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Semi-stable models for rigid-analytic spaces

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Abstract. Let R be a complete discrete valuation ring with field of fractions K and let X_K be a smooth, quasi-compact rigid-analytic space over $\operatorname{Sp} K$. We show that there exists a finite separable field extension K' of K, a rigid-analytic space $X'_{K'}$ over $\operatorname{Sp} K'$ having a strictly semi-stable formal model over the ring of integers of K', and an étale, surjective morphism $f : X'_{K'} \longrightarrow X_K$ of rigid-analytic spaces over $\operatorname{Sp} K$. This is different from the alteration result of A.J. de Jong [dJ] who does not obtain that f is étale. To achieve this property we have to work locally on X_K, i.e. our f is not proper and hence not an alteration.

Introduction

Let R be a complete discrete valuation ring with uniformizing parameter π, field of fractions K and residue field k. When studying R-models X of smooth K-varieties X_K, i.e. schemes X which are flat, separated and of finite type over $\operatorname{Spec} R$ with $X_K \cong X \times_R K$, one in general must allow singularities on the special fiber $X_0 := X \times_R k$. Therefore one is interested in models with "mild" singularities. For example if X_K is a proper, smooth, geometrically connected curve, P. Deligne and D. Mumford [DM] have proved, that after allowing a finite extension of discrete valuation rings, there exists a *semi-stable model* X of X_K. This means that X is proper and flat over $\operatorname{Spec} R$ and its special fiber X_0 is a geometrically reduced and connected curve with ordinary double points as the only singularities. See [Ab] for a survey on different proofs for this theorem. The concept of semi-stability can be generalized to higher dimensions, cf. [dJ, 2.16]. There again semi-stable varieties possess the mildest sort of singularities one may hope for in the degeneration of smooth varieties. If the residue characteristic $\operatorname{char}(k)$ is zero, G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat [KKMS] established the existence of semi-stable models for every dimension using the technique of toroidal embeddings. However in arbitrary characteristic this remains an open question. On the other hand A.J. de Jong has shown in [dJ, Theorem 6.5] that every variety X over R admits an alteration $X' \longrightarrow X$, i.e. a proper, surjective, generically finite morphism, by a regular, strictly semi-stable R-variety

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X' after a finite extension of discrete valuation rings. Although this is less than the desired general existence of semi-stable models it suffices for many applications.

The present article contains a different result in this direction, which we are going to describe. The field K is complete with respect to its non-archimedean valuation. So we can do rigid-analytic geometry over K; cf. [Ta] and [BGR]. Due to M. Raynaud [Ra] every quasi-compact and quasi-separated rigid-analytic space X_K over K has a formal model over the formal spectrum Spf R of R. See [FRG, I & II] for a detailed account on Raynaud's theorem. In this setting one can also define the notion of a semi-stable formal R-scheme, cf. Definition 1.1. And one may again ask if every smooth rigid-analytic space X_K over K has such a formal model. We will show that after a finite separable field extension of K it does étale locally on X_K; cf. Theorem 1.4.

The difference to the results of de Jong is that we work locally on X_K instead of globally, i.e. we do not obtain a proper dominant morphism $f : X'_{K'} \longrightarrow X_K$. Our f is only surjective and hence not an alteration. However in our theorem we can find an f that is étale, whereas in [dJ] it can and does happen that f is not even flat; cf. [dJ, 6.13] where the blowing-up has center in the generic fiber. So our result applies to other questions than de Jong's. For instance it may help proving universal properties. Namely using faithfully flat descent these only have to be tested on rigid-analytic spaces, which have semi-stable models; cf. Theorem 3.2. In fact this technique was used for demonstrating the representability of the rigid-analytic Picard functor in [HL]. However the argument for the existence of the semi-stable model given there is somewhat unsatisfactory; cf. [HL, Proof of Lemma 3.12]. The aim of this article is to fill this gap and to give a complete proof for the étale-local existence of semi-stable models.

We would like to give a brief sketch of our proof. It is inspired by the technique of de Jong's article [dJ]. Let X be a formal model of our rigid-analytic space X_K. The idea is to fiber X over a formal scheme Y such that all fibers are one-dimensional and to use induction on the dimension d of X_K. Namely since X is rig-smooth, it can locally be fibered over a $(d-1)$-dimensional unit ball $Y := \mathbb{D}_R^{d-1}$ such that $X \longrightarrow Y$ is a rig-smooth curve-fibration. We are free to perform rig-étale base changes to Y. So using a result of W. Lütkebohmert [L2, Theorem 5.3], this curve-fibration can be embedded into a projective, rig-smooth curve fibration. Applying the Reduced Fiber Theorem [FRG, IV], we may replace $X \longrightarrow Y$ by a projective, rig-smooth curve fibration with geometrically reduced fibers. Again after a rig-étale base change we can assume that $X \longrightarrow Y$ is punctured by sections making X_{rig} into a stable pointed curve over Y_{rig}. By the existence of proper moduli spaces of stable pointed curves we can replace Y by a rig-étale cover such that X_{rig} extends to a stable pointed curve C over Y, i.e. $C_{\mathrm{rig}} \cong X_{\mathrm{rig}}$. As in [dJ] an important step is to show that the morphism $C_{\mathrm{rig}} \xrightarrow{\sim} X_{\mathrm{rig}}$ extends to a morphism $C \longrightarrow X$ of formal schemes after an admissible blowing-up of Y. Thus we may replace X by C. Now the induction hypothesis allows us to replace Y by a strictly semi-stable formal R-scheme. This means that the special fiber Y_0 is a divisor with strict normal crossings. Since C_{rig} is smooth over Y_{rig} we see that the singularities of C lie in the special fiber and are given by equations of the form

$$uv = t_1^{n_1} \cdot \ldots \cdot t_r^{n_r} \, .$$

These are resolved explicitly as in [dJ].

We remark that for our proof it is necessary to work with rigid-analytic spaces instead of algebraic K-varieties, since the result on the smooth compactification of curve-fibrations does not hold in the coarse Zariski-topology but only in the finer Grothendieck-topology of rigid-analytic geometry.

1. Announcement of the Theorem

For a general introduction to formal and rigid-analytic geometry we refer the reader to the standard literature [FRG], [Ta] and [BGR]. Due to M. Raynaud [Ra] every quasi-compact and quasi-separated rigid-analytic space X_K over $\operatorname{Sp} K$ has a flat formal model over $\operatorname{Spf} R$ which we usually denote by X. Conversely to every quasi-compact admissible formal R-scheme X is associated a rigid-analytic space X_{rig}, cf. [FRG, I, Theorem 4.1]. So if X is a model of X_K we have an isomorphism $X_K \cong X_{\text{rig}}$. If X_K is proper over $\operatorname{Sp} K$, then X is proper over $\operatorname{Spf} R$ and vice versa; cf. [L1, Theorem 3.1]. In this article all admissible formal R-shemes will be quasi-compact. When considering a morphism f of quasi-compact admissible formal R-schemes we will say that f is *rig-smooth*, *rig-étale*, etc. if the corresponding rigid-analytic morphism f_{rig} is smooth, étale, etc. respectively. We repeat the notion of semi-stable formal schemes. See [dJ, 2.16] for the algebraic analog.

