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Semi-stable models for curves with cusps

Urs T. Hartl*

Department of Mathematics, ETH–Zentrum, CH – 8092 Zürich, Switzerland
(e-mail: hartl@math.ethz.ch)

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Abstract. : Let R be a complete discrete valuation ring with residue characteristic zero, and let X be an integral regular flat curve over R with smooth generic fiber. Assume that the special fiber of X is smooth outside a single point where it has a cusp as singularity. We explicitly determine the structure of the minimal semi-stable model of X. In particular, we give an algebraic proof for the fact that the special fiber of any semi-stable model of X is treelike. This is equivalent to the finiteness of the monodromy of X over R. These two results were obtained in the 1970's by Lê Dũng Tráng and A. Durfee using analytic methods.

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1 Introduction

Let R be a complete discrete valuation ring with uniformizing parameter π, field of fractions K, and algebraically closed residue field k of characteristic zero, thus $R = k[[\pi]]$. By a curve over R we mean an integral scheme X, flat, separated and of finite type over $\operatorname{Spec} R$, whose fibers are geometrically connected and one dimensional. It is called regular if X is regular. We will only consider curves X over R whose generic fiber $X_K := X \otimes_R K$ is smooth over $\operatorname{Spec} K$.

Now let X be a regular curve over R. We assume that the special fiber $C := X_k := X \otimes_R k$ of X is smooth outside a single point P. The completion of the local ring of X at P is isomorphic to $R[[x, y]]/(f)$ for some

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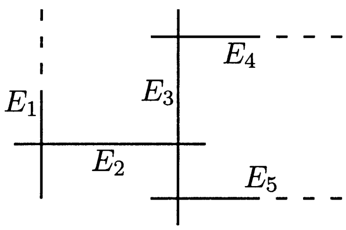
element $f \in R[[x,y]]$. E. Brieskorn studied the monodromy of X over R in the case $k = \mathbb{C}$, the field of complex numbers, and conjectured it to be of finite order. This was proved by Lê Dũng Tráng in [Lê], if the singularity of X_k is a *cusp*, i.e. if $\overline{f} := f \mod \pi$ is irreducible in $k[[x,y]]$, in which case we also say that f is *analytically irreducible in* 0. Later N. A'Campo [AC] and D.W. Sumners and J.M. Woods [SW] gave different proofs for this result. Also N. A'Campo [AC] constructed a counterexample to Brieskorn's conjecture for general f. Furthermore, there is an unpublished proof by P. Deligne. All the articles mentioned use the analytic methods developed by Milnor [Mr] and study the knot associated to the singularity. Still by analytic means A. Durfee [Du] gave a criterion for the finiteness of the monodromy: The monodromy is of finite order if and only if any semi-stable model of X is *treelike*. Being treelike means that the graph associated to its special fiber is a tree. There, the vertices of this graph are the irreducible components of the special fiber and the edges are the intersection points of these components.

The aim of the present paper is to give an algebraic proof of these results. If f is analytically irreducible, we explicitly construct the minimal semi-stable model of X, which exists after an extension of discrete valuation rings. We also determine this extension. From this we read off the structure of the semi-stable model and find that its special fiber is treelike. The usual argument with vanishing cycles then gives the finiteness of the monodromy using the algebraic monodromy theory [SGA 7].

The construction of the semi-stable model follows the usual line. We first blow up all singular points of the special fiber, which are not yet ordinary double points. This process terminates after finitely many blowing-ups. The special fiber of the resulting curve is in general not reduced. It is a tree in which every irreducible component intersects at most three other components. Next we change the base to an extension of R, whose ramification index is a common multiple of all the multiplicities of the components of the special fiber. Afterwards, the normalization of our curve has reduced special fiber. The problem thereby is to control the effect of the normalization on the configuration. Above the components which meet at most two other components nothing bad can happen. A loop can only come from a component meeting three other components. For example consider the following configuration of projective lines in $\mathrm{V}(\pi)$ just before the normalization.



The component E_1 is linked to the original component C by a chain of further projective lines. Let the multiplicity of E_i in $\mathrm{V}(\pi)$ be r_i. If now

$$d \ := \ \gcd(r_2, r_3, r_4, r_5) \ < \ \gcd(r_1, r_2, r_3) \ =: \ d',$$

then there will be d' components above E_2, but only d components above E_3 (cf. Lemma 2.1). This causes $d' - d$ loops to appear in the normalization.

It is not obvious that this scenario is impossible in our situation. However by using the Puiseux expansion of the cusp and a careful analysis of the combinatorial data involved, we can show that the scenario is in fact impossible and that the semi-stable model is treelike.

2 Preliminaries

An essential ingredient in our proof is the Puiseux expansion. We briefly recall it in this section together with other facts needed later on.

Puiseux expansions

Let X be a regular curve over R and let the special fiber of X be smooth outside a single point P, where X_k has a cusp as singularity. The completion of the local ring of X at P is isomorphic to $R[[x,y]]/(f)$ for some element $f \in R[[x,y]]$. Being regular implies that its maximal ideal is generated by x and y. Since $k \subseteq R$ we can thus write

$$f(x,y) \ = \ \overline{f}(x,y) - \pi \cdot \varepsilon$$

for $\overline{f} \in k[[x,y]]$ and a unit $\varepsilon \in R[[x,y]]^{\times}$. Having a cusp, $\overline{f}$ is irreducible in $k[[x,y]]$. So we may assume, that

$$\overline{f} \ = \ y^n + \overline{f}_{n+1} + \ldots$$

for $\overline{f}_i$ homogeneous in x and y of degree i. By the Weierstraß Preparation Theorem there is a unit $u \in k[[x,y]]^{\times}$, such that $u\overline{f}$ is a polynomial in $k[[x]][y]$ monic and of degree n in y. Hence y is integral over $k[[x]]$. Now it is a classical fact due to I. Newton [Ne] and rediscovered by V. Puiseux [Pu], that the roots y of this polynomial are *Puiseux series* in x, i.e. formal series in $x^{1/n}$ with coefficients in k

$$y \ = \ \sum_{j \geq 0} a_j x^{j/n} \quad \text{in} \quad k[[x^{1/n}]] \, .$$

For a modern account of Puiseux' theorem see [Ei, p. 118]. We will write out the Puiseux series expansion in increasing order of the exponent of

x. Thereby we only write down the monomials with non-zero coefficients and we write their exponents as reduced fractions. The Puiseux series may start with a polynomial $P(x)$ in x. But there comes a term whose exponent does not lie in $\mathbb{Z}$, we call it $a_{j_1}\, x^{m_1/n_1}$. Certainly $n_1|n$. If $n_1 < n$, we continue until we get to the first term whose exponent does not lie in $\frac{1}{n_1}\mathbb{Z}$. Its denominator must be divisible by n_1, we call it $a_{j_2}\, x^{m_2/n_1 n_2}$. Again $n_1 n_2|n$. We proceed in this manner and obtain

$$\begin{aligned} y \;\; = \;\; & P(x) + a_{j_1}\, x^{m_1/n_1} + \sum_{l=1}^{l_1} a_{1,l}\, x^{(m_1+l)/n_1} + \\ & \qquad + a_{j_2}\, x^{m_2/n_1 n_2} + \sum_{l=1}^{l_2} a_{2,l}\, x^{(m_2+l)/n_1 n_2} + \\ + \;\; & \ldots + a_{j_N}\, x^{m_N/n_1\cdot\ldots\cdot n_N} + \sum_{j>m_N} a_j\, x^{j/n} \;\; , \end{aligned} \tag{2.1}$$

where $n_1 \cdot \ldots \cdot n_N = n$. The $a_{j_1}, \ldots, a_{j_N}$ are non-zero and $P(x)$ is a polynomial without constant and linear terms. The pairs (m_i, n_i) are called the *Puiseux pairs* of f. They are numerical invariants of the singularity of X_k at $\mathrm{V}(x, y)$. We have $m_1 > n_1$ and $m_i > n_i m_{i-1}$ for all $i \geq 2$. The Puiseux expansion yields a factorization of $\overline{f}$ over $k[\![x^{1/n}]\!]$

$$f(x,y) \;\; = \;\; \overline{f}(x,y) - \pi\cdot\varepsilon \;\; = \;\; u^{-1} \prod_{\xi\in\mu_n(k)} \left(y - \sum_{j\geq 0} a_j\, \xi^j\, x^{j/n} \right) - \pi\cdot\varepsilon \, ,$$

where by $\mu_r(A)$ we denote the group of r-th roots of unity in a ring A. See [Lê] for more details.

