal{A}b\text{-}\mathcal{S}h^{nr',1}. \end{tikzcd}$$

Finally let us turn to Drinfeld-Anderson shtuka.

Definition 1.7. A *right (left) Drinfeld-Anderson shtuka* $\underline{\mathcal{E}} = (\mathcal{E}, \widetilde{\mathcal{E}}, j, \tau, b, c)$ *on* C *of rank* r *and dimension* d *over* S consists of two $\mathbb{F}_q$-morphisms $b, c: S \to C$ and a diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "j"] & \widetilde{\mathcal{E}} \\ \sigma^*\mathcal{E} \arrow[ur, "\tau"'] & \end{tikzcd} \qquad \left(\text{resp.} \quad \begin{tikzcd} & \mathcal{E} \\ \widetilde{\mathcal{E}} \arrow[ur, "\tau"] \arrow[r, "j"'] & \sigma^*\mathcal{E} \end{tikzcd} \right)$$

of locally free sheaves $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ of rank r on C_S such that $\operatorname{coker} j$, respectively $\operatorname{coker} \tau$, are locally free of rank d as $\mathcal{O}_S$-modules and supported on the graphs of b, respectively c. The morphism b is called the *pole of* $\underline{\mathcal{E}}$ and c is called the *zero of* $\underline{\mathcal{E}}$.

Remark. Every abelian sheaf $(\mathcal{F}_i, \Pi_i, \tau_i)$ on C of rank r, dimension d, and characteristic c over S gives rise to a right Drinfeld-Anderson shtuka on C over S by setting for any $i \in \mathbb{Z}$

$$\mathcal{E} := \mathcal{F}_i, \quad \widetilde{\mathcal{E}} := \mathcal{F}_{i+1}, \quad j := \Pi_i, \quad \tau := \tau_i.$$

This defines a faithful functor from abelian sheaves to Drinfeld-Anderson shtuka on C over S. Together with the functor from Drinfeld-A-modules to elliptic sheaves on C one obtains a fully faithful functor from Drinfeld-A-modules of rank r over S to Drinfeld-Anderson shtuka on C of rank r and dimension 1 over S, see Drinfeld [7, §1]

The argument of Proposition 1.6 also shows

Proposition 1.8. *Relaxing the conditions on $\pi : C' \to C$ assume only that π is finite of degree n. Then the push forward along π defines a* restriction of coefficients functor

$$\pi_* : \ (\mathcal{E}\,,\,\widetilde{\mathcal{E}}\,,\,j\,,\,\tau\,,\,b\,,\,c) \ \longmapsto \ (\pi_*\mathcal{E}\,,\,\pi_*\widetilde{\mathcal{E}}\,,\,\pi_*j\,,\,\pi_*\tau\,,\,\pi\circ b\,,\,\pi\circ c)$$

from Drinfeld-Anderson shtuka on C' of rank r' and dimension d' to Drinfeld-Anderson shtuka on C of rank nr' and dimension d' over S. □

Remark. Consider the contravariant moduli functor

$$\begin{aligned}\underline{C\text{-DA-Sht}}^{r,d} : \ \mathcal{S}ch_{/C\times C} \quad &\longrightarrow \quad \mathcal{S}ets \\ \big((b,c) : S \to C \times_{\mathbb{F}_q} C\big) \quad &\mapsto \quad \left\{\begin{array}{l}\text{Isomorphism classes of Drinfeld-Anderson shtuka on} \\ C \text{ of rank } r, \text{ dimension } d, \text{ pole } b, \text{ and zero } c \text{ over } S\end{array}\right\}\end{aligned}$$

Also this functor is not representable but the restriction of coefficients functor defines a *restriction of coefficients morphism*

$$\pi_* : \ \underline{C'\text{-DA-Sht}}^{r',d'} \ \longrightarrow \ \underline{C\text{-DA-Sht}}^{nr',d'}\,, \quad \underline{\mathcal{F}'} \ \mapsto \ \pi_*\underline{\mathcal{F}'}\,.$$

Here again the category of Drinfeld-Anderson shtuka of rank r and dimension d over varying $\mathbb{F}_q$-schemes S is an algebraic stack $C\text{-}\mathcal{DA}\text{-}\mathcal{S}ht^{\,r,d}$ for the fppf topology in the sense of Deligne-Mumford [4] and the restriction of coefficients functor defines a *restriction of coefficients 1-morphism* $\pi_* : C'\text{-}\mathcal{DA}\text{-}\mathcal{S}ht^{\,r',d'} \to C\text{-}\mathcal{DA}\text{-}\mathcal{S}ht^{\,nr',d'}$.

2 Restriction of Coefficients for Drinfeld Modules

Theorem 2.1. *The restriction of coefficient morphism $\pi_* : \underline{\text{Dr-}A'\text{-Mod}}^{r'} \to \underline{\text{Dr-}A\text{-Mod}}^{nr'}$ for Drinfeld modules is in general not relatively representable.*

Proof. We give a counterexample to relative representability. Let $q = 3$, $A = \mathbb{F}_3[x]$, $A' = \mathbb{F}_3[y]$ and $\pi^* \colon A \to A', x \mapsto y^2$. Let $S = \operatorname{Spec}\mathbb{F}_3$ and $c^* \colon A \to \mathbb{F}_3, x \mapsto 0$. Consider the Drinfeld-$A$-module (E,φ) of rank 2 over S given by $E = \mathbb{G}_{a,S}$ and

$$\varphi : \ A \longrightarrow \mathbb{F}_3\{\tau\}\,, \quad \varphi(x) = \tau^2\,.$$

Let $\underline{T} := \underline{\text{Dr-}A'\text{-Mod}}^1 \times_{\underline{\text{Dr-}A\text{-Mod}}^2} S$ be the fiber product of functors. Then $\underline{T}$ is the contravariant functor

$$\begin{aligned}\underline{T} : \ \mathcal{S}ch_{/\operatorname{Spec}A'\times_{\operatorname{Spec}A}S} \quad &\longrightarrow \quad \mathcal{S}ets \\ \left(\begin{array}{l}S', c' : S' \to \operatorname{Spec}A' \\ \quad f : S' \to S\end{array}\right) \quad &\mapsto \quad \left\{\begin{array}{l}\text{Isomorphism classes of Drinfeld-}A'\text{-modules } (E',\varphi') \\ \text{of rank 1 over } S', \text{ such that } f^*(E,\varphi) \cong \pi_*(E',\varphi')\end{array}\right\}.\end{aligned}$$

We show that $\underline{T}$ is not representable. For this purpose make S into a $\operatorname{Spec}A'$-scheme by $(c')^* : A' \to \mathbb{F}_3, y \mapsto 0$. Then $\underline{T}(S)$ contains two isomorphism classes given by $E'_1 = E'_2 = \mathbb{G}_{a,S}$ and

$$\varphi'_1 : \ y \mapsto \tau \qquad \text{and} \qquad \varphi'_2 : \ y \mapsto -\tau\,.$$

These two isomorphism classes are different because otherwise there were an isomorphism

$$\varepsilon \;\in\; \mathrm{Isom}\big((E'_1,\varphi'_1),(E'_2,\varphi'_2)\big) \;=\; \big\{\, \varepsilon\in\mathcal{O}_S^\times :\; -\tau\circ\varepsilon=\varepsilon\circ\tau \,\big\}\,.$$

That is, $\varepsilon\in\mathbb{F}_3^\times$ must satisfy $-\varepsilon^3\tau=\varepsilon\tau$, whence $\varepsilon^2=-1$. This is impossible for $\varepsilon\in\mathbb{F}_3^\times$.