Definition 1.1. *Let X be a quasi-compact, admissible formal R-scheme and let X_0^σ for $\sigma = 1, \ldots, s$ be the irreducible components of the special fiber X_0 of X. For all $M \subset N := \{1, \ldots, s\}$, $M \neq \emptyset$ we define*

$$X_0^M := \bigcap_{\sigma \in M} X_0^\sigma$$

as the scheme-theoretic intersection. X is called strictly semi-stable over $\operatorname{Spf} R$ *if*

(a) *X_0 is geometrically reduced,*
(b) *X_0^σ is a Cartier divisor on X for all $\sigma \in N$ and*
(c) *X_0^M is smooth over k for all $M \subset N$, $M \neq \emptyset$ and equidimensional of dimension* $\dim X - \#M$.

Remark 1.1.1. If X is strictly semi-stable over $\operatorname{Spf} R$, then X is a regular scheme due to condition (b). Further X_{rig} is then smooth over $\operatorname{Sp} K$. These properties can be read off from [HL, Proposition 1.3] which we repeat here.

Proposition 1.2. *Let X be a quasi-compact, admissible formal R-scheme. The following are equivalent:*

(a) *X is strictly semi-stable.*
(b) *Every closed point $x \in X_0$ of the special fiber has an open neighborhood, which for some $r \in \mathbb{N}$ is formally smooth over the formal scheme*

$$\operatorname{Spf} R \langle \xi_1, \ldots, \xi_r \rangle / (\xi_1 \cdot \ldots \cdot \xi_r - \pi) \, .$$

To state our result on the existence of semi-stable models, we define the following class of morphisms.

Definition 1.3. *Let $f : S' \longrightarrow S$ be a morphism of quasi-compact, admissible formal R-schemes. f is called a* rig-étale cover *if it is a composition $S' \longrightarrow S^\dagger \longrightarrow S$ of an admissible formal blowing-up $S^\dagger \longrightarrow S$ with a morphism $S' \longrightarrow S^\dagger$ which is quasi-finite, flat and surjective, as well as* rig-étale*, i.e. étale on the associated rigid spaces $S'_{\mathrm{rig}} \longrightarrow S^\dagger_{\mathrm{rig}}$. This implies that $f_{\mathrm{rig}} : S'_{\mathrm{rig}} \longrightarrow S_{\mathrm{rig}}$ is étale and surjective. Conversely by [FRG, Theorem 5.2 and Corollary 5.3b)] every étale, surjective morphism $f_K : S'_K \longrightarrow S_K$ of quasi-compact and quasi-separated rigid-analytic spaces has a model f, which is a rig-étale cover.*

Such morphisms are stable under composition and play the role of surjective, étale morphisms of finite type in rigid-analytic geometry; cf. [FRG, III & IV].

Now we can formulate our main theorem.

Theorem 1.4. *Let X be a quasi-compact admissible formal R-scheme such that the associated rigid-analytic space X_{rig} is smooth over* Sp K*. Then there exists a finite separable field extension K' of K with ring of integers R', a quasi-compact, strictly semi-stable, formal R'-scheme X' and a rig-étale cover $X' \longrightarrow X$ of formal R-schemes.*

This immediately implies

Corollary 1.5. *Let X_K be a smooth rigid-analytic space over* Sp K*. Chose an admissible covering $\{X^i_K\}$ of X_K by quasi-compact and quasi-separated rigid-analytic subspaces. Then for each i there exists a finite separable field extension K' of K with ring of integers R', a quasi-compact, strictly semi-stable, formal R'-scheme X' and an étale, surjective morphism $X'_{\mathrm{rig}} \longrightarrow X^i_K$ of rigid-analytic K-varieties.*

Proof. We chose formal models for the X^i_K and apply Theorem 1.4 to each of these formal models. □

Remark 1.5.1. Corollary 1.5 applies in particular to every scheme, locally of finite type and smooth over Spec K, by taking X_K as its analytification ([BGR, Example 9.3.4/2]).

For the proof of Theorem 1.4 we need the following lemmas. The first Lemma is [dJ, Lemma 5.6] except for the additional property of being étale.

Lemma 1.6. *Let Y be a normal and integral scheme and let $U \subset Y$ be a dense open subset. Let $f : X \longrightarrow Y$ be a proper morphism and let $Z \subset X$ be a closed subscheme, finite and flat over Y and étale over U. Then there exists a morphism $Y' \longrightarrow Y$, which over U is finite and étale, and sections $\sigma_1, \ldots, \sigma_n : Y' \longrightarrow X \times_Y Y' =: X'$ such that $\varphi^{-1}(Z) = \bigcup_i \sigma_i(Y')$ (set theoretically), where $\varphi : X' \longrightarrow X$ is the projection onto the first factor.*

Proof (cf. [dJ, Lemma 5.6]). Let $Z_{\text{red}} = \bigcup_j Z_j$ be the decomposition of Z_{red} into its irreducible components. The field extensions $R(Y) \subset R(Z_j)$ are finite and separable. Let L be the normal hull of the composition of all the $R(Z_j)$ inside an algebraic closure of $R(Y)$. Let Y' be the normalization of Y in L.

For $e \in \mathbb{N}$ big enough, L is a direct summand of the e-fold tensor product $\big(R(Z_1) \otimes_{R(Y)} \ldots \otimes_{R(Y)} R(Z_n)\big)^{\otimes e}$. Since U is normal, this makes $Y' \times_Y U$ into a connected component of the e-fold fiber product $U \times_Y \big(Z_1 \times_Y \ldots \times_Y Z_n\big)^e$, which is finite and étale over U. Hence $Y' \times_Y U$ is also finite and étale over U.

Since $Z \times_Y Y'$ is finite and flat over Y', all its irreducible components are finite over Y'. Further by our choice of L they are birational to Y'. Thus they are isomorphic to Y', since Y' is normal. Hence the irreducible components of $Z \times_Y Y'$ are sections $\sigma_i : Y' \longrightarrow X'$. □

The next Lemma is an analogue of [dJ, Lemma 5.2]

Lemma 1.7. *Let A be an admissible formal R-algebra which is integral and normal. Let $Y := \operatorname{Spf} A$ and $f : X \longrightarrow Y$ be a rig-smooth, flat, projective morphism such that all fibers are nonempty, equidimensional of dimension 1, containing a dense open part on which f is smooth. Then there exists a morphism $Y' \longrightarrow Y$ of quasi-compact, admissible formal schemes, which is étale and surjective and sections $\sigma_1, \ldots, \sigma_n : Y' \longrightarrow X \times_Y Y' =: X'$ into the smooth locus of f with the property that for every geometric point $\overline{y}$ of Y' and every irreducible component C of $X'_{\overline{y}}$ there exist $i, j, k \in \{1, \ldots, n\}$ such that $\sigma_i(\overline{y})$, $\sigma_j(\overline{y})$ and $\sigma_k(\overline{y})$ are three distinct points of C.*

Proof. The proof given by de Jong also works in our case. Instead of [dJ, Lemma 5.6] we use Lemma 1.6. We briefly indicate the argument.