Intersection theory

From Lichtenbaum [Li] we recall the basic facts about Intersection Theory on arithmetic surfaces.

Let X be a regular curve over R and $C \subseteq X_k$ a proper reduced, not necessarily irreducible curve over $\operatorname{Spec} k$. Further let D be a positive Cartier divisor on X with defining sheaf of ideals $\mathcal{I}$. We define the *intersection number* of C and D as

$$(C.D) \;\; := \;\; \deg_k \left(\mathcal{I}^{-1} \otimes_{\mathcal{O}_X} \mathcal{O}_C \right) .$$

It has the following properties:

- It is bilinear, hence we can define it for arbitrary Cartier divisors C and D, with C supported on X_k.

- It is symmetric if both C and D are supported on X_k.
- If D is principal, we have $(C.D) = 0$.
- Let C and D be positive and have no common component, i.e $C \cap D = \{P_1, \ldots, P_n\}$. Let locally at P_i the divisors be given as $C = \mathrm{V}(c_i)$ and $D = \mathrm{V}(d_i)$. Then

$$(C.D) \;=\; \sum_{i=1}^{n} \dim_k\, \mathcal{O}_{P_i}/(c_i, d_i)\,.$$

Let $P = \mathrm{V}(\mathfrak{m}_P) \in X_k$ be a closed point and D a positive Cartier divisor on X, locally at P given as $D = \mathrm{V}(g)$. We define the *multiplicity of* P *on* D as

$$m_P(D) \;:=\; \max\{r \in \mathbb{N}_0 : g \in \mathfrak{m}_P^r\}\,.$$

If $P \in C$ is a point of an irreducible and reduced curve C over k we define the δ-*invariant* of C at P as

$$\delta_P(C) \;:=\; \mathrm{length}\big(\widetilde{\mathcal{O}}_{C,P}/\mathcal{O}_{C,P}\big),$$

where $\widetilde{\mathcal{O}}_{C,P}$ denotes the normalization of the local ring $\mathcal{O}_{C,P}$ in its field of fractions. The curve C has a singularity at P if and only if $\delta_P(C) > 0$. In this case the δ-invariant is a measure for the singularity. If C is further projective over k and $\widetilde{C}$ is its normalization, the arithmetic genus can be computed by the following formula

$$p_a(C) \;=\; p_a(\widetilde{C}) + \sum_{P \in C} \delta_P(C)\,. \tag{2.2}$$

In relation with blowing-ups the intersection number satisfies the following rules. Let $P \in X_k$ be a closed point and let $f : \widetilde{X} \longrightarrow X$ be the blowing-up of X in P. We denote by $E = f^{-1}(P)$ the exceptional line of the blowing-up. It is isomorphic to $\mathbb{P}^1_k$, since X is regular. For each Cartier divisor D on X we denote by $f^*(D)$, respectively $\widetilde{D}$ the total, respectively the strict transform of D under f. Then

- $f^*(D) \;=\; \widetilde{D} + m_P(D) \cdot E$,
- $\big(f^*(C).f^*(D)\big) \;=\; (C.D)$,
- $\big(f^*(D).E\big) = 0$ and $(E.E) = -1$, hence $(\widetilde{D}.E) \;=\; m_P(D)$,
- $(\widetilde{C}.\widetilde{D}) \;=\; (C.D) - m_P(C)\, m_P(D)$.

During that blowing-up the δ-invariant at P of an irreducible and reduced curve $C \subseteq X$ passing through P drops according to

$$\delta_P(C) - \sum_{Q \in E} \delta_Q(\widetilde{C}) \;=\; \frac{1}{2} m_P(C)\bigl(m_P(C) - 1\bigr)\,. \tag{2.3}$$

This is due to the fact that $\delta_P(C)$ can be computed as the sum over $m_Q(\widetilde{C})\bigl(m_Q(\widetilde{C}) - 1\bigr)$ for all infinitely near points $Q \in \widetilde{C}$ of $P \in C$, i.e. points Q on blowing-ups of X such that Q lies above P (including $Q = P \in X$).

Semi-stable models

We call a curve X over R *semi-stable* if its special fiber is reduced with only ordinary double points as singularities. A closed subscheme of a regular scheme is called a *normal crossings divisor* if locally for the étale topology it is defined as $\mathrm{V}(f_1 \cdot \ldots \cdot f_r)$ for a regular system of parameters $(f_1, \ldots, f_r)$. A normal crossings divisor is reduced. In particular the special fiber X_k of a regular curve X over R is a normal crossings divisor if and only if X is semi-stable. We say that X_k has *transversal crossings* if the reduced special fiber $X_{k,\mathrm{red}}$ is a normal crossings divisor, i.e. if $X_{k,\mathrm{red}}$ has only ordinary double points as singularities. We recall the construction of a semi-stable model for a regular curve X over R in $\operatorname{char} k = 0$ (cf. [Ab]). By such a model we mean a semi-stable curve $\widetilde{X}$ over R with $X_K \cong \widetilde{X}_K$. It exists after an eventual extension $R \subseteq R'$ of discrete valuation rings. The construction proceeds in two steps.

During the blowing-up process we blow up all the closed points at which the special fiber does not yet have transversal crossings. This process terminates after finitely many blowing-ups (use the formulas on intersection numbers). We obtain a regular curve X' over R whose special fiber X'_k has transversal crossings, but is in general not reduced. All irreducible components of X'_k that appeared during the blowing-up process are divisors on X'_k isomorphic to $\mathbb{P}^1_k$, since X was regular.

During the normalization process we change the base ring from R to $R' := R[\pi']/(\pi'^e - \pi)$, where e is a common multiple of all the multiplicities of the components in the special fiber X'_k. Let $\widetilde{X}$ be the normalization of $X' \otimes_R R'$ in its field of fractions. By the Abhyankar-Lemma the special fiber $\widetilde{X}_k$ is reduced. Further it has only ordinary double points as singularities, hence $\widetilde{X}$ is semi-stable. However it may not be regular at the double points. Blowing up the non-regular points yields after finitely many steps a regular semi-stable curve over R'. We want to explicitly determine the effect of the normalization on the irreducible components of X'_k.

Lemma 2.1 *Let X be a regular curve over R and let X_k have transversal crossings. Let $D \subseteq X_k$ be a closed subscheme, isomorphic to $\mathbb{P}^1_k$ with multiplicity r in X_k. Denote the irreducible components of X_k which meet D by $D_1, \ldots, D_n$ and their multiplicities in X_k by $r_1, \ldots, r_n$ respectively. Let $d_i := \gcd(r, r_i)$ and $d := \gcd(r, r_1, \ldots, r_n)$ and let e be a common multiple of all the r, r_i. Consider the normalization $\widetilde{X}$ of the base change $X \otimes_R R'$, where $R' := R[\pi']/(\pi'^e - \pi)$.*

Then the component D splits in $\widetilde{X}_k$ into d irreducible components which are all isomorphic. We call one of them $\widetilde{D}$. The induced morphism $\widetilde{D} \longrightarrow D$ is of degree r/d and ramified exactly above the points in which D meets the D_i. The ramification indices there are r/d_i.