On the other hand such an element exists in $\mathbb{F}_9^\times$. So if $S'=\mathrm{Spec}\,\mathbb{F}_9$ the two isomorphism classes become equal in $\underline{T}(S')$. But this implies that $\underline{T}$ is not representable. Since if it were representable by a scheme T we had two different morphisms from S to T which yield the same morphism from S' to T

$$\mathrm{Spec}\,\mathbb{F}_9 \;\longrightarrow\; \mathrm{Spec}\,\mathbb{F}_3 \;\rightrightarrows\; T\,.$$

As $\mathrm{Spec}\,\mathbb{F}_9\to\mathrm{Spec}\,\mathbb{F}_3$ is a homeomorphism and $\mathbb{F}_3\subset\mathbb{F}_9$ this is impossible. □

Remark. The reason why $\underline{T}$ is not representable is that the isomorphism $\alpha: f^*(E,\varphi)\xrightarrow{\;\sim\;}\pi_*(E',\varphi')$ in the definition of $\underline{T}(S)$ is only supposed to exist but is not added to the data. More precisely we have

Theorem 2.2. *Let $c:S\to\mathrm{Spec}\,A$ be a morphism of $\mathbb{F}_q$-schemes and let (E,φ) be a Drinfeld-A-module of rank nr' and characteristic c over S. Then the contravariant functor*

$$\begin{array}{llll} \underline{T}: & \mathcal{S}ch_{/\,\mathrm{Spec}\,A'\times_{\mathrm{Spec}\,A}S} & \longrightarrow & \mathcal{S}ets \\ & \left(\begin{array}{c} S',\,c':S'\to\mathrm{Spec}\,A' \\ f:S'\to S \end{array}\right) & \mapsto & \left\{\begin{array}{l} \textit{Isomorphism classes of tripples } (E',\varphi',\alpha) \textit{ where} \\ \bullet\ (E',\varphi') \textit{ is a Drinfeld-}A'\textit{-module of rank } r' \\ \quad \textit{and characteristic } c' \textit{ over } S' \textit{ and} \\ \bullet\ \alpha: f^*(E,\varphi)\xrightarrow{\;\sim\;}\pi_*(E',\varphi') \textit{ is a fixed isomorphism} \end{array}\right\} \end{array}$$

is representable by an affine S-scheme of finite presentation.

Proof. Since the question is local on S we may by Proposition 1.3 assume that $S=\mathrm{Spec}\,B$, $E=\mathbb{G}_{a,B}$ and φ is given by $\varphi:A\to B\{\tau\}$ such that the highest coefficient of every $\varphi(a)$ is a unit in B.

Let the A-algebra A' be generated by $a'_1,\ldots,a'_N$. In order to extend φ to $\varphi':A'\to B\{\tau\}$ we must define $\varphi'(a'_1),\ldots,\varphi'(a'_N)$. Set $d_\nu:=-r'\,\mathrm{ord}_{\infty'}(a'_\nu)\deg(\infty')$ for all ν. Define

$$B' \;:=\; B\otimes_A A'\,[\,\delta_{i,\nu},\delta_{d_\nu,\nu}^{-1}:\;\nu=1,\ldots,N\,,\,i=0,\ldots,d_\nu\,]$$

and the morphism $c':\mathrm{Spec}\,B'\to\mathrm{Spec}\,A'$ by the natural map $A'\to B'$. Define

$$\varphi'(a'_\nu)\;:=\;\sum_{i=0}^{d_\nu}\delta_{i,\nu}\tau^i\;\in\;B'\{\tau\}\,,$$

and $\varphi'|_A:=\varphi$, and let $\alpha=\mathrm{id}_{\mathbb{G}_{a,B'}}$. In order that the so defined φ' is a Drinfeld-A'-module of rank r' and characteristic c' over $\mathrm{Spec}\,B'$ we must require several conditions which are all represented by finitely presented closed subschemes of $\mathrm{Spec}\,B'$. Namely consider successively for $\nu=1,\ldots,N$ the minimal polynomial of a'_ν over $A(a'_1,\ldots,a'_{\nu-1})$

$$(a'_\nu)^m+b_{\nu,m-1}(a'_\nu)^{m-1}+\ldots+b_{\nu,1}a'_\nu+b_{\nu,0}\;=\;0$$

with $b_{\nu,k} \in A(a'_1, \ldots, a'_{\nu-1})$. The fact that $\varphi' : A' \to B'\{\tau\}$ is a ring homomorphism is now expressed by the vanishing of

$$\varphi'(a'_\nu)^m + \varphi'(b_{\nu,m-1})\varphi'(a'_\nu)^{m-1} + \ldots + \varphi'(b_{\nu,1})\varphi'(a'_\nu) + \varphi'(b_{\nu,0}) \;=\; 0$$

in $B'\{\tau\}$. Looking at the coefficients of this τ-polynomial we get a finitely generated ideal of B' which we must require to vanish, that is, must divide out. Likewise the commutation of $\varphi'(a'_\nu)$ with a (finite) generating system of the $\mathbb{F}_q$-algebra $A(a'_1, \ldots, a'_{\nu-1})$ yields a finitely generated ideal of B'. Finally the condition on the characteristic means that $(c')^*(a'_\nu) = \delta_{0,\nu}$. Putting everything together the sum of these ideals defines a closed subscheme $T \subset \operatorname{Spec} B'$ which is of finite presentation and affine over S.

We claim that T represents $\underline{T}$. So let (E', φ', α) be an element of $\underline{T}(S')$. The isomorphism $\alpha : f^*\mathbb{G}_{a,S} \xrightarrow{\sim} E'$ yields an isomorphism $\alpha : (\mathbb{G}_{a,S'}, \psi') \xrightarrow{\sim} (E', \varphi')$ of Drinfeld-A'-modules over S' where $\psi'(a) := \alpha^{-1} \circ \varphi'(a) \circ \alpha$ for all $a \in A'$. Since $\psi'(a) = f^*\varphi(a)$ for $a \in A$, ψ' is of the form described in Proposition 1.3. In particular

$$\psi'(a'_\nu) \;=\; \sum_{i=0}^{d_\nu} \delta_i(a'_\nu)\tau^i \;\in\; \Gamma(S', \mathcal{O}_{S'})\{\tau\}\,.$$

Mapping $\delta_{i,\nu}$ to $\delta_i(a'_\nu)$ defines the desired uniquely determined morphism $B' \to \Gamma(S', \mathcal{O}_{S'})$, whence $S' \to T$. $\square$

Proposition 2.3. *In the situation of Theorem 2.2 the scheme T representing $\underline{T}$ is finite over S.*

Proof. We already know that T is separated and of finite presentation over S. We use the valuative criterion of properness to show that it is proper. Since it is also affine over S it must be finite.