Let $y \in Y_0$ be a closed point. We solve the problem in a neighborhood of y. Let $\mathcal{L}$ be a very ample line bundle on X over Y. Let $n \geq 3$ be so large that $f_*\mathcal{L}^{\otimes n} \longrightarrow \mathrm{H}^0\big(X_y, \mathcal{L}^{\otimes n}|_{X_y}\big)$ is surjective. We use Bertini's Theorem on the generically smooth curve X_y; see for instance [Jo, Théorème I.6.6]. So after replacing Y by an étale neighborhood of y we obtain a section $t \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that the zero divisor $(t)_0 \times_Y \operatorname{Spec} \kappa(y)$ is finite étale over $\kappa(y)$. By shrinking Y we may assume that $(t)_0$ is finite, étale over Y. After applying Lemma 1.6 the divisor $(t)_0$ is the union of sections $\sigma_i : Y \longrightarrow X$.

Let $\overline{y}$ be a geometric point of Y and let C be an irreducible component of $X_{\overline{y}}$. Since the degree of $\mathcal{L}|_C$ is at least 1, the degree of $\mathcal{L}^{\otimes n}|_C$ is at least 3. So $(t)_0$ intersects C in at least 3 points as claimed. □

Lemma 1.8. *Let A be an admissible formal R-algebra which is integral and normal. Let $Y := \operatorname{Spf} A$ and $f : X \longrightarrow Y$ be a projective, semi-stable curve which is rig-smooth. Then there exists a rig-étale cover $Y' \longrightarrow Y$ and sections $\sigma_1, \ldots, \sigma_n : Y' \longrightarrow X' := X \times_Y Y'$ with the property that for every geometric point $\overline{y}$ of Y' and every singular point $\overline{x}$ of $X'_{\overline{y}}$ there is an $i \in \{1, \ldots, n\}$ such that $\overline{x} = \sigma_i(\overline{y})$.*

Proof. Let $y \in Y_0$ be a closed point. We will solve the problem in a neighborhood U of y. This suffices since Y is quasi-compact. If the fiber X_y over y is smooth, there is nothing to do. Otherwise let $x \in X_y$ be a singular point. After an étale base change of Y the point x is rational over y. So we may assume that a neighborhood of x in X is isomorphic to a neighborhood of $\mathrm{V}(u, v, \mathfrak{m}_y)$ in

$$\operatorname{Spf} A\langle u, v\rangle/\big(h + L(u, v) + Q(u, v) + R(u, v)\big)\,.$$

Here h lies in the maximal ideal $\mathfrak{m}_y \subset A$ of y and $L(u, v) = L_1\,u + L_2\,v$ is a linear polynomial over A, congruent to zero modulo $\mathfrak{m}_y$. Further $Q(u, v) = Q_0\,uv + Q_1\,u^2 + Q_2\,v^2$ is a quadratic polynomial over A with discriminant $Q_0^2 - 4Q_1Q_2$ being nonzero modulo $\mathfrak{m}_y$ and $R(u, v)$ is a polynomial in u and v without homogeneous terms of degree less than 3. We want to remove the linear term L by a transformation of the coordinates u and v. Setting $u = \widetilde{u} + a$ and $v = \widetilde{v} + b$, we have to find a and b satisfying the equation

$$F(a, b) \;:=\; \frac{\partial(L + Q + R)}{\partial(u, v)}\bigg|_{(u,v)=(a,b)} \;=\; 0\,.$$

The base change $\operatorname{Spf} A\langle a, b\rangle/(F) \to \operatorname{Spf} A$ is étale in a neighborhood of the point $\mathrm{V}(a, b, \mathfrak{m}_y)$, since the Jacobi-matrix of F is

$$dF \;=\; \begin{pmatrix} 2Q_1 & Q_0 \\ Q_0 & 2Q_2 \end{pmatrix} + \text{Terms in the ideal } (a, b)$$

with determinant $Q_0^2 - 4Q_1Q_2$ being nonzero modulo $(a, b, \mathfrak{m}_y)$. We replace Y by this neighborhood and thus we may assume, that $L = 0$. This implies that $\mathrm{V}(h)$ is contained in the special fiber Y_0, since f is rig-smooth.

Now we chose an integer $r \geq 3$, prime to the characteristic of K and consider the base change

$$\begin{aligned} Y' &:= \operatorname{Spf} A\langle \tilde{h}, w\rangle/\Big(\tilde{h}^r - h\,,\; Q(1, w) + \tilde{h}^{-2}R(\tilde{h}, \tilde{h}w) + \tilde{h}^{r-2}\Big) \;\longrightarrow\; Y \\ &= \;\operatorname{Spf} A\,. \end{aligned}$$

It is quasi-finite, flat, surjective and rig-étale on an open neighborhood of y. We shrink Y to this neighborhood. Setting $u = \tilde{h}$ and $v = \tilde{h}w$ we obtain a section $Y' \longrightarrow X \times_Y Y'$. We do this for all singular points of the fiber X_y. Summarizing we have found an open neighborhood U of y in Y, a rig-étale cover $U' \longrightarrow U$ and sections $U' \longrightarrow X \times_Y U'$, such that every singular point of every fiber over U' lies on one of these sections. (Note that for a semi-stable curve the formation of singular locus commutes with base change.) □

2. Proof of Theorem 1.4

2.1. The theorem is proved by induction on the relative dimension d of X over $\operatorname{Spf} R$. In case $d = 0$ the rigid space $X_{\mathrm{rig}} = \operatorname{Sp} \bigoplus_{i\in I} K'_i$ is the finite disjoint union of finite separable field extensions K'_i of K. Let K' denote the composition of all the K'_i inside an algebraic closure of K and take X' to be the normalization of X in K'. Thus X' is the $\#I$-fold disjoint union of the formal spectrum of the ring of integers R' of K'. Then $X' \longrightarrow X$ is a rig-étale cover. We remark that the case $d = 1$ does not trivially follow from the stable reduction theorem.

2.2. Let $d \geq 1$. We divide the proof into several steps. At each step we reduce to a case where we have additional conditions on X numbered (a), (b), etc.

2.3. Since X_{rig} is smooth over $\operatorname{Sp} K$ of relative dimension d there exists an admissible formal blowing-up $X' \longrightarrow X$, an open covering $\{X'_i : i \in I\}$ of X' and a commutative diagram

$$\begin{array}{ccc} X'_i & \xrightarrow{g_i} & \mathbb{D}^d_R \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spf} R & = & \operatorname{Spf} R \end{array}$$

where $\mathbb{D}^d_R := \operatorname{Spf} R\langle \xi_1, \ldots, \xi_d \rangle$ is the d-dimensional unit ball over R and g_i is *rig-étale*; cf. [FRG, III, Proposition 3.7]. Observe that the index set I can be chosen finite since X is quasi-compact.