Proof. (cf. [Ab, Théorème 1.2])
Let $x \in D$ be a not necessarily closed point and let A be the completion of the local ring $\mathcal{O}_{X,x}$ of X in x. Further let $x_1, \ldots, x_s$ be the points of $\widetilde{X}$ lying above x and for every i let $\widehat{\mathcal{O}}_{\widetilde{X},x_i}$ be the completion of the local ring of $\widetilde{X}$ in x_i. Denote by $\widetilde{A}$ the normalization of $A \otimes_R R'$ in its ring of total fractions. Since X is excellent we have by [EGA, IV, Scholie 7.8.3]

$$\widetilde{A} \;\cong\; \prod_{i=1}^{s} \widehat{\mathcal{O}}_{\widetilde{X},x_i}\,.$$

We show that D splits into d irreducible components in $\widetilde{X}$. Let $\mathcal{I}$, $\mathcal{I}_i$ for $i = 1, \ldots, n$ be the invertible ideals of $\mathcal{O}_X$ with $\mathrm{V}(\mathcal{I}) = D$ and $\mathrm{V}(\mathcal{I}_i) = D_i$. Then in a neighborhood of D in X we have an equality of invertible $\mathcal{O}_X$-ideals

$$\pi\,\mathcal{O}_X \;=\; \mathcal{I}^r \cdot \mathcal{I}_1^{r_1} \cdot \ldots \cdot \mathcal{I}_n^{r_n}\,.$$

We tensor it with $\mathcal{O}_D$ and obtain an equality of invertible $\mathcal{O}_D$-modules. Now let $y \in D$ be a point not lying on any of the D_i. Then

$$D - \{y\} \;\cong\; \mathbb{A}^1_k \;=\; \operatorname{Spec} k[t]$$

and so all the above invertible sheaves are principal on $D - \{y\}$, say $\mathcal{I}_i \otimes \mathcal{O}_D$ generated by $t - t_i$ and $\mathcal{I} \otimes \mathcal{O}_D = \mathcal{I}/\mathcal{I}^2$, the conormal sheaf of D, generated by u. So the above equality reads on $D - \{y\}$

$$\pi \;=\; c\,u^r\,(t - t_1)^{r_1} \cdot \ldots \cdot (t - t_n)^{r_n}$$

for a global unit c on $D - \{y\}$, i.e. $c \in k[t]^\times = k^\times$. If we lift these generators to local sections of $\mathcal{O}_X$, they still satisfy the same equation with c replaced by some section C of $\mathcal{O}_X$. Now we consider this equation in the completion A of the local ring $\mathcal{O}_{X,\eta}$ at the generic point η of D. Since the d-th root of c

exists in the residue field of A, the d-th root of C lies in A and the equation splits in $A \otimes_R R'$ into irreducible factors

$$\begin{aligned} 0 &= \pi'^e - C\, u^r\, (t-t_1)^{r_1} \cdot \ldots \cdot (t-t_n)^{r_n} \\ &= \prod_{\xi \in \mu_d(R)} \left(\pi'^{e/d} - \xi\, C^{1/d}\, u^{r/d}\, (t-t_1)^{r_1/d} \cdot \ldots \cdot s(t-t_n)^{r_n/d} \right). \end{aligned}$$

Hence there are d generic points of irreducible components lying above η. This means that D splits in $\widetilde{X}$ into d irreducible components which are all conjugate by $\mu_d(R)$ and one of which we call $\widetilde{D}$.

It remains to investigate the ramification of the morphism $\widetilde{D} \longrightarrow D$. So let x be a closed point of D. Now we distinguish two cases. First let x not be lying on any of the D_i. Then

$$A \;\cong\; R[\![u,v]\!]/(u^r - \pi)\,.$$

Since r divides e, we set $e = e'\, r$ and compute

$$\widetilde{A} \;\cong\; \prod_{\xi \in \mu_r(R)} R'[\![u,v]\!]/(\pi'^{e'} - \xi u) \;\cong\; \prod_{\xi \in \mu_r(R)} R'[\![v]\!]\,.$$

So there are r smooth points lying above x. Thus the degree of the morphism $\widetilde{D} \longrightarrow D$ is r/d.

Next let x be the intersection point of D and D_i. Then

$$A \;\cong\; R[\![u,v]\!]/(u^r v^{r_i} - \pi)\,.$$

Set $r = a\, d_i$, $r_i = b\, d_i$ and $e = e'\, abd_i$,

$$\widetilde{A} \;\cong\; \prod_{\xi \in \mu_{d_i}(R)} \left\{ R'[\![u,v]\!]/(\pi'^{e'ab} - \xi u^a v^b) \right\}^{\sim},$$

where $\{\ \}^{\sim}$ denotes the normalization. Fixing $\xi \in \mu_{d_i}(R)$ and choosing $\theta \in R$ such that $\theta^{ab} = \xi$, we claim that

$$\left\{ R'[\![u,v]\!]/(\pi'^{e'ab} - \xi u^a v^b) \right\}^{\sim} \;\cong\; R'[\![z,y]\!]/(\pi'^{e'} - \theta z y) \;\;=:\;\; B\,.$$

Namely z and y should satisfy the integral equations $z^b = u$ and $y^a = v$ along with the equations

$$z^a \;=\; \frac{\pi'^{e'a}}{v\, \theta^a} \qquad \text{and} \qquad y^b \;=\; \frac{\pi'^{e'b}}{u\, \theta^b}$$

in the field of fractions of $R'[\![u,v]\!]/(\pi'^{e'ab} - \xi u^a v^b)$. Since $(a,b) = 1$ we indeed find z and y in this field of fractions satisfying the above equations. Since B is normal, our claim is proved. Hence, above the intersection point x of D and D_i there are d_i ordinary double points, which are all conjugate by the action of $\mu_{d_i}(R)$. The morphism $\widetilde{D} \longrightarrow D$ is thus ramified above x with ramification index r/d_i. $\square$

3 Theorems and proofs

In this section we give an algebraic proof for the following

Theorem 3.1 *Let X be a regular curve over R and let the special fiber X_k of X be smooth outside a single point, where X_k has a cusp. Then*

a) *every semi-stable model of X is treelike,*

b) *the monodromy of X over R is of finite order.*

Remark 3.1.1 As mentioned earlier an analytic proof of this theorem was given by Lê Dũng Tráng [Lê, 3.3] and A. Durfee [Du, Theorem 2].

Proof. Part a) is a direct consequence of Theorem 3.5 below.

Assertion b) is equivalent to assertion a) by the usual argument with vanishing cycles (cf. [Du, Theorem 1] for the analytic version). For simplicity we give the argument in the case, where X is proper over S. It follows the same line in the general case.

The *monodromy of X over R* is by definition the action of the inertia group $I := \mathrm{Gal}(\overline{K}/K)$ of the ring R on the ℓ-adic cohomology group $\mathrm{H}^1(X_K \otimes_K \overline{K}, \mathbb{Z}_\ell)$. Let now $\widetilde{X}$ be a regular semi-stable model of X, which exists after a finite extension $R \subseteq R'$ of discrete valuation rings. Since $\widetilde{X}_K \cong X_K$ and the inertia group $I' := \mathrm{Gal}(\overline{K}/K')$ of R' is of finite index in I, we see that the monodromy of X over R is of finite order if and only if the monodromy of $\widetilde{X}$ over R' is. Let $Z \subseteq \widetilde{X}_k$ be the singular locus of $\widetilde{X}_k$. Since all points of Z are ordinary double points, the monodromy of $\widetilde{X}$ over R' can be computed by the "*Formula of Picard-Lefschetz*" [SGA 7, Exposé XV, Théorème 3.4].

Namely to every double point $x \in Z$ is associated a *vanishing cycle* $\delta_x \in \mathrm{H}^1(\widetilde{X}_K \otimes_K \overline{K}, \mathbb{Z}_\ell)$. Strictly speaking the δ_x are generators of $\mathrm{H}^1_{\{x\}}(\widetilde{X}_k, \mathbb{Z}_\ell)$, the cohomology groups with support in $\{x\}$, uniquely determined up to multiplication with -1 (cf. [SGA 7, Exposé XV, 3.3.4]). We also denote by δ_x all its images under the maps

$$\mathrm{H}^1_{\{x\}}(\widetilde{X}_k, \mathbb{Z}_\ell) \longrightarrow \mathrm{H}^1(\widetilde{X}_k, \mathbb{Z}_\ell) \hookrightarrow \mathrm{H}^1(\widetilde{X}_K \otimes_K \overline{K}, \mathbb{Z}_\ell)\,.$$

By [SGA 7, Exposé XV, Théorème 3.4] the map on the right is injective. Further the vanishing cycles are pairwise orthogonal and orthogonal to themselves under the cup-product pairing on $\mathrm{H}^1(\widetilde{X}_K \otimes_K \overline{K}, \mathbb{Z}_\ell)$. In $\mathrm{H}^1(\widetilde{X}_k, \mathbb{Z}_\ell)$ the vanishing cycle δ_x can be represented by the following cocycle: Let U be an open neighborhood of x contained in the two components of $\widetilde{X}_k$ that intersect in x, such that $U \cap Z = \{x\}$, and let $V := \widetilde{X}_k - \{x\}$. The intersection $U \cap V$ has two connected components. Then $\pm\delta_x$ is represented by the (alternating) Čech-cocycle in $\check{\mathrm{H}}^1\big(\{U, V\}, \mathbb{Z}_\ell\big)$ given by the element of $\mathbb{Z}_\ell(U \cap V)$ that is 1 on one connected component and 0 on the other.