So let R be a valuation ring with fraction field K and consider the diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "{(E',\varphi',\alpha)}"] \arrow[d, "g"'] & T \arrow[d, "\pi_*"] \\
\operatorname{Spec} R \arrow[r, "f"'] \arrow[ur, dashed, "{(\widetilde{E},\widetilde{\varphi},\widetilde{\alpha})}"'] & S
\end{tikzcd}$$

where the horizontal arrow on top is given by a Drinfeld-A'-module (E', φ') of rank r' and characteristic $c' : \operatorname{Spec} K \to \operatorname{Spec} A'$ together with an isomorphism $\alpha : (fg)^*(E, \varphi) \xrightarrow{\sim} \pi_*(E', \varphi')$ over $\operatorname{Spec} K$. We must exhibit the dashed arrow which corresponds to a Drinfeld-A'-module $(\widetilde{E}, \widetilde{\varphi})$ of rank r' and characteristic $\tilde{c} : \operatorname{Spec} R \to \operatorname{Spec} A'$ (note that c' factors through a unique morphism $\tilde{c}$ satisfying $\pi \circ \tilde{c} = c \circ f : \operatorname{Spec} R \to \operatorname{Spec} A$ because $\operatorname{Spec} A'$ is proper over $\operatorname{Spec} A$) together with an isomorphism $\widetilde{\alpha} : f^*(E, \varphi) \xrightarrow{\sim} \pi_*(\widetilde{E}, \widetilde{\varphi})$ over $\operatorname{Spec} R$. The commutativity of the diagram means that there exists an isomorphism $\beta : g^*(\widetilde{E}, \widetilde{\varphi}) \xrightarrow{\sim} (E', \varphi')$ with $\pi_*\beta \circ g^*\widetilde{\alpha} = \alpha$.

Since R is a local ring $f^*E = \mathbb{G}_{a,R}$ without loss of generality and $f^*\varphi : A \to R\{\tau\}$. We use the isomorphism α to replace (E', φ') by $(\mathbb{G}_{a,K}, \psi')$ with $\psi'(a) := \alpha^{-1} \circ \varphi'(a) \circ \alpha \in K\{\tau\}$ for all $a \in A'$. Thus α is replaced by $\mathrm{id}_{\mathbb{G}_{a,K}}$ and $\psi'(a) = (fg)^*\varphi(a)$ for all $a \in A$. If we show that $\psi'(a)$ belongs to $R\{\tau\}$ for all $a \in A'$, then we may take $\widetilde{E} = \mathbb{G}_{a,R}$ and $\widetilde{\varphi} = \psi' : A' \to R\{\tau\}$, as well as $\widetilde{\alpha} = \mathrm{id}_{\mathbb{G}_{a,R}}$ and $\beta = \mathrm{id}_{\mathbb{G}_{a,K}}$, and we are done.

So let $a \in A'$ and let $a^m + b_{m-1}a^{m-1} + \ldots + b_1 a + b_0 = 0$ be an equation of integral dependence of a over A. In particular

$$\psi'(a)^m + \psi'(b_{m-1})\psi'(a)^{m-1} + \ldots + \psi'(b_1)\psi'(a) + \psi'(b_0) \;=\; 0\,. \tag{2.1}$$

Over an algebraic closure of K the polynomial $\psi'(a)(x)$, where we use $\tau(x) = x^q$, splits as $\psi'(a)(x) = \prod_i (x - \lambda_i)$ with $\lambda_i \in K^{\mathrm{alg}}$. From equation (2.1) we see that each λ_i is a root of $\psi'(b_0) = (fg)^*\varphi(b_0)$. Since $f^*\varphi(b_0)$ has coefficients in R with the highest coefficient in $R^\times$, all λ_i must be integral over R. Therefore the coefficients of $\psi'(a)$ which are symmetric polynomials in the λ_i are integral over R and belong to K, hence they lie in R as desired. This proves the proposition. $\square$

Theorem 2.4. *The restriction of coefficients morphism* $\pi_* : \underline{\text{Dr-}A'\text{-Mod}}^{r'} \to \underline{\text{Dr-}A\text{-Mod}}^{nr'}$ *satisfies the valuative criterion for properness.*

Proof. Let R be a valuation ring with fraction field K and consider the diagram

$$\begin{tikzcd}
 & \underline{T} \arrow[d] & \\
\operatorname{Spec} K \arrow[r, "{(E',\varphi')}"] \arrow[d, "f"'] \arrow[ur, bend left, "{(E',\varphi',\alpha)}"] & \underline{\text{Dr-}A'\text{-Mod}}^{r'} \times_{\underline{\text{Dr-}A\text{-Mod}}^{nr'}} \operatorname{Spec} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\Box"] & \underline{\text{Dr-}A'\text{-Mod}}^{r'} \arrow[d, "\pi_*"] \\
\operatorname{Spec} R \arrow[r, equal] \arrow[ur, dashed] & \operatorname{Spec} R \arrow[r, "{(E,\varphi)}"] & \underline{\text{Dr-}A\text{-Mod}}^{nr'}
\end{tikzcd}$$

where the horizontal morphisms are induced by a Drinfeld-A'-module (E', φ') of rank r' and characteristic $c' : \operatorname{Spec} K \to \operatorname{Spec} A'$ over $\operatorname{Spec} K$ and a Drinfeld-A-module (E, φ) of rank nr' and characteristic $c : \operatorname{Spec} R \to \operatorname{Spec} A$ over $\operatorname{Spec} R$ and where $\underline{T}$ is the representable functor from Theorem 2.2 for $S = \operatorname{Spec} R$. The commutativity of the square on the left means that $f^*(E, \varphi) \cong \pi_*(E', \varphi')$. The choice of any such isomorphism α defines a morphism $\operatorname{Spec} K \to \underline{T}$. By Proposition 2.3 we find a unique morphism $\operatorname{Spec} R \to \underline{T}$ fitting into the diagram which induces the dashed morphism. It remains to show that the dashed morphism is uniquely determined (independent of the choice of α) and this is proved in the following lemma. $\square$

Lemma 2.5. *Let R be a valuation ring with fraction field K and let* $f : \operatorname{Spec} K \to \operatorname{Spec} R$ *be the induced morphism. Let (E'_1, φ'_1) and (E'_2, φ'_2) be two Drinfeld-A'-modules of rank r' and characteristic* $c' : \operatorname{Spec} R \to \operatorname{Spec} A'$ *over* $\operatorname{Spec} R$ *and let* $\alpha : f^*(E'_1, \varphi'_1) \xrightarrow{\sim} f^*(E'_2, \varphi'_2)$ *be an isomorphism over* $\operatorname{Spec} K$. *Then $\alpha = f^*\beta$ for a unique isomorphism* $\beta : (E'_1, \varphi'_1) \xrightarrow{\sim} (E'_2, \varphi'_2)$ *over* $\operatorname{Spec} R$.