Consider the rig-smooth morphism $f_i : X'_i \longrightarrow \mathbb{D}^{d-1}_R$ obtained by composing g_i with the natural projection $\mathbb{D}^d_R \longrightarrow \mathbb{D}^{d-1}_R$. Let $Y := \coprod_{i \in I} \mathbb{D}^{d-1}_R$ and $X'' := \coprod_{i \in I} X'_i$ and define $f := \coprod_{i \in I} f_i : X'' \longrightarrow Y$. After performing an admissible blowing-up of Y and replacing X'' by the strict transform under this blowing-up we may assume that f is flat; cf. [FRG, II, Theorem 5.2]. It suffices to prove the theorem for the formal R-scheme X''. Thus we may assume that

(a) there exists a quasi-compact, admissible formal R-scheme Y, rig-smooth over $\operatorname{Sp} K$, and a rig-smooth morphism $f : X \longrightarrow Y$ which is quasi-compact and flat, such that the fibers of f are nonempty and equidimensional of dimension 1.

We call such an f a *rig-smooth curve fibration*. Property (a) is preserved if we replace Y by a rig-étale cover $Y' \longrightarrow Y$.

2.4. We want to compactify f. According to [L2, Theorem 5.3] there exists a rig-étale cover $Y' \longrightarrow Y$, an admissible covering $\{X^i_K : i \in I\}$ of $\left(X \times_Y Y'\right)_{\mathrm{rig}}$ with I finite and a commutative diagram

$$\begin{array}{ccccccc} X_{\mathrm{rig}} & \longleftarrow & \left(X \times_Y Y'\right)_{\mathrm{rig}} & \longleftarrow & X^i_K & \xrightarrow{g^i_K} & W^i_K \\ {\scriptstyle f}\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle p_K} \\ Y_{\mathrm{rig}} & \longleftarrow & Y'_{\mathrm{rig}} & = & Y'_{\mathrm{rig}} & = & Y'_{\mathrm{rig}} \end{array}$$

where g^i_K is an open immersion and p_K makes W^i_K into a smooth, projective relative curve over Y'_{rig}. After an admissible blowing-up $X' \longrightarrow X \times_Y Y'$ there is an open covering $\{X^i : i \in I\}$ of X' with $X^i_K \cong X^i_{\mathrm{rig}}$; cf. [FRG, II, Theorem 5.5]. We chose formal models W^i of W^i_K and set $W := \coprod_{i \in I} W^i$ and $X'' := \coprod_{i \in I} X^i$. After blowing up X'' the morphism $g_K := \coprod_{i \in I} g^i_K$ extends to a morphism $g : X'' \longrightarrow W$ due to [FRG, II, Lemma 5.6]. Blowing up W and replacing X'' by the strict transform we can achieve that g is an open immersion by [FRG, II, Corollary 5.4] and that

p_K extends to a morphism $p : W \longrightarrow Y'' := \coprod_{i \in I} Y'$. This morphism will be flat after a blowing-up of Y''; cf. [FRG, II, Theorem 5.2]. Hence we obtain a diagram of admissible formal R-schemes

$$\begin{tikzcd} X & X' \arrow[l] & X'' \arrow[l] \arrow[r, "g"] & W \\ Y \arrow[from=u, "f"'] & Y' \arrow[l] \arrow[from=u] & Y'' \arrow[l] \arrow[from=u] \arrow[r, equal] & Y'' \arrow[from=u, "p"'] \end{tikzcd}$$

where $X'' \longrightarrow X$ is a rig-étale cover, g is an open immersion and W_{rig} is projective over Y''_{rig}. If we prove the theorem for W it follows for X. Therefore we may replace Y by Y'' and X by W and assume that we have (a) and

(b) f_{rig} is projective.

2.5. Property (b) implies that f is proper; cf. [L1, Lemma 2.6]. Since Y is quasi-compact, it has a finite covering by affine, open subsets $\{Y_i : i \in I\}$. Replacing Y by $\coprod_{i \in I} Y_i$ we may assume that $Y = \operatorname{Spf} A$ is itself affine. Let $\overline{Y} := \operatorname{Spec} A$. Then Y is the formal completion of $\overline{Y}$ along its special fiber $\overline{Y}_0 := \operatorname{Spec}\big(A \otimes_R k\big)$.

From now on for every scheme $\overline{X}$ of finite type over Spec A, which is πA-torsion free, we denote by $\overline{X}_0 := \overline{X} \times_{\operatorname{Spec} R} \operatorname{Spec} k$ its special fiber and by $\overline{X}_K := \overline{X} \times_{\operatorname{Spec} R} \operatorname{Spec} K$ its generic fiber over Spec R. The completion of $\overline{X}$ along $\overline{X}_0$ is an admissible formal R-scheme which we denote by X and call the *analytification* of $\overline{X}$. If $\overline{X}$ is proper over $\overline{Y}$ the rigid-analytic space X_{rig} is the analytification of $\overline{X}_K$.

By property (b) we have a rigid-analytic morphism $\iota : X_{\mathrm{rig}} \hookrightarrow \mathbb{P}^N_{Y_{\mathrm{rig}}}$ for some $N \in \mathbb{N}$, which is a closed immersion. Thus there exists an admissible blowing-up $X' \longrightarrow X$ such that ι extends to a formal morphism $X' \longrightarrow \mathbb{P}^N_Y$. After an admissible blowing-up $P \longrightarrow \mathbb{P}^N_Y$ this gives a closed immersion $X' \hookrightarrow P$; cf. [FRG, II, Corollary 5.4c)].

Now the formal projective space $\mathbb{P}^N_Y$ can be thought of as the analytification of the algebraic projective space $\mathbb{P}^N_{\overline{Y}}$ of finite type over $\overline{Y}$. The blowing-up $P \longrightarrow \mathbb{P}^N_Y$ takes place in a sheaf $\mathcal{I}$ of open ideals, i.e. $\mathcal{I} \supset \pi^m \mathcal{O}_{\mathbb{P}^N_Y}$ for m sufficiently large. This sheaf $\mathcal{I}$ can be approximated by an algebraic sheaf $\overline{\mathcal{I}} \subset \mathcal{O}_{\mathbb{P}^N_{\overline{Y}}}$ of open ideals, i.e. $\mathcal{I} = \overline{\mathcal{I}} \cdot \mathcal{O}_{\mathbb{P}^N_Y}$. Therefore P is the analytification of the blowing-up $\overline{P}$ of $\mathbb{P}^N_{\overline{Y}}$ in $\overline{\mathcal{I}}$ and $\overline{P}$ is projective of finite type over $\overline{Y}$. By the formal GAGA [EGA, III, Corollaire 5.1.8] we conclude that X' is the analytification of a projective scheme $\overline{X}'$ of finite type over $\overline{Y}$. After an admissible blowing-up of Y the scheme X' is again flat over Y. Thus replacing X by X', we reduce to a situation where we have (a), (b) and

(c) $Y = \operatorname{Spf} A$ is affine and there exists a projective scheme $\overline{X}$ over $\overline{Y} := \operatorname{Spec} A$ such that X is the completion of $\overline{X}$ along the closed subscheme $\overline{X}_0 := \overline{X} \times_{\operatorname{Spec} R} \operatorname{Spec} k$.