According to the "Formula of Picard-Lefschetz" the monodromy action of $\sigma \in I'$ on $a \in \mathrm{H}^1(\widetilde{X}_K \otimes_K \overline{K}, \mathbb{Z}_\ell)$ is given by

$$\sigma(a) \;=\; a \,-\, \varepsilon(\sigma) \sum_{x \in Z} (a \cup \delta_x)\delta_x ,$$

where $\varepsilon : I' = \widehat{\mathbb{Z}}(1) \longrightarrow \mathbb{Z}_\ell(1)$ is the canonical character (cf. [SGA 7, Exposé XV, 3.3.2], using that X is regular). Let

$$V \;:=\; \langle\, \delta_x : x \in Z \,\rangle_{\mathbb{Z}_\ell} \;\subseteq\; \mathrm{H}^1(\widetilde{X}_k, \mathbb{Z}_\ell)$$

and let $p : Y_k \longrightarrow \widetilde{X}_k$ be the normalization of the special fiber. We claim that

$$V \;=\; \ker\Big(p^* :\; \mathrm{H}^1(\widetilde{X}_k, \mathbb{Z}_\ell) \longrightarrow \mathrm{H}^1(Y_k, \mathbb{Z}_\ell)\Big)\,.$$

Indeed in the following diagram where the horizontal sequences are exact (cf. [Mi, Proposition III.1.25]) V is the image of the map α, since $\mathrm{H}^1_Z(\widetilde{X}_k, \mathbb{Z}_\ell)$ is generated by the δ_x.

$$\begin{array}{ccccc}
\mathrm{H}^1_Z(\widetilde{X}_k, \mathbb{Z}_\ell) & \xrightarrow{\ \alpha\ } & \mathrm{H}^1(\widetilde{X}_k, \mathbb{Z}_\ell) & \longrightarrow & \mathrm{H}^1(\widetilde{X}_k - Z, \mathbb{Z}_\ell) \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle p^*} & & \| \\
\mathrm{H}^1_{p^{-1}Z}(Y_k, \mathbb{Z}_\ell) & \longrightarrow & \mathrm{H}^1(Y_k, \mathbb{Z}_\ell) & \longrightarrow & \mathrm{H}^1(Y_k - p^{-1}Z, \mathbb{Z}_\ell)
\end{array}$$

The explicit description of δ_x above shows that $p^*(\delta_x) = 0$. (Also β is zero by [SGA 7, Exposé XV, 3.3.4].) Thus the claim follows.

We deduce that $V = 0$ if and only if $p^* : \mathrm{H}^1(\widetilde{X}_k, \mathbb{Z}_\ell) \longrightarrow \mathrm{H}^1(Y_k, \mathbb{Z}_\ell)$ is injective, which in turn is the case if and only if $\widetilde{X}$ is treelike. So if $\widetilde{X}$ is treelike, the monodromy of $\widetilde{X}$ over R' is trivial. On the other hand if $\widetilde{X}$ is not treelike, there exist $0 \neq \delta_x \in \mathrm{H}^1(X_K \otimes_K \overline{K}, \mathbb{Z}_\ell)$. The relations among the δ_x are of the form

$$\sum_{x \in Z} \lambda_x \delta_x \;=\; 0 \qquad \text{with } \lambda_x \in \{-1, 0, 1\}\,.$$

Therefore there exists an $a \in \mathrm{H}^1(X_K \otimes_K \overline{K}, \mathbb{Z}_\ell)$ such that

$$\sum_{x \in Z} (a \cup \delta_x)\delta_x \;\neq\; 0\,.$$

So the monodromy is of infinite order in this case. □

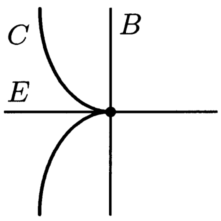
In order to prove part a) of Theorem 3.1 we will explicitly construct a semi-stable model of X. The Puiseux expansion will serve as a tool to control the effect of the blowing-up process.

The blowing-up process

Proposition 3.2 *For $r, s \in \mathbb{N}_0$ consider the scheme*

$$Y \;=\; \operatorname{Spec} R[[x,y]]/(y^r x^s f - \pi\,\varepsilon)\,,$$

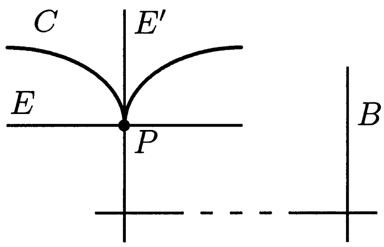
where $f = y^n + f_{n+1} + \ldots \in k[[x,y]]$ is analytically irreducible, f_ν is homogeneous of degree ν and $\varepsilon \in R[[x,y]]^\times$ is a global unit. Let the Puiseux pairs of f be $(m_1,n_1),\ldots,(m_N,n_N)$. In particular $n = n_1 \cdot \ldots \cdot n_N$. Let $m_1 = qn_1 + p$ for integers p,q with $0 < p < n_1$. The special fiber $\mathrm{V}(\pi)$ of Y has the displayed configuration with the components



$$\begin{aligned} C &= \mathrm{V}(f)\,,\\ E &= \mathrm{V}(y)\,,\\ B &= \mathrm{V}(x)\,. \end{aligned}$$

The component E meets C tangentially in the common intersection point of all three components.

Then q times blowing up the singular point of C leads to the following configuration of projective lines in $\mathrm{V}(\pi)$.

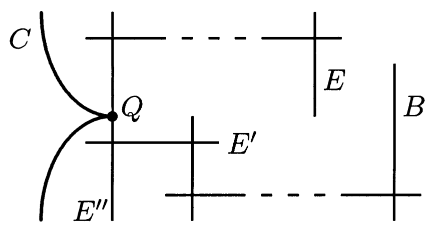


All the indicated components are strict transforms under the various blowing-ups. The components C, E and the exceptional line E' of the last blowing-up all intersect in the point P. Thereby E' is tangential to C in P. From E' proceeds a chain of the previous $q-1$ exceptional lines, the first of which intersects B.

The multiplicities of the components in the chain from E' to B decrease by $n + r$ at each component. The completion of the local ring at P is $\widehat{\mathcal{O}}_P \cong R[[u,x]]/(x^{\widetilde{s}} u^r \widetilde{f} - \pi\,\widetilde{\varepsilon})$ with $\widetilde{s} = s + q(n+r)$, $\widetilde{n} = n\,p/n_1$ and $\widetilde{f} = x^{\widetilde{n}} + \widetilde{f}_{\widetilde{n}+1} + \ldots \in k[[u,x]]$ analytically irreducible, $\widetilde{f}_\nu$ homogeneous *of degree ν and $\widetilde{\varepsilon} \in R[\![u,x]\!]^\times$ a global unit. The Puiseux pairs of $\widetilde{f}$ are*

$$(\widetilde{m}_i, \widetilde{n}_i) \;=\; \begin{cases} (n_1, p) & \textit{for } i = 1 \\ \big(m_i - n_2 \cdot \ldots \cdot n_i \cdot (m_1 - n_1), n_i\big) & \textit{for } i > 1 \end{cases}$$

In case $p = 1$, blowing up the singular point of C another n_1 times leads to the following configuration of projective lines in $\mathrm{V}(\pi)$.



All the indicated components are strict transforms under the various blowing-ups. The component C intersects the exceptional line E'' of the last blowing-up in the point Q. Out of a second point of E'' proceeds a chain of the previous $n_1 - 1$ exceptional lines, the first of which intersects E. In a third point E'' intersects E', from which proceeds the chain with end B mentioned in the previous diagram.