Proof. Since R is a local ring we have without loss of generality $E'_1 = E'_2 = \mathbb{G}_{a,R}$. Let $a \in A' \smallsetminus \mathbb{F}_q$ and write for $j = 1, 2$

$$\varphi'_j(a) \;=\; \sum_{i=0}^{m} \delta_{i,j}\tau^i\,.$$

The isomorphism over $\operatorname{Spec} K$ is given by an element $\alpha \in K^\times$ which satisfies $\varphi_2'(a) \circ \alpha = \alpha \circ \varphi_1'(a)$, whence $\delta_{2,m}\, \alpha^{q^m} = \alpha\, \delta_{1,m}$. Since $\delta_{1,m}$ and $\delta_{2,m}$ are units in R the same is true for α. So the isomorphism α is already defined over $\operatorname{Spec} R$. □

Remark. Phrased in the language of stacks [13], Theorems 2.1 and 2.4 say that the restriction of coefficients 1-morphism $\pi_* : \mathcal{D}r\text{-}A'\text{-}\mathcal{M}od^{\,r'} \to \mathcal{D}r\text{-}A\text{-}\mathcal{M}od^{\,nr'}$ is proper but in general not representable. Namely by the arguments of Theorem 2.2 the stack $\mathcal{T}$ classifying data $\big((E,\varphi),\, (E',\varphi,),\, \alpha\big)$ where (E,φ), respectively (E',φ'), is a Drinfeld-A-module of rank nr', respectively a Drinfeld-A'-module of rank r' over the same scheme S together with a fixed isomorphism $\alpha : (E,\varphi) \xrightarrow{\sim} \pi_*(E',\varphi')$ over S is relatively representable over $\mathcal{D}r\text{-}A\text{-}\mathcal{M}od^{\,nr'}$ by a finite and finitely presented morphism of schemes. The projection $\mathcal{T} \to \mathcal{D}r\text{-}A'\text{-}\mathcal{M}od^{\,r'}$ onto (E',φ') is an étale epimorphism and makes $\mathcal{T}$ into a torsor under the finite relative group scheme $\operatorname{Aut}\big(\pi_*(E',\varphi')\big)$ over $\mathcal{D}r\text{-}A'\text{-}\mathcal{M}od^{\,r'}$. In particular $\mathcal{D}r\text{-}A'\text{-}\mathcal{M}od^{\,r'}$ is of finite presentation over $\mathcal{D}r\text{-}A\text{-}\mathcal{M}od^{\,nr'}$ since $\mathcal{T}$ is and it satisfies the valuative criterion for properness by the arguments of Theorem 2.4.

3 Restriction of Coefficients for Abelian Sheaves

Theorem 3.1. *The restriction of coefficients morphism $\pi_* : \underline{C'\text{-}\mathrm{Ab}\text{-}\mathrm{Sh}}^{r',d'} \to \underline{C\text{-}\mathrm{Ab}\text{-}\mathrm{Sh}}^{nr',d'}$ for abelian sheaves is in general not relatively representable.*

Proof. This follows directly from Theorem 2.1 and the remark after Definition 1.5. The example from Theorem 2.1 yields the following abelian sheaf on $C = \mathbb{P}^1_{\mathbb{F}_3}$ over $S = \operatorname{Spec} \mathbb{F}_3$. Let $\mathcal{F}_i = \mathcal{O}_{C_S}\big(\lfloor\frac{i+1}{2}\rfloor \cdot \infty\big) \oplus \mathcal{O}_{C_S}\big(\lfloor\frac{i}{2}\rfloor \cdot \infty\big)$ where $\lfloor\frac{i}{2}\rfloor$ is the largest integer $\le \frac{i}{2}$. Let Π_i be the natural inclusion $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ and let $\tau_i : \sigma^*\mathcal{F}_i \to \mathcal{F}_{i+1}$ be given by the matrix $\begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix}$ where $\mathbb{P}^1_{\mathbb{F}_3} \smallsetminus \{\infty\} = \operatorname{Spec} \mathbb{F}_3[x]$.

Let $\pi : C' = \mathbb{P}^1_{\mathbb{F}_3} \to C$ be given by $A \to A' = \mathbb{F}_3[y]$, $x \mapsto y^2$. Then $(\mathcal{F}_i, \Pi_i, \tau_i)$ is isomorphic to $\pi_*(\mathcal{F}_i', \Pi_i', \tau_i')$ for $\mathcal{F}_i' = \mathcal{O}_{C_S'}(i \cdot \infty')$, Π_i' the natural inclusion, and $\tau_i' = \pm y$. The two abelian sheaves for $\tau_i' = +y$ and $\tau_i' = -y$ are not isomorphic over $\operatorname{Spec} \mathbb{F}_3$ but become isomorphic over $\operatorname{Spec} \mathbb{F}_9$. □

Theorem 3.2. *Let S be a locally noetherian $\mathbb{F}_q$-scheme and let $c : S \to C$ be an $\mathbb{F}_q$-morphism. Let $\underline{\mathcal{F}}$ be an abelian sheaf on C of rank nr', dimension d' and characteristic c over S. Then the contravariant functor*

$$\begin{array}{rcl} \underline{\mathcal{T}} :\ \mathcal{S}ch_{/C'\times_C S} & \longrightarrow & \mathcal{S}ets \\ \left(\begin{array}{r} S',\, c' : S' \to C' \\ f : S' \to S \end{array}\right) & \mapsto & \left\{ \begin{array}{l} \textit{Isomorphism classes of pairs } (\underline{\mathcal{F}}', \alpha) \textit{ where} \\ \bullet\ \underline{\mathcal{F}}' \textit{ is an abelian sheaf of rank } r', \textit{ dimension } d', \\ \quad \textit{and characteristic } c' \textit{ over } S' \textit{ and} \\ \bullet\ \alpha : f^*\underline{\mathcal{F}} \xrightarrow{\sim} \pi_*\underline{\mathcal{F}}' \textit{ is a fixed isomorphism} \end{array} \right\} \end{array}$$

is representable by a (quasi-affine) S-scheme of finite type.

For the proof we need the following

Lemma 3.3. *Let S be a locally noetherian scheme, let $\rho : Y \to S$ be a flat projective morphism, and let $\pi : X \to Y$ be a finite faithfully flat morphism of degree n. For an S-scheme S' set $Y' := Y \times_S S'$ and $X' := X \times_S S'$. Let $\mathcal{F}$ be a locally free sheaf on Y of rank rn. Then the contravariant functor*

$$\begin{aligned} \underline{U} : \ \mathcal{S}ch_{/S} \ &\longrightarrow \ \mathcal{S}ets \\ (f : S' \to S) \ &\mapsto \ \Big\{ \text{Isomorphism classes of pairs } (\mathcal{F}', \alpha) \text{ where} \\ &\qquad \bullet\ \mathcal{F}' \text{ is a locally free sheaf of rank } r \text{ on } X' \text{ and} \\ &\qquad \bullet\ \alpha : f^*\mathcal{F} \xrightarrow{\ \sim\ } \pi_*\mathcal{F}' \text{ is a fixed isomorphism} \Big\} \end{aligned}$$

is representable by a (quasi-affine) S-scheme of finite type.

Proof. Since the question is local on S we may assume that S is affine. By [EGA, II, Proposition 1.4.3] the functor $\underline{U}$ is isomorphic to the functor

$$\underline{U}' : \ (f : S' \to S) \ \mapsto \ \mathrm{Hom}_{\mathcal{O}_{Y'}\text{-algebras}}\big(\pi_*\mathcal{O}_{X'} \, , \, \mathcal{E}nd_{\mathcal{O}_{Y'}}(f^*\mathcal{F})\big)$$

the set of $\mathcal{O}_{Y'}$-algebra homomorphisms $\pi_*\mathcal{O}_{X'} \to \mathcal{E}nd_{\mathcal{O}_{Y'}}(f^*\mathcal{F})$. Fix an ample invertible sheaf $\mathcal{L}$ on Y. For any integer N define $\mathcal{H}_N := \mathcal{H}om_{\mathcal{O}_Y}\big(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes N}\big) = \mathcal{E}nd_{\mathcal{O}_Y}(f^*\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes N}$. Then for the homomorphisms of $\mathcal{O}_{Y'}$*-modules* we obtain

$$\mathrm{Hom}_{\mathcal{O}_{Y'}\text{-modules}}\big(\pi_*\mathcal{O}_{X'} \, , \, \mathcal{E}nd_{\mathcal{O}_{Y'}}(f^*\mathcal{F})\big) \ = \ \mathrm{Hom}_{\mathcal{O}_{Y'}\text{-mod}}\big(\pi_*\mathcal{O}_{X'} \otimes_{\mathcal{O}_{Y'}} \mathcal{L}^{\otimes N} \, , \, f^*\mathcal{H}_N\big) \, .$$

There is an integer N such that

- $\pi_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes N}$ is generated by global sections and
- $\rho_*(\pi_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes N})$ and $\rho_*\mathcal{H}_N$ are locally free on S

since these conditions are achieved for $N \gg 0$ by the Theorem on Cohomology and Base Change [10, Theorem III.12.11].