This property is preserved if we apply a rig-étale cover $Y' \longrightarrow Y$, since we may replace Y' by an affine covering. Further it is preserved by admissible blowing-ups

$X' \longrightarrow X$, since every blowing-up in a sheaf of open ideals on X can be defined algebraically on $\overline{X}$.

2.6. We replace Y by its normalization Y'. Since Y is rig-smooth over $\mathrm{Sp}\, K$, the normalization morphism $Y' \longrightarrow Y$ is a rig-isomorphism, hence an admissible blowing-up; cf. [FRG, Corollary 5.4b)]. Thus we have (a)–(c) and

(d) Y is normal.

This property is also preserved if we apply a rig-étale cover $Y' \longrightarrow Y$, since we may replace Y' by its normalization.

2.7. There is a formal morphism $Y' \longrightarrow Y$, which is étale and surjective such that all connected components of all fibers of f over generic points of $\overline{Y}$ have a rational point; for instance use Lemma 1.7. Observe that also Y' is normal due to (d). Now let $X' \longrightarrow X$ be the normalization morphism. Since X is rig-smooth this is a rig-isomorphism, hence an admissible blowing-up. Let $\overline{Y}^i$ for $i \in I$ be the connected components of $\overline{Y}$. For each $i \in I$ let $\overline{X}^{i,j}$, $j \in J_i$ be the connected components of $\overline{X} \times_{\overline{Y}} \overline{Y}^i$. Then the $\overline{X}_K^{i,j}$ are the connected components of $\overline{X}_K$, because $\overline{X}$ is normal. We replace $\overline{X}$ by $\coprod_{i\in I} \coprod_{j\in J_i} \overline{X}^{i,j}$ and $\overline{Y}$ by $\coprod_{i\in I} \coprod_{j\in J_i} \overline{Y}^i$. Since $\overline{X}$ is smooth over $\overline{Y}$, the fibers over the generic points of $\overline{Y}$ are now geometrically irreducible. After an admissible blowing-up of Y the morphism f is flat again. We are allowed to replace Y by an affine covering and therefore to assume that $Y = \mathrm{Spf}\, A$ is affine. We replace $\overline{Y}$ by $\mathrm{Spec}\, A$. Therefore we may assume that in addition to (a) – (d) we have

(e) The fibers over the generic points of $\overline{Y}$ are geometrically irreducible.

This is unchanged if we replace Y by a rig-étale cover and X by an admissible blowing-up.

2.8. So far the fibers of f need not be geometrically reduced. To achieve this is our next aim. We use [FRG, IV, Theorem 2.1'] and deduce that there is a commutative diagram

$$\begin{tikzcd}
\overline{X} \arrow[d, "f"'] & \overline{X} \times_{\overline{Y}} \overline{Y}' \arrow[l] \arrow[d] & \overline{X}' \arrow[l, "g"'] \arrow[d] \\
\overline{Y} & \overline{Y}' \arrow[l, "h"'] & \overline{Y}' \arrow[l, equal]
\end{tikzcd}$$

such that $\overline{X}'$ is flat over $\overline{Y}'$ with geometrically reduced fibers. Thereby h is a composition of a blowing-up $\overline{Y}^\dagger \longrightarrow \overline{Y}$ with center in $\overline{Y}_0$ and a flat surjective map $\overline{Y}' \longrightarrow \overline{Y}^\dagger$ of finite type which is étale over $\overline{Y}_K$, hence its analytification h^{an} is a rig-étale cover. Further g is a finite morphism which is an isomorphism over $\overline{X}_K$, so g^{an} is an admissible blowing-up. So we reduce to the case where we have (a) – (e) and

(f) The fibers of f are geometrically reduced.

Again this property is preserved if we apply a rig-étale cover to Y.

2.9. We apply Lemma 1.7 to the curve $f : \overline{X} \longrightarrow \overline{Y}$ and arrive at a situation, where we have (a) – (f) and

(g) There are disjoint sections $\sigma_1, \dots, \sigma_n : \overline{Y} \longrightarrow \overline{X}$ such that for every geometric point $\overline{y} \in \overline{Y}$ and every irreducible component C of $\overline{X}_{\overline{y}}$ there exist $i, j, k \in \{1, \dots, n\}$ such that $\sigma_i(\overline{y}), \sigma_j(\overline{y})$ and $\sigma_k(\overline{y})$ are three distinct points of C lying in the smooth locus of f.

2.10. In our next step we will find a semi-stable curve $\mathcal{C}$ over Y with $\mathcal{C}_{\mathrm{rig}} \cong X_{\mathrm{rig}}$. The argument follows [dJ]. A general reference is Knudsen [Kn]. See also Harris and Mumford [HM] for the case $g = 0$.

Let $g \geq 0$ and $n \geq 3$ and let $\overline{\mathcal{M}}_{g,n}$ denote the algebraic stack over $\mathbb{Z}$ classifying stable, n-pointed curves of genus g. Denote by $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ the open substack that classifies smooth, n-pointed curves. Let $\ell \geq 3$ be a prime different from the residue characteristic $\mathrm{char}(k)$ of R and let

$$_\ell\mathcal{M}_{g,n} \longrightarrow \mathcal{M}_{g,n}[1/\ell] = \mathcal{M}_{g,n} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[1/\ell]$$

be the representable, finite, étale cover given by trivialising the ℓ-torsion of the Jacobian of the universal genus g curve over $\mathcal{M}_{g,n}[1/\ell]$. By the usual arguments $_\ell\mathcal{M}_{g,n} = {}_\ell M_{g,n}$ is a scheme. Finally let

$$_\ell\overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}[1/\ell] = \overline{\mathcal{M}}_{g,n} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[1/\ell]$$

be the normalization of $\overline{\mathcal{M}}_{g,n}[1/\ell]$ in the function field of $_\ell\mathcal{M}_{g,n}$. We remark that $_\ell\overline{\mathcal{M}}_{g,n} = {}_\ell\overline{M}_{g,n}$ is a projective scheme over $\mathrm{Spec}\,\mathbb{Z}[1/\ell]$, compare [De].

Now fix a connected component $\overline{Y}^i$ of $\overline{Y}$ and let g be the genus of the fiber of f over the generic point of $\overline{Y}^i$. Further let $n \geq 3$ be the number of sections as in (g). Then $\big(\overline{X}_K|_{\overline{Y}^i_K}, \sigma_1|_{\overline{Y}^i_K}, \dots, \sigma_n|_{\overline{Y}^i_K}\big)$ is a stable, n-pointed curve of genus g over $\overline{Y}^i_K$. This defines a 1-morphism

$$\overline{Y}^i_K \longrightarrow \mathcal{M}_{g,n}\,.$$

Let $\overline{U}^i_K \subset \overline{Y}^i_K \times_{\mathcal{M}_{g,n}} {}_\ell M_{g,n}$ be an irreducible component. Then $\overline{U}^i_K$ is finite étale over $\overline{Y}^i_K$, since ℓ is prime to $\mathrm{char}(k)$. Further $\overline{U}^i_K$ maps surjectively onto $\overline{Y}^i_K$. We let $\overline{U}^i$ be the schematic closure of

$$\mathrm{im}\big(\overline{U}^i_K \longrightarrow \overline{Y}^i \times_{\mathrm{Spec}\,R} {}_\ell\overline{M}_{g,n}\big) \;\subset\; \overline{Y}^i \times_{\mathrm{Spec}\,R} {}_\ell\overline{M}_{g,n}.$$

Then $\overline{U}^i$ is projective over $\overline{Y}^i$ and étale over $\overline{Y}^i_K$.