The completion of the local ring at Q is $\widehat{\mathcal{O}}_Q \cong R[\![u,v]\!]/(u^{\widetilde{r}} \widetilde{f} - \pi\, \widetilde{\varepsilon})$ with $\widetilde{r} = n_1 s + m_1(n+r)$, $\widetilde{n} = n/n_1$ and $\widetilde{f} = v^{\widetilde{n}} + \widetilde{f}_{\widetilde{n}+1} + \ldots \in k[\![u,v]\!]$ analytically irreducible, $\widetilde{f}_\nu$ homogeneous of degree ν and $\widetilde{\varepsilon} \in R[\![u,v]\!]^\times$ a global unit. The first Puiseux pair of f has disappeared. The new Puiseux pairs of $\widetilde{f}$ are

$$(\widetilde{m}_i, \widetilde{n}_i) \;=\; (m_{i+1} - n_2 \cdot \ldots \cdot n_{i+1} \cdot m_1, n_{i+1}) \qquad \textit{for } i = 1, \ldots, N-1\,.$$

Proof. We study the Puiseux expansion of y in terms of x as written out in equation (2.1) on page 4. Now q times blowing up the singular point $\mathrm{V}(x,y) = \mathrm{V}\big(x, y - P(x)\big)$ of C introduces the new coordinate

$$\begin{aligned} u \;:=\; \frac{y - P(x)}{x^q} \;&=\; a_{j_1}\, x^{p/n_1} + \sum_{l=1}^{l_1} a_{1,l}\, x^{(p+l)/n_1} + \\ &+\; a_{j_2}\, x^{(m_2 - q n_1 n_2)/n_1 n_2} + \sum_{l=1}^{l_2} a_{2,l}\, x^{(m_2 - q n_1 n_2 + l)/n_1 n_2} + \\ &+\; \ldots \;+\; a_{j_N}\, x^{(m_N - qn)/n} + \sum_{j > m_N} a_j\, x^{(j - qn)/n}\;\; . \end{aligned}$$

Since $p < n_1$ we can expand x into a Puiseux series $x = \sum_{j\geq 0} b_j u^{j/\widetilde{n}}$ of u. We plug in and compare coefficients to obtain

$$\begin{aligned} x &= b_{j_1}\, u^{n_1/p} + \sum_{l=1}^{l_1} b_{1,l}\, u^{(n_1+l)/p} + \\ &+ b_{j_2}\, u^{\big(m_2-n_2(m_1-n_1)\big)/pn_2} + \sum_{l=1}^{l_2} b_{2,l}\, u^{\big(m_2-n_2(m_1-n_1)+l\big)/pn_2} + \\ &+ \ \ldots \ + \\ &+ b_{j_N}\, u^{\big(m_N-n_2\cdot\ldots\cdot n_N(m_1-n_1)\big)/\widetilde{n}} + \sum_{j\,>\,m_N-n_2\cdot\ldots\cdot n_N(m_1-n_1)} b_j\, u^{j/\widetilde{n}} \ . \end{aligned}$$

Indeed the first non-zero term is $a_{j_1}^{-n_1/p}\, u^{n_1/p}$. Further if

$$j \ < \ m_i\, n_{i+1}\cdot\ldots\cdot n_N - \frac{n}{n_1} m_1 + n \qquad \text{and} \qquad n_{i+1}\cdot\ldots\cdot n_N \nmid j$$

we show inductively that b_j has to be zero. Namely, say x is of the form

$$\begin{aligned} x \ = \ b_{j_1}\, u^{n_1/p} \ \big(& 1 + Q(u^{1/pn_1\cdot\ldots\cdot n_i}) + b_j/b_{j_1}\, u^{(j-n)/\widetilde{n}} \\ &+\text{Terms of higher order}\big)\ , \end{aligned}$$

where Q is a polynomial without constant term. Now the first assumption on j implies $1 + \ (j-n)/\widetilde{n} \ < \ m_i/pn_2\cdot\ldots\cdot n_i \ - \ qn_1/p$. So plugging into the series expansion of u we get

$$\begin{aligned} u \ = \ u\cdot \big(& 1 + \widetilde{Q}(u^{1/pn_1\cdot\ldots\cdot n_i}) + a_{j_1}\cdot p/n_1\cdot b_j/b_{j_1}\, u^{(j-n)/\widetilde{n}} \\ &+\text{Terms of higher order}\big)\ , \end{aligned}$$

for some polynomial $\widetilde{Q}$ without constant term. The term

$$a_{j_1}\cdot p/n_1\cdot b_j/b_{j_1}\, u^{(j-n)/\widetilde{n}}$$

is not subsumed under this polynomial, because $n_{i+1}\cdot\ldots\cdot n_N \nmid j$. This implies $b_j = 0$ as claimed.

On the other hand we compute

$$b_{j_i} \ = \ -\frac{n_1}{p}\, a_{j_i}\, a_{j_1}^{-\frac{m_i}{pn_2\cdot\ldots\cdot n_i}+\frac{qn_1}{p}} \ \neq \ 0\ .$$

This proves our claim about the Puiseux expansion of x in terms of u. In particular the new Puiseux pairs are

$$(\widetilde{m}_i, \widetilde{n}_i) \ = \ \begin{cases} (n_1, p) & \text{for } i = 1 \\ \big(m_i - n_2\cdot\ldots\cdot n_i\cdot(m_1-n_1), n_i\big) & \text{for } i > 1 \end{cases}$$

Since the multiplicity of the singular point on C is n and on E is r at each step, the multiplicity of the components in the chain ending in B decreases by $n + r$. Thus E' has the multiplicity $\widetilde{s} = s + q(n + r)$.

In case $p = 1$ the first terms in the u-expansion of x are in fact a polynomial

$$P(u) \;=\; b_{j_1}\, u^{n_1} + \sum_{l=1}^{l_1} b_{1,l}\, u^{n_1+l}$$

of u. Blowing up the singular point $\mathrm{V}(x, u)$ of C another n_1 times introduces the new coordinate $v = x/u^{n_1}$. The component C intersects $E'' = \mathrm{V}(u)$ in $Q = \mathrm{V}(u, v - b_{j_1})$, whereas $E' = \mathrm{V}(v)$ intersects E'' in $\mathrm{V}(u, v)$. Thus the configuration is as claimed.

Since the multiplicity of the singular point on C is n/n_1 and on E' is $\widetilde{s}$ at each step, the multiplicity of the components in the chain ending in E decreases by $n/n_1 + \widetilde{s}$. Thus the component E'' has the multiplicity

$$\widetilde{r} \;=\; r + n_1 \big(s + q(n + r) + \frac{n}{n_1}\big) \;=\; n_1 s + m_1(n + r)\;.$$

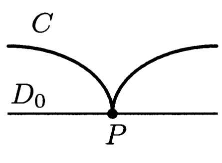
Finally the first Puiseux pair has disappeared. Subtracting n_1 from the exponents in the u-expansion of x yields the remaining Puiseux pairs as claimed. □

With this proposition we can give a description of the blowing-up process. It is a repeated application of the following proposition.

Proposition 3.3 *For $r \in \mathbb{N}_0$ consider the scheme*

$$Y \;=\; \operatorname{Spec} R[\![x, y]\!]/(x^r f - \pi\,\varepsilon)\,,$$

where $n|r$, $f = y^n + f_{n+1} + \ldots \in k[\![x, y]\!]$ is analytically irreducible, f_ν is homogeneous of degree ν and $\varepsilon \in R[\![x, y]\!]^\times$ is a global unit. Let the Puiseux pairs of f be $(m_1, n_1), \ldots, (m_N, n_N)$. So $n = n_1 \cdot \ldots \cdot n_N$. The special fiber $\mathrm{V}(\pi)$ of Y has the displayed configuration with the components

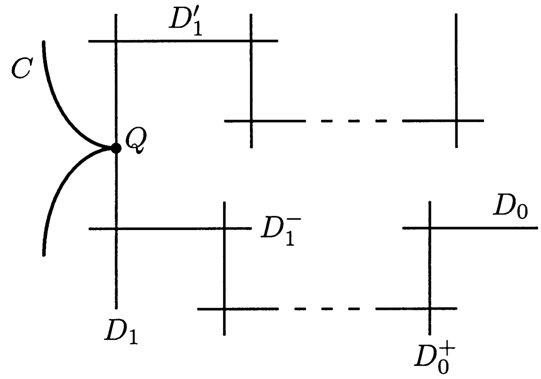


$$C \;=\; \mathrm{V}(f)\,,$$
$$D_0 \;=\; \mathrm{V}(x)\,.$$

Set $p_0 := n_1$ and $p_{-1} := m_1$ and let p_μ and q_μ be integers with

$$\begin{aligned} p_{\mu-2} &= q_\mu p_{\mu-1} + p_\mu\,, \qquad 0 \le p_\mu < p_{\mu-1} \quad (1 \le \mu \le e)\,,\\ p_{e-1} &= 1 \qquad \textit{and} \qquad p_e \;=\; 0\,. \end{aligned}$$

Then $q_1 + \ldots + q_e$ times blowing up the singular point of C leads to the following configuration of projective lines in $\mathrm{V}(\pi)$.