Shrinking S we let $x_1, \ldots, x_m$ be an $\mathcal{O}_S$-basis of $\rho_*(\pi_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes N})$. We must specify their images in $\rho_*\mathcal{H}_N$. Let $U_1 := \underline{\mathrm{Spec}}_S \, \mathrm{Sym}_{\mathcal{O}_S}(\rho_*\mathcal{H}_N)^{\vee}$ and $U_2 := U_1 \times_S \ldots \times_S U_1$ the m-fold fiber product. Let $f : U_2 \to S$ be the induced morphism and set $Y_2 := Y \times_S U_2$ and $X_2 := X \times_S U_2$. Then for any S-scheme S'

$$\begin{aligned} \mathrm{Hom}_S(S', U_1) \ &= \ \mathrm{Hom}_{\mathcal{O}_S\text{-algebras}}\big(\mathrm{Sym}_{\mathcal{O}_S}(\rho_*\mathcal{H}_N)^{\vee} \, , \, \mathcal{O}_{S'}\big) \\ &= \ \mathrm{Hom}_{\mathcal{O}_S\text{-modules}}\big((\rho_*\mathcal{H}_N)^{\vee} \, , \, \mathcal{O}_{S'}\big) \\ &= \ \Gamma\big(S' \, , \, \mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \rho_*\mathcal{H}_N\big) \, . \end{aligned}$$

So on U_2 there exist m universal global sections of $\rho_*\mathcal{H}_N$ which we use as the images of our $x_1, \ldots, x_m$ to obtain a universal homomorphism of $\mathcal{O}_{U_2}$-modules

$$\rho_*\big(\pi_*\mathcal{O}_{X_2} \otimes_{\mathcal{O}_{Y_2}} f^*\mathcal{L}^{\otimes N}\big) \ \longrightarrow \ \rho_* f^*\mathcal{H}_N \, . \tag{3.2}$$

Next we take care of the $\mathcal{O}_Y$-algebra structures. Every x_i has a minimal polynomial over $\Gamma(Y, \mathcal{L}^{\otimes N})[x_1, \ldots, x_{i-1}]$ of the form

$$P(x_i) \ := \ x_i^k + a_{k-1}x_i^{k-1} + \ldots + a_0 \ = \ 0$$

inside $\Gamma(Y, \pi_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes Nk})$. Using our homomorphism (3.2) and the $\mathcal{O}_{U_2}$-module structure of $f^*\mathcal{F}$ we view $P(x_i)$ as an element of $\Gamma(U_2, f^*\rho_*\mathcal{H}_{Nk})$. The requirement that this element vanishes defines a closed subscheme of U_2 by Lemma 3.4 below. Let U_3 be the closed subscheme of U_2 obtained in this way for $i = 1, \ldots, m$. This yields a $\pi_*\mathcal{O}_{X_3}$-module structure on $f^*\mathcal{F}$, whence (an isomorphism class of) a coherent sheaf $\mathcal{F}_3$ on $X_3 := X \otimes_S U_3$ together with an isomorphism $\alpha : f^*\mathcal{F} \xrightarrow{\sim} \pi_*\mathcal{F}_3$.

It remains to represent the condition that $\mathcal{F}_3$ is locally free. Let $V \subset X_3$ be the open subscheme on which $\mathcal{F}_3$ is flat, see [EGA, IV$_3$, Theorem 11.1.1]. Define $U := U_3 \smallsetminus \pi(X_3 \smallsetminus V)$. Since $\rho\pi : X_3 \to U_3$ is proper $U \subset U_3$ is open. Since $(\rho\pi)^{-1}U \subset V$ the coherent sheaf $\mathcal{F}_3$ is locally free on $(\rho\pi)^{-1}U$ of rank r. We claim that U represents the functor $\underline{U}$. Indeed, let S' be an S-scheme and $(\mathcal{F}', \alpha) \in \underline{U}(S')$. Then the $\pi_*\mathcal{O}_{X'}$-module structure on $\pi_*\mathcal{F}'$ defines a uniquely determined morphism $S' \to U_3$. Since above every point $s \in S'$ the fiber $\mathcal{F}'_s$ is flat on $X \times_S s$, the image of s in U_3 lands in U by [EGA, IV$_3$, Theorem 11.3.10]. (This is the only place where we use the assumption that π is flat.) This proves the lemma. $\square$

Lemma 3.4. *Let S be a scheme and let $\mathcal{H}$ be a locally free sheaf on S. Let I be a set and let $h_i \in \Gamma(S, \mathcal{H})$ for all $i \in I$. Then the condition $h_i = 0$ for all $i \in I$ is represented by a closed subscheme of S.*

Proof. This is [EGA, 0_{new}, Proposition 5.5.1] taking into account that on a locally noetherian topological space the set of global sections of an arbitrary direct sum equals the direct sum of the global sections. $\square$

Proof of Theorem 3.2. Let $\underline{\mathcal{F}} = (\mathcal{F}_i, \Pi_i, \tau_i)$ and let ℓ' and k' be relatively prime positive integers with $\frac{k'}{\ell'} = \frac{d'}{r' \deg(\infty')}$. For $i = 0, \ldots, \ell'$ let U_i be the scheme from Lemma 3.3 classifying the pairs $(\mathcal{F}'_i, \alpha_i)$ of locally free sheaves $\mathcal{F}'_i$ on $X = C'_S$ and isomorphisms $\alpha_i : \mathcal{F}_i \xrightarrow{\sim} \pi_*\mathcal{F}'_i$. Over $T := U_0 \times_S \ldots \times_S U_{\ell'}$ we have the universal sheaves $\mathcal{F}'_0, \ldots, \mathcal{F}'_{\ell'}$ on C'_T. We need that the morphisms of $\mathcal{O}_{C_T}$-modules

$$\begin{aligned} \Pi'_i &:= \alpha_{i+1} \circ \Pi_i \circ \alpha_i^{-1} : \ \pi_*\mathcal{F}'_i \longrightarrow \pi_*\mathcal{F}'_{i+1} \quad \text{and} \\ \tau'_i &:= \alpha_{i+1} \circ \tau_i \circ \sigma^*\alpha_i^{-1} : \ \pi_*\sigma^*\mathcal{F}'_i \longrightarrow \pi_*\mathcal{F}'_{i+1} \end{aligned}$$

are actually morphisms of $\pi_*\mathcal{O}_{C'_T}$-modules and thus by [EGA, II, Proposition 1.4.3] morphisms $\Pi' : \mathcal{F}'_i \to \mathcal{F}'_{i+1}$ and $\tau'_i : \sigma^*\mathcal{F}'_i \to \mathcal{F}'_{i+1}$.