Now we pull back the universal curve over $_\ell\overline{M}_{g,n}$ to obtain a stable, n-pointed curve $\big(\overline{C}, \tau_1, \dots, \tau_n\big)$ over $\overline{U}^i$ which extends the curve

$$\big(\overline{X}_K \times_{\overline{Y}_K} \overline{U}^i_K, \sigma_1|_{\overline{U}^i_K}, \dots, \sigma_n|_{\overline{U}^i_K}\big).$$

We define $\overline{Y}' := \coprod_i \overline{U}^i$. If we denote by Y' the formal completion of $\overline{Y}'$ along the special fiber $\overline{Y}'_0$, we see that the map $Y' \longrightarrow Y$ is a rig-étale cover. Replacing $\overline{Y}$ by $\overline{Y}'$ and $\overline{X}$ by $\overline{X} \times_{\overline{Y}} \overline{Y}'$ we reduce to the case where we have (a)–(g) and in addition

(h) There exists a stable, n-pointed curve $\left(\overline{C}, \tau_1, \ldots, \tau_n\right)$ over $\overline{Y}$ and an isomorphism $\beta_K : \overline{C}_K \xrightarrow{\sim} \overline{X}_K$ mapping the section $\tau_i|_{\overline{Y}_K}$ to the section $\sigma_i|_{\overline{Y}_K}$.

2.11. As in [dJ] a crucial step is to extend the morphism $\beta_K : \overline{C}_K \xrightarrow{\sim} \overline{X}_K$ of schemes over $\overline{Y}_K$ to a birational morphism $\beta : \overline{C} \longrightarrow \overline{X}$ of schemes over $\overline{Y}$. For this we use the procedure described in [dJ, 4.18.–4.22]. Define $\overline{T}$ as the closure of the graph Γ_β of β_K inside $\overline{C} \times_{\overline{Y}} \overline{X}$. If we perform an admissible blowing-up of Y and replace $\overline{X}, \overline{C}, \overline{T}$ by their strict transforms, we may assume that $\overline{T}$ is flat over $\overline{Y}$ in addition to (a)–(h). Observe that the projections $pr_1 : \overline{T} \longrightarrow \overline{C}$ and $pr_2 : \overline{T} \longrightarrow \overline{X}$ are blowing-ups since they are isomorphisms over $\overline{Y}_K$. It is shown in [dJ] that $pr_1 : \overline{T} \longrightarrow \overline{C}$ is an isomorphism. We recall the argument.

Take a point $y \in \overline{Y}$ and denote by $\overline{X}_y, \overline{C}_y, \overline{T}_y$ the fibers over y. These fibers have pure dimension one. Let

$$\begin{aligned} \overline{X}_y &= \overline{X}_1 \cup \ldots \cup \overline{X}_r\,, \\ \overline{C}_y &= \overline{C}_1 \cup \ldots \cup \overline{C}_s\,, \\ \overline{T}_y &= \overline{T}_1 \cup \ldots \cup \overline{T}_t \end{aligned}$$

be the decomposition into irreducible components. Then we have

Lemma 2.12. ([dJ, Lemma 4.20])

a) *For each i, $1 \le i \le r$ there is exactly one $j = j_i$ such that $pr_2(\overline{T}_j) = \overline{X}_i$. There is an open subscheme $\overline{V} = \overline{V}_i \subset \overline{X}$ such that $\overline{V} \cap \overline{X}_i$ is nonempty, and $pr_2^{-1}(\overline{V}) \longrightarrow \overline{V}$ is an isomorphism. The morphism $pr_1 : \overline{T}_j \longrightarrow \overline{C}_y$ is nonconstant.*

b) *For each i, $1 \le i \le s$ there is exactly one $j = j_i$ such that $pr_1(\overline{T}_j) = \overline{C}_i$. There is an open subscheme $\overline{V} = \overline{V}_i \subset \overline{C}$ such that $\overline{V} \cap \overline{C}_i$ is nonempty, and $pr_1^{-1}(\overline{V}) \longrightarrow \overline{V}$ is an isomorphism.*

Proof. This is due to the fact that the smooth locus of f is dense in every $\overline{X}_i$, hence so is the locus where $\overline{X}$ is normal. The assertion that $pr_1 : \overline{T}_j \longrightarrow \overline{C}_y$ is nonconstant relies on the fact that $\overline{C}_y$ is a semi-stable curve. See [dJ] for the details. □

The Lemma implies that $pr_1 : \overline{T} \longrightarrow \overline{C}$ is a finite morphism. Further $\overline{C}$ is normal being a generically smooth, semi-stable curve over a normal base. Thus pr_1 is an isomorphism. So β_K extends to a morphism $\beta : \overline{C} \longrightarrow \overline{X}$. This is an isomorphism over $\overline{Y}_K$, hence an admissible blowing-up.

2.13. Replacing $\overline{X}$ by $\overline{C}$ we may assume that $\overline{X}$ is a semi-stable curve over $\overline{Y}$. However we may have lost property (g) since the morphism $\overline{C} \longrightarrow \overline{X}$ blows up outside $\overline{X}_K$.

So we apply Lemma 1.7 and Lemma 1.8 and increase the number n of our sections. Thereby we regain (g). Further for every geometric point $\overline{y}$ every singular point $\overline{x}$ of the fiber $\overline{X}_{\overline{y}}$ is equal to $\sigma_i(\overline{y})$ for some $i \in \{1, \ldots, n\}$. Going through the arguments of 2.10 and 2.11 once again, we see that after replacing Y by a rig-étale cover, there is a stable, n-pointed curve $\big(\overline{C}, \tau_1, \dots, \tau_n\big)$ over $\overline{Y}$ and a morphism $\beta : \overline{C} \longrightarrow \overline{X}$ mapping the section τ_i to the section σ_i.

Fix a geometric point $\overline{y}$ of Y and a singular point $\overline{x} = \sigma_i(\overline{y}) \in \overline{X}_{\overline{y}}$. Since the genus of $\overline{C}_{\overline{y}}$ is equal to the genus of $\overline{X}_{\overline{y}}$, we see that $\beta^{-1}(\overline{x})$ is either a point, or a string of smooth projective lines. But $\tau_i(\overline{y}) \in \beta^{-1}(\overline{x})$ is a smooth point of $\overline{C}_{\overline{y}}$. Therefore $\beta^{-1}(\overline{x})$ is not a point. Hence all components of $\overline{C}_{\overline{y}}$ are smooth: the components contracted under β are smooth projective lines, the components mapping onto components of $\overline{X}_{\overline{y}}$ are smooth by the preceding argument.

