Product Formulas for Periods of CM Abelian Varieties and the Function Field Analog

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Abstract

We survey Colmez's theory and conjecture about the Faltings height and a product formula for the periods of abelian varieties with complex multiplication, along with the function field analog developed by the authors. In this analog, abelian varieties are replaced by Drinfeld modules and A-motives. We also explain the necessary background on abelian varieties, Drinfeld modules and A-motives, including their cohomology theories and comparison isomorphisms and their theory of complex multiplication.
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1 Introduction

One purpose of this survey is to give a brief introduction to abelian varieties with complex multiplication over number fields, some of their cohomology theories with comparison isomorphisms, and to explain Colmez's conjectures [Col93] on a product formula for the periods and on the Faltings height of these abelian varieties. The second purpose is to explain the function field analog of this theory. There abelian varieties are replaced by Drinfeld modules [Dri76, Gos96] and their higher dimensional generalizations, so-called A-motives. So we give a brief introduction to Drinfeld modules and A-motives with complex multiplication, some of their cohomology theories with comparison isomorphisms, and explain the conjecture [HS20] of the authors on periods of these A-motives. We point out that recently other surveys on Colmez's conjectures were written by Gross [Gro18], by Yuan [Yua19], and by Gao, van Känel and Mocz [GvKM19] based on a lecture of Shou-Wu Zhang. However, these do not discuss the function field analog that we are discussing in Part II. In [Gro18] it is explained how Colmez's conjectures generalize the Chowla-Selberg formula. And in [Yua19] the consequences of the recently proved averaged Colmez Conjecture for the André-Oort Conjecture are explained. In [GvKM19] in addition to these aspects, the proof of Yuan and Shou-Wu Zhang [YZ18] of the averaged Colmez conjecture, and the work of Yun and Wei Zhang [YZ17, YZ19] on the Gross-Zagier formula for intersection numbers in the Chow group of moduli spaces of PGL_2-shtukas is discussed.

1.1. We begin with a review of product formulas for global fields. For a rational number $\alpha \in \mathbb{Q}^\times$, all of its absolute values $|\alpha|_v$ are linked by the *product formula* $\prod_v |\alpha|_v = 1$ where only finitely many factors are different from 1. Here v runs through the set $\mathcal{P}$ of *places* of $\mathbb{Q}$ consisting of all prime numbers p together with ∞, and the *p-adic absolute values* $|\,.\,|_p$ are normalized such that $|p|_p = p^{-1}$. This product formula extends to *number fields*, i.e. finite extensions of $\mathbb{Q}$, as follows. Let $\mathbb{Q}^{\mathrm{alg}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and if p is a prime number let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to $|\,.\,|_p$ and let $\mathbb{Q}_p^{\mathrm{alg}}$ be an algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value $|\,.\,|_p$ extends canonically to $\mathbb{Q}_p^{\mathrm{alg}}$. We denote by $|\,.\,|_\infty$ the usual absolute value on $\mathbb{C}$. In addition to the embedding $\mathbb{Q}^{\mathrm{alg}} \subset \mathbb{C}$ we fix once and for all an embedding of $\mathbb{Q}^{\mathrm{alg}}$ in $\mathbb{Q}_p^{\mathrm{alg}}$ for every p and consider the induced absolute value $|\,.\,|_p$ on $\mathbb{Q}^{\mathrm{alg}}$. For a finite field extension K of $\mathbb{Q}$ we set $H_K := \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{Q}^{\mathrm{alg}})$. Then the product formula [Lan94, Chapter V, § 1, bottom of page 99] for $0 \neq \alpha \in K$ can be written as

$$\prod_{p \in \mathcal{P}} \prod_{\eta \in H_K} |\eta(\alpha)|_p = 1. \tag{1.1}$$

1.2. The product formula also holds for *function fields*. More precisely, let Q be a finitely generated field of transcendence degree one over the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let $\mathbb{F}_q := \{a \in Q \colon a \text{ is algebraic over } \mathbb{F}_p\} \subset Q$ be the *field of constants*, see [VS06, Definition 2.1.3], which is a finite field with q elements. Then Q is the field of rational functions on a smooth, projective curve C over $\mathbb{F}_q$ by [Liu02, Chapter 7.3, Proposition 3.13] which is geometrically irreducible by [Gro65, IV$_2$, 4.3.1 and Proposition 4.5.9c)]. Every closed point v of C is called a *place*. We denote its residue field by $\mathbb{F}_v$ and set $q_v := \#\mathbb{F}_v = q^{[\mathbb{F}_v:\mathbb{F}_q]}$. The local ring $\mathcal{O}_{C,v}$ is a discrete valuation ring by [Sil86, Proposition 1.1]. We denote the corresponding valuation also by v and the corresponding absolute value on Q by $|\,.\,|_v$. Both are normalized such that $v(z_v) = 1$ and $|z_v|_v = q_v^{-1}$ for a uniformizing parameter $z_v \in Q$ at v. Then every $a \in Q \smallsetminus \{0\}$ satisfies $\prod_v |a|_v = 1$ where again only finitely many factors are different from 1, see [Cas67, Chapter II, § 12, Theorem]. This can be reinterpreted in terms of *divisors* on C. Namely, since $|a|_v = q_v^{-v(a)}$ we have $-\log\prod_v |a|_v = \sum_v v(a)\cdot[\mathbb{F}_v : \mathbb{F}_q]\cdot\log q = 0$, because $\sum_v v(a)\cdot[\mathbb{F}_v : \mathbb{F}_q]$ is the degree of the principal divisor of a which is zero, see [VS06, Corollary 3.2.9].

Let Q^{alg} be a fixed algebraic closure of Q. For every place v of Q let Q_v be the completion of Q with respect to $|\,.\,|_v$ and let Q_v^{alg} be an algebraic closure of Q_v. The v-adic absolute value $|\,.\,|_v$ extends canonically to Q_v^{alg}. We fix once and for all an embedding of Q^{alg} in Q_v^{alg} for every v and consider the induced absolute value $|\,.\,|_v$ on Q^{alg}. For a finite field extension K of Q we set $H_K := \mathrm{Hom}_Q(K, Q^{\mathrm{alg}})$. Then by transformations of equations as in [Lan94, Chapter V, § 1, bottom of page 99] the product formula [Cas67, Chapter II, § 12, Theorem] for $0 \neq a \in K$ can be written as

$$\prod_{\text{all } v}\ \prod_{\eta\in H_K} |\eta(a)|_v = 1. \tag{1.2}$$

1.3. In [Col93] P. Colmez considers product formulas for periods of abelian varieties. Let X be an abelian variety defined over a number field K with complex multiplication by the ring of integers in a CM-field E and of CM-type Φ, see Section 6 for explanations. Assume that K contains $\psi(E)$ for every $\psi \in H_E$. For a $\psi \in H_E$ let $\omega_\psi \in \mathrm{H}^1_{\mathrm{dR}}(X, K)$ be a non-zero cohomology class such that $b^*\omega_\psi = \psi(b)\cdot\omega_\psi$ for all $b \in E$, see Section 4.3. For every embedding $\eta\colon K \hookrightarrow \mathbb{Q}^{\mathrm{alg}}$, let $X^\eta := X \times_{\operatorname{Spec} K,\operatorname{Spec}\eta} \operatorname{Spec}\eta(K)$ and $\omega_\psi^\eta \in \mathrm{H}^1_{\mathrm{dR}}(X^\eta, \eta(K))$ be deduced from X and ω_ψ by base extension. Let $(u_\eta)_\eta \in \prod_{\eta\in H_K} \mathrm{H}_1(X^\eta(\mathbb{C}), \mathbb{Z})$ be a family of cycles compatible with complex conjugation, see Section 4.1. Let v be a place of $\mathbb{Q}$. If $v = \infty$ the de Rham isomorphism between Betti and de Rham cohomology (Theorem 4.4) yields a complex number $\int_{u_\eta}\omega_\psi^\eta$ and its absolute value $\big|\int_{u_\eta}\omega_\psi^\eta\big|_\infty \in \mathbb{R}$. If v corresponds to a prime number $p \in \mathbb{Z}$, Colmez [Col93] associates a period $\int_{u_\eta}\omega_\psi^\eta$ in Fontaine's p-adic period field $\mathbb{B}_{p,\mathrm{dR}}$, see Notation 5.4, and an absolute value $\big|\int_{u_\eta}\omega_\psi^\eta\big|_v \in \mathbb{R}$. He considers the product $\prod_v\prod_{\eta\in H_K}\big|\int_{u_\eta}\omega_\psi^\eta\big|_v$ and (after some modifications which we explain in Section 8) conjectures that this product evaluates to 1; see Conjecture 8.6 for the precise formulation. This conjecture implies a conjectural formula for the Faltings height of a CM abelian variety in terms of the logarithmic derivatives at $s = 0$ of certain Artin L-functions. Colmez proves the conjectures when E is an abelian extension of $\mathbb{Q}$, see Theorem 8.10. On the way, he computes $\prod_{\eta\in H_K}\big|\int_{u_\eta}\omega_\psi^\eta\big|_v$ at a finite place v in terms of the local factor at v of the Artin L-series associated with an Artin character $a^0_{E,\psi,\Phi}\colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q}) \to \mathbb{C}$ that only depends on E, ψ and Φ but not on X and v; see Theorem 8.3. There has been further progress on Colmez's conjecture on which we report in Section 8.

We point out that Colmez's formulation generalizes various previous results. Namely, when $[E : \mathbb{Q}] = 2$ his Theorem 8.10 is equivalent to the formula proved by Lerch [Ler97] and rediscovered by Chowla-Selberg [SC67]

$$\frac{\zeta_E'(0)}{\zeta_E(0)} \;=\; \frac{1}{12\,\#\,\mathrm{Pic}(\mathcal{O}_E)}\sum_{[I]\in\mathrm{Pic}(\mathcal{O}_E)}\log\big(\Delta(I)\Delta(I^{-1})\big)\,, \tag{1.3}$$

where $\Delta(I)$ is the modular discriminant of the lattice $I \subset E \subset \mathbb{C}$. A new geometric proof of (1.3) was given by Gross [Gro78], who together with Deligne conjectured a generalization to a formula for the archimedean periods of certain CM motives up to multiplication by algebraic numbers. Anderson [And82] reformulated the Gross-Deligne conjecture in terms of the logarithmic derivative of an L-function at $s = 0$ and proved it when the CM field E is abelian over $\mathbb{Q}$. Colmez added the consideration of the non-archimedean periods and thus removed the ambiguity of the algebraic factors in Anderson's theorem.

1.4. There is a beautiful analog to the theory of elliptic curves and abelian varieties in the "Arithmetic of function fields". Namely, Drinfeld [Dri76] invented the analog of elliptic curves under the name "elliptic modules". These are today called *Drinfeld modules*, see Section 9. Since then, the arithmetic of function fields has evolved into an equally rich parallel world to the arithmetic of number fields. As higher dimensional generalizations of Drinfeld modules and analogs of abelian varieties, Anderson [And86] has defined *abelian t-modules* and the dual notion of *t-motives*, which are a kind of "global Dieudonné-modules" for abelian t-modules, see Remark 9.3. They can be slightly generalized to *A-motives* as follows. In the notation of § 1.2 let ∞ be a fixed closed point on C and let $A = \Gamma(C \smallsetminus \{\infty\}, \mathcal{O}_C) = \{a \in A\colon v(a) \geq 0 \text{ for all } v \neq \infty\}$. Let $K \subset Q^{\mathrm{alg}}$ be a finite field extension of Q. We write $A_K := A \otimes_{\mathbb{F}_q} K$ and consider the endomorphism $\sigma^* := \mathrm{id}_A \otimes \mathrm{Frob}_{q,K}$ of A_K, where $\mathrm{Frob}_{q,K}(b) = b^q$ for $b \in K$. For an A_K-module M we set $\sigma^*M := M \otimes_{A_K,\sigma^*} A_K$ and for a homomorphism $f\colon M \to N$ of

A_K-modules we set $\sigma^* f := f \otimes \mathrm{id}_{A_K} : \sigma^* M \to \sigma^* N$. Let $\gamma\colon A \to K$ be the inclusion $A \subset Q \subset K$, and set $\mathcal{J} := (a \otimes 1 - 1 \otimes \gamma(a)\colon a \in A) \subset A_K$. Then γ can be recovered as the homomorphism $A \to A_K/\mathcal{J} = K$.

Definition 1.5. An *(effective) A-motive of rank r and dimension d over K* is a pair $\underline{M} = (M, \tau_M)$ consisting of a locally free A_K-module M of rank r and an A_K-homomorphism $\tau_M : \sigma^* M \to M$ such that

(a) $\dim_K(\operatorname{coker} \tau_M) = d$.

(b) $(a - \gamma(a))^d \cdot \operatorname{coker} \tau_M = 0$ for all $a \in A$.

We write $\operatorname{rk} \underline{M} := r$ and $\dim \underline{M} := d$.

A-motives possess cohomology realizations in analogy with abelian varieties, see Section 13.

1.6. Let us now explain the analog of Colmez's theory from § 1.3 which was developed by the authors in [HS20]. Let $\underline{M}$ be a uniformizable A-motive defined over a finite extension $K \subset Q^{\mathrm{alg}}$ of Q with complex multiplication by the ring of integers in a CM-algebra E and of CM-type Φ, see Sections 13.1 and 15 for explanations. Assume that K contains $\psi(E)$ for every $\psi \in H_E := \mathrm{Hom}_Q(E, Q^{\mathrm{alg}})$. For a $\psi \in H_E$ let $\omega_\psi \in \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K[\![z-\zeta]\!])$ be a non-zero cohomology class such that $b^*\omega_\psi = \psi(b) \cdot \omega_\psi$ for all $b \in E$, see Section 13.3. For every embedding $\eta\colon K \hookrightarrow Q^{\mathrm{alg}}$, let $\underline{M}^\eta := \underline{M} \otimes_{K,\eta} \eta(K)$ and $\omega_\psi^\eta \in \mathrm{H}^1_{\mathrm{dR}}(\underline{M}^\eta, \eta(K)[\![z-\zeta]\!])$ be deduced from $\underline{M}$ and ω_ψ by base extension. Let $(u_\eta)_\eta \in \prod_{\eta \in H_K} \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta, A)$ be a family of cycles, see Section 13.1. Let v be a place of Q. If $v = \infty$ the comparison isomorphism between Betti and de Rham cohomology (Theorem 13.18) yields an element $\int_{u_\eta} \omega_\psi^\eta$ in the completion $\mathbb{C}_\infty$ of Q_∞^{alg} with respect to $|\,.\,|_\infty$ and its absolute value $\big|\int_{u_\eta} \omega_\psi^\eta\big|_\infty \in \mathbb{R}$. If v corresponds to a maximal ideal of A, the period isomorphism between v-adic and de Rham cohomology (Theorem 14.12) gives a period $\int_{u_\eta} \omega_\psi^\eta$ in the analog $\mathbb{C}_v(\!(z_v - \zeta_v)\!)$ of Fontaine's p-adic period field $\mathbb{B}_{p,\mathrm{dR}}$ and an absolute value $\big|\int_{u_\eta} \omega_\psi^\eta\big|_v \in \mathbb{R}$, see Definition 14.14. We consider the product $\prod_v \prod_{\eta \in H_K} \big|\int_{u_\eta} \omega_\psi^\eta\big|_v$ and (after some modifications analogous to Colmez's which we explain in Section 17) we conjecture that this product evaluates to 1; see Conjecture 17.6 for the precise formulation. In [HS20] we have computed $\prod_{\eta \in H_K} \big|\int_{u_\eta} \omega_\psi^\eta\big|_v$ at all finite places $v \neq \infty$ in terms of the local factor at v of the Artin L-series associated with an Artin character $a^0_{E,\psi,\Phi}\colon \mathrm{Gal}(Q^{\mathrm{alg}}/Q) \to \mathbb{C}$ that only depends on E, ψ and Φ but not on $\underline{M}$ and v; see Theorem 17.3.

If $\underline{M}$ is the A-motive associated with a Drinfeld module $\underline{G}$, then Conjecture 17.6 is equivalent to a formula for the Taguchi height (Definition 16.3) of $\underline{G}$ in terms of the logarithmic derivatives at $s = 0$ of an Artin L-function. This formula was established by Fu-Tsun Wei [Wei20] by first proving the function field analogs of Kronecker's limit theorem and Lerch's formula 1.3, see Theorem 17.10 below. Previously, formulas of Chowla-Selberg type expressing the periods at ∞ of CM Drinfeld modules in terms of Γ-values were obtained by Thakur [Tha91] for certain CM-fields. Also when proving his results in [And82] Anderson had considered the analogous case of A-motives, but without publishing his results.

This survey contains no new results, except for Theorems 13.20 and 17.8 which give a formula for the Taguchi height of a Drinfeld module with complex multiplication. Our presentation summarizes material from various sources. But all shortcomings of the exposition are solely due to the authors. We describe the content of the individual sections of this survey. In Part I we first define elliptic curves and abelian varieties and discuss their torsion points in Section 2. Section 3 is concerned with simple and semi-simple abelian varieties and their endomorphism rings. In Section 4 we review the singular (co-)homology, Tate modules and the ℓ-adic (co-)homology, and the de Rham (co-)homology of abelian varieties and period isomorphisms between these (co-)homologies. The period isomorphism between ℓ-adic and de Rham (co-)homology is explained in Section 5. It is based on the concept of p-divisible groups, which we also review in this section. The definition of complex multiplication of abelian varieties, of CM-fields, CM-algebras and CM-types is explained in Section 6. A short review of the Faltings height of an abelian variety fills Section 7. Finally, in Section 8 we discuss Colmez's conjecture alluded to in § 1.3 above.