All the indicated components are strict transforms under the various blowing-ups. The component C intersects the exceptional line D_1 of the last blowing-up in the point Q. In a second point D_1 intersects D_1', which is the beginning of a chain of previous exceptional lines. In a third point D_1 intersects D_1^-, which is the beginning of a chain of the other previous exceptional lines. This chain ends in D_0^+ which in turn meets the component D_0 we started with.

For any component D of the special fiber let $r(D)$ be its multiplicity in $\mathrm{V}(\pi)$. *Then we have*

$$r(C) \;=\; 1\,, \qquad r(D_0) \;=\; r\,, \qquad r(D_1) \;=\; nm_1 + n_1 r\,,$$

$$\begin{aligned} \gcd\bigl(r(D_0), r(D_0^+)\bigr) &= n\,, \\ \gcd\bigl(r(D_1), r(D_1^-)\bigr) &= n\,, \\ \gcd\bigl(r(D_1), r(D_1')\bigr) &= \frac{n}{n_1}\, m_1 + r\,. \end{aligned}$$

The greatest common divisor of the multiplicities of any two consecutive components in the chain from D_1 to D_0 is n. Further the value of the δ-invariant of C at its singular point has dropped after these blowing-ups by

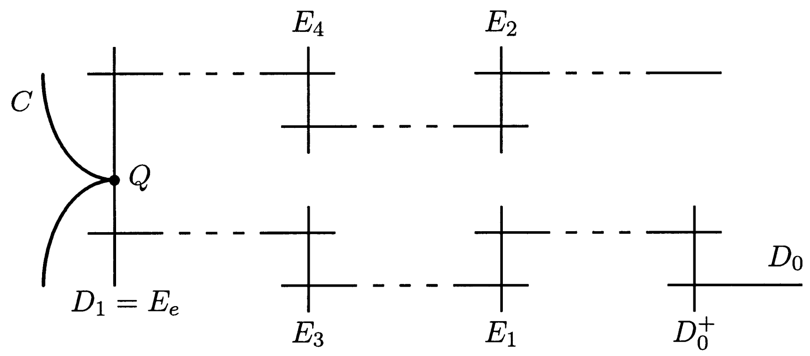
$$\delta_P(C) \,-\, \delta_Q(C) \;=\; \frac{1}{2}\,\frac{n}{n_1}\,(nm_1 - n_1 - m_1 + 1)\,.$$

The completion of the local ring at Q is $\widehat{\mathcal{O}}_Q \cong R[[x,y]]/(x^{r(D_1)}\widetilde{f} - \pi\,\widetilde{\varepsilon})$ with $\widetilde{n} = n/n_1$ and $\widetilde{f} = y^{\widetilde{n}} + f_{\widetilde{n}+1} + \ldots \in k[[x,y]]$ analytically irreducible,

f_ν homogeneous of degree ν and $\widetilde{\varepsilon} \in R[[x,y]]^\times$ a global unit. In particular $\widetilde{n}$ divides $r(D_1)$. The Puiseux pairs of $\widetilde{f}$ are

$$(\widetilde{m}_i, \widetilde{n}_i) \;=\; (m_{i+1} - n_2 \cdot \ldots \cdot n_{i+1} \cdot m_1, n_{i+1}) \qquad \textit{for } i = 1, \ldots, N-1$$

Proof. We divide the blowing-ups into e phases such that the μ-th phase consists of q_μ blowing-ups. In other words, each phase corresponds to the process described in Proposition 3.2. We denote the exceptional line of the last blowing-up of phase μ by E_μ and its multiplicity in $\mathrm{V}(\pi)$ by r_μ. By Proposition 3.2 the configuration of the special fiber after the $q_1 + \ldots + q_e$ blowing-ups is as follows.



The exceptional lines E_μ lie on the chain ending in D_0 for odd μ and on the other chain for even μ. One of the neighbors D_1' or D_1^- of $D_1 = E_e$ is the exceptional line E_{e-1}.

The following statements are true in Phase μ for $1 \le \mu \le e$:

- Before each blowing-up the multiplicity of the singular point on C is
$$p_{\mu-1}\,\frac{n}{n_1}\;.$$
- The multiplicity of the exceptional line of each blowing-up is by
$$d_\mu \;:=\; \begin{cases} r_{\mu-1} + p_{\mu-1}\,\dfrac{n}{n_1} & \text{for } 2 \le \mu \le e \\ n & \text{for } \mu = 1 \end{cases}$$
greater than the multiplicity of the exceptional line of the previous blowing-up.
- The self-intersection number $\left(E_{\mu-1}\right)^2$ of the component $E_{\mu-1}$ decreases at each blowing-up by 1 since the blowing-up is centered in a smooth point of $E_{\mu-1}$.

– The self-intersection number $\big(E_{\mu-2}\big)^2$ of the component $E_{\mu-2}$ decreases at the first blowing-up by 1 since this blowing-up is centered in a smooth point of $E_{\mu-1}$ and stays constant afterwards since $E_{\mu-2}$ does not meet C then.

This implies that the self-intersection number of E_μ at the end of phase e is

$$\big(E_\mu\big)^2 \;=\; \begin{cases} -q_{\mu+1}-2 & \text{for } 1 \le \mu \le e-2\,, \\ -q_e - 1 & \text{for } \mu = e-1\,, \\ -1 & \text{for } \mu = e\,. \end{cases}$$

The multiplicities of the two components intersecting E_μ are $r_\mu - d_\mu$ and $r_\mu + d_{\mu+2}$. Hence pairing $\big(E_\mu \,.\, \mathrm{V}(\pi)\big) = 0$ yields the equation

$$d_{\mu+2} \;=\; q_{\mu+1} r_\mu + d_\mu \qquad \text{for } 1 \le \mu \le e-2\,.$$

Further there are q_μ blowing-ups in phase μ, therefore setting $r_0 := 0$ and $r_{-1} := r$ we have

$$r_\mu \;=\; q_\mu d_\mu + r_{\mu-2} \qquad \text{for } 1 \le \mu \le e\,.$$

This implies by induction that $p_{\mu-1} r_\mu + p_\mu d_\mu \;=\; n m_1 + n_1 r$ for every $1 \le \mu \le e$. In particular we obtain $r(D_1) = r_e = n m_1 + n_1 r$ using $p_e = 0$ and $p_{e-1} = 1$.

The neighbor D_0^+ of D_0 has multiplicity $r + n$. Therefore

$$\gcd\big(r(D_0), r(D_0^+)\big) \;=\; n\,.$$

The neighbors of $D_1 = E_e$, one of them being E_{e-1}, have multiplicities r_{e-1} and $r_e - d_e$. We obtain

$$\gcd(r_e, r_{e-1}) \;=\; \gcd(q_e p_{e-1} \frac{n}{n_1} + r_{e-2}, r_{e-1}) \;=\; \gcd(r_{e-1}, d_{e-1})\,,$$

because $q_e p_{e-1} = p_{e-2}$. So using $\gcd(r_\mu, d_\mu) = \gcd(r_{\mu-2}, d_{\mu-2})$ we obtain

$$\big\{\gcd(r_e, r_{e-1})\,,\, \gcd(r_e, r_e - d_e)\big\} \;=\; \big\{\gcd(r_2, d_2)\,,\, \gcd(r_1, d_1)\big\}\,.$$

Since D_1^- belongs to the chain ending in D_0^+, i.e. the chain with odd indices, we see

$$\gcd\big(r(D_1), r(D_1^-)\big) \;=\; \gcd(r_1, d_1) \;=\; n\,,$$

$$\gcd\big(r(D_1), r(D_1')\big) \;=\; \gcd(r_2, d_2) \;=\; \frac{n}{n_1} m_1 + r\,.$$

Further the $\gcd$ of the multiplicities of any two consecutive components in the chain from D_1 to D_0 is $\gcd(r_1, d_1) = n$.