So we again replace $\overline{X}$ by $\overline{C}$. Finally we apply Lemma 1.7 and Lemma 1.8 once more to $\overline{X}$. Now all singular points of all fibers $\overline{X}_y$ are rational over $\kappa(y)$ since they are hit by our sections. Further all components are defined over $\kappa(y)$ since each component has a rational point. Hence we have reduced to the case where we have the properties (a)–(g) and in addition

(i) $\overline{X}$ is a *split semi-stable curve* over $\overline{Y}$, i.e. in every fiber $\overline{X}_y$ all singular points are rational and all irreducible components are defined over $\kappa(y)$ and smooth.

2.14. We apply our induction hypothesis to Y. Thus we find a finite separable field extension K' of K, a strictly semi-stable formal scheme Y' over the ring of integers R' of K' and a rig-étale cover $Y' \longrightarrow Y$. The split semi-stable curve $\overline{X}$ pulls back to a split semi-stable curve $\overline{X}'$ over Y'. We replace Y by Y' and $\overline{X}$ by $\overline{X}'$ and therefore assume (a)–(g), (i) and

(j) Y is strictly semi-stable over R.

2.15. Let us describe the local situation at a singular point x of X. If x were in the smooth locus of f, then X would be regular at x since Y is regular. Hence x lies in the singular locus of f, i.e. x is a double point of the fiber over $y = f(x) \in Y$. Let $\widehat{\mathcal{O}}_y$ be the complete local ring of Y at y and $\widehat{\mathcal{O}}_x$ be the complete local ring of X at x. Due to Proposition 1.2 the ring $\widehat{\mathcal{O}}_y$ is formally smooth over

$$R[[t_1, \dots, t_r]]/(t_1 \cdot \ldots \cdot t_r - \pi)\,,$$

where the irreducible components of Y_0 are locally at y given as $\mathrm{V}(t_i)$. Since the singular locus of X lies in the special fiber, i.e. over Y_0,we see that

$$\widehat{\mathcal{O}}_x \cong \widehat{\mathcal{O}}_y[[u, v]]/(uv - t_1^{n_1} \cdot \ldots \cdot t_r^{n_r})$$

with $n_i \geq 0$. This follows as in [dJ, 3.3] (observe that X is split). If $\sum n_i = 1$ then X is strictly semi-stable over R in a neighborhood of x, hence regular at x. Thus we must have $\sum n_i \geq 2$.

2.16. We apply the blowing-up procedures of de Jong [dJ, Lemma 3.2] and [dJ, Lemma 3.6]. This means the following.

As long as there are irreducible components T of the singular locus Sing X which have codimension 2 in X we blow up X in the ideal sheaf of such a component T. In terms of 2.15 the component T is given locally at x as say $\mathrm{V}(u, v, t_1)$. Therefore T maps isomorphically onto an irreducible component D of Y_0, which locally at y is given as $\mathrm{V}(t_1)$. Since X is singular at the generic point of T, the integer n_1 must be at least 2. Blowing up T lowers n_1 by two. So finally all exponents n_i are 0 or 1. This means that Sing X has codimension at least 3. Compare [dJ, Lemma 3.2].

Now we follow the lines of [dJ, Lemma 3.6]. Let D be an irreducible component of Y_0 and let E be an irreducible component of $f^{-1}(D)$. We denote by $\varphi : X' \longrightarrow X$ the blowing-up of X in the ideal sheaf of E. If $x \in E$ is a regular point of X, then φ is an isomorphism at x as E can be defined by one equation. So let x be a singular point of X. By the above the complete local ring of X at x looks like

$$\widehat{\mathcal{O}}_x \cong \widehat{\mathcal{O}}_y[\![u, v]\!]/(uv - t_1 \cdot \ldots \cdot t_\rho)$$

for some $2 \le \rho \le r$. Say D corresponds to $\mathrm{V}(t_i)$. If $i > \rho$, then E is given by the principal ideal t_i, hence φ is an isomorphism at x. If $i \le \rho$, then E is given as $\mathrm{V}(u, t_i)$ or as $\mathrm{V}(v, t_i)$. This follows from the fact that all irreducible components of all fibers of f are smooth; cf. [dJ, Lemma 3.6]. Therefore we study the blowing-up of the algebra

$$R\langle t_1, \ldots , t_r, u, v\rangle/(t_1 \cdot \ldots \cdot t_r - \pi \ , \ uv - t_1 \cdot \ldots \cdot t_\rho)$$

in the ideal (u, t_ρ). Note that a neighborhood of x in X is smooth over the formal spectrum of this algebra. We obtain two charts.

Chart: "$|t_\rho| \le |u|$" . We get the algebra

$$R\langle t_1, \ldots , t_r, u, v, t'_\rho\rangle/(t_1 \cdot \ldots \cdot t_r - \pi \ , \ t_\rho - ut'_\rho \ , \ v - t_1 \cdot \ldots \cdot t_{\rho-1}\, t'_\rho)\, .$$

It is isomorphic to $R\langle t_1, \ldots , t'_\rho, \ldots , t_r, u\rangle/(u\, t_1 \cdot \ldots \cdot t'_\rho \cdot \ldots \cdot t_r - \pi)$. Since X is smooth over the formal spectrum of this algebra, we see by Proposition 1.2 that X is strictly semi-stable in this chart.

Chart: "$|u| \le |t_\rho|$" . We get the algebra

$$R\langle t_1, \ldots , t_r, u, v, u'\rangle/(t_1 \cdot \ldots \cdot t_r - \pi \ , \ u - t_\rho u' \ , \ u'v - t_1 \cdot \ldots \cdot t_{\rho-1})\, .$$

Thus the blowing-up φ lowers the number ρ by 1. If $\rho = 2$, then our algebra from the second chart is isomorphic to

$$R\langle t_2, \ldots , t_r, u', v\rangle/(u'vt_2 \cdot \ldots \cdot t_r - \pi)\, .$$

Since X is smooth over the formal spectrum of this algebra, we see that X is strictly semi-stable also in this chart. Thus repeatedly blowing up components of $f^{-1}(D)$ for all irreducible components D of Y_0 we arrive at the situation where X is strictly semi-stable over R. This finishes the proof of Theorem 1.4. □

3. Application

As an application we want to demonstrate that Theorem 1.4 may help proving the representability of functors. Namely using faithfully flat descent, it suffices to test universal properties on rigid-analytic spaces which possess semi-stable models. See [HL, Lemma 3.12], where this technique was used to show the representability of the rigid-analytic Picard functor. We begin with a lemma concerning faithfully flat descent for rigid-analytic spaces.

Lemma 3.1. *Let X_K, X'_K and Y_K be rigid-analytic spaces over* $\operatorname{Sp} K$ *and let $f_K : X'_K \longrightarrow X_K$ be a quasi-compact morphism, which is faithfully flat, i.e. flat and surjective. Then the canonical sequence of sets is exact*

$$\operatorname{Hom}_K(X_K, Y_K) \longrightarrow \operatorname{Hom}_K(X'_K, Y_K) \rightrightarrows \operatorname{Hom}_K\left(X'_K \times_{X_K} X'_K, Y_K\right).$$

Remark 3.1.1. We remark that this lemma does not trivially follow from the faithfully flat descent on schemes, since the fiber product $X'_K \times_{X_K} X'_K$ is built using the *complete* tensor product instead of the ordinary tensor product.