It suffices to work on an affine covering of T. Let $pr : C_T \to T$ be the projection onto the second factor. Let $\mathcal{L}$ be an ample invertible sheaf on C and let N be an integer such that for $i = 0, \ldots, \ell' - 1$

- $\pi_*\mathcal{O}_{C'_T} \otimes_{\mathcal{O}_{C_T}} \mathcal{L}^{\otimes N}$ is generated by global sections $x_1, \ldots, x_m$,
- $\pi_*\mathcal{F}'_i \otimes_{\mathcal{O}_{C_T}} \mathcal{L}^{\otimes N}$ is generated by global sections $y_1, \ldots, y_n$, and
- $\mathcal{H}_{i+1} := pr_*(\pi_*\mathcal{F}'_{i+1} \otimes_{\mathcal{O}_{C_T}} \mathcal{L}^{\otimes 2N})$ is locally free on T.

Then $\mathcal{G}_i := \pi_*\mathcal{O}_{C'_T} \otimes_{\mathcal{O}_{C_T}} \pi_*\mathcal{F}'_i \otimes_{\mathcal{O}_{C_T}} \mathcal{L}^{\otimes 2N}$ is generated by the $x_\mu \otimes y_\nu$. There are two morphisms of $\mathcal{O}_{C_T}$-modules

$$\mathcal{G}_i \rightrightarrows \pi_*\mathcal{F}'_{i+1} \otimes_{\mathcal{O}_{C_T}} \mathcal{L}^{\otimes 2N}$$

depending on the order in which Π'_i is composed with the contraction $\pi_*\mathcal{O}_{C'_T} \otimes_{\mathcal{O}_{C_T}} \pi_*\mathcal{F}'_i \to \pi_*\mathcal{F}'_i$ (coming from the $\mathcal{O}_{C'_T}$-module structure on $\mathcal{F}'_i$). Whether the difference of these two morphisms is the zero morphism can be tested on the images of the global sections $x_\mu \otimes y_\nu$ inside $\mathcal{H}_{i+1}$. By Lemma 3.4 this condition is represented by a closed subscheme of T.

We proceed analogously for the τ_i and obtain a closed subscheme $T_1 \subset T$ and for $i = 0, \ldots, \ell' - 1$ universal morphisms $\Pi'_i : \mathcal{F}'_i \to \mathcal{F}'_{i+1}$ and $\tau'_i : \sigma^*\mathcal{F}'_i \to \mathcal{F}'_{i+1}$ on C'_{T_1} which satisfy axiom 1 of Definition 1.5. Since $pr_*\pi_* \operatorname{coker} \Pi'_i = pr_* \operatorname{coker} \Pi_i$, and the same for τ_i, also axioms 3 and 4 hold except for the condition on the support. For this condition let $T_2 := C' \times_C T_1$, let $c' : T_2 \to C'$ be the projection and let $\mathcal{J}'$ be the ideal defining the graph of c'. Similarly to the above argument let $\mathcal{L}$ and N be such that $(\mathcal{J}')^{\otimes d'} \otimes_{\mathcal{O}_{C'_{T_2}}} \mathcal{L}^{\otimes N}$ is generated by global sections. Again by Lemma 3.4 the condition that the multiplication morphism

$$pr_*\big((\mathcal{J}')^{\otimes d'} \otimes_{\mathcal{O}_{C'_{T_2}}} \mathcal{L}^{\otimes N} \otimes_{\mathcal{O}_{C'_{T_2}}} \operatorname{coker} \tau'_i\big) \;\longrightarrow\; pr_*\big(\mathcal{L}^{\otimes N} \otimes_{\mathcal{O}_{C'_{T_2}}} \operatorname{coker} \tau'_i\big)$$

is zero is represented by a closed subscheme T_3 of T_2.

Finally for axiom 2 consider the morphism

$$\Pi'_{\ell'-1} \circ \ldots \circ \Pi'_0 : \; \mathcal{F}'_0 \;\longrightarrow\; \mathcal{F}'_{\ell'} \;\longrightarrow\; \mathcal{F}'_{\ell'} \otimes_{\mathcal{O}_{C'_{T_3}}} \mathcal{O}_{C'_{T_3}} / \mathcal{O}_{C'_{T_3}}(-k' \cdot \infty')\,. \tag{3.3}$$

Since $\operatorname{coker} \Pi'_i$ has rank d' axiom 2 is satisfied if and only if the morphism (3.3) is the zero morphism. Using that the target is locally free on T_3 and reasoning as above the later condition is represented by a closed subscheme T_4 of T_3. Over T_4 we define $\mathcal{F}'_{i+m\ell'} := \mathcal{F}'_i(k'm \cdot \infty')$ for all $i = 0, \ldots, \ell' - 1$ and all $m \in \mathbb{Z}$. Then T_4 represents the functor $\underline{T}$. $\square$

Proposition 3.5. *In the situation of Theorem 3.2 the scheme T representing $\underline{T}$ is finite over S.*

Proof. By Theorem 3.2 it is separated, of finite type, and quasi-affine over S. It remains to show that T is proper over S. So let R be a discrete valuation ring with fraction field K and consider the diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "(\underline{\mathcal{F}}'{,}\alpha)"] \arrow[d, "g"'] & T \arrow[d, "\pi_*"] \\ \operatorname{Spec} R \arrow[r, "f"'] \arrow[ur, dashed, "(\underline{\widetilde{\mathcal{F}}}{,}\widetilde{\alpha})"'] & S \end{tikzcd}$$

where the horizontal arrow is given by an abelian sheaf $\underline{\mathcal{F}}'$ on C' over $\operatorname{Spec} K$ of rank r', dimension d' and characteristic $c' : \operatorname{Spec} K \to C'$ together with an isomorphism $\alpha : (fg)^*\underline{\mathcal{F}} \xrightarrow{\sim} \pi_*\underline{\mathcal{F}}'$ on C_K. We need to construct an abelian sheaf $\underline{\widetilde{\mathcal{F}}}$ on C' over $\operatorname{Spec} R$ of rank r', dimension d', and characteristic $\tilde{c} : \operatorname{Spec} R \to C'$ (again the properness of π implies that c' factors through a unique morphism $\tilde{c}$ with $\pi \circ \tilde{c} = c \circ f : \operatorname{Spec} R \to C$) together with an isomorphism $\widetilde{\alpha} : f^*\underline{\mathcal{F}} \xrightarrow{\sim} \pi_*\underline{\widetilde{\mathcal{F}}}$ on C_R and an isomorphism $\beta : g^*\underline{\widetilde{\mathcal{F}}} \xrightarrow{\sim} \underline{\mathcal{F}}'$ on C'_K satisfying $\pi_*\beta \circ g^*\widetilde{\alpha} = \alpha$.

We begin by constructing for all $i \in \mathbb{Z}$ the locally free sheaf $\widetilde{\mathcal{F}}_i$ on C'_R and the isomorphism $\widetilde{\alpha}_i : f^*\mathcal{F}_i \xrightarrow{\sim} \pi_*\widetilde{\mathcal{F}}_i$. Let $\mathcal{L}$ be an ample invertible sheaf on C such that $\pi_*\mathcal{O}_{C'} \otimes_{\mathcal{O}_C} \mathcal{L}$ is generated by global sections $x_1, \ldots, x_m$.