In Part II we discuss the analog of Colmez's theory in the "Arithmetic of function fields". We define Drinfeld modules and A-motives in Section 9, and isogenies and semi-simplicity in Section 10, where we also describe the endomorphism rings of semi-simple A-motives. The analytic theory of Drinfeld modules via lattices is explained in Section 11. Section 12 is devoted to torsion points and Tate modules of Drinfeld modules. In Section 13 we review the singular (co-)homology, Tate modules and the v-adic (co-)homology, and the de Rham (co-)homology of A-motives and period isomorphisms between these (co-)homologies. The period isomorphism between v-adic and de Rham (co-)homology is explained in Section 14. It is based on the concept of z-divisible local Anderson modules and local shtukas, which we also review in this section. In Section 15 we introduce the concept of complex multiplication of A-motives and of their CM-types. Section 16 contains a brief review of the Taguchi height of a Drinfeld module. Then in Section 17 we present the theory of the authors on the product formula for periods of A-motives analogous to Colmez's conjecture. In the last Section 18 we compute an interesting example for this product formula where Q and C have genus 1.

Contents

Part I
Abelian Varieties and Elliptic Curves

Our exposition of the theory of abelian varieties and elliptic curves follows [Mum70, Mil86, Mil08, Sil86, DS05], which serve as background material for this article.

2 Basic Definitions

Notation 2.1. As usual we denote by $\mathbb{Q}$ and $\mathbb{R}$ the fields of rational and real numbers, respectively, by $\mathbb{Z}$ the ring of integers and by $\mathbb{N}_0$, respectively $\mathbb{N}_{>0}$ the set of non-negative, respectively positive integers. By a *place* of $\mathbb{Q}$ we mean either ∞ or a maximal ideal $v = (p) \subset \mathbb{Z}$ for a prime number $p \in \mathbb{N}_{>0}$. It defines a normalized absolute value $|\,.\,|_v \colon \mathbb{Q} \to \mathbb{R}_{\geq 0}$ given for $v = \infty$ by the usual absolute value $|x|_\infty = x$ if $x \geq 0$ and $|x|_\infty = -x$ if $x \leq 0$, and for $v = (p)$ by the p-adic absolute value $|x|_v := |x|_p = p^{-v_p(x)}$ where $v_p(x) = n$ if $x = p^n \frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $p \nmid ab$. Let $\mathbb{Q}_v$ be the completion of $\mathbb{Q}$ with respect to the valuation v, that is $\mathbb{Q}_\infty = \mathbb{R}$ and $\mathbb{Q}_v = \mathbb{Q}_p$ for $v = (p)$. Let $\mathbb{Q}_v^{\mathrm{alg}}$ be a fixed algebraic closure of $\mathbb{Q}_v$ and let $\mathbb{C}_v$ be the completion of $\mathbb{Q}_v^{\mathrm{alg}}$ with respect to the canonical extension of the absolute value $|\,.\,|_v$ to $\mathbb{Q}_v^{\mathrm{alg}}$. Note that $\mathbb{C}_v$ is algebraically closed. It equals the field of complex numbers $\mathbb{C}$ when $v = \infty$, and is usually denoted $\mathbb{C}_p$ when $v = (p)$. We also fix an algebraic closure $\mathbb{Q}^{\mathrm{alg}}$ of $\mathbb{Q}$ and an embedding $\mathbb{Q}^{\mathrm{alg}} \hookrightarrow \mathbb{Q}_v^{\mathrm{alg}}$ for every place v of $\mathbb{Q}$. We let $\mathcal{O}_{\mathbb{C}_p}$ be the ring of integers of $\mathbb{C}_p$.

Definition 2.2. Let K be an arbitrary field, let K^{alg} be a fixed algebraic closure and let K^{sep} be the separable closure of K in K^{alg}, and $\mathscr{G}_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$. We mean by a (smooth) *group variety* over K an irreducible smooth separated scheme G of finite type over K with a group law $mult : G \times_K G \to G$, an inverse map $inv : G \to G$ and a K-rational point $0 \in G(K)$, the identity element, such that $mult$ and inv are morphisms of varieties satisfying the usual axioms, see [Mum70, Chapter III, § 11]. A morphism of group varieties is a morphism of varieties which is also a homomorphism of groups.

For a group variety G over K, let $\mathrm{Lie}(G) = \mathrm{T}_0 G$ be the tangent space to G at the identity element 0. It is also called the *Lie algebra* of G. For every endomorphism f of G we let $\mathrm{Lie}(f)$ be the induced endomorphism of $\mathrm{Lie}\, G$.

Definition 2.3. An *elliptic curve* over a field K is a smooth projective curve E of genus 1, together with a distinguished point $0 \in E(K)$. Every such can be written as a smooth projective plane curve which is the zero locus of an equation

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3 \quad \text{with } a_i \in K \tag{2.1}$$

and with distinguished point $0 = (0 : 1 : 0)$. It carries a group law making it into a commutative group variety with identity element 0 (see [Sil86, Hus04]).

Let E be an elliptic curve over $\mathbb{C}$. Then $E(\mathbb{C})$ inherits a complex structure as a sub-manifold of $\mathbb{P}^2(\mathbb{C})$. It is a complex manifold (because E is nonsingular) and compact (because it is closed in the compact space $\mathbb{P}^2(\mathbb{C})$). It is connected and carries a commutative group structure. Therefore, E is a compact connected complex Lie group of dimension 1. Let $\mathrm{T}_0 E(\mathbb{C})$ be the tangent space of $E(\mathbb{C})$ at the identity element 0. It is also called the Lie algebra of $E(\mathbb{C})$ and denoted $\mathrm{Lie}\, E$. Then there is a unique homomorphism

$$\exp : \mathrm{T}_0 E(\mathbb{C}) \to E(\mathbb{C})$$

of complex Lie groups such that, for each $v \in \mathrm{T}_0 E(\mathbb{C})$, $z \mapsto \exp(zv)$ is the one parameter subgroup[1] $f_v : \mathbb{C} \to E(\mathbb{C})$ corresponding to v. The differential of exp at 0 is the identity map

$$\mathrm{T}_0 E(\mathbb{C}) \to \mathrm{T}_0 E(\mathbb{C}),$$

[1] For a complex Lie group G, a one parameter subgroup of G is a holomorphic homomorphism $f : \mathbb{C} \to G$. In complex analysis one proves that for every tangent vector v to G at e, there is a unique one-parameter subgroup $f_v : \mathbb{C} \to G$ such that $f_v(0) = e$ and $(df_v)(1) = v$, see [Hoc65, pp. 79 and 195].

and the map exp is surjective, and its kernel is a lattice $\Lambda = \Lambda(E)$ in the complex vector space $\mathrm{T}_0E(\mathbb{C})$. So $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ as a complex Lie group (for more details see [Mum70, Chapter I, § 1]).

Now we explain how one associates an elliptic curve with a lattice. Let Λ be a lattice in $\mathbb{C}$, that is, a discrete $\mathbb{Z}$-module $\Lambda \subset \mathbb{C}$ which is free of rank 2. With Λ, we associate its Weierstrass $\wp$-function

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \smallsetminus \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}. \tag{2.2}$$

Then $\wp_\Lambda(z)$ is Λ-invariant and meromorphic on $\mathbb{C}$ with poles of order 2 at all $\omega \in \Lambda$. It satisfies the equation

$$\wp'_\Lambda(z)^2 = 4\wp^3_\Lambda(z) - g_2(\Lambda)\wp_\Lambda(z) - g_3(\Lambda) \tag{2.3}$$

where $g_2(\Lambda) = 60G_4(\Lambda)$ and $g_3(\Lambda) = 140G_6(\Lambda)$, and

$$G_k(\Lambda) = \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^k},$$

is the Eisenstein series of the lattice Λ for $k > 2$ even. g_2 and g_3 satisfy the relation

$$\Delta := g_2^3 - 27g_3^2 \neq 0. \tag{2.4}$$

This means $\big(\wp_\Lambda(z), \wp'_\Lambda(z)\big) \in \mathbb{C}^2$ for $z \notin \Lambda$ is a point on the smooth affine curve E_Λ^{aff} (since $\Delta \neq 0$) with equation

$$Y^2 = 4X^3 - g_2X - g_3 \tag{2.5}$$

and $\big(\wp_\Lambda(z) : \wp'_\Lambda(z) : 1\big) \in \mathbb{P}^2(\mathbb{C})$ for all $z \in \mathbb{C}$ is a point on the projective model of the above curve with equation

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3. \tag{2.6}$$

The above yields a biholomorphic isomorphism of the complex torus $\mathbb{C}/\Lambda$ with $E_\Lambda(\mathbb{C})$, well-defined through its restriction to $(\mathbb{C}\smallsetminus\Lambda)/\Lambda$ by $z \mapsto \big(\wp_\Lambda(z) : \wp'_\Lambda(z) : 1\big)$. Note that $E_\Lambda(\mathbb{C})$ inherits a group structure from $\mathbb{C}/\Lambda$, which may however be defined in purely algebraic terms on the algebraic curve E_Λ, and which turns E_Λ into an elliptic curve. This is the elliptic curve associated with the lattice Λ. In fact, each elliptic curve E over $\mathbb{C}$ has the form $E = E_\Lambda$ for some lattice Λ as above, and two such, E_Λ and $E_{\Lambda'}$, are isomorphic as elliptic curves (i.e., as algebraic curves through some isomorphism preserving the group structures) if and only if Λ' and Λ are homothetic, that is, $\Lambda' = c\Lambda$ for some $c \in \mathbb{C}^\times$.

Definition 2.4. An *abelian variety* over a field K is a smooth projective connected group variety. The group law is automatically commutative; see [Mum70, Chapter II, § 4, Corollary 2]. Abelian varieties are higher-dimensional generalizations of elliptic curves, which in turn are abelian varieties of dimension 1.

A *homomorphism* $f\colon X \to Y$ between abelian varieties over K is a morphism of varieties over K which is compatible with the group structure. The abelian group of homomorphisms $f\colon X \to Y$ over K is denoted $\mathrm{Hom}_K(X,Y)$ and we write $\mathrm{End}_K(X) = \mathrm{Hom}_K(X,X)$. We also write $\mathrm{QHom}_K(X,Y) = \mathrm{Hom}_K(X,Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{QEnd}_K(X) = \mathrm{QHom}_K(X,X) = \mathrm{End}_K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. For an abelian variety X over K and an integer $m \in \mathbb{Z}$, there is an endomorphism $[m] \in \mathrm{End}_K(X)$ given as the multiplication by m on the points. Thus if $m > 0$, then

$$[m](P) = P + P + \cdots + P \quad (m \text{ times})$$

For $m < 0$, we set $[m](P) = [-m](-P)$, and we define $[0](P) = 0$.

A morphism $f\colon X \to Y$ between abelian varieties is an *isogeny* if it is surjective with finite kernel. Every isogeny is finite, flat, surjective, see [Mil86, Proposition 8.2]. The *degree* of an isogeny $f : X \to Y$ is its degree as a regular map, i.e., the degree of the field extension $[K(X) : f^*K(Y)]$. If there exists an isogeny $X \to Y$ defined over K we will say that X and Y are *isogenous over* K and write $X \approx_K Y$. Note that $\approx_K$ is an equivalence relation. In fact, for every isogeny $f : X \to Y$ there is an isogeny $g : Y \to X$ such that $g \circ f = [n]$ on X for some $n \in \mathbb{Z}$, see [Mil08, Remark 6.5]. This means that f becomes invertible in $\mathrm{QHom}_K(X,Y)$, in the sense that $f^{-1} := g \otimes \frac{1}{n} \in \mathrm{QHom}_K(Y,X)$ is its inverse.

Remark 2.5. (a) Let X and Y be abelian varieties over K. If X and Y are isogenous over K via an isogeny f, then

$$\mathrm{QEnd}_K(X) \cong \mathrm{QHom}_K(X,Y) \cong \mathrm{QEnd}_K(Y), \quad h \mapsto f \circ h \mapsto f \circ h \circ f^{-1}.$$

More precisely, $\mathrm{QHom}_K(X,Y)$ is a free right $\mathrm{QEnd}_K(X)$-module of rank 1 and a free left $\mathrm{QEnd}_K(Y)$-module of rank 1. If X and Y are not isogenous then $\mathrm{QHom}_K(X,Y) = (0)$.

(b) The homomorphism $[m] \in \mathrm{End}_K(X)$ is an isogeny of degree m^{2g}, where $g = \dim X$. It is always étale when K has characteristic zero, and when K has characteristic $p > 0$ it is étale if and only if p does not divide m, see [Mum70, Chapter II, § 6].

(c) The kernel $X[m] := \ker([m]\colon X \to X)$ is a finite group scheme over K of order m^{2g}.

Definition 2.6. Let X be an abelian variety and let $m \in Z$ with $m \geq 1$. The *m-torsion subgroup* of X, denoted by $X[m](K^{\mathrm{alg}})$, is the subgroup of points of $X(K^{\mathrm{alg}})$ of order m,

$$X[m](K^{\mathrm{alg}}) = \{P \in X(K^{\mathrm{alg}}) : [m]P = 0\}.$$

It equals the group of K^{alg}-valued points of the finite group scheme $X[m]$.

Remark 2.7. For any m not divisible by the characteristic of K, $X[m](K^{\mathrm{alg}})$ has order m^{2g} and is contained in $X(K^{\mathrm{sep}})$. Since this is also true for any n dividing m, $X[m](K^{\mathrm{alg}})$ must be a free $\mathbb{Z}/m\mathbb{Z}$-module of rank $2g$.

Finally, if X is an abelian variety over $\mathbb{C}$ of dimension g, then $X(\mathbb{C})$ is isomorphic to a complex torus $\mathbb{C}^g/\Lambda$,

$$X(\mathbb{C}) \cong \mathbb{C}^g/\Lambda$$

for some lattice $\Lambda = \Lambda(X)$ in $\mathbb{C}^g$ under an isomorphism of complex manifolds which preserves the group structures. Here $\Lambda \subset \mathbb{C}^g$ is a discrete $\mathbb{Z}$-submodule which is free of rank $2g$. However, when $g > 1$, not every lattice $\Lambda \subset \mathbb{C}^g$ arises from an abelian variety, that is, the quotient $\mathbb{C}^g/\Lambda$ of $\mathbb{C}^g$ by an arbitrary lattice Λ does not always arise from an abelian variety. There is a criterion on Λ for when $\mathbb{C}^g/\Lambda$ is an algebraic (hence abelian) variety, namely, that $(\mathbb{C}^g, \Lambda)$ admits a Riemannian form[2], see [Mum70, Chapter I, § 3].

3 Semi-simple Abelian Varieties

Theorem 3.1. *For two abelian varieties X and Y over a field K the $\mathbb{Z}$-module $\mathrm{Hom}_K(X,Y)$ is finite projective of rank $\leq (2\dim X)\cdot(2\dim Y)$.*

Proof. See for example [Mum70, Chapter IV, § 19, Corollary 1]. ☐

Definition 3.2. Let X be an abelian variety over K. Then X is called

(a) *simple over K* if X is non-trivial and there does not exist an abelian subvariety $Y \subset X$ over K other than (0) and X.

(b) *semi-simple over K* if X is isogenous over K to a direct product of simple abelian varieties, i.e. $X \approx_K X_1 \times_K \ldots \times_K X_n$ with X_i simple.

Remark 3.3. The Theorem of Poincaré and Weil [Mil08, Proposition 9.1] states that any abelian variety is semi-simple over K. More precisely, for any abelian variety X over K, there are simple abelian subvarieties $X_1, \cdots, X_n \subset X$ such that the map $X_1 \times_K \cdots \times_K X_n \to X$, $(a_1, \cdots, a_n) \to a_1 + \cdots + a_n$ is an isogeny. The proof of this is analogous with a standard proof for the semi-simplicity of a representation of a finite group G on a finite-dimensional vector space over $\mathbb{Q}$, see [Mil08, Remark 9.2].

Let X be a simple abelian variety, and let $0 \neq f \in \mathrm{End}_K(X)$. Then f is an isogeny, because by the simplicity of X, the image of f equals X and the connected component of $\ker f$ equals $\{0\}$, as both are abelian subvarieties. So f is surjective with finite kernel. From this it follows that $\mathrm{QEnd}_K(X)$ is a division algebra or equivalently a skew-field, i.e., a ring, possibly non commutative, in which every nonzero element has an inverse.

Remark 3.4. Let X be a simple abelian variety over K, and let $D = \mathrm{QEnd}_K(X)$. Then $\mathrm{QEnd}_K(X^n) = M_n(D)$ is the ring of $n \times n$ matrices with coefficients in D.

Now consider an arbitrary abelian variety X. Then X is isogenous over K to a product $X_1^{n_1} \times_K \cdots \times_K X_r^{n_r}$, where each X_i is simple, and X_i is not isogenous to X_j for $i \neq j$ over K. The above remarks show that

$$\mathrm{QEnd}_K(X) \cong \prod M_{n_i}(D_i), \quad D_i = \mathrm{QEnd}_K(X_i).$$

[2]For a complex torus V/Λ where V is a complex vector space and Λ is a full lattice in V, a skew-symmetric form $F : \Lambda \times \Lambda \to \mathbb{Z}$, that is $F(w,v) = -F(v,w)$, extended to a skew-symmetric $\mathbb{R}$-bilinear form $F_{\mathbb{R}} : V \times V \to \mathbb{R}$ is a *Riemannian form* if $F_{\mathbb{R}}(iv, iw) = F_{\mathbb{R}}(v,w)$ and the associated Hermitian form $H\colon V \times V \to \mathbb{C}$ with $H(v,w) := F_{\mathbb{R}}(iv,w) + i\,F_{\mathbb{R}}(v,w)$ and $F_{\mathbb{R}}(v,w) = \mathfrak{Im}(H(v,w))$ is positive definite.