We compute the Puiseux pairs. According to Proposition 3.2 the Puiseux pairs at the end of phase μ for $0 \leq \mu \leq e-1$ are

$$(p_{\mu-1}, p_\mu) \text{ for } i = 1\,, \; \big(m_i - n_2 \cdot \ldots \cdot n_i\,(m_1 - p_{\mu-1}),\, n_i\big) \text{ for } 2 \leq i \leq N\,.$$

After phase $e-1$ the first pair disappears because $p_{e-1} = 1$. During the remaining $q_e = p_{e-2}$ blowing-ups the exponents in the Puiseux expansion decrease by p_{e-2} and thus the Puiseux pairs are as claimed.

The value of the δ-invariant of C at its singular point drops according to formula (2.3) on page 6 in phase μ at each blowing-up by

$$\frac{1}{2}\, p_{\mu-1}\, \frac{n}{n_1} \Big(p_{\mu-1}\, \frac{n}{n_1} - 1 \Big)\,.$$

Thus in total it drops by

$$\frac{1}{2} \Big(\frac{n}{n_1}\Big)^2 \sum_{\mu=1}^{e} q_\mu p_{\mu-1}^2 - \frac{1}{2}\, \frac{n}{n_1} \sum_{\mu=1}^{e} q_\mu p_{\mu-1} \;=\; \frac{1}{2}\, \frac{n}{n_1}\, (n m_1 - n_1 - m_1 + 1)\,.$$

This proves the proposition. □

The proposition enables us to determine the result of the whole blowing-up process. It describes the way in which the number of the Puiseux pairs is reduced by performing the blowing-ups.

We start with the situation $\mathrm{V}(\pi) = C$ at the beginning of the blowing-up process. In terms of Proposition 3.3 the component D_0 is not there, i.e. $r = 0$. Let the Puiseux pairs at the beginning be

$$\big(m_1^{(0)}, n_1^{(0)}\big) \;:=\; \big(m_1, n_1\big)\,, \quad \ldots\,, \quad \big(m_N^{(0)}, n_N^{(0)}\big) \;:=\; \big(m_N, n_N\big)\,.$$

So the multiplicity of the singular point of C is $n^{(0)} = n_1 \cdot \ldots \cdot n_N$ in the beginning. Now we apply Proposition 3.3 N times. After the ν-th application a new component has appeared which meets the other components in three distinct points. We call it D_ν. Let its multiplicity in $\mathrm{V}(\pi)$ be $r^{(\nu)}$. In the chain leading from D_ν to $D_{\nu-1}$ let the neighbor of D_ν be D_ν^- with multiplicity $r^{(\nu)-}$ and the neighbor of $D_{\nu-1}$ be $D_{\nu-1}^+$ with multiplicity $r^{(\nu-1)+}$. In the third chain proceeding from D_ν let the neighbor of D_ν be D_ν' with multiplicity $r^{(\nu)\prime}$. (In Proposition 3.3 these were called D_1, D_1^-, D_0^+, D_1' respectively.) After the ν-th application let the Puiseux pairs be $\big(m_i^{(\nu)}, n_i^{(\nu)}\big)$ for $1 \leq i \leq N - \nu$. We have the following

Lemma 3.4 *Setting $r^{(0)} = 0$ the data described above can be computed as follows* $(1 \le \nu \le N\,,\ \ 1 \le i \le N-\nu)$:

$$\begin{aligned}
\big(m_i^{(\nu)}\,,\,n_i^{(\nu)}\big) &= \big(m_{i+\nu} - n_{\nu+1}\cdot\ldots\cdot n_{i+\nu}m_\nu\,,\,n_{i+\nu}\big)\\
n^{(\nu)} \;:=\; n_1^{(\nu)}\cdot\ldots\cdot n_{N-\nu}^{(\nu)} &= n_{\nu+1}\cdot\ldots\cdot n_N\,,\\
r^{(\nu)} &= n_\nu\cdot\ldots\cdot n_N\,\Big(m_\nu + n_\nu\sum_{j=1}^{\nu-1}(n_{j+1}\cdot\ldots\cdot n_{\nu-1})^2(n_j-1)m_j\Big)\,,\\
\gcd\big(r^{(\nu)}, r^{(\nu)+}\big) &= n_{\nu+1}\cdot\ldots\cdot n_N\,,\\
\gcd\big(r^{(\nu)}, r^{(\nu)-}\big) &= n_\nu\cdot\ldots\cdot n_N\,,\\
\gcd\big(r^{(\nu)}, r^{(\nu)\prime}\big) &= \frac{n^{(\nu-1)}}{n_1^{(\nu-1)}}m_1^{(\nu-1)} + r^{(\nu-1)}\,.
\end{aligned}$$

The value of the δ-invariant of C at its singular point drops during the whole blowing-up process by

$$\frac{1}{2}\Big(1 - n + \sum_{\nu=1}^{N}(n_{\nu+1}\cdot\ldots\cdot n_N)^2(n_\nu-1)m_\nu\Big)\,.$$

Proof. Setting $m_0 := 0$ the first three formulas also hold for $\nu = 0$, i.e. for the situation before the first blowing-up. We prove them by induction on ν observing that by Proposition 3.3 we have

$$\begin{aligned}
m_i^{(\nu+1)} &= m_{i+1}^{(\nu)} - n_2^{(\nu)}\cdot\ldots\cdot n_{i+1}^{(\nu)}m_1^{(\nu)} \qquad \text{and}\\
r^{(\nu+1)} &= n^{(\nu)}m_1^{(\nu)} + n_1^{(\nu)}r^{(\nu)}\,.
\end{aligned}$$

The expressions for the greatest common divisors can be read off from Proposition 3.3 directly. Furthermore, the value of the δ-invariant of C at its singular point drops during the ν-th application of Proposition 3.3 by

$$\frac{1}{2}\,\frac{n^{(\nu-1)}}{n_1^{(\nu-1)}}\big(n^{(\nu-1)}m_1^{(\nu-1)} - n_1^{(\nu-1)} - m_1^{(\nu-1)} + 1\big).$$

□

The normalization process

Let the model of X_K obtained at the end of the blowing-up process be X'. Now we extend the base from R to $R' := R[\pi']/(\pi'^m - \pi)$, where m is the least common multiple of all the multiplicities of the components of X'_k. Then a semi-stable model $\widetilde{X}$ is obtained by normalizing $X' \otimes_R R'$. In general it is not regular. We describe its configuration using Lemma 2.1 and the numerical data obtained in Lemma 3.4.

First let $E \subseteq X'_k$ be a projective line which meets the other components of X'_k in at most *two* points. Let $\widetilde{E}$ be a component of $\widetilde{X}_k$ lying above E. Then the map $\widetilde{E} \longrightarrow E$ is ramified in at most two points. Hence by the Riemann-Hurwitz formula $\widetilde{E}$ is a projective line ($\operatorname{char} k = 0$). In the chain leading from $D_{\nu+1}$ to D_ν the greatest common divisor of the multiplicities of two consecutive projective lines is always $n^{(\nu)} = n_{\nu+1} \cdot \ldots \cdot n_N$ by Proposition 3.3. Therefore by Lemma 2.1 we see that above this chain there are exactly $n_{\nu+1} \cdot \ldots \cdot n_N$ chains of projective lines in $\widetilde{X}_k$.

Now consider the component $D_\nu \subseteq X'_k$. It meets *three* of the other components. By Lemma 2.1 there are exactly $n_{\nu+1} \cdot \ldots \cdot n_N$ components of $\widetilde{X}_k$ lying above D_ν. They are all isomorphic. Let $\widetilde{D}_\nu$ be one of them. Then the map $\widetilde{D}_\nu \longrightarrow D_\nu$ is of degree

$$\frac{r^{(\nu)}}{n_{\nu+1} \cdot \ldots \cdot n_N}$$

and ramified above three points with ramification indices

$$\frac{r^{(\nu)}}{n_{\nu+1} \cdot \ldots \cdot n_N}\,, \quad \frac{r^{(\nu)}}{n_{\nu} \cdot \ldots \cdot n_N} \quad \text{and} \quad \frac{r^{(\nu)}}{\gcd(r^{(\nu)}, r^{(\nu)\prime})}\,.$$

The degree of the ramification divisor $R \subset \widetilde{D}$ is therefore

$$\begin{aligned} \deg R \;=\;& \frac{r^{(\nu)}}{n_{\nu+1} \cdot \ldots \cdot n_N} - 1 \quad + \quad \frac{r^{(\nu)}}{n_{\nu+1} \cdot \ldots \cdot n_N} - n_\nu \quad + \\ &+ \quad \frac{r^{(\nu)} - r^{(\nu-1)}}{n_{\nu+1} \cdot \ldots \cdot n_N} - (m_\nu - n_\nu m_{\nu-1})\,. \end{aligned}$$

We use the Riemann-Hurwitz formula to compute the genus of $\widetilde{D}_\nu$:

$$2\,p_a(\widetilde{D}_\nu) - 2 \;=\; \frac{r^{(\nu)}}{n_{\nu+1} \cdot \ldots \cdot n_N} \cdot (-2) + \deg R\,,$$

hence the arithmetic genus $p_a(\widetilde{D}_\nu)$ of $\widetilde{D}_\nu$ is

$$\frac{1}{2}\Big(1 - m_\nu + \frac{r^{(\nu)}}{n_{\nu+1} \cdot \ldots \cdot n_N}\Big) \;-\; \frac{1}{2}\, n_\nu \Big(1 - m_{\nu-1} + \frac{r^{(\nu-1)}}{n_\nu \cdot \ldots \cdot n_N}\Big) \quad (3.1)$$

Plugging in the values for $r^{(\nu)}$ and $r^{(\nu-1)}$ from Lemma 3.4 this is equal to

$$\frac{1}{2}\Big(1 - m_\nu + n_\nu m_\nu + n_\nu^2 \sum_{j=1}^{\nu-1} (n_{j+1} \cdot \ldots \cdot n_{\nu-1})^2 (n_j - 1) m_j + n_\nu m_{\nu-1} -$$

$$- n_\nu - n_\nu n_{\nu-1} m_{\nu-1} - n_\nu n_{\nu-1}^2 \sum_{j=1}^{\nu-2} (n_{j+1} \cdot \ldots \cdot n_{\nu-2})^2 (n_j - 1) m_j\Big)$$

$$= \frac{1}{2}(n_\nu - 1)\Big(m_\nu - 1 + n_\nu \sum_{j=1}^{\nu-1} (n_{j+1} \cdot \ldots \cdot n_{\nu-1})^2 (n_j - 1) m_j\Big) .$$

Further we see that the $n_{\nu+1} \cdot \ldots \cdot n_N$ components of $\widetilde{X}_k$ lying above D_ν are the ends of the $n_{\nu+1} \cdot \ldots \cdot n_N$ chains of projective lines coming from the components lying above $D_{\nu+1}$. Vice versa, from the components above D_ν originate $n_\nu \cdot \ldots \cdot n_N$ chains of projective lines, each ending in a different component of $\widetilde{X}_k$ lying above $D_{\nu-1}$. The component $\widetilde{C}$ of $\widetilde{X}_k$ lying above C is isomorphic to the normalization of C and intersects $\widetilde{D}_N$ in one point. This shows that $\widetilde{X}_k$ is treelike.

If X is proper over R, we could equivalently deduce this last statement by the following argument. Since the arithmetic genus is constant in flat families, we see that

$$p_a(X_K) \;=\; p_a(C) \,.$$

All our blowing-ups and the normalization induce isomorphisms on the generic fiber. So we have by the genus formula [BL, Theorem 4.6]

$$p_a(X_K) \;=\; \sum_{\widetilde{D} \subseteq \widetilde{X}_k} p_a(\widetilde{D}) \,+\, g(\mathcal{T}) \,,$$

where the sum runs over all irreducible components $\widetilde{D}$ of $\widetilde{X}_k$ and $g(\mathcal{T})$ denotes the genus of the graph $\mathcal{T}$ associated to $\widetilde{X}_k$. One of these components $\widetilde{D}$ is the normalization $\widetilde{C}$ of C. Now formula (2.2) on page 5 reads for C

$$p_a(C) \;=\; p_a(\widetilde{C}) + \delta_P(C) \,,$$

where P is the singular point of $C = X_k$. So all together we obtain the equality

$$\delta_P(C) \;=\; \sum_{\widetilde{D} \subseteq \widetilde{X}_k,\, \widetilde{D} \neq \widetilde{C}} p_a(\widetilde{D}) + g(\mathcal{T}) \;=\; \sum_{\nu=1}^{N} n_{\nu+1} \cdot \ldots \cdot n_N \, p_a(\widetilde{D}_\nu) + g(\mathcal{T}) \,.$$

Using formula (3.1) on page 20 the sum of the arithmetic genera on the right side equals

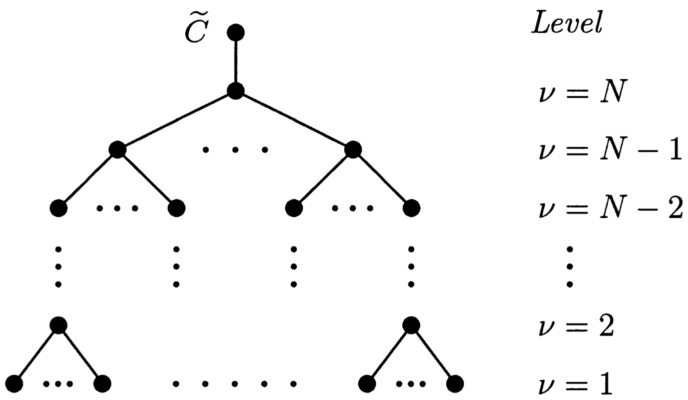
$$\begin{aligned}
&\frac{1}{2}\big(1 - m_N + r^{(N)}\big) - \frac{1}{2} n_1 \cdot \ldots \cdot n_N \\
&= \frac{1}{2}\left(1 - n + \sum_{j=1}^{N} (n_{j+1} \cdot \ldots \cdot n_N)^2 (n_j - 1) m_j\right).
\end{aligned}$$

Thus by Lemma 3.4 we see that the genus $g(\mathcal{T}) = 0$, hence $\mathcal{T}$ is a tree.

We summarize our considerations in the following

Theorem 3.5 *Let X be a regular curve over* $\operatorname{Spec} R$. *Let the special fiber $C := X_k$ be smooth outside a single point, where it has a cusp as singularity. Hence C is irreducible. Let the Puiseux pairs of C at the singularity be (m_i, n_i) for $1 \le i \le N$.*

Then the special fiber of the minimal (not necessarily regular) semi-stable model of X over a suitable base extension is tree-like. Its associated graph has the following configuration.



On level ν there are $n_{\nu+1} \cdot \ldots \cdot n_N$ curves each of genus

$$\frac{1}{2}(n_\nu - 1)\Big(m_\nu - 1 + \sum_{j=1}^{\nu-1} (n_{j+1} \cdot \ldots \cdot n_{\nu-1})^2 (n_j - 1) m_j\Big).$$

Each of these curves intersects n_ν curves of the next lower level $\nu - 1$.

The curve $\widetilde{C}$ is the normalization of C. The δ-invariant of C at its singular point is

$$\delta_P(C) \;=\; \frac{1}{2}\left(1 - n + \sum_{\nu=1}^{N} (n_{\nu+1} \cdot \ldots \cdot n_N)^2 (n_\nu - 1) m_\nu\right).$$

Proof. The minimal semi-stable model is obtained from the model $\widetilde{X}$ we constructed above by contracting all projective lines in $\widetilde{X}_k$. Each such projective line intersects the other components of $\widetilde{X}_k$ at most in two points. Therefore the model obtained by the contraction is still semi-stable but not necessarily regular. □

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