For the next step in the proof we need to introduce some notation. Let ϖ be the generic point of the special fiber of C_R over the residue field of R and let $\mathcal{O}_\varpi := \mathcal{O}_{C_R,\varpi}$ be the local ring at ϖ. It is a discrete valuation ring and every uniformizing parameter of R is a uniformizing parameter of $\mathcal{O}_\varpi$. Let further $K(C)$ be the fraction field of $\mathcal{O}_\varpi$. It equals the function field of C_K. Similarly let $\mathcal{O}_{\varpi'}$ and $K(C')$ be the rings associated with the curve C'. Since the

$f^*\Pi_i$ are invertible over $\mathcal{O}_\varpi$ we get τ-modules $\bigl(f^*\mathcal{F}_i\otimes_{\mathcal{O}_{C_R}}\mathcal{O}_\varpi, f^*(\Pi_i^{-1}\circ\tau_i)\bigr)$ over $\mathcal{O}_\varpi$ with $f^*(\Pi_i^{-1}\circ\tau_i)$ being isomorphisms. Now the argument of Gardeyn [8, Proposition 2.13(i)] shows that $\bigl(f^*\mathcal{F}_i\otimes_{\mathcal{O}_{C_R}}\mathcal{O}_\varpi, f^*(\Pi_i^{-1}\circ\tau_i)\bigr)$ is the unique maximal $f^*(\Pi_i^{-1}\circ\tau_i)$-invariant $\mathcal{O}_\varpi$-lattice in $\bigl(f^*\mathcal{F}_i\otimes_{\mathcal{O}_{C_R}}K(C), f^*(\Pi_i^{-1}\circ\tau_i)\bigr)$. Since every x_μ is an endomorphism of

$$\bigl(f^*\mathcal{F}_i\otimes_{\mathcal{O}_{C_R}}K(C)\,,\,f^*(\Pi_i^{-1}\circ\tau_i)\bigr)$$

it must map $f^*\mathcal{F}_i\otimes_{\mathcal{O}_{C_R}}\mathcal{O}_\varpi$ into itself. This makes $f^*\mathcal{F}_i\otimes_{\mathcal{O}_{C_R}}\mathcal{O}_\varpi$ into a free $\mathcal{O}_{\varpi'}$-module. Now we can apply Lafforgue's [11, Lemme 2.7] which says that to give a locally free sheaf $\widetilde{\mathcal{F}}_i$ on C'_R is equivalent to giving its restrictions $\widetilde{\mathcal{F}}_i\otimes_{\mathcal{O}_{C'_R}}\mathcal{O}_{C'_K}$ and $\widetilde{\mathcal{F}}_i\otimes_{\mathcal{O}_{C'_R}}\mathcal{O}_{\varpi'}$. Thus out of $\mathcal{F}'_i$ and the $\mathcal{O}_{\varpi'}$-module $f^*\mathcal{F}_i\otimes_{\mathcal{O}_{C_R}}\mathcal{O}_\varpi$ we may construct the locally free sheaf $\widetilde{\mathcal{F}}_i$ together with the isomorphism $\beta_i: g^*\widetilde{\mathcal{F}}_i\xrightarrow{\sim}\mathcal{F}'_i$. Since by construction

$$\alpha_i:\ \Bigl((fg)^*\mathcal{F}_i\,,\,f^*\mathcal{F}_i\otimes_{\mathcal{O}_{C_R}}\mathcal{O}_\varpi\Bigr)\ \xrightarrow{\ \sim\ }\ \Bigl(\pi_*(\widetilde{\mathcal{F}}_i\otimes_{\mathcal{O}_{C'_R}}\mathcal{O}_{C'_K})\,,\,\pi_*(\widetilde{\mathcal{F}}_i\otimes_{\mathcal{O}_{C'_R}}\mathcal{O}_{\varpi'})\Bigr)$$

is an isomorphism on the two restrictions we obtain the isomorphism $\widetilde{\alpha}_i: f^*\mathcal{F}_i\xrightarrow{\sim}\pi_*\widetilde{\mathcal{F}}_i$ from Lafforgue's lemma.

Since the Π'_i and the τ'_i are commuting homomorphisms of $\mathcal{O}_{C'_K}$-modules they restrict to commuting homomorphisms $\widetilde{\Pi}_i$ and $\widetilde{\tau}_i$ of $\mathcal{O}_{C'_R}$-modules. Altogether we have shown that $\underline{\widetilde{\mathcal{F}}}=(\widetilde{\mathcal{F}}_i,\widetilde{\Pi}_i,\widetilde{\tau}_i)$ satisfies axioms 1, 3, and 4 from Definition 1.5 except for the condition on the support of $\operatorname{coker}\widetilde{\tau}_i$. Let $\widetilde{\mathcal{J}}$ be the ideal sheaf on C'_R defining the graph of $\tilde{c}$. Then $\widetilde{\mathcal{J}}^{d'}$ annihilates the generic fiber of the free R-module $\operatorname{coker}\widetilde{\tau}_i$, so it annihilates all of $\operatorname{coker}\widetilde{\tau}_i$. Likewise if z' is a uniformizing parameter on C' at ∞' then $(z')^{k'}$ annihilates the generic fiber of the free R-module $\operatorname{coker}(\widetilde{\Pi}_{i+\ell'-1}\circ\ldots\circ\widetilde{\Pi}_i)$, so it annihilates this whole cokernel. Now all axioms are verified and $\underline{\widetilde{\mathcal{F}}}$ is the desired abelian sheaf on C' over $\operatorname{Spec}R$. □

Theorem 3.6. *The restriction of coefficients morphism* $\pi_*:\underline{C'\text{-Ab-Sh}}^{r',d'}\to\underline{C\text{-Ab-Sh}}^{nr',d'}$ *satisfies the valuative criterion for properness.*

Proof. Since the stacks $C\text{-}\mathcal{A}b\text{-}\mathcal{S}h^{r,d}$ are locally noetherian by [9, Theorem 3.1] it suffices to test the valuative criterion only for *discrete* valuation rings. For those the argument proceeds as in Theorem 2.4 using Lemma 3.7 below instead of Lemma 2.5. □

Lemma 3.7. *Let R be a valuation ring with fraction field K and let $f:\operatorname{Spec}K\to\operatorname{Spec}R$ be the induced morphism. Let $\underline{\mathcal{F}}$ and $\underline{\mathcal{F}}'$ be two abelian sheaves on C over* $\operatorname{Spec}R$ *of rank r, dimension d, and characteristic $c:\operatorname{Spec}R\to C$. Let $\alpha:f^*\underline{\mathcal{F}}\to f^*\underline{\mathcal{F}}'$ be an isomorphism over* $\operatorname{Spec}K$. *Then $\alpha=f^*\beta$ for a unique isomorphism $\beta:\underline{\mathcal{F}}\xrightarrow{\sim}\underline{\mathcal{F}}'$ over* $\operatorname{Spec}R$.