Since $\mathrm{End}_K(X)$ is a free $\mathbb{Z}$-module of finite rank $\leq (2\dim X)^2$ we know that $\mathrm{QEnd}_K(X)$ is a finite dimensional $\mathbb{Q}$-algebra.

In the following we recall a few facts about semi-simple algebras. Let Q be a field, let B be a semi-simple Q-algebra of finite dimension, and let $B = \prod B_i$ be its decomposition into a product of simple algebras B_i. A simple Q-algebra is isomorphic to a matrix algebra over a division Q-algebra. The center of each B_i is a field F_i, and each degree $[B_i : F_i]$ is a square. The *reduced degree* of B over Q is defined to be

$$[B:Q]_{red} = \sum_i [B_i : F_i]^{1/2}[F_i : Q].$$

For any field Q' containing Q,

$$[B:Q] = [B \otimes_Q Q' : Q'] \quad \text{and} \quad [B:Q]_{red} = [B \otimes_Q Q' : Q']_{red}.$$

Proposition 3.5 ([Mil06, Proposition 1.2]). *Let B be a semi-simple Q-algebra which is finite dimensional over Q. For any faithful B-module M,*

$$\dim_Q M \geq [B:Q]_{red};$$

and there exists a faithful module for which equality holds if and only if the simple factors of B are matrix algebras over their centers.

Proposition 3.6 ([Mil06, Proposition 1.3]). *Let $\mathrm{char}(Q) = 0$ and let B be a semi-simple Q-algebra. Every maximal étale Q-subalgebra of B has degree $[B:Q]_{red}$ over Q. Here we mean by an* étale Q-algebra *a finite product of finite separable field extensions of Q.*

4 Cohomology

4.1 Singular Cohomology

Let X be an abelian variety of dimension g over $\mathbb{C}$. Let V be the tangent space of X at the identity element and let Λ be the kernel of the exponential map $\exp : V \to X$. Now the space $V \cong \mathbb{C}^g$ is simply connected, and $\exp : V \to X$ is a covering map, therefore it realizes V as the universal covering space of X, and so $\pi_1(X)$ is its group of covering transformations, which is Λ. In particular, it is abelian. As for any good topological space we obtain for the *singular cohomology* of X

$$\mathrm{H}^1(X,\mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{groups}}(\pi_1(X),\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Z}).$$

Since we have seen that X is a complex torus of dimension g, it is isomorphic to $(\mathbb{R}/\mathbb{Z})^{2g} = (S^1)^{2g}$ as a real Lie group, where S^1 is the circle group. We claim that for all $r \in \mathbb{N}_{>0}$

$$\bigwedge^r \mathrm{H}^1(X,\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^r(X,\mathbb{Z})$$

under the natural map defined by the cup product. Indeed, by the Künneth formula if the above map is an isomorphism for spaces X_1 and X_2 with finitely generated cohomologies, then it is an isomorphism for $X_1 \times_K X_2$. Since it is an isomorphism for S^1 for all $r \geq 0$, where the module is (0) for $r \geq 2$, the result for X follows.

Since X is compact and orientable and $\mathrm{H}^r(X,\mathbb{Z})$ is torsion free, the duality theorems gives us for the *singular homology* of X

$$\mathrm{H}_r(X,\mathbb{Z}) \cong \mathrm{H}^r(X,\mathbb{Z})^\vee \qquad \text{and in particular} \qquad \mathrm{H}_1(X,\mathbb{Z}) = \Lambda\,.$$

4.2 ℓ-adic Cohomology

We follow [Mil86, § 15]. Let X be an abelian variety of dimension g over a field K, and let ℓ be a prime different from $\mathrm{char}(K)$. Recall that, for any m not divisible by the characteristic of K, $X[m](K^{\mathrm{sep}})$ has order m^{2g}. Define the *ℓ-adic Tate module* of X as

$$T_\ell(X) = \varprojlim \big(X[\ell^n](K^{\mathrm{sep}}), [\ell]\big).$$

It follows that $T_\ell(X)$ is a free $\mathbb{Z}_\ell$-module of rank $2g$. There is a continuous action of $\mathscr{G}_K$ on this module.

Let X and Y be two abelian varieties over K. A homomorphism $f : X \to Y$ induces a homomorphism $X[\ell^n] \to Y[\ell^n]$, and hence a homomorphism

$$T_\ell(f) : T_\ell(X) \to T_\ell(Y), \quad (a_1, a_2, \cdots) \mapsto \big(f(a_1), f(a_2), \cdots\big).$$

Therefore, T_ℓ is a functor from abelian varieties to $\mathbb{Z}_\ell$-modules. It is easy to see that for any prime $\ell \neq \mathrm{char}(K)$, the natural map

$$\mathrm{Hom}_K(X, Y) \to \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(X), T_\ell(Y))$$

is injective. From this one obtains that the $\mathbb{Z}$-algebra $\mathrm{Hom}_K(X, Y)$ of morphisms $X \to Y$ of group varieties is torsion-free. The following theorem was conjectured by Tate [Tat66] and proved by him for finite fields K. It was proved by Zarhin [Zar75] for fields of positive characteristic and by Faltings [Fal83, Fal84b] for fields of characteristic zero.

Theorem 4.1 (Tate conjecture for abelian varieties)**.** *Let X and Y be two abelian varieties over a finitely generated field K and let ℓ be a prime different from the characteristic of K. Then the natural map*

$$\mathrm{Hom}_K(X, Y) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathbb{Z}_\ell[\mathscr{G}_K]}(T_\ell X, T_\ell Y), \quad f \otimes a \mapsto a \cdot T_\ell(f)$$

is an isomorphism of $\mathbb{Z}_\ell$-modules.

The theorem is known to fail for some classes of fields which are not finitely generated (e.g. local fields and of course algebraically closed fields).

Now we write $X_{K^{\mathrm{alg}}} := X \times_K \mathrm{Spec}\, K^{\mathrm{alg}}$ and denote by $\pi_1^{\text{ét}}(X_{K^{\mathrm{alg}}}, 0)$ the étale fundamental group, then

$$\mathrm{H}^1_{\text{ét}}(X_{K^{\mathrm{alg}}}, \mathbb{Z}_\ell) \cong \mathrm{Hom}^{\mathrm{cont}}(\pi_1^{\text{ét}}(X_{K^{\mathrm{alg}}}, 0), \mathbb{Z}_\ell).$$

For each n the map $[\ell^n] : X \to X$ is a finite étale covering of X with group of covering transformations $X[\ell^n](K^{\mathrm{sep}})$. By definition $\pi_1^{\text{ét}}(X_{K^{\mathrm{alg}}}, 0)$ classifies such coverings, and therefore there is a canonical epimorphism $\pi_1^{\text{ét}}(X_{K^{\mathrm{alg}}}, 0) \twoheadrightarrow X[\ell^n]$. On passing to the inverse limit, we get an epimorphism $\pi_1^{\text{ét}}(X_{K^{\mathrm{alg}}}, 0) \twoheadrightarrow T_\ell(X)$, and consequently an injection

$$\mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(X), \mathbb{Z}_\ell) \hookrightarrow \mathrm{H}^1_{\text{ét}}(X_{K^{\mathrm{alg}}}, \mathbb{Z}_\ell),$$

which actually is an isomorphism, see [Mil86, Theorem 15.1]. So we obtain for the first ℓ-adic homology group of X

$$\mathrm{H}_{1,\text{ét}}(X_{K^{\mathrm{alg}}}, \mathbb{Z}_\ell) = T_\ell(X) \text{ and } \mathrm{H}_{1,\text{ét}}(X_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell) = T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell\,,$$

and for the first ℓ-adic cohomology group of X

$$\mathrm{H}^1_{\text{ét}}(X_{K^{\mathrm{alg}}}, \mathbb{Z}_\ell) = \mathrm{H}_{1,\text{ét}}(X_{K^{\mathrm{alg}}}, \mathbb{Z}_\ell)^\vee \text{ and } \mathrm{H}^1_{\text{ét}}(X_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell) = \mathrm{H}_{1,\text{ét}}(X_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell)^\vee.$$

By [Mil86, Theorem 15.1] the cup product pairings define isomorphisms

$$\mathrm{H}_{r,\text{ét}}(X_{K^{\mathrm{alg}}}, \mathbb{Z}_\ell) \cong \bigwedge\nolimits^r \mathrm{H}_{1,\text{ét}}(X_{K^{\mathrm{alg}}}, \mathbb{Z}_\ell) \text{ and } \mathrm{H}^r_{\text{ét}}(X_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell) \cong \bigwedge\nolimits^r \mathrm{H}^1_{\text{ét}}(X_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell).$$

Now, over the field $K = \mathbb{C}$ the choice of an isomorphism $X(\mathbb{C}) \cong \mathbb{C}^g/\Lambda$ determines $X[m](\mathbb{C}) \cong m^{-1}\Lambda/\Lambda$. Then

$$\begin{aligned} T_\ell(X) = \varprojlim\big(X[\ell^n](\mathbb{C}), [\ell]\big) &\cong \varprojlim\big(\ell^{-n}\Lambda/\Lambda, \text{multiplication with } \ell\big) \\ &\cong \varprojlim\big(\Lambda \otimes_{\mathbb{Z}} (\mathbb{Z}/\ell^n\mathbb{Z}), \bmod \ell^n\big) \\ &\cong \Lambda \otimes_{\mathbb{Z}} \varprojlim(\mathbb{Z}/\ell^n\mathbb{Z}), \text{ because } \Lambda \text{ is a free } \mathbb{Z}\text{-module} \\ &\cong \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_\ell. \end{aligned}$$

Taking duals and exterior powers, we can summarize the results as a

Theorem 4.2. *For every abelian variety X over $\mathbb{C}$ there are canonical* comparison isomorphisms between singular and ℓ-adic (co-)homology

$$\mathrm{H}^r_{\text{ét}}(X, \mathbb{Z}_\ell) \cong \mathrm{H}^r(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \qquad \textit{and} \qquad \mathrm{H}_{r,\text{ét}}(X, \mathbb{Z}_\ell) \cong \mathrm{H}_r(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell.$$

Example 4.3. Also for the multiplicative group scheme $\mathbb{G}_m := \mathbb{G}_{m,\mathbb{Q}} = \operatorname{Spec}\mathbb{Q}[x,x^{-1}]$ there is a period isomorphism between $\mathrm{H}_1(\mathbb{G}_m(\mathbb{C}),\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}_\ell$ and $\mathrm{H}_{1,\text{ét}}(\mathbb{G}_{m,\mathbb{C}},\mathbb{Z}_\ell)$. Namely, $\mathrm{H}_1(\mathbb{G}_m(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}\cdot u$, where $u\colon [0,1]\to \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$ is the cycle given by $u(s)=\exp(2\pi i s)$. Also let $\varepsilon_\ell^{(n)} := \exp(2\pi i/\ell^n)\in\mathbb{Q}^{\text{alg}}\subset\mathbb{C}$. It is a primitive ℓ^n-th root of unity with $(\varepsilon_\ell^{(n+1)})^\ell = \varepsilon_\ell^{(n)}$ for all n. Let $\varepsilon_\ell := (\varepsilon_\ell^{(n)})_{n\in\mathbb{N}}\in T_\ell\mathbb{G}_m$. Then $\mathrm{H}_{1,\text{ét}}(\mathbb{G}_{m,\mathbb{C}},\mathbb{Z}_\ell) = T_\ell\mathbb{G}_m = \varepsilon_\ell^{\mathbb{Z}_\ell}$ and the comparison isomorphism

$$\mathrm{H}_1(\mathbb{G}_m(\mathbb{C}),\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{H}_{1,\text{ét}}(\mathbb{G}_{m,\mathbb{C}},\mathbb{Z}_\ell)$$

sends u to ε_ℓ. This can be seen from the exact sequence $0\to\mathbb{Z}=\pi_1(\mathbb{C}^\times)\to\mathbb{C}\xrightarrow{\exp(2\pi i\,\bullet\,)}\mathbb{C}^\times\to 0$ and the induced comparison isomorphism $\pi_1(\mathbb{C}^\times)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell\xrightarrow{\sim}T_\ell\mathbb{G}_m$, $1\mapsto\big(\exp(2\pi i/\ell^n)\big)_{n\in\mathbb{N}}$.

4.3 De Rham Cohomology

We will now explain the construction of the Dolbeault complex associated with X which is an analog of the de Rham complex for complex manifolds. Let X be an abelian variety over $\mathbb{C}$.

Let $\mathscr{C}^n = \oplus_{p+q=r}\mathscr{C}^{p,q}$ be the sheaf of C^∞ complex valued differential n-forms, where $\mathscr{C}^{p,q}$ is the sheaf of C^∞ complex valued differential forms of type (p,q). In terms of local coordinates, let $(z_1,\cdots,z_g)$ be a holomorphic coordinate system. First we decompose the complex coordinates into their real and imaginary parts: $z_j = x_j+iy_j$ for each j. Letting $dz_j = dx_j+idy_j$, $d\bar z_j = dx_j-idy_j$, one sees that any differential 1-form with complex coefficients can be written uniquely as a sum

$$\sum_{j=1}^n (f_j dz_j + g_j d\bar z_j),$$

for $\mathbb{C}$-valued C^∞-functions f_j and g_j. Let $\mathscr{C}^{1,0}$ be the sheaf of C^∞ complex valued differential 1-forms where all g_j are zero, and let $\mathscr{C}^{0,1}$ be the sheaf of C^∞ complex valued differential 1-forms where all f_j are zero. Then the space $\mathscr{C}^{p,q}$ of type (p,q)-forms is defined by taking linear combinations of the wedge products of p elements from $\mathscr{C}^{1,0}$ and q elements from $\mathscr{C}^{0,1}$. Symbolically,

$$\mathscr{C}^{p,q} = \bigwedge^p \mathscr{C}^{1,0}\wedge\bigwedge^q\mathscr{C}^{0,1}$$

In particular for each n and each p and q with $p+q=n$, there are canonical projection maps which we denote by

$$\pi^{(p,q)}:\mathscr{C}^n\to\mathscr{C}^{p,q}.$$

The exterior derivative defines a map $d:\mathscr{C}^n\to\mathscr{C}^{n+1}$ i.e. if $\varphi\in\mathscr{C}^{p,q}$ then $d(\varphi)\in\mathscr{C}^{p+1,q}\oplus\mathscr{C}^{p,q+1}$. Using d and the projections maps, it is possible to define the operators:

$$\partial = \pi^{p+1,q}\circ d:\mathscr{C}^{p,q}\to\mathscr{C}^{p+1,q},\quad \bar\partial = \pi^{p,q+1}\circ d:\mathscr{C}^{p,q}\to\mathscr{C}^{p,q+1}$$

In terms of local coordinates $z=(z_1,\cdots z_g)$ we can write $\varphi\in\mathscr{C}^{p,q}$ as

$$\varphi = \sum_{\#I=p,\,\#J=q} f_{IJ}\,dz_I\wedge d\bar z_J\in\mathscr{C}^{p,q}$$

where I and J are multi-indices and $dz_I=\bigwedge_{i\in I}dz_i$ and $d\bar z_I = \bigwedge_{i\in I}d\bar z_i$. Then

$$\partial\varphi = \sum_{\#I=p,\,\#J=q}\sum_i\frac{\partial f_{IJ}}{\partial z_i}dz_i\wedge dz_I\wedge d\bar z_J \qquad\text{and}\qquad \bar\partial\varphi \quad = \sum_{\#I=p,\,\#J=q}\sum_i\frac{\partial f_{IJ}}{\partial\bar z_i}d\bar z_i\wedge dz_I\wedge d\bar z^J.$$

It is not difficult to see the following properties:

$$d=\partial+\bar\partial$$
$$\partial^2=\bar\partial^2=\partial\bar\partial+\bar\partial\partial=0.$$

Then the Poincaré lemma gives that the complex

$$0\to\mathbb{C}\to\mathscr{C}^0\xrightarrow{d}\mathscr{C}^1\xrightarrow{d}\cdots$$

is a fine resolution of the constant sheaf $\mathbb{C}$. It is called the *de Rham resolution.* We define the de Rham cohomology as the cohomology of this complex i.e.

$$\mathrm{H}^n_{\mathrm{dR}}(X,\mathbb{C}) = \frac{\{\text{global } n\text{-forms } \varphi \in \mathscr{C}^n(X) \text{ on } X \text{ which are } d\text{-closed, i.e. } d\varphi = 0\}}{\{d\psi : \text{where } \psi \in \mathscr{C}^{n-1}(X) \text{ is a global } (n-1)\text{-form on } X\}}.$$

Let $V = \mathrm{T}_0 X$ be the tangent space to X at 0 (regarded as a complex vector space). Let $T^\vee = \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ be the complex cotangent space to X at 0 and $\overline{T}^\vee = \mathrm{Hom}_{\mathbb{C}\text{-antilinear}}(V,\mathbb{C})$. Then from linear algebra

$$\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) \oplus \mathrm{Hom}_{\mathbb{C}\text{-antilinear}}(V,\mathbb{C}) \ \text{ i.e. } \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) \cong T^\vee \oplus \overline{T}^\vee,$$

and

$$\bigwedge^r \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) \cong \bigoplus_{p+q=r} \bigwedge^p T^\vee \otimes \bigwedge^q \overline{T}^\vee$$

By translation under the group law on X every complex co-vector $\varphi \in \wedge^p T^\vee \otimes \wedge^q \overline{T}^\vee$ extends to a unique translation invariant $\omega_\varphi \in \mathscr{C}^{p,q}$, and therefore every complex co-vector $\varphi \in \wedge^r \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$ extends to a unique translation invariant form ω_φ belonging to $\mathscr{C}^n$. For all d-closed n-forms ω, there is unique translation invariant ω_φ for $\varphi \in \wedge^n \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$, such that

$$\omega - \omega_\varphi = d\eta, \text{ for some } (n-1)\text{-form } \eta.$$

Therefore, $\mathrm{H}^r_{\mathrm{dR}}(X,\mathbb{C}) \cong \bigwedge^r \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$, and has the decomposition

$$\mathrm{H}^r_{\mathrm{dR}}(X,\mathbb{C}) \cong \bigoplus_{p+q=r} \bigwedge^p T^\vee \otimes \bigwedge^q \overline{T}^\vee.$$

For the sheaf $\Omega^p := \ker(\bar\partial \colon \mathscr{C}^{p,0} \to \mathscr{C}^{p,1})$ of holomorphic p-forms on X we know from [Mum70, Chapter I, § 1, Theorem] that

$$\mathrm{H}^q(X,\mathcal{O}_X) \cong \bigwedge^q \overline{T}^\vee \quad \text{and} \quad \mathrm{H}^q(X,\Omega^p) \cong \bigwedge^p T^\vee \otimes \bigwedge^q \overline{T}^\vee,$$

so

$$\mathrm{H}^r_{\mathrm{dR}}(X,\mathbb{C}) \cong \bigoplus_{p+q=r} \mathrm{H}^{p,q}(X), \quad \text{where} \quad \mathrm{H}^{p,q}(X) := \mathrm{H}^q(X,\Omega^p).$$

This is the famous Hodge decomposition.

Now we obtain the *de Rham isomorphism*

$$\mathrm{H}^1(X,\mathbb{C}) = \mathrm{H}^1(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) \cong \mathrm{H}^1_{\mathrm{dR}}(X,\mathbb{C}).$$

Then, $\mathrm{H}^n_{\mathrm{dR}}(X,\mathbb{C}) \cong \mathrm{H}^n(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$. Note that complex conjugation on the right tensor factor of the target defines a conjugate-linear automorphism of $\mathrm{H}^n_{\mathrm{dR}}(X,\mathbb{C})$. For more details see [Mum70, Chapter I, § 1]. Taking also exterior powers, we can summarize the results as a

Theorem 4.4 (De Rham isomorphism). *For every abelian variety X over $\mathbb{C}$ there are canonical* comparison isomorphisms between singular and de Rham cohomology

$$\mathrm{H}^r_{\mathrm{dR}}(X,\mathbb{C}) \cong \mathrm{H}^r(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Example 4.5. Also for the multiplicative group scheme $\mathbb{G}_m := \mathbb{G}_{m,\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,x^{-1}]$ there is a de Rham isomorphism between $\mathrm{H}^1(\mathbb{G}_m(\mathbb{C}),\mathbb{Z})$ and $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m,\mathbb{C}) = \mathbb{C}\frac{dx}{x}$. As in Example 4.3, the singular homology $\mathrm{H}_1(\mathbb{G}_m(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}\cdot u$, where $u \colon [0,1] \to \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$ is the cycle given by $u(s) = \exp(2\pi i s)$. The de Rham isomorphism is given as the pairing

$$\mathrm{H}_1(\mathbb{G}_m,\mathbb{Z}) \times \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m,\mathbb{C}) \longrightarrow \mathbb{C}, \qquad (nu,\omega) \longmapsto n\textstyle\int_u \omega\,, \qquad (u,\frac{dx}{x}) \longmapsto \int_u \frac{dx}{x} \;=\; 2\pi i.$$

The corresponding isomorphism $\mathrm{H}^1(\mathbb{G}_m,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m,\mathbb{C})$ sends the generator of $\mathrm{H}^1(\mathbb{G}_m,\mathbb{Z})$, which is dual to u, to $(2\pi i)^{-1}\cdot\frac{dx}{x}$.

5 p-divisible Groups and the p-adic Period Isomorphism

Let R be a commutative ring. Let p be a prime number, and h an integer ≥ 0. A p-divisible group G over R of height h is an inductive system
$$(G_n, i_n), \quad n \geq 0$$
where

(a) G_n is a finite flat commutative group scheme of finite presentation over R of order p^{nh},

(b) for each $n \geq 0$,
$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{[p^n]} G_{n+1}$$
is exact (i.e., G_n can be identified via i_n with the kernel of multiplication by p^n in G_{n+1}).

These axioms for ordinary abelian groups would imply
$$G_n \cong (\mathbb{Z}/p^n\mathbb{Z})^h \quad \text{and} \quad G = \varinjlim G_n = (\mathbb{Q}_p/\mathbb{Z}_p)^h.$$
A homomorphism $f : G \to H$ of p-divisible groups is defined in the obvious way: if $G = (G_n, i_n)$, $H = (H_n, i_n)$ then f is a system of homomorphisms $f_n : G_n \to H_n$ of group schemes over R, satisfying $i_n \circ f_n = f_{n+1} \circ i_n$ for all $n \geq 1$.

Example 5.1. Let G be a commutative group variety over a field K, which is either an abelian variety or $\mathbb{G}_m$. We can associate a p-divisible group with G:

Define $G[m]$ as the kernel of multiplication by m. Then $(G[p^n], i_n)$ is a p-divisible group, where i_n denotes the obvious inclusion. This p-divisible group is sometimes denoted $G[p^\infty]$.

(a) If $G = X$ is an abelian variety, then the height of $G[p^\infty]$ is $2 \dim X$.

(b) If $G = \mathbb{G}_m$ is the multiplicative group scheme, then $\mathbb{G}_m[p^\infty] = \mu_{p^\infty} := (\mu_{p^n}, i_n)$ with height 1. Here $\mu_{p^n} = \operatorname{Spec} K[x]/(x^{p^n} - 1)$ is the group scheme of p^n-th roots of unity.

Let us see how p-divisible groups generalize Tate modules. Suppose $p \neq \operatorname{char}(K)$. Then for a p-divisible group (G_n, i_n) of height h over K each G_n is a finite étale group scheme over K and each $M_n := G_n(K^{\text{sep}})$ is a discrete $\mathscr{G}_K$-module of size p^{nh} annihilated by p^n and $M_{n+1}[p^n] = M_n$. It follows that $M_n = (\mathbb{Z}/p^n\mathbb{Z})^h$. We can form two kinds of limits:
(i) the direct limit $M_\infty = \varinjlim M_n$ is $(\mathbb{Q}_p/\mathbb{Z}_p)^h$ with a continuous $\mathscr{G}_K$-action for the discrete topology, and
(ii) multiplication by p on M_{n+1} provides a quotient map $M_{n+1} \twoheadrightarrow M_n$ of discrete $\mathscr{G}_K$-modules yielding an inverse limit $T_p(M) = \varprojlim M_n$ that is a finite free $\mathbb{Z}_p$-module of rank h equipped with a continuous action of $\mathscr{G}_K$ for the p-adic topology.

We can recover the direct system (M_n, i_n) from both limits, namely $M_n = M_\infty[p^n]$ and $M_n = T_p(M)/(p^n)$. The viewpoint of M_∞ explains the p-divisible aspect of the situation (since multiplication by p is surjective on $(\mathbb{Q}_p/\mathbb{Z}_p)^h$), whereas $T_p(M)$ has a nicer $\mathbb{Z}_p$-module structure. Since the étale group scheme G_n is uniquely determined by the $\mathscr{G}_K$-module M_n, this proves:

Proposition 5.2. *If K is a field with $p \neq \operatorname{char}(K)$, then the functor $G \to T_p(G)$ is an equivalence from the category of p-divisible groups over K to the category of finite free $\mathbb{Z}_p$-modules with continuous $\mathscr{G}_K$-action.* □

On the other hand let K be a finite extension of $\mathbb{Q}_p$ and let X be an abelian variety over K. Assume that X has *good reduction*, i.e. there exists a smooth projective commutative group scheme $\mathcal{X}$ over $\mathcal{O}_K$ with $X \cong \mathcal{X} \times_{\mathcal{O}_K} \operatorname{Spec} K$. Then $X[p^n]$ admits an integral model $\mathcal{G}_n := \mathcal{X}[p^n]$ with $\mathcal{G}_n = \mathcal{G}_{n+1}[p^n]$ for all $n \geq 1$ and $\mathcal{G} = (\mathcal{G}_n, i_n)$ is a p-divisible group over $\mathcal{O}_K$ with $\mathcal{G}_K := \mathcal{G} \times_{\mathcal{O}_K} \operatorname{Spec} K \cong X[p^\infty]$.

Now due to Tate [Tat67] we know that if $\mathcal{G}$ and $\mathcal{H}$ are p-divisible groups over $\mathcal{O}_K$ then
$$\operatorname{Hom}_{\mathcal{O}_K}(\mathcal{G}, \mathcal{H}) \xrightarrow{\sim} \operatorname{Hom}_K(\mathcal{G}_K, \mathcal{H}_K).$$

Remark 5.3. p-divisible groups over a perfect field k of characteristic p have a description via semi-linear algebra by their *Dieudonné module*. The latter is a finite free module M over the ring $W(k)$ of p-typical Witt-vectors over k, equipped with a Frob_p-semi-linear morphism $F : M \to M$, called *Frobenius*, and a $\operatorname{Frob}_p^{-1}$-semi-linear morphism $V : M \to M$, called *Verschiebung*, satisfying $FV = p = VF$.

This was generalized by Fontaine [Fon77] to p-divisible groups G over the ring of integers $\mathcal{O}_K$ of a finite field extension K of $\mathbb{Q}_p$. Those p-divisible groups are described by the Dieudonné module D of the special fiber $G \times_{\mathcal{O}_K} \operatorname{Spec} k$ together with a decreasing exhaustive and separated filtration $\mathrm{Fil}^\bullet$ on $D_K = D \otimes_{K_0} K$ satisfying $\mathrm{Fil}^0(D_K) = D_K$, $\mathrm{Fil}^2(D_K) = (0)$, where $K_0 = W(k)[p^{-1}]$ is the maximal unramified subextension of K.

Notation 5.4. Let $\mathcal{O}^\flat_{\mathbb{C}_p} := \varprojlim(\mathcal{O}_{\mathbb{C}_p}, \mathrm{Frob}_p) = \{x = (x^{(n)})_{n\in\mathbb{N}_0} \in (\mathcal{O}_{\mathbb{C}_p})^{\mathbb{N}_0} \colon (x^{(n+1)})^p = x^{(n)}\}$ and $A_{\mathrm{inf}} := W(\mathcal{O}^\flat_{\mathbb{C}_p})$ be the ring of Witt vectors. Every element of A_{inf} can be written in the form $\sum_{i=0}^\infty [x_i]p^i$ where $[x]$ denotes the Teichmüller lift of the element $x = (x^{(n)})_n \in \mathcal{O}^\flat_{\mathbb{C}_p}$. Let $\Theta : A_{\mathrm{inf}}[\frac{1}{p}] \twoheadrightarrow \mathbb{C}_p$ be the morphism sending $\sum_i [x_i]p^i$ to $\sum_i x_i^{(0)} p^i$. The *de Rham period ring* $\mathbb{B}^+_{p,\mathrm{dR}}$ is the completion of $A_{\mathrm{inf}}[\frac{1}{p}]$ at the maximal ideal $\ker\Theta$ and $\mathbb{B}_{p,\mathrm{dR}} := \mathrm{Frac}(\mathbb{B}^+_{p,\mathrm{dR}})$ is *the field of p-adic periods*. The de Rham period ring $\mathbb{B}^+_{p,\mathrm{dR}}$ is a complete discrete valuation ring with residue field $\mathbb{C}_p$ and maximal ideal $\ker\Theta$. One can show that the ideal $\ker\Theta \subset A_{\mathrm{inf}}$ is principal and generated by an element $[p^\flat] - p \in A_{\mathrm{inf}}$, where $p^\flat = (p, p^{1/p}, p^{1/p^2}, \cdots) \in \mathcal{O}^\flat_{\mathbb{C}_p}$. Any other generator is of the form $([p^\flat] - p) \cdot u$ for $u \in A^\times_{\mathrm{inf}}$. For more details see [Fon77]. There is a filtration on $\mathbb{B}_{p,\mathrm{dR}}$ defined by putting $\mathrm{Fil}^i(\mathbb{B}_{p,\mathrm{dR}}) = ([p^\flat] - p)^i \cdot \mathbb{B}^+_{p,\mathrm{dR}}$ for $i \in \mathbb{Z}$, and we define $\hat{v}_p(x)$ for $x \in \mathbb{B}_{p,\mathrm{dR}} \smallsetminus \{0\}$ by $\hat{v}_p(x) = i$ if $x \in \mathrm{Fil}^i(\mathbb{B}_{p,\mathrm{dR}}) \smallsetminus \mathrm{Fil}^{i+1}(\mathbb{B}_{p,\mathrm{dR}})$. For $x \in \mathbb{B}_{p,\mathrm{dR}} \smallsetminus \{0\}$, the quantity

$$v_p(x) \;:=\; v_p\big(\Theta(x \cdot ([p^\flat] - p)^{-\hat{v}_p(x)})\big) \in \mathbb{Q} \tag{5.1}$$

does not depend on the choice of the generator $[p^\flat] - p$ of $A_{\mathrm{inf}} \cap \ker\Theta$. Indeed, if we replace the generator $[p^\flat] - p$ of $\ker\Theta \subset A_{\mathrm{inf}}$ by another generator $([p^\flat] - p) \cdot u$ with $u \in A^\times_{inf}$, because then $v_p\big(\Theta(x \cdot (([p^\flat] - p) \cdot u)^{-\hat{v}_p(x)}\big) = v_p\big(\Theta(x \cdot ([p^\flat] - p)^{-\hat{v}_p(x)}\big) + v_p(\Theta(u))^{-\hat{v}_p(x)} = v_p\big(\Theta(x \cdot ([p^\flat] - p)^{-\hat{v}_p(x)}\big)$ as $\Theta(u) \in \mathcal{O}^\times_{\mathbb{C}_p}$. If x and y are two elements of $\mathbb{B}_{p,\mathrm{dR}}$, then $\hat{v}_p(xy) = \hat{v}_p(x) + \hat{v}_p(y)$, and hence $v_p(xy) = v_p(x) + v_p(y)$. But note that v_p does not satisfy the triangle inequality.

Finally, if $K \subset \mathbb{C}_p$ is a finite field extension of $\mathbb{Q}_p$, then there is an action of $\mathscr{G}_K$ on $\mathbb{B}_{p,\mathrm{dR}}$ which respects the filtration, and $(\mathbb{B}_{p,\mathrm{dR}})^{\mathscr{G}_K} = K$. Also note that there does not exist a lift of the absolute Frobenius φ_p on $\mathbb{B}_{p,\mathrm{dR}}$.

The *p-adic period isomorphism* is provided by the following theorem which was proved by Fontaine and Messing [FM87] using the associated p-divisible group.

Theorem 5.5. *Let $K_p \subset \mathbb{Q}_p^{\mathrm{alg}}$ be a finite extension of $\mathbb{Q}_p$ and let X be an abelian variety over K_p. Then for every $n \geq 0$ there is a period isomorphism from p-adic Hodge theory*

$$h_{p,\mathrm{dR}} : \mathrm{H}^n_{\acute{e}t}(X \times_{K_p} \operatorname{Spec} \mathbb{Q}_p^{\mathrm{alg}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{p,\mathrm{dR}} \xrightarrow{\;\sim\;} \mathrm{H}^n_{\mathrm{dR}}(X, K_p) \otimes_{K_p} \mathbb{B}_{p,\mathrm{dR}},$$

which is $\mathscr{G}_{K_p}$-equivariant and compatible with the filtrations, where on the left hand side, $\mathscr{G}_{K_p}$ acts on both factors and the filtration is induced only by $\mathbb{B}_{p,\mathrm{dR}}$, and on the right hand side $\mathscr{G}_{K_p}$ acts only on $\mathbb{B}_{p,\mathrm{dR}}$ and the filtration is induced by the Hodge filtration on $\mathrm{H}^1_{\mathrm{dR}}(X, K_p)$ and the filtration on $\mathbb{B}_{p,\mathrm{dR}}$, i.e. $\mathrm{Fil}^k\big(\mathrm{H}^1_{\mathrm{dR}}(X, K_p) \otimes_{K_p} \mathbb{B}_{p,\mathrm{dR}}\big) := \sum\limits_{i+j=k} \mathrm{Fil}^i \mathrm{H}^1_{\mathrm{dR}}(X, K_p) \otimes_{K_p} \mathrm{Fil}^j \mathbb{B}_{p,\mathrm{dR}}$.

It was conjectured by Fontaine [Fon82, A.6] and proved by Faltings [Fal89, Theorem 8.1], Niziol [Niz98] and Tsuji [Tsu99], that the theorem also holds for arbitrary smooth proper schemes over K_p.

Example 5.6. Also for the multiplicative group scheme $\mathbb{G}_m := \mathbb{G}_{m,\mathbb{Q}_p} = \operatorname{Spec} \mathbb{Q}_p[x, x^{-1}]$ there is a period isomorphism between $\mathrm{H}^1_{\acute{e}t}(\mathbb{G}_{m,\mathbb{Q}_p^{\mathrm{alg}}}, \mathbb{Z}_p)$ and $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathbb{Q}_p) = \mathbb{Q}_p \frac{dx}{x}$, see Example 4.5. As in Example 4.3 let $\varepsilon_p^{(n)} \in \mathbb{Q}^{\mathrm{alg}} \subset \mathbb{Q}_p^{\mathrm{alg}}$ be a primitive p^n-th root of unity with $(\varepsilon_p^{(n+1)})^p = \varepsilon_p^{(n)}$, such that $\varepsilon_p = (\varepsilon_p^{(n)})_n \in \mathcal{O}^\flat_{\mathbb{C}_p}$. Then $\mathrm{H}_{1,\acute{e}t}(\mathbb{G}_{m,\mathbb{Q}_p^{\mathrm{alg}}}, \mathbb{Z}_p) = T_p\mathbb{G}_m = \varepsilon_p^{\mathbb{Z}_p}$ and $\mathrm{H}^1_{\acute{e}t}(\mathbb{G}_{m,\mathbb{Q}_p^{\mathrm{alg}}}, \mathbb{Z}_p) = (T_p\mathbb{G}_m)^\vee = (\varepsilon_p^{-1})^{\mathbb{Z}_p}$. On the latter $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{alg}}/\mathbb{Q}_p)$ acts through the inverse of the cyclotomic character. The series $t_p := \log[\varepsilon_p] := -\sum_{n>0} \frac{1}{n}(1 - [\varepsilon_p])^n$ converges in $\mathbb{B}_{p,\mathrm{dR}}$. Under the period isomorphism

$$h_{p,\mathrm{dR}} : \mathrm{H}^1_{\acute{e}t}(\mathbb{G}_{m,\mathbb{Q}_p^{\mathrm{alg}}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{p,\mathrm{dR}} \xrightarrow{\;\sim\;} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{p,\mathrm{dR}},$$

of $\mathbb{G}_m$ the element $\frac{dx}{x} \otimes 1$ is mapped to $\varepsilon_p^{-1} \otimes t_p$. Therefore t_p can be viewed as the p-adic analog of the complex period $2\pi i$ from Example 4.5. It satisfies $\hat{v}_p(t_p) = 1$ and $v_p(t_p) = v_p\big(\Theta(t_p \cdot ([p^\flat] - p)^{-1})\big) = \frac{1}{p-1}$, see [Col93, § 0.2].

6 Complex Multiplication

We follow [Mil06]. Complex conjugation on $\mathbb{C}$ (or a subfield) is denoted by c or simply by $a \to \bar{a}$. A complex conjugation on a field K is an involution induced by an embedding of K into $\mathbb{C}$ and by complex conjugation on $\mathbb{C}$.

A number field E is a *CM-field* if it is a quadratic extension E/F where the base field F is totally real but E is totally imaginary. i.e., every embedding of $F \hookrightarrow \mathbb{C}$ lies entirely within $\mathbb{R}$, but there is no embedding of $E \hookrightarrow \mathbb{R}$ or equivalently there exists an automorphism $c_E \neq \mathrm{id}$ of E such that $\rho \circ c_E = c \circ \rho$ for all homomorphisms $\rho : E \hookrightarrow \mathbb{C}$. In other words, there is a subfield F of E such that $E = F[\sqrt{\alpha}]$, F totally real, $\alpha \in F$ and $\rho(\alpha) < 0$ for all homomorphisms $\rho : F \hookrightarrow \mathbb{C}$.

Remark 6.1. A finite composite of CM-subfields of a field is CM. In particular, the Galois closure of a CM-field in any larger field is CM.

A *CM-algebra* is a finite product of CM-fields. Equivalently, it is a finite product of number fields admitting an automorphism c_E that is of order 2 on each factor and such that $\rho \circ c_E = c \circ \rho$ for all homomorphisms $\rho : E \to \mathbb{C}$. The fixed algebra of c_E is a product of the largest totally real subfields of the factors.

Let E be a CM-algebra. The set $\mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C})$ of $\mathbb{Q}$-homomorphisms $E \to \mathbb{C}$ is a union in complex conjugate pairs $\{\varphi, c \circ \varphi\}$. A *CM-type* on E is the choice of one element from each such pair. More formally:

Definition 6.2. A *CM-type* on a CM-algebra E is a subset $\Phi \subset \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C})$ such that

$$\mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C}) = \Phi \sqcup c\Phi \quad \text{(disjoint union)}.$$

Here $c\Phi := \{c \circ \varphi \mid \varphi \in \Phi\}$).

Let X be an abelian variety over the complex numbers $\mathbb{C}$. We have seen that $\mathrm{QEnd}_{\mathbb{C}}(X)$ is a semi-simple $\mathbb{Q}$-algebra which acts faithfully on the $(2 \dim X)$-dimensional $\mathbb{Q}$-vector space $\mathrm{H}_1(X, \mathbb{Q})$. Therefore, by Proposition 3.5

$$2 \dim X \geq [\mathrm{QEnd}_{\mathbb{C}}(X) : \mathbb{Q}]_{red}$$

and when equality holds, $\mathrm{QEnd}_{\mathbb{C}}(X)$ is a product of matrix algebras over fields.

Definition 6.3. An abelian variety X over a subfield $K \subset \mathbb{C}$ is said to have *complex multiplication* (or be of *CM-type*, or be a *CM abelian variety*) over K if

$$2 \dim X = [\mathrm{QEnd}_K(X) : \mathbb{Q}]_{red}.$$

By Proposition 3.6 this definition is equivalent to the statement that $\mathrm{QEnd}_K(X)$ contains an étale $\mathbb{Q}$-subalgebra of degree $2 \dim X$ over $\mathbb{Q}$. Indeed, if the latter holds then $2 \dim X$ is less or equal to the degree of a maximal étale $\mathbb{Q}$-subalgebra. By Proposition 3.6 the latter degree equals $[\mathrm{QEnd}_K(X) : \mathbb{Q}]_{red}$. And the inequality $[\mathrm{QEnd}_K(X) : \mathbb{Q}]_{red} \leq 2 \dim X$ proves the claim.

Note that when X is a CM abelian variety over a field $K \subset \mathbb{C}$ then $\mathrm{QEnd}_K(X) \subset \mathrm{QEnd}_{\mathbb{C}}(X)$ implies that this inclusion is an equality.

Remark 6.4. Let $X \approx_K \prod_i X_i^{n_i}$ be the decomposition of X (up to isogeny) into a product of isotypic abelian varieties over K. Then $D_i = \mathrm{QEnd}_K(X_i)$ is a division algebra, and $\mathrm{QEnd}_K(X) \cong \prod M_{n_i}(D_i)$ is the decomposition of $\mathrm{QEnd}_K(X)$ into a product of simple $\mathbb{Q}$-algebras. From the above definition and Proposition 3.5 we see that X has complex multiplication if and only if D_i is a commutative field of degree $2 \dim X_i$ for all i. In particular, a simple abelian variety X has complex multiplication if and only if $\mathrm{QEnd}_K(X)$ is a field of degree $2 \dim X$ over $\mathbb{Q}$, and an arbitrary abelian variety has complex multiplication if and only if each simple isogeny factor does.

Let X be an abelian variety over $\mathbb{C}$. An endomorphism α of X defines an endomorphism of the vector space $\mathrm{H}_1(X, \mathbb{Q})$ of dimension $2 \dim X$ over $\mathbb{Q}$. Therefore, the characteristic polynomial P_α of α is defined as

$$P_\alpha(T) := \det\big(\alpha - T |\, \mathrm{H}_1(X, \mathbb{Q})\big).$$

It is monic, of degree $2 \dim X$, and has coefficients in $\mathbb{Z}$. More generally, we define the characteristic polynomial of any element of $\mathrm{QEnd}(X)$ by the same formula.

Consider an endomorphism α of an abelian variety X over $\mathbb{C}$, and write $X = \mathbb{C}^g/\Lambda$ with $\Lambda = \mathrm{H}_1(X,\mathbb{Z})$. If α is an isogeny, then $\alpha : \Lambda \to \Lambda$ is injective, and it defines an isomorphism

$$\ker(\alpha) = \alpha^{-1}\Lambda/\Lambda \xrightarrow{\sim} \Lambda/\alpha\Lambda.$$

Therefore, for an isogeny $\alpha : X \to X$

$$\deg \alpha = \#\ker(\alpha) = \left|\det\left(\alpha|\ \mathrm{H}_1(X,\mathbb{Q})\right)\right| = |P_\alpha(0)|.$$

More generally, for any integer r we have $\deg(\alpha - r) = \left|\det\left(\alpha - r|\ \mathrm{H}_1(X,\mathbb{Q})\right)\right| = |P_\alpha(r)|$; compare [CS86, Chap 5 § 12].

For the convenience of the reader we reproduce the proof from [Mil06] of the following results.

Lemma 6.5 ([Mil06, Lemma 3.7]). *Let F be a subfield of $\mathrm{QEnd}(X)$, where X is an abelian variety over $\mathbb{C}$. If F has a real prime, then $[F:\mathbb{Q}]$ divides $\dim X$.*

Proof. First note that $\mathrm{H}_1(X,\mathbb{Q})$ is a vector space over F of dimension $m := 2\dim X/[F:\mathbb{Q}]$. So for any $\alpha \in \mathrm{End}(X) \cap F$, the characteristic polynomial $P_\alpha(T)$ is the m-th-power of the characteristic polynomial of α in $F/\mathbb{Q}$. In particular,

$$\mathrm{Norm}_{F/\mathbb{Q}}(\alpha)^m = \deg\alpha \geq 0.$$

However, if F has a real prime, then from the weak approximation theorem α can be chosen to be large and negative at that prime and close to 1 at the remaining primes so that $\mathrm{Norm}_{F/\mathbb{Q}}(\alpha) < 0$. This gives a contradiction unless m is even. □

For the next proposition recall the definition of a Rosati involution on $\mathrm{QEnd}_K(X)$. By [Mum70, Chapter III, § 13, Corollary 5] there exist *polarizations* on X, that is, isogenies $\lambda\colon X \to X^\vee = \mathrm{Pic}^0(X)$ which over K^{alg} are of the form $\lambda(x) = x^*\mathcal{L} \otimes \mathcal{L}^{-1}$ for an ample line bundle $\mathcal{L}$ on $X_{K^{\mathrm{alg}}}$. Every polarization λ has an inverse $\lambda^{-1} \in \mathrm{QHom}_K(X^\vee, X)$. The *Rosati involution* on $\mathrm{QEnd}_K(X)$ corresponding to λ is

$$\alpha \mapsto \alpha^\dagger = \lambda^{-1} \circ \alpha^\vee \circ \lambda \tag{6.1}$$

Proposition 6.6 ([Mil06, Proposition 3.6]). *(a) A simple abelian variety X has complex multiplication if and only if $\mathrm{QEnd}(X)$ is a CM-field of degree $2\dim X$ over $\mathbb{Q}$.*

(b) An isotypic abelian variety X has complex multiplication if and only if $\mathrm{QEnd}(X)$ contains a field of degree $2\dim X$ over $\mathbb{Q}$ (which can be chosen to be a CM-field invariant under some Rosati involution).

(c) An abelian variety X has complex multiplication if and only if $\mathrm{QEnd}(X)$ contains an étale $\mathbb{Q}$-algebra E (which can be chosen to be a CM-algebra invariant under some Rosati involution) of degree $2\dim X$ over $\mathbb{Q}$. In this case $\mathrm{H}_1(X,\mathbb{Q})$ is free over E of rank 1.

Proof. (a) $\mathrm{QEnd}_K(X)$ is a field extension of $\mathbb{Q}$ of degree $2\dim X$ by Remark 6.4. We know that it is either totally real or CM because it is stable under the Rosati involutions (6.1). Now Lemma 6.5 shows that $\mathrm{QEnd}_K(X)$ is a CM-field.

For (b) and (c) see [Mil06, Proposition 3.6]. □

Definition 6.7. Let X be an abelian variety with complex multiplication, so that $\mathrm{QEnd}(X)$ contains a CM-algebra E for which $\mathrm{H}_1(X,\mathbb{Q})$ is a free E-module of rank 1, and let Φ be the set of homomorphisms $E \to \mathbb{C}$ occurring in the representation of E on $\mathrm{T}_0(X)$, i.e., $\mathrm{T}_0(X) \cong \bigoplus_{\varphi\in\Phi}\mathbb{C}_\varphi$ where $\mathbb{C}_\varphi$ is the one-dimensional $\mathbb{C}$-vector space on which $a \in E$ acts as $\varphi(a)$. Then, because

$$\mathrm{H}_1(X,\mathbb{R}) \cong \mathrm{T}_0(X) \oplus \overline{\mathrm{T}_0(X)}, \tag{6.2}$$

Φ is a CM-type on E, and we say that, X together with the injective homomorphism $E \to \mathrm{QEnd}(X)$ is *of CM-type* (E,Φ).

Let e be a basis vector for $\mathrm{H}_1(X,\mathbb{Q})$ as an E-module, and let $\mathfrak{a}$ be the $\mathcal{O}_E$-lattice in E such that $\mathfrak{a}e = \mathrm{H}_1(X,\mathbb{Z})$. Under the above isomorphism

$$\begin{aligned} \mathrm{H}_1(X,\mathbb{R}) &\xrightarrow{\sim} \bigoplus_{\varphi\in\Phi}\mathbb{C}_\varphi \oplus \bigoplus_{\varphi\in c\Phi}\mathbb{C}_\varphi, \\ e\otimes 1 &\longmapsto (\cdots, e_\varphi, \cdots; \cdots, e_{c\circ\varphi}, \cdots) \end{aligned} \tag{6.3}$$

where each e_φ is a $\mathbb{C}$-basis for $\mathbb{C}_\varphi$. The e_φ determine an isomorphism

$$\mathrm{T}_0(X) \cong \bigoplus_{\varphi\in\Phi} \mathbb{C}_\varphi .$$

Next we state two important results on abelian varieties with complex multiplication from [ST61] and [ST68] which we will need later.

Proposition 6.8. *[ST61, Prop 26 §12.4] Let X be an abelian variety over $K = K^{\mathrm{sep}} \subset \mathbb{C}$ with complex multiplication, then there exists an abelian variety isogenous to X defined over a field which is a finite extension of $\mathbb{Q}$.*

Theorem 6.9. *[ST68, Thm 6] Let X be an abelian variety over a finite extension $K/\mathbb{Q}$ with complex multiplication, then there exists a finite extension L/K such that X has good reduction at every place of $\mathcal{O}_L$.*

7 The Faltings Height of an Abelian Variety

We recall the definition of the *Faltings height* of an abelian variety. It was introduced by Faltings in his proof of the Mordell Conjecture and the Tate Conjecture 4.1 for abelian varieties; see [Fal83] or [CS86, Chapter 2, § 3] for the English translation. Let K be a number field, $\mathcal{O}_K$ the ring of integers in K. We define a metrized line bundle on $\mathrm{Spec}(\mathcal{O}_K)$ to be a projective $\mathcal{O}_K$-module P of rank 1, together with norms $\|\,.\,\|_v$ on $P \otimes_{\mathcal{O}_K} K_v$ for all infinite places v of K, where K_v denotes the completion of K at v. We define $\varepsilon_v = 1$ or 2 according to whether $K_v \cong \mathbb{R}$ or $K_v \cong \mathbb{C}$. The *degree* of the metrized line bundle is defined as

$$\deg(P, \|\,.\,\|) = \log(\#(P/\mathcal{O}_K \cdot x)) - \sum_{v|\infty} \varepsilon_v \log \|x\|_v ,$$

where x is a nonzero element of P and the sum runs over all infinite places of K. The right-hand side is of course independent of x because of the product formula (1.1).

Let now X be an abelian variety of dimension g over K, and let $\mathcal{X}$ be the relative identity component of the Néron model of X over $\mathcal{O}_K$. Assume that $\mathcal{X}$ is semi-abelian, i.e. a smooth algebraic group $q : \mathcal{X} \to \mathrm{Spec}\,\mathcal{O}_K$, whose fibers are connected of dimension g, and are extensions of an abelian variety by a torus. Let $s : \mathcal{X} \to \mathrm{Spec}\,\mathcal{O}_K$ be the zero section. Let $\omega_{X/\mathcal{O}_K} = s^*(\Omega^g_{\mathcal{X}/\mathcal{O}_K})$, $\omega_{X/\mathcal{O}_K}$ is a line bundle on $\mathcal{O}_K$. The metrics at the infinite places v of K are given by

$$\|\alpha\|_v^2 := \frac{1}{(2\pi)^g} \int_{X_v(\mathbb{C})} |\alpha \wedge \bar{\alpha}| \qquad \text{for} \qquad \alpha \in \omega(X_v) = \Gamma(X_v, \Omega^g_{X_v}) \,,$$

where X_v denotes the base change of X under the map $K \to K_v$. Then Faltings [CS86, Chapter 2, § 3] defines a moduli-theoretic height as follows.

Definition 7.1. The (*stable*) *Faltings height* $ht^{\mathrm{st}}_{\mathrm{Fal}}(X)$ of X is defined as

$$ht^{\mathrm{st}}_{\mathrm{Fal}}(X) \;:=\; \frac{1}{[K:\mathbb{Q}]} \deg(\omega_{X/\mathcal{O}_K}, \|\,.\,\|). \tag{7.1}$$

It is easy to check that $ht^{\mathrm{st}}_{\mathrm{Fal}}(X)$ is invariant under extension of the ground field. Since every abelian variety is potentially semi-stable by Grothendieck [SGA 7, Exposé IX, Théorème 3.6], the Faltings height is defined for every abelian variety over a number field. It measures the arithmetic complexity of the abelian variety and is "not far" from an actual height on the moduli space of principally polarized abelian varieties.

8 Colmez's Conjecture on Periods of CM Abelian Varieties

In [Col93] P. Colmez considers product formulas for periods of abelian varieties in the following

Situation 8.1. Let X be an abelian variety defined over a number field K with complex multiplication by the ring of integers $\mathcal{O}_E$ in a CM-field E and of CM-type (E, Φ). Let $H_E := \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{Q}^{\mathrm{alg}})$ be the set of all ring homomorphisms $E \hookrightarrow \mathbb{Q}^{\mathrm{alg}}$ and assume that K contains $\psi(E)$ for every $\psi \in H_E$. By Theorem 6.9 we may assume moreover, that K is a finite Galois extension of $\mathbb{Q}$ and that X has good reduction at every prime of $\mathcal{O}_K$. For a fixed $\psi \in H_E$ let $\omega_\psi \in \mathrm{H}^1_{\mathrm{dR}}(X, K)$ be a non-zero cohomology class such that $b^*\omega_\psi = \psi(b) \cdot \omega_\psi$ for all $b \in E$.

For every embedding $\eta\colon K\hookrightarrow\mathbb{Q}^{\rm alg}$, let $X^\eta:=X\times_{\operatorname{Spec}K,\operatorname{Spec}\eta}\operatorname{Spec}K$ and $\omega_\psi^\eta\in \mathrm{H}^1_{\rm dR}(X^\eta,K)$ be deduced from X and ω_ψ by base extension. Let $(u_\eta)_\eta\in\prod_{\eta\in H_K}\mathrm{H}_1(X^\eta(\mathbb{C}),\mathbb{Z})$ be a family of cycles compatible with complex conjugation c, that is $u_{c\eta}=c(u_\eta)$. Let v be a place of $\mathbb{Q}$.

If $v=\infty$ the de Rham isomorphism (Theorem 4.4) between Betti and de Rham cohomology yields a pairing

$$\langle\,.\,,\,.\,\rangle_\infty\colon\ \mathrm{H}_1(X^\eta(\mathbb{C}),\mathbb{Z})\times\mathrm{H}^1_{\rm dR}(X^\eta,K)\longrightarrow\mathbb{C},\quad (u_\eta,\omega_\psi^\eta)\longmapsto\langle u_\eta,\omega_\psi^\eta\rangle_\infty\,.$$

We define the complex absolute value $\big|\int_{u_\eta}\omega_\psi^\eta\big|_\infty:=|\langle u_\eta,\omega_\psi^\eta\rangle_\infty|_\infty\in\mathbb{R}$.

If v corresponds to a prime number $p\in\mathbb{Z}$, the comparison isomorphism $\mathrm{H}^1(X^\eta(\mathbb{C}),\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}_p\xrightarrow{\sim}\mathrm{H}^1_{\rm \acute et}(X^\eta_{\mathbb{Q}_p^{\rm alg}},\mathbb{Z}_p)$ together with the comparison isomorphism from p-adic Hodge theory (Theorems 4.2 and 5.5) yield a pairing

$$\langle\,.\,,\,.\,\rangle_p\colon\ \mathrm{H}_1(X^\eta(\mathbb{C}),\mathbb{Z})\times\mathrm{H}^1_{\rm dR}(X^\eta,K)\longrightarrow\mathbb{B}_{p,\rm dR}\,,\quad (u_\eta,\omega_\psi^\eta)\longmapsto\langle u_\eta,\omega_\psi^\eta\rangle_p\,.$$

We define the absolute value $\big|\int_{u_\eta}\omega_\psi^\eta\big|_p:=|\langle u_\eta,\omega_\psi^\eta\rangle_p|_p:=p^{-v_p(\langle u_\eta,\omega_\psi^\eta\rangle_p)}\in\mathbb{R}$, where the "valuation" v_p on $\mathbb{B}_{p,\rm dR}$ was defined in (5.1) in Notation 5.4.

Colmez [Col93] now considers the product $\prod\limits_v\prod\limits_{\eta\in H_K}\big|\int_{u_\eta}\omega_\psi^\eta\big|_v$, or equivalently $\frac{1}{\#H_K}$ times its logarithm

$$\tfrac{1}{\#H_K}\sum_v\sum_{\eta\in H_K}\log\big|\textstyle\int_{u_\eta}\omega_\psi^\eta\big|_v\;=\;\tfrac{1}{\#H_K}\displaystyle\sum_{\eta\in H_K}\log\big|\langle u_\eta,\omega_\psi^\eta\rangle_\infty\big|_\infty-\tfrac{1}{\#H_K}\sum_{v=v_p\neq\infty}\ \sum_{\eta\in H_K}v_p\big(\langle u_\eta,\omega_\psi^\eta\rangle_p\big)\log p\,. \tag{8.1}$$

The right sum over all $v=v_p$ does not converge. Namely, Colmez [Col93, Theorem II.1.1] proves the following Theorem 8.3 below. To formulate the theorem we need a

Definition 8.2. In this definition we denote by Q the function field from the introduction or the field $\mathbb{Q}$, and by Q_v the completion of Q at a place $v\neq\infty$. The case $Q=\mathbb{Q}$ is relevant in the present section, and the other case will be relevant in Section 17. For $F=Q$ or $F=Q_v$ let $F^{\rm sep}$ be the separable closure of F in $F^{\rm alg}$ and let $\mathscr{G}_F:=\operatorname{Gal}(F^{\rm sep}/F)$. Let $\mathcal{C}(\mathscr{G}_F,\mathbb{Q})$ be the $\mathbb{Q}$-vector space of locally constant functions $a\colon\mathscr{G}_F\to\mathbb{Q}$ and let $\mathcal{C}^0(\mathscr{G}_F,\mathbb{Q})$ be the subspace of those functions which are constant on conjugacy classes, that is, which satisfy $a(h^{-1}gh)=a(g)$ for all $g,h\in\mathscr{G}_F$. Then the $\mathbb{C}$-vector space $\mathcal{C}^0(\mathscr{G}_F,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$ is spanned by the traces of representations $\rho\colon\mathscr{G}_F\to\mathrm{GL}_n(\mathbb{C})$ with open kernel for varying n by [Ser77, § 2.5, Theorem 6]. Via the fixed embedding $Q^{\rm sep}\hookrightarrow Q_v^{\rm sep}$ we consider the induced inclusion $\mathscr{G}_{Q_v}\subset\mathscr{G}_Q$ and morphism $\mathcal{C}(\mathscr{G}_Q,\mathbb{Q})\to\mathcal{C}(\mathscr{G}_{Q_v},\mathbb{Q})$. If χ is the trace of a representation $\rho\colon\mathscr{G}_Q\to\mathrm{GL}_n(\mathbb{C})$ with open kernel we let $L(\chi,s):=\prod_{\text{all }v}L_v(\chi,s)$, respectively $L^\infty(\chi,s):=\prod_{v\neq\infty}L_v(\chi,s)$ be the Artin L-function of ρ with, respectively without the factor at ∞. Note that the latter factor involves the Gamma-function if $Q=\mathbb{Q}$. These L-functions only depend on χ and converge for all $s\in\mathbb{C}$ with $\mathcal{R}e(s)>1$; see [Lan94, Chapter XII, § 2] for $Q=\mathbb{Q}$ and [Ros02, pp. 126ff] for the function field case. We also let q_v be the cardinality of the residue field of Q_v (this means $q_v=p$ if $Q=\mathbb{Q}$ and $Q_v=\mathbb{Q}_p$) and we set

$$Z^\infty(\chi,s)\;:=\;\frac{\frac{d}{ds}L^\infty(\chi,s)}{L^\infty(\chi,s)}\;=\;-\sum_{v\neq\infty}Z_v(\chi,s)\log q_v\qquad\text{with} \tag{8.2}$$

$$Z_v(\chi,s)\;:=\;\frac{\frac{d}{ds}L_v(\chi,s)}{-L_v(\chi,s)\cdot\log q_v}\;=\;\frac{\frac{d}{dq_v^{-s}}L_v(\chi,s)}{q_v^s\cdot L_v(\chi,s)}\,. \tag{8.3}$$

Moreover, we let $\mathfrak{f}_\chi$ be the Artin conductor of χ. If $Q=\mathbb{Q}$, it is a positive integer $\mathfrak{f}_\chi=\prod_p p^{\mu_{{\rm Art},p}(\chi)}\in\mathbb{Z}$, and if Q is the function field of the curve C it is an effective divisor $\mathfrak{f}_\chi=\sum_v\mu_{{\rm Art},v}(\chi)\cdot[v]$ on C; see [Ser79, Chapter VI, §§ 2,3], where $\mu_{{\rm Art},v}(\chi)$ is denoted $f(\chi,v)$. In particular, only finitely many values $\mu_{{\rm Art},v}(\chi)$ are non-zero. We set

$$\mu_{\rm Art}^\infty(\chi)\;:=\;\log(\mathfrak{f}_\chi)\;=\;\sum_{v\neq\infty}\mu_{{\rm Art},v}(\chi)\log q_v\qquad\text{if } Q=\mathbb{Q},\ \text{respectively} \tag{8.4}$$

$$\mu_{\rm Art}(\chi)\;:=\;\deg(\mathfrak{f}_\chi)\log q\;:=\;\sum_{\text{all }v}\mu_{{\rm Art},v}(\chi)[\mathbb{F}_v:\mathbb{F}_q]\log q\;=\;\sum_{\text{all }v}\mu_{{\rm Art},v}(\chi)\log q_v\qquad\text{and}$$

$$\mu_{\rm Art}^\infty(\chi)\;:=\;\sum_{v\neq\infty}\mu_{{\rm Art},v}(\chi)\log q_v\qquad\text{if } Q \text{ is a function field}\,. \tag{8.5}$$

By linearity we extend $Z^\infty(\,.\,,s)$ and $\mu_{\rm Art}^\infty$ to all $a\in\mathcal{C}^0(\mathscr{G}_Q,\mathbb{Q})$ and $Z_v(\,.\,,s)$ and $\mu_{{\rm Art},v}$ to all $a\in\mathcal{C}^0(\mathscr{G}_{Q_v},\mathbb{Q})$. The map $Z_v(\,.\,,s)$ takes values in $\mathbb{Q}(q_v^{-s})$.

For our CM-type (E,Φ) and for every $\psi\in H_E$ we define the functions

$$a_{E,\psi,\Phi}\colon\ \mathscr{G}_{\mathbb{Q}}\to\mathbb{Z},\quad g\mapsto\begin{cases}1 & \text{when } g\psi\in\Phi\\ 0 & \text{when } g\psi\notin\Phi\end{cases}\qquad\text{and}$$

$$a^0_{E,\psi,\Phi}\colon\ \mathscr{G}_{\mathbb{Q}}\to\mathbb{Q},\quad g\mapsto\tfrac{1}{\#H_K}\sum_{\eta\in H_K}a_{E,\eta\psi,\eta\Phi}(g)\ =\ \frac{\#\{\eta\in H_K\colon \eta^{-1}g\eta\psi\in\Phi\}}{\#H_K}\tag{8.6}$$

which factor through $\mathrm{Gal}(K/\mathbb{Q})$ by our assumption that $\psi(E)\subset K$ for all $\psi\in H_E$. In particular, $a_{E,\psi,\Phi}\in\mathcal{C}(\mathscr{G}_{\mathbb{Q}},\mathbb{Q})$ and $a^0_{E,\psi,\Phi}\in\mathcal{C}^0(\mathscr{G}_{\mathbb{Q}},\mathbb{Q})$ is independent of K.

We also define integers $v_p(\omega^\eta_\psi)$ which are all zero except for finitely many. Let K_p be the p-adic completion of $K\subset\mathbb{Q}^{\rm alg}\subset\mathbb{Q}_p^{\rm alg}\subset\mathbb{C}_p$ and let $\mathcal{X}^\eta$ be an abelian scheme over $\mathcal{O}_{K_p}$ with $\mathcal{X}^\eta\times_{\mathcal{O}_{K_p}}\operatorname{Spec}K_p\cong X^\eta\times_K\operatorname{Spec}K_p$. Then there is an element $x\in K_p^\times$, unique up to multiplication by $\mathcal{O}_{K_p}^\times$, such that $x^{-1}\omega^\eta_\psi$ is an $\mathcal{O}_{K_p}$-generator of the free $\mathcal{O}_{K_p}$-module of rank one

$$\mathrm{H}^{\eta\psi}(\mathcal{X}^\eta,\mathcal{O}_{K_p})\ :=\ \big\{\omega\in\mathrm{H}^1_{\rm dR}(\mathcal{X}^\eta,\mathcal{O}_{K_p})\colon b^*\omega=\eta\psi(b)\cdot\omega\ \ \forall\, b\in\mathcal{O}_E\big\}\,,$$

and we set

$$v_p(\omega^\eta_\psi)\ :=\ v_p(x)\ \in\ \mathbb{Z}\,.\tag{8.7}$$

This value does not depend on the choice of the model $\mathcal{X}^\eta$ with good reduction, because all such models are isomorphic over $\mathcal{O}_{K_p}$. Now Colmez [Col93, Theorem II.1.1] computed the terms in (8.1) as follows.

Theorem 8.3. *If the image of u_η in $\mathrm{H}_1(X^\eta(\mathbb{C}),\mathbb{Q}_p)=\mathrm{H}_{1,\acute{e}t}(X^\eta_{\mathbb{Q}_p^{\rm alg}},\mathbb{Z}_p)$ is a generator of the $\mathcal{O}_E\otimes_{\mathbb{Z}}\mathbb{Z}_p$-module $\mathrm{H}_{1,\acute{e}t}(X^\eta_{\mathbb{Q}_p^{\rm alg}},\mathbb{Z}_p)=T_pX^\eta$, then*

$$\tfrac{1}{\#H_K}\sum_{\eta\in H_K}v_p(\langle\omega^\eta_\psi,u_\eta\rangle_v)\ =\ Z_p(a^0_{E,\psi,\Phi},1)-\mu_{\mathrm{Art},p}(a^0_{E,\psi,\Phi})+\tfrac{1}{\#H_K}\sum_{\eta\in H_K}v_p(\omega^\eta_\psi)\,.\tag{8.8}$$

Since $-\mu_{\mathrm{Art},p}(a^0_{E,\psi,\Phi})+\frac{1}{\#H_K}\sum\limits_{\eta\in H_K}v_p(\omega^\eta_\psi)$ vanishes for all but finitely many primes p and $\sum\limits_p Z_p(a^0_{E,\psi,\Phi},1)$ diverges, the sum (8.1) diverges. Colmez [Col93, Convention 0] assigns to this divergent sum a value by the following

Convention 8.4. Let $(x_p)_{p\neq\infty}$ be a tuple of complex numbers indexed by the prime numbers p in $\mathbb{Z}$. We will give a sense to the (divergent) series $\Sigma\overset{?}{=}\sum_{p<\infty}x_p$ in the following situation. We suppose that there exists an element $a\in\mathcal{C}^0(\mathscr{G}_{\mathbb{Q}},\mathbb{Q})$ such that $x_p=-Z_p(a,1)\log p$ for all p except for finitely many. Then we let $a^*\in\mathcal{C}^0(\mathscr{G}_{\mathbb{Q}},\mathbb{Q})$ be defined by $a^*(g):=a(g^{-1})$. We further assume that $Z^\infty(a^*,s)$ does not have a pole at $s=0$, and we define the limit of the series $\sum_{p<\infty}x_p$ as

$$\Sigma\ :=\ -Z^\infty(a^*,0)-\mu^\infty_{\rm Art}(a)+\sum_{p<\infty}\big(x_p+Z_p(a,1)\log p\big)\tag{8.9}$$

inspired by the functional equation relating $L(a,s)$ with $L(a^*,1-s)$ deprived of the terms at ∞.

Example 8.5. The convention allows to prove the product formula for the multiplicative group $\mathbb{G}_m:=\mathbb{G}_{m,\mathbb{Q}}=\operatorname{Spec}\mathbb{Q}[x,x^{-1}]$. Namely, for the generator $\omega=\frac{dx}{x}$ of $\mathrm{H}^1_{\rm dR}(\mathbb{G}_m,\mathbb{Q})=\mathbb{Q}\cdot\omega$ and for the cycle $u\colon[0,1]\to\mathbb{G}_m(\mathbb{C})=\mathbb{C}^\times$ given by $u(s)=\exp(2\pi is)$ with $\mathrm{H}_1(\mathbb{G}_m(\mathbb{C}),\mathbb{Z})=\mathbb{Z}\cdot u$, we have computed in Examples 4.3, 4.5 and 5.6

$$\begin{aligned}\langle\omega,u\rangle_\infty\ &=\ 2\pi i &&\text{and}\qquad \log\big|\langle\omega,u\rangle_\infty\big|_\infty\ =\ \log(2\pi)\,,\\ \langle\omega,u\rangle_p\ &=\ t_p &&\text{and}\qquad \log\big|\langle\omega,u\rangle_p\big|_p\ =\ \log|t_p|_p\ =\ -\tfrac{\log p}{p-1}\ =\ -Z_p(\mathbb{1},1)\log p\,,\end{aligned}$$

where $\mathbb{1}(g)=1$ for every $g\in\mathscr{G}_{\mathbb{Q}}$. So Convention 8.4 implies $\sum_{p<\infty}\log|\langle\omega,u\rangle_p|_p=-\frac{\zeta'_{\mathbb{Z}}(0)}{\zeta_{\mathbb{Z}}(0)}=-\log(2\pi)$ for the Riemann Zeta-function $\zeta_{\mathbb{Z}}$ and $\sum_v\log|\langle\omega,u\rangle_v|_v=0$. Therefore $\prod\limits_v|\langle\omega,u\rangle_v|_v=1$.

The Convention 8.4 and the Theorem 8.3 allow us to give to the divergent sum (8.1) a convergent interpretation. In order to remove the dependency on the chosen cycles $(u_\eta)_\eta \in \prod_{\eta\in H_K} \mathrm{H}_1(X^\eta(\mathbb{C}),\mathbb{Z})$, Colmez considers the value

$$\langle \omega_\psi^\eta, \omega_{c\psi}^\eta, u_\eta\rangle_v \;:=\; \left(t_v \cdot \frac{\langle \omega_\psi^\eta, u_\eta\rangle_v}{\langle \omega_{c\psi}^\eta, u_\eta\rangle_v}\right)^{\frac{1}{2}}, \tag{8.10}$$

where $t_\infty = 2\pi i$ and for $v = v_p \neq \infty$, $t_v = t_p$ is the p-adic analog of $2\pi i$ from Examples 5.6 and 8.5. Note that $\Phi \sqcup c\Phi = H_E$ implies $a^0_{E,\psi,\Phi} + a^0_{E,c\psi,\Phi} = \mathbb{1}$, and hence $Z_p(a^0_{E,\psi,\Phi},1) + Z_p(a^0_{E,c\psi,\Phi},1) = Z_p(\mathbb{1},1)$ and $\mu_{\mathrm{Art},p}(a^0_{E,\psi,\Phi}) + \mu_{\mathrm{Art},p}(a^0_{E,c\psi,\Phi}) = \mu_{\mathrm{Art},p}(\mathbb{1}) = 0$. Therefore, Theorem 8.3 implies

$$\begin{aligned}
\tfrac{1}{\#H_K} \sum_{\eta\in H_K} v_p(\langle \omega_\psi^\eta, \omega_{c\psi}^\eta, u_\eta\rangle_v) \;&=\; \frac{1}{2}\Big(Z_p(\mathbb{1},1) + Z_p(a^0_{E,\psi,\Phi},1) - \mu_{\mathrm{Art},p}(a^0_{E,\psi,\Phi}) + \tfrac{1}{\#H_K}\sum_{\eta\in H_K} v_p(\omega_\psi^\eta) \\
&\qquad - Z_p(a^0_{E,c\psi,\Phi},1) + \mu_{\mathrm{Art},p}(a^0_{E,c\psi,\Phi}) - \tfrac{1}{\#H_K}\sum_{\eta\in H_K} v_p(\omega_{c\psi}^\eta)\Big) \\
&= Z_p(a^0_{E,\psi,\Phi},1) - \mu_{\mathrm{Art},p}(a^0_{E,\psi,\Phi}) + \frac{1}{2}\Big(\tfrac{1}{\#H_K}\sum_{\eta\in H_K} v_p(\omega_\psi^\eta) - v_p(\omega_{c\psi}^\eta)\Big)\,.
\end{aligned}$$

Using Convention 8.4 one thus obtains

$$\begin{aligned}
&\tfrac{1}{\#H_K}\sum_v \sum_{\eta\in H_K} \log\big|\langle \omega_\psi^\eta, \omega_{c\psi}^\eta, u_\eta\rangle_v\big|_v \\
&\quad=\; -Z^\infty((a^0_{E,\psi,\Phi})^*,0) + \tfrac{1}{\#H_K}\sum_{\eta\in H_K}\Big(\log\big|\langle \omega_\psi^\eta, \omega_{c\psi}^\eta, u_\eta\rangle_\infty\big|_\infty - \frac{1}{2}\sum_{p<\infty}\big(v_p(\omega_\psi^\eta) - v_p(\omega_{c\psi}^\eta)\big)\log p\Big)
\end{aligned} \tag{8.11}$$

which is independent of the chosen u_η. Colmez formulated the following

Conjecture 8.6 ([Col93, Conjecture 0.1]). *The sum* (8.11) *is zero, or equivalently the* product formula *holds:*

$$\prod_v \prod_{\eta\in H_K} \big|\langle \omega_\psi^\eta, \omega_{c\psi}^\eta, u_\eta\rangle_v\big|_v = 1\,.$$

He then proved

Lemma 8.7 ([Col93, Lemme II.2.9]). *In Situation 8.1 the value*

$$ht(E,\psi,\Phi) \;:=\; \tfrac{1}{\#H_K}\sum_{\eta\in H_K}\Big(\log\big|\langle \omega_\psi^\eta, \omega_{c\psi}^\eta, u_\eta\rangle_\infty\big|_\infty - \frac{1}{2}\sum_{p<\infty}\big(v_p(\omega_\psi^\eta) - v_p(\omega_{c\psi}^\eta)\big)\log p\Big) \tag{8.12}$$

only depends on E,ψ and Φ and not on the choice of X,ω_ψ,u_η and K.

Colmez also relates the product formula to the Faltings height, see Definition 7.1.

Theorem 8.8 ([Col93, Théorème II.2.10(ii)]). *In Situation 8.1 the Faltings height $ht^{\mathrm{st}}_{\mathrm{Fal}}(X)$ of X satisfies*

$$ht^{\mathrm{st}}_{\mathrm{Fal}}(X) \;=\; -\sum_{\psi\in\Phi}\Big(ht(E,\psi,\Phi) + \tfrac{1}{2}\mu^\infty_{\mathrm{Art}}(a^0_{E,\psi,\Phi})\Big) \tag{8.13}$$

This immediately implies the following

Corollary 8.9. *In Situation 8.1 the following assertions are equivalent.*

(a) $ht(E,\psi,\Phi) = Z^\infty((a^0_{E,\psi,\Phi})^*,0)$.

(b) The product formula holds, that is, the expression (8.11) *is zero and* $\prod_v \prod_{\eta\in H_K} \big|\langle \omega_\psi^\eta, \omega_{c\psi}^\eta, u_\eta\rangle_v\big|_v = 1$.

If (a) and (b) hold for all $\psi\in\Phi$ then $ht^{\mathrm{st}}_{\mathrm{Fal}}(X) \;=\; -\sum_{\psi\in\Phi}\big(Z^\infty((a^0_{E,\psi,\Phi})^*,0) + \tfrac{1}{2}\mu^\infty_{\mathrm{Art}}(a^0_{E,\psi,\Phi})\big)$. □

Colmez [Col93, Conjecture II.2.11] conjectures that statements (a) and (b) of Corollary 8.9 hold for all E,ψ,Φ. There are various partial results in this direction. The first is due to Colmez himself who was able to prove the following theorem up to a rational multiple of $\log 2$, which was then removed by Obus:

Theorem 8.10 ([Col93, Théorème 0.5], [Obu13, Theorem 4.9]). *If E is abelian over $\mathbb{Q}$, then the product formula holds true for every ψ, Φ, and hence*

$$ht^{\mathrm{st}}_{\mathrm{Fal}}(X) \;=\; -\sum_{\psi\in\Phi}\Big(Z^{\infty}((a^0_{E,\psi,\Phi})^*,0) + \tfrac{1}{2}\mu^{\infty}_{\mathrm{Art}}(a^0_{E,\psi,\Phi})\Big)\,. \tag{8.14}$$

There has been much further work and progress on Colmez's conjecture by many people. For example, Yang [Yan13] proved it for a large class of CM-fields E of degree $[E:\mathbb{Q}]=4$, including the first known cases when $E/\mathbb{Q}$ is non-abelian. Let us also mention the most recent results by Andreatta, Goren, Howard, Madapusi Pera [AGHMP18], Yuan, Shou-Wu Zhang [YZ18] and Barquero-Sanchez, Masri [BSM18].

Theorem 8.11 ([AGHMP18, Theorem A], [YZ18, Theorem 1.1]). *For every CM-field E Colmez's conjecture holds true on average over all CM-types Φ, that is*

$$\sum_{\Phi}\sum_{\psi\in\Phi} ht(E,\psi,\Phi) \;=\; \sum_{\Phi}\sum_{\psi\in\Phi} Z^{\infty}((a^0_{E,\psi,\Phi})^*,0)\,.$$

Remark 8.12. In [YZ18] the averaged Colmez conjecture (Theorem 8.11) follows from a generalized Chowla-Selberg formula [YZ18, Theorem 1.7]. Moreover, (generalized) Chowla-Selberg formulas are special cases of generalized Gross-Zagier formulas. In the case when $[E:\mathbb{Q}]=2$, the generalized Chowla-Selberg formula [YZ18, Theorem 1.7] is actually equivalent to the classical Lerch-Chowla-Selberg formula (1.3), and it is also equivalent to the Colmez conjecture for E, by using a result of Faltings [Fal84a, Theorem 7.b)]. See [Col93, § 0.6] and [GvKM19, § 4.3] for additional explanations.

As a consequence of Theorem 8.11, Barquero-Sanchez and Masri [BSM18, Theorem 1.1] proved that for any fixed totally real number field F of degree $[F:\mathbb{Q}]\geq 3$ there are infinitely many effective, "positive density" sets of CM extensions E/F such that $E/\mathbb{Q}$ is non-abelian and Colmez's conjecture (8.14) on the Faltings height holds true for E and any Φ. Moreover, they prove

Theorem 8.13 ([BSM18, Theorem 1.4]). *In Situation 8.1 if the Galois closure of E has degree $2^{\dim X}\cdot(\dim X)!$ over $\mathbb{Q}$, then*

$$ht^{\mathrm{st}}_{\mathrm{Fal}}(X) \;=\; -\sum_{\psi\in\Phi} Z^{\infty}((a^0_{E,\psi,\Phi})^*,0) - \tfrac{1}{2}\mu^{\infty}_{\mathrm{Art}}(a^0_{E,\psi,\Phi})\,.$$

As another consequence of Theorem 8.11 and of previous work by Edixhoven [EMO01, Problem 14], Pila, Wilkie, Yafaev, Zannier and many others [EY03, PT14, PW06, PZ08], Tsimerman [Tsi18] proved the *André-Oort-Conjecture* for the Siegel modular varieties:

Theorem 8.14 ([Tsi18, Theorem 1.3]). *Let $\mathcal{A}_g$ be the Siegel modular variety parameterizing principally polarized abelian varieties of dimension g over $\mathbb{C}$. Let $X\subset\mathcal{A}_g$ be an irreducible closed subvariety which contains a Zariski dense subset of special points of $\mathcal{A}_g$. Then X is a special subvariety.*

The averaged Colmez conjecture (Theorem 8.11) enters in this result by implying that the Galois orbit of a special point, that is a CM abelian variety, is large. This result and the André-Oort-Conjecture were previously obtained in several cases conditionally under assumption of the generalized Riemann Hypothesis.

Part II
Drinfeld Modules and A-motives

9 Basic Definitions

Following the general philosophy about similarities between number fields and function fields, we now transfer the contents of Part I to characteristic p. Here Drinfeld modules replace elliptic curves and A-motives replace abelian varieties. We follow the expositions in [Gos96, Ch.4], [Tha04, Ch.2] and begin with the analog of Notation 2.1

Notation 9.1. Let $\mathbb{F}_q$ be a finite field with q elements and characteristic p. Let C be a smooth projective, geometrically irreducible curve over $\mathbb{F}_q$ with function field $Q=\mathbb{F}_q(C)$. Let $\infty\in C$ be a fixed closed point and let

$A := \Gamma(C \smallsetminus \{\infty\}, \mathcal{O}_C)$ be the $\mathbb{F}_q$-algebra of those rational functions on C which are regular outside ∞. Let v_∞ be the valuation associated with the prime ∞.

By a *place* of C we mean a closed point $v \in C$. So either $v = \infty$ or v is a maximal ideal of A. It defines a normalized valuation on Q which we also denote by v, respectively by v_∞ and which takes the value $v(z_v) = 1$ on a uniformizing parameter $z_v \in Q$ at v. We now fix such a uniformizer z_v at every v and if $v = \infty$ we abbreviate z_∞ to z. We denote the residue field of v by $\mathbb{F}_v$, its degree over $\mathbb{F}_q$ by $d_v = [\mathbb{F}_v : \mathbb{F}_q]$ and its cardinality by $q_v := \#\mathbb{F}_v = q^{d_v}$. Thus if $a \in A \smallsetminus \mathbb{F}_q$ then $v_\infty(a) < 0$, because $\mathbb{F}_q$ is the field of constants in Q as C is geometrically irreducible, see [Gro65, IV_2, 4.3.1 and Proposition 4.5.9c)]. The ring A and its fraction field Q play the role of $\mathbb{Z}$ and $\mathbb{Q}$ in the arithmetic of function fields.

Let Q_v be the completion of Q with respect to the valuation v and let $A_v \subset Q_v$ be the valuation ring of v. Then there is a canonical isomorphism $A_v \cong \mathbb{F}_v[\![z_v]\!]$. Let Q_v^{alg} be a fixed algebraic closure of Q_v and let $\mathbb{C}_v$ be the completion of Q_v^{alg} with respect to the canonical extension of v. We also use v to denote this extension to Q_v^{alg} and thus to $\mathbb{C}_v$. However, we denote the image of z_v in $\mathbb{C}_v$ by ζ_v and abbreviate ζ_∞ to ζ. Note that $\mathbb{C}_v$ is algebraically closed. On $\mathbb{C}_v$ and all its subrings we consider the normalized absolute value $|\,.\,|_v \colon \mathbb{C}_v \to \mathbb{R}_{\geq 0}$ given by $|x|_v = q_v^{-v(x)}$. We let $\mathcal{O}_{\mathbb{C}_v} = \{x \in \mathbb{C}_v : |x|_v \leq 1\}$ be the valuation ring of $\mathbb{C}_v$. We also fix an algebraic closure Q^{alg} of Q and an embedding $Q^{\mathrm{alg}} \hookrightarrow Q_v^{\mathrm{alg}}$ for every place v of Q.

Let K be a field extension of $\mathbb{F}_q$ and fix an $\mathbb{F}_q$-morphism $\gamma : A \to K$. We will call the pair $(K, \gamma : A \to K)$ an *A-field.* The prime ideal $\ker(\gamma) \subset A$ is called the *A-characteristic* of K and is denoted A-char(K,γ) or simply A-char(K). If A-char$(K) = (0)$ we say K has *generic A-characteristic.* Then γ is injective and K is via γ a field extension of Q. If A-char$(K) = v \subset A$ is a maximal ideal, we say that A-char(K) is *finite* and K has *finite A-characteristic* v. Then K is via γ a field extension of $\mathbb{F}_v$.

Let $\mathbb{G}_{a,K} = \mathrm{Spec}(K[X])$ be the additive group scheme over K and let $\tau \in \mathrm{End}_K(\mathbb{G}_{a,K})$ be the q-th power Frobenius endomorphism given by $\tau^*(X) = X^q$. Also every $b \in K$ induces an endomorphism $\psi_b \in \mathrm{End}_K(\mathbb{G}_{a,K})$ given by $\psi_b^*(X) = bX$. These endomorphisms satisfy $\tau \circ \psi_b = \psi_{b^q} \circ \tau$. Then the ring $\mathrm{End}_{K,\mathbb{F}_q}(\mathbb{G}_{a,K})$ of $\mathbb{F}_q$-linear endomorphisms of group schemes over K equals the non-commutative polynomial ring over K in τ:

$$K\{\tau\} \;:=\; \Big\{ \sum_{i=0}^{n} b_i \tau^i \colon n \in \mathbb{N}_0, b_i \in K \Big\} \qquad \text{with} \quad \tau b = b^q \tau\,.$$

For $\sum_{i=0}^{n} b_i \tau^i \in K\{\tau\}$ we set $\deg_\tau\Big(\sum_{i=0}^{n} b_i \tau^i\Big) = \max\{i \colon b_i \neq 0\}$.

Definition 9.2. Let $(K, \gamma : A \to K)$ be an A-field. A *Drinfeld A-module* over K is a pair $\underline{G} = (G, \varphi)$ with $G \cong \mathbb{G}_{a,K}$ and φ is an $\mathbb{F}_q$-algebra homomorphism $\varphi : A \to \mathrm{End}_{K,\mathbb{F}_q}(G) \cong K\{\tau\}$, $a \mapsto \varphi_a$, such that

(a) $\mathrm{Lie}(\varphi_a) = \gamma(a)$ i.e. $(a - \gamma(a)) \cdot \mathrm{Lie}(G) = 0$ in K for all $a \in A$.

(b) There exists an $a \in A$ such that $\varphi_a \in K\{\tau\} \smallsetminus K$ i.e. $\varphi_a \neq \gamma(a) \cdot \tau^0$ i.e. $\deg_\tau(\varphi_a) > 0$.

Then there is an integer $r > 0$ such that $\deg_\tau(\varphi_a) = -r d_\infty v_\infty(a)$ for every $a \in A$, see [Gos96, § 4.5]. It is called the *rank* of (G, φ) and is denoted $\mathrm{rk}\,\underline{G}$ or $\mathrm{rk}\,\varphi$. Also sometimes a Drinfeld A-module $\underline{G} = (G, \varphi)$ is simply denoted by φ.

A *morphism* between Drinfeld A-modules (G, φ) and (G', φ') over K is a homomorphism $f : G \to G'$ of group schemes such that $\varphi'_a \circ f = f \circ \varphi_a$ for every $a \in A$. We denote the set of morphisms between $\underline{G}$ and $\underline{G}'$ by $\mathrm{Hom}_K(\underline{G}, \underline{G}')$ and we write $\mathrm{End}_K(\underline{G}) := \mathrm{Hom}_K(\underline{G}, \underline{G})$.

In particular, for every $c \in A$ the commutation $\varphi_a \circ \varphi_c = \varphi_{ac} = \varphi_{ca} = \varphi_c \circ \varphi_a$ implies that $\varphi_c \in \mathrm{End}_K(\underline{G})$. Thus $\mathrm{End}_K(\underline{G})$ is an A-algebra via $A \to \mathrm{End}_K(\underline{G})$, $c \mapsto \varphi_c$ and $\mathrm{Hom}_K(\underline{G}, \underline{G}')$ is an A-module. So we may also define $\mathrm{QHom}_K(\underline{G}, \underline{G}') := \mathrm{Hom}_K(\underline{G}, \underline{G}') \otimes_A Q$ and write $\mathrm{QEnd}_K(\underline{G}) := \mathrm{QHom}_K(\underline{G}, \underline{G}) = \mathrm{End}_K(\underline{G}) \otimes_A Q$.

Remark 9.3. Drinfeld A-modules possess higher dimensional generalizations, which are called *abelian Anderson A-modules*, see [Har17, Definition 1.2]. They were originally defined by Anderson [And86] for $A = \mathbb{F}_q[t]$ under the name *abelian t-modules.* These are group schemes which carry an action of the ring A subject to certain conditions. Abelian Anderson A-modules are the function field analogs of abelian varieties. Although Anderson worked over a field, abelian Anderson A-modules also exist naturally over arbitrary A-algebras as base rings. They possess an (anti-)equivalent description by semi-linear algebra objects called *A-motives*, which we will define next. Through the work of Drinfeld and Anderson it was realized very early on that a Drinfeld module or abelian Anderson A-module over a field is completely described by its A-motive. The same is true over an arbitrary A-algebra R, as is shown for example in [Har17]. So in a way the situation in function field arithmetic is much better than in the arithmetic of abelian varieties (which only have a local p-adic semi-linear algebra description via the Dieudonné module of the associated p-divisible group, see Remark 5.3): the A-motive is a "global" Dieudonné module which integrates the "local" Dieudonné modules for every prime in a single object. We will return to this in Section 14 and Proposition 14.7.

Before we define A-motives we have to fix some

Notation 9.4. For an A-field (K,γ) we write $A_K := A \otimes_{\mathbb{F}_q} K$ and set $\mathcal{J} := (a\otimes 1 - 1\otimes\gamma(a)\colon a\in A) \subset A_K$. We consider the endomorphism $\sigma^* := \mathrm{id}_A \otimes \mathrm{Frob}_{q,K}$ of A_K, where $\mathrm{Frob}_{q,K}(b) = b^q$ for $b \in K$. For an A_K-module M we set $\sigma^* M := M \otimes_{A_K,\sigma^*} A_K$ and we write $\sigma^*_M : M \to \sigma^* M$, $m \mapsto m\otimes 1$ for the natural σ^*-semilinear map. For a homomorphism $f\colon M \to N$ of A_K-modules we set $\sigma^* f := f \otimes \mathrm{id}_{A_K} : \sigma^* M \to \sigma^* N$. Note that the endomorphism σ^* corresponds to a morphism of schemes

$$\sigma := \mathrm{id}_C \times \mathrm{Spec}(\mathrm{Frob}_{q,K})\colon\; C_K := C \times_{\mathbb{F}_q} \mathrm{Spec}\, K \to C_K \tag{9.1}$$

which is the identity on points and on sections of $\mathcal{O}_C$ and the q-Frobenius on K. It satisfies $\sigma|_{\mathrm{Spec}\, A_K} = \mathrm{Spec}(\sigma^*)\colon\ \mathrm{Spec}\, A_K \to \mathrm{Spec}\, A_K$.

Example 9.5. Before we give the general definition of A-motives, we define the A-motive associated to a Drinfeld A-module $\underline{G} = (G,\varphi)$ over K as in [And86]. Namely, we set

$$M := M(\underline{G}) := M(\varphi) := \mathrm{Hom}_{K,\mathbb{F}_q}(G, \mathbb{G}_{a,K}),$$

where $\mathrm{Hom}_{K,\mathbb{F}_q}(-,-)$ is the group of $\mathbb{F}_q$-linear homomorphisms of group schemes over K. Every choice of an isomorphism $G \cong \mathbb{G}_{a,K}$ induces an isomorphism $M(\underline{G}) \cong K\{\tau\}$. We make $M(\underline{G})$ into an $A_K\{\tau\} = A \otimes_{\mathbb{F}_q} K\{\tau\}$ module in the fashion given below:

$$\begin{aligned}
&(a,m) \mapsto m \circ \varphi_a && \text{for} \quad m \in M,\ a \in A; && (9.2)\\
&(b,m) \mapsto \psi_b \circ m && \text{for} \quad m \in M,\ b \in K; && (9.3)\\
&(\tau,m) \mapsto \tau m = \mathrm{Frob}_{q,\mathbb{G}_a} \circ m && \text{for} \quad \mathbb{G}_{a,K} \to \mathbb{G}_{a,K} :\ m \in M. && (9.4)
\end{aligned}$$

Since the actions of $a \in A$ and of $b \in K$ commute, i.e. $a(b\cdot m) = \psi_b \circ m \circ \varphi_a = b(a\cdot m)$, this makes M into a module over $A_K := A \otimes_{\mathbb{F}_q} K$. It is not difficult to see that M is a locally free A_K-module of rank $r := \mathrm{rk}\,\underline{G}$, see [Gos96, Lemma 5.4.1]. Now for $a \in A$ and $b \in K$ we have

$$\tau \circ (a\otimes b)(m) = \tau \circ (\psi_b \circ m \circ \varphi_a) = \psi_{b^q} \circ \tau \circ m \circ \varphi_a = (a \otimes b^q) \circ \tau m\,.$$

Since the action of τ is not A_K-linear but σ^*-semi linear, it induces an A_K-linear map $\tau_M : \sigma^* M \to M$ defined by $\tau_M(m\otimes 1) = \tau m$. Sending $m \in M := \mathrm{Hom}_{K,\mathbb{F}_q}(G,\mathbb{G}_{a,K})$ to $\mathrm{Lie}\, m \in \mathrm{Hom}_K(\mathrm{Lie}\, G, \mathrm{Lie}\,\mathbb{G}_{a,K}) = \mathrm{Hom}_K(\mathrm{Lie}\, G, K)$ defines a canonical isomorphism of A_K-modules

$$\mathrm{coker}\,\tau_M \;=\; M/\tau_M(\sigma^* M) \;\xrightarrow{\;\sim\;}\; \mathrm{Hom}_K(\mathrm{Lie}\, G, K), \quad m \bmod \tau_M(\sigma^* M) \;\longmapsto\; \mathrm{Lie}\, m\,, \tag{9.5}$$

where $a \in A$ acts on $\mathrm{Lie}\, E$ via $\mathrm{Lie}\,\varphi_a$; see [And86, Lemma 1.3.4]. This implies $\dim_K(\mathrm{coker}\,\tau_M) = 1$, which can also be seen directly from $M \cong K\{\tau\}$ and $\tau_M(\sigma^* M) \cong K\{\tau\}\cdot\tau$.

The above construction motivates the definition of A-motives:

Definition 9.6. An *(effective) A-motive of rank r and dimension d over K* is a pair $\underline{M} = (M, \tau_M)$ consisting of a locally free A_K-module M of rank r and an A_K-homomorphism $\tau_M\colon \sigma^* M \to M$ such that

(a) $\dim_K(\mathrm{coker}\,\tau_M) = d$.

(b) $(a - \gamma(a))^d \cdot \mathrm{coker}\,\tau_M = 0$ for all $a \in A$.

We write $\mathrm{rk}\,\underline{M} := r$ and $\dim \underline{M} := d$.

A *morphism* between A-motives $f\colon (M,\tau_M) \to (N,\tau_N)$ over K is an A_K-homomorphism $f\colon M \to N$ with $f\circ\tau_M = \tau_N \circ \sigma^* f$. We denote the set of morphisms between $\underline{M}$ and $\underline{N}$ by $\mathrm{Hom}_K(\underline{M},\underline{N})$ and we write $\mathrm{End}_K(\underline{M}) := \mathrm{Hom}_K(\underline{M},\underline{M})$. Since $\sigma^*(a) = a$ for all $a \in A$ and τ_M is A_K-linear, we have $a\cdot \mathrm{id}_M \in \mathrm{End}_K(\underline{M})$. Thus $\mathrm{End}_K(\underline{M})$ is an A-algebra via $A \to \mathrm{End}_K(\underline{M})$, $a \mapsto a\cdot\mathrm{id}_M$ and $\mathrm{Hom}_K(\underline{M},\underline{N})$ is an A-module. So we may also define $\mathrm{QHom}_K(\underline{M},\underline{N}) := \mathrm{Hom}_K(\underline{M},\underline{N}) \otimes_A Q$ and write $\mathrm{QEnd}_K(\underline{M}) := \mathrm{QHom}_K(\underline{M},\underline{M}) = \mathrm{End}_K(\underline{M}) \otimes_A Q$.

On the relation with Drinfeld A-modules we have the following theorem, see [And86] or [Gos96, §5.4].

Theorem 9.7. *The contravariant functor $\underline{G} \mapsto \underline{M}(\underline{G})$ from Drinfeld A-modules to A-motives over K is fully faithful. Its essential image consists of all $\underline{M} = (M, \tau_M)$ such that M is free over $K\{\tau\}$ of rank 1. The latter implies that* $\dim \underline{M} = 1$.

In this sense we view A-motives as higher dimensional generalizations of Drinfeld A-modules. As an illustration of the claim that A-motives (and abelian Anderson A-modules) play the role of abelian varieties, see for example [BH09] where the theory of A-motives over finite fields is developed in analogy with [Tat66].

Example 9.8. Let $C = \mathbb{P}^1_{\mathbb{F}_q}$, and set $A = \mathbb{F}_q[t]$. Then $A_K = K[t]$. Let $K = \mathbb{F}_q(\theta)$ be the rational function field in the variable θ and let $\gamma : A \to K$ be given by $\gamma(t) = \theta$. The *Carlitz module* over K is given by $\underline{G} = (\mathbb{G}_{a,K}, \varphi)$ with $\varphi \colon \mathbb{F}_q[t] \to K\{\tau\}$ defined by $\varphi_t = \theta + \tau$. The A-motive associated with the Carlitz module is given by $\underline{\mathcal{C}} = (\mathcal{C} = K[t], \tau_{\mathcal{C}} = t - \theta)$ and is called the *Carlitz motive*. Both $\underline{G}$ and $\underline{\mathcal{C}}$ have rank 1. As we will see in Examples 12.3 and 14.10 below, the Carlitz module is the function field analog of the multiplicative group $\mathbb{G}_{m,\mathbb{Q}}$ from Example 4.3.

10 Isogenies and Semi-simple A-Motives

If we define the *rank* of an abelian variety X ad $\operatorname{rk} X := 2 \cdot \dim X$, see Remark 12.5 below, the analog of Theorem 3.1 is the following

Theorem 10.1. *For two A-motives $\underline{M}$ and $\underline{N}$ over an A-field K the A-module $\operatorname{Hom}_k(\underline{M}, \underline{N})$ is finite projective of rank $\leq (\operatorname{rk} \underline{M}) \cdot (\operatorname{rk} \underline{N})$. The same is true for Drinfeld A-modules over K.*

Proof. For A-motives this was proved by Anderson [And86, Corollary 1.7.2] and for Drinfeld A-modules it can be found in [Gos96, Theorem 4.7.8]. □

Definition 10.2. Let $\underline{G} = (G, \varphi)$ and $\underline{G}' = (G', \varphi')$ be two Drinfeld A-modules over K. A non zero morphism $f \in \operatorname{Hom}_K(\underline{G}, \underline{G}')$ is called an *isogeny.* If there is an isogeny $f : \underline{G} \to \underline{G}'$, then $\underline{G}$ and $\underline{G}'$ are *isogenous.*

From [Gos96, 4.7.13], we know that if there is an isogeny $f : \underline{G} \to \underline{G}'$, then there exists a some nonzero $a \in A$ and an isogeny $\hat{f} : \underline{G}' \to \underline{G}$ such that

$$\hat{f} f = \varphi_a \quad \text{and} \quad f \hat{f} = \varphi'_a.$$

In particular, if $0 \neq f \in \operatorname{End}_K(\underline{G})$, then f is invertible in $\operatorname{QEnd}(\underline{G}) := \operatorname{End}_K(\underline{G}) \otimes_A Q$, so $\operatorname{QEnd}(\underline{G})$ is a finite dimensional division algebra over Q.

Definition 10.3. Let $\underline{M}$ and $\underline{N}$ be two A-motives over K. A morphism $f \in \operatorname{Hom}_K(\underline{M}, \underline{N})$ is called an *isogeny* if f is injective and $\operatorname{coker} f$ is a finite dimensional K-vector space. If there exists an isogeny $f \in \operatorname{Hom}_K(\underline{M}, \underline{N})$ then $\underline{M}$ and $\underline{N}$ are said to be *isogenous* over K and we write $\underline{M} \approx_K \underline{N}$. This defines an equivalence relation by Remark 10.4(d) below.

Remark 10.4. (a) Two Drinfeld A-modules are isogenous if and only if their associated A-motives are isogenous, see [Har17, Theorem 5.9 and Proposition 5.4].

(b) If two A-motives $\underline{M}$ and $\underline{N}$ are isogenous then $\operatorname{rk} \underline{M} = \operatorname{rk} \underline{N}$ and $\dim \underline{M} = \dim \underline{N}$, see [Har17, Proposition 5.8].

(c) Conversely, let $f \colon \underline{M} \to \underline{N}$ be a morphism of A-motives with $\operatorname{rk} \underline{M} = \operatorname{rk} \underline{N}$. Then f is injective if and only if $\operatorname{coker} f$ is a finite dimensional K-vector space, and in this case f is an isogeny. Indeed, since M is locally free over A_K, it is contained in $M \otimes_{A_K} \operatorname{Quot}(A_K)$ where $\operatorname{Quot}(A_K)$ denotes the fraction field of A_K. Since $\operatorname{rk} \underline{M} = \operatorname{rk} \underline{N}$ the injectivity of f is equivalent to f inducing an isomorphism $M \otimes_{A_K} \operatorname{Quot}(A_K) \to N \otimes_{A_K} \operatorname{Quot}(A_K)$, and this in turn is equivalent to $\operatorname{coker} f$ being torsion, and hence finite.

(d) If $f \colon \underline{M} \to \underline{N}$ is an isogeny between A-motives, then there exists non-canonically an isogeny $\hat{f} \colon \underline{N} \to \underline{M}$ and a non-zero element $a \in A$ with $\hat{f} f = a \cdot \mathrm{id}_{\underline{M}}$ and $f \hat{f} = a \cdot \mathrm{id}_{\underline{N}}$ by [Har17, Corollary 5.15]

(e) Let $\underline{M}$ and $\underline{N}$ be A-motives over K. If $\underline{M}$ and $\underline{N}$ are isogenous over K via an isogeny f, then

$$\operatorname{QEnd}_K(\underline{M}) \cong \operatorname{QHom}_K(\underline{M}, \underline{N}) \cong \operatorname{QEnd}_K(\underline{N}), \quad h \mapsto f \circ h \mapsto f \circ h \circ f^{-1}.$$

More precisely, $\operatorname{QHom}_K(\underline{M}, \underline{N})$ is a free right $\operatorname{QEnd}_K(\underline{M})$-module of rank 1 and a free left $\operatorname{QEnd}_K(\underline{N})$-module of rank 1. If $\underline{M}$ and $\underline{N}$ are not isogenous then $\operatorname{QHom}_K(\underline{M}, \underline{N}) = (0)$.

Definition 10.5. Let $\underline{M}$ be an A-motive over K.

(a) An *A-factor-motive over K* of $\underline{M}$ is an A-motive $\underline{M}'$ together with a surjective morphism $\underline{M} \twoheadrightarrow \underline{M}'$ of A-motives over K.

(b) $\underline{M}$ is called *simple over K* if $\underline{M}$ is non trivial and $\underline{M}$ has no A-factor-motives over K other than (0) and $\underline{M}$.

(c) $\underline{M}$ is called *semi-simple over K* if $\underline{M}$ is isogenous to a direct sum of simple A-motives over K, i.e. $\underline{M} \approx_K \oplus_i \underline{M}_i$ with $\underline{M}_i$ simple.

Remark 10.6. (a) In comparison to the analogous Definition 3.2 for abelian varieties, A-motives behave dually. This is due to the fact that the functor from Drinfeld A-modules to A-motives is contravariant.

(b) For any Drinfeld A-module φ over K the A-motive $\underline{M}(\varphi)$ is simple by [BH11, Corollary 7.5].

(c) But in contrast to abelian varieties (Remark 3.3) not every A-motive is semi-simple up to isogeny. This was observed in [BH09, Examples 6.1 and 6.13].

(d) Let $\underline{M}$ and $\underline{N}$ be two A-motives over K of same rank and let $\underline{M}$ be simple over K. Then every non-zero morphism $f \in \mathrm{Hom}_K(\underline{M}, \underline{N})$ is an isogeny. Namely, the image of f is a non-zero A-factor-motive of $\underline{M}$, and hence isomorphic to $\underline{M}$ via f, because $\underline{M}$ is simple. So f is injective and hence an isogeny by Remark 10.4(c).

In particular, if $\underline{M}$ is simple over K then every non-zero endomorphism $0 \neq f \in \mathrm{End}_K(\underline{M})$ is an isogeny and therefore invertible in $\mathrm{QEnd}_K(\underline{M})$ by Remark 10.4(d). This implies that $\mathrm{QEnd}_K(\underline{M})$ is a division algebra over Q.

Moreover, if $\underline{M}$ is semi-simple over K with decomposition $\underline{M} \approx_K \underline{M}_1 \oplus \cdots \oplus \underline{M}_n$ up to isogeny into simple A-motives $\underline{M}_i$ over K, then $\mathrm{QEnd}_K(\underline{M})$ decomposes into a finite direct product of full matrix algebras over the division algebras $\mathrm{QEnd}_K(\underline{M}_i)$ over Q, compare Remark 3.4.

11 Analytic Theory of Drinfeld Modules

In this section we consider Drinfeld A-modules over $\mathbb{C}_\infty$, which is an A-field via the natural inclusion $A \subset Q \subset Q_\infty \subset \mathbb{C}_\infty$ denoted by γ.

If $\underline{G} = (\mathbb{G}_{a,\mathbb{C}_\infty}, \varphi)$ with $\varphi\colon A \to \mathbb{C}_\infty\{\tau\}$ is a Drinfeld A-module over $\mathbb{C}_\infty$ then there is a uniquely determined power series $\exp_{\underline{G}}(z) = \sum_{i=0}^{\infty} e_i z^{q^i}$ with $e_i \in \mathbb{C}_\infty$, $e_0 = 1$ satisfying

$$\varphi_a(\exp_{\underline{G}}(z)) = \exp_{\underline{G}}(\gamma(a)\cdot z)$$

for all $a \in A$, see [Gos96, 4.6.7]. It is called the *exponential function of $\underline{G}$*. The power series $\exp_{\underline{G}}$ converges for every $z \in \mathbb{C}_\infty$ and its kernel $\Lambda(\underline{G})$ is an *A-lattice* in $\mathbb{C}_\infty$ (that is