Proof. Since f_K is flat and quasi-compact, it is open. Indeed if $U_K \subset Y_K$ is open and if $\{X^i_K\}$ is an admissible covering by affinoid subvarieties, then $f_K(U_K) \cap X^i_K$ is admissible open in X^i_K due to [FRG, II, Corollary 5.11]. Hence the question is local on Y_K and X_K. Therefore we may assume that Y_K and X_K are affinoid. Thus we can regard Y_K as a closed subvariety of some N-dimensional unit ball $\mathbb{D}^N_K$. By [FRG, I, Theorem 4.1] and [FRG, II, Theorem 5.2] there exist formal models X and X' of X_K and X'_K such that f_K extends to a flat and quasi-compact morphism $f : X' \longrightarrow X$ and all morphisms in question mapping to $Y_K \subset \mathbb{D}^N_K$ extend to formal morphisms into $\mathbb{D}^N_R$. Since f_K is surjective, also f is surjective and hence faithfully flat. Now we may replace X by an affine open subvariety $\operatorname{Spf} A$. Since f is quasi-compact, X' is a finite union of affine open subvarieties. Replacing X' by the disjoint union of these subvarieties, we may assume that $X' = \operatorname{Spf} A'$ is affine and faithfully flat over X. Thus we are in the formal descent situation

$$A \quad\longrightarrow\quad A' \quad\rightrightarrows\quad A'\widehat{\otimes}_A A' \;=:\; A'' \;.$$

This sequence is exact. Namely, after tensoring with $R_n := R/(\pi^{n+1})$ over R, the sequence is exact for all $n \in \mathbb{N}$, since $f \otimes_R R_n$ is faithfully flat; cf. [FGA, no. 190, Corollaire B.1]. Tensoring with K we obtain the exact sequence

$$A_K \quad\longrightarrow\quad A'_K \quad\rightrightarrows\quad A'_K\widehat{\otimes}_{A_K} A'_K \;=:\; A''_K \;.$$

Now all morphisms in question into $\mathbb{D}^N_K$ are given by N-tuples of functions $(a_1, \ldots, a_N) \in (A_K)^N$, etc. As Y_K is a closed subvariety of $\mathbb{D}^N_K$ these functions have to satisfy certain equations. Then all N-tuples produced by descent or pullback automatically satisfy these equations also. Hence all morphisms factor through Y_K. Therefore the exactness of the later sequence implies our lemma. □

Now we can prove the following theorem.

Theorem 3.2. *Let $\mathfrak{C}$ be the category of smooth rigid-analytic spaces over* Sp K *together with all rigid-analytic morphisms and let $\mathcal{F}$ be a contravariant functor from $\mathfrak{C}$ to* (Sets). *Further let F_K be an object of $\mathfrak{C}$ and $\alpha \in \mathcal{F}(F_K)$. Assume that*

(a) *for every morphism $f : T'_K \longrightarrow T_K$ in $\mathfrak{C}$, which is either an admissible covering, or étale, quasi-compact and surjective, the morphism of sets $\mathcal{F}(f)$ is injective,*

(b) *for every finite, separable field extension K' of K and every quasi-compact, strictly semi-stable, formal scheme X over the ring of integers of K', the natural map*

$$\mathrm{Hom}_{\mathfrak{C}}(X_{\mathrm{rig}}, F_K) \longrightarrow \mathcal{F}(X_{\mathrm{rig}}) , \qquad \varphi \longmapsto \mathcal{F}(\varphi)(\alpha)$$

is bijective.

Then the pair (F_K, α) represents the functor $\mathcal{F}$ on the category $\mathfrak{C}$.

Proof. Let X_K be an object of $\mathfrak{C}$ and $\xi \in \mathcal{F}(X_K)$. We have to show that there is a unique morphism $\varphi : X_K \longrightarrow F_K$ such that $\xi = \mathcal{F}(\varphi)(\alpha)$. We chose an admissible covering $\{X^i_K\}$ of X_K by affinoid subspaces. First we fix an i and work on X^i_K. By Theorem 1.4 there is a finite, separable field extension K'/K, a quasi-compact, strictly semi-stable, formal scheme X' over the ring of integers of K' and an étale, surjective morphism $f : X'_{\mathrm{rig}} \longrightarrow X^i_K$. We set $\xi' := \mathcal{F}(f)(\xi|_{X^i_K})$. By assumption (b) there exists a uniquely defined morphism $\varphi' : X'_{\mathrm{rig}} \longrightarrow F_K$ such that $\xi' = \mathcal{F}(\varphi')(\alpha)$. We have to descend φ' to X^i_K. By the two projections

$$p_i : \; X''_K \;:=\; X'_{\mathrm{rig}} \times_{X^i_K} X'_{\mathrm{rig}} \;\longrightarrow\; X'_{\mathrm{rig}} , \quad i = 1, 2$$

we obtain two morphism $\varphi''_i := \varphi' \circ p_i : X''_K \longrightarrow F_K$ satisfying

$$\mathcal{F}(\varphi''_1)(\alpha) \;=\; \mathcal{F}(f \circ p_1)(\xi|_{X^i_K}) \;=\; \mathcal{F}(f \circ p_2)(\xi|_{X^i_K}) \;=\; \mathcal{F}(\varphi''_2)(\alpha) .$$

Now X''_K being étale over X^i_K is smooth over Sp K. Hence there is another finite, separable field extension $\widetilde{K}/K$, a strictly semi-stable formal scheme X''' over the ring of integers of $\widetilde{K}$ and an étale, surjective morphism $f'' : X'''_{\mathrm{rig}} \longrightarrow X''_K$. This morphism induces two morphism $\varphi'''_i := \varphi''_i \circ f'' : X'''_{\mathrm{rig}} \longrightarrow F_K$ satisfying

$$\mathcal{F}(\varphi'''_1)(\alpha) \;=\; \mathcal{F}(\varphi'''_2)(\alpha) .$$

Therefore assumption (b) implies that $\varphi'''_1 = \varphi'''_2$. By Lemma 3.1 it follows that $\varphi''_1 = \varphi''_2$. Again by Lemma 3.1 this shows that the morphism $\varphi' : X'_{\mathrm{rig}} \longrightarrow F_K$ descends to a uniquely defined morphism

$$\varphi_i : \; X^i_K \;\longrightarrow\; F_K ,$$

i.e. $\varphi' = \varphi_i \circ f$. By assumption (a) we see that $\xi|_{X^i_K} = \mathcal{F}(\varphi_i)(\alpha)$. Since φ_i is uniquely determined by ξ, the φ_i glue to give a morphism

$$\varphi : \; X_K \;\longrightarrow\; F_K .$$

It satisfies $\xi|_{X^i_K} = \mathcal{F}(\varphi)(\alpha)|_{X^i_K}$ for all i. Hence by assumption (a) we obtain $\xi = \mathcal{F}(\varphi)(\alpha)$ and so the theorem is proved. $\square$

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