Proof. Recall the rings $\mathcal{O}_\varpi$ and $K(C)$ introduced in the proof of Proposition 3.5 and consider the τ-modules $\bigl(\mathcal{F}_i\otimes_{\mathcal{O}_{C_R}}\mathcal{O}_\varpi,\Pi_i^{-1}\circ\tau_i\bigr)$ and $\bigl(\mathcal{F}'_i\otimes_{\mathcal{O}_{C_R}}\mathcal{O}_\varpi,\Pi_i'^{-1}\circ\tau'_i\bigr)$ over $\mathcal{O}_\varpi$. By the arguments of Gardeyn [8, Proposition 2.13(i)] these are the unique maximal $\Pi_i^{-1}\circ\tau_i$-invariant $\mathcal{O}_\varpi$-modules in $\mathcal{F}_i\otimes_{\mathcal{O}_{C_R}}K(C)$, respectively $\mathcal{F}'_i\otimes_{\mathcal{O}_{C_R}}K(C)$. Hence they are mapped isomorphically into each other under the isomorphism α. Now the lemma follows from [11, Lemme 2.7]. □

Remark. Like for Drinfeld modules these results say in the language of stacks that the restriction of coefficients 1-morphism $\pi_*:C'\text{-}\mathcal{A}b\text{-}\mathcal{S}h^{r',d'}\to C\text{-}\mathcal{A}b\text{-}\mathcal{S}h^{nr',d'}$ is proper but in general not representable.

4 Restriction of Coefficients for Drinfeld-Anderson Shtuka

Theorem 4.1. *The restriction of coefficients morphism* $\pi_* : \underline{C'\text{-DA-Sht}}^{r',d'} \to \underline{C\text{-DA-Sht}}^{nr',d'}$ *for Drinfeld-Anderson shtuka is in general not relatively representable.*

Proof. The abelian sheaf from Theorem 3.1 provides the counter example also for Drinfeld-Anderson shtuka. □

The same reasoning as in Theorem 3.2 and Proposition 3.5 yields the following

Theorem 4.2. *Let S be a locally noetherian $\mathbb{F}_q$-scheme and let $b, c : S \to C$ be two $\mathbb{F}_q$-morphisms. Let $\underline{\mathcal{E}} = (\mathcal{E}, \widetilde{\mathcal{E}}, j, \tau, b, c)$ be a Drinfeld-Anderson shtuka on C of rank nr' and dimension d' over S. Then the contravariant functor*

$$\begin{array}{rcl} \underline{T}: \ \mathcal{S}ch_{/(C'\times C')\times_{(C\times C)}S} & \longrightarrow & \mathcal{S}ets \\ \begin{pmatrix} S', f : S' \to S \\ (b', c') : S' \to C' \times_{\mathbb{F}_q} C' \end{pmatrix} & \mapsto & \left\{ \begin{array}{l} \textit{Isomorphism classes of pairs } (\underline{\mathcal{E}}', \alpha) \textit{ where} \\ \bullet\ \underline{\mathcal{E}}' \textit{ is a Drinfeld-Anderson shtuka of rank } r', \\ \quad \textit{dimension } d', \textit{ pole } b', \textit{ and zero } c' \textit{ over } S' \textit{ and} \\ \bullet\ \alpha : f^*\underline{\mathcal{E}} \xrightarrow{\ \sim\ } \pi_*\underline{\mathcal{E}}' \textit{ is a fixed isomorphism} \end{array} \right\} \end{array}$$

is representable by a finite S-scheme. □

The following results are proved analogously to Theorem 3.6 and Lemma 3.7.

Theorem 4.3. *The restriction of coefficients morphism* $\pi_* : \underline{C'\text{-DA-Sht}}^{r',d'} \to \underline{C\text{-DA-Sht}}^{nr',d'}$ *satisfies the valuative criterion for properness.* □

Lemma 4.4. *Let R be a valuation ring with fraction field K and let $f : \operatorname{Spec} K \to \operatorname{Spec} R$ be the induced morphism. Let $\underline{\mathcal{E}}$ and $\underline{\mathcal{E}}'$ be two Drinfeld-Anderson shtuka on C over $\operatorname{Spec} R$ of rank r, dimension d, pole $b : \operatorname{Spec} R \to C$, and zero $c : \operatorname{Spec} R \to C$. Let $\alpha : f^*\underline{\mathcal{E}} \to f^*\underline{\mathcal{E}}'$ be an isomorphism over $\operatorname{Spec} K$. Then $\alpha = f^*\beta$ for a unique isomorphism $\beta : \underline{\mathcal{E}} \xrightarrow{\ \sim\ } \underline{\mathcal{E}}'$ over $\operatorname{Spec} R$.* □

Remark. Again these results say in the language of stacks that the restriction of coefficients 1-morphism $\pi_* : C'\text{-}\mathcal{DA}\text{-}\mathcal{S}ht^{\,r',d'} \to C\text{-}\mathcal{DA}\text{-}\mathcal{S}ht^{\,nr',d'}$ is proper but in general not representable.

References

[EGA] A. Grothendieck: *Élements de Géométrie Algébrique*, Publ. Math. IHES **4**, **8**, **11**, **17**, **20**, **24**, **28**, **32**, Bures-Sur-Yvette, 1960–1967; see also Grundlehren **166**, Springer-Verlag, Berlin etc. 1971.

[1] A. Blum, U. Stuhler: Drinfeld Modules and Elliptic Sheaves, in *Vector Bundles on Curves – New Directions*, pp. 110–188, LNM **1649**, Springer-Verlag, Berlin etc. 1991.

[2] M. Bornhofen, U. Hartl: *Abelian Sheaves over Finite Fields*, in preparation.

[3] N. Bourbaki: *Algèbre Commutative*, Hermann, Paris 1967.

[4] P. Deligne, D. Mumford: The Irreducibility of the Space of Curves of Given Genus, *Publ. Math. IHES* **36** (1969), 75–110.

[5] V.G. Drinfeld: Elliptic Modules, *Math. USSR-Sb.* **23** (1976), 561–592.

[6] V.G. Drinfeld: Commutative Subrings of Certain Noncommutative Rings, *Funct. Anal. Appl.* **11** (1977), 9–12.

[7] V.G. Drinfeld: Moduli variety of F-sheaves, *Funct. Anal. Appl.* **21** (1987), no. 2, 107–122.

[8] F. Gardeyn: The structure of analytic τ-sheaves, *J. Number Th.* **100** (2003), 332–362.

[9] U. Hartl: Uniformizing the Stacks of Abelian Sheaves, in *Number Fields and Function fields - Two Parallel Worlds, Papers from the 4th Conference held on Texel Island, April 2004*, G. van der Geer, B. Moonen, R. Schoof, Editors, pp. 167–222, Progress in Math. 239, Birkhäuser-Verlag, Basel 2005. See also `arXiv:math.NT/0409341`.

[10] R. Hartshorne: *Algebraic Geometry*, GTM **52**, Springer-Verlag, Berlin etc. 1977.

[11] L. Lafforgue: Une compactification des champs classifiant les chtoucas de Drinfeld, *J. Amer. Math. Soc.* **11** (1998), no. 4, 1001–1036.

[12] G. Laumon: *Cohomology of Drinfeld Modular Varieties I*, Cambridge Studies in Advanced Mathematics **41**, Cambridge University Press, Cambridge, 1996.

[13] G. Laumon, L. Moret-Bailly: *Champs algébriques*, Ergebnisse **39**, Springer-Verlag, Berlin etc. 2000.

[14] H. Matzat: Introduction to Drinfeld modules, in *Drinfeld modules, modular schemes and applications (Alden-Biesen, 1996)*, pp. 3–16, World Sci. Publishing, River Edge, NJ, 1997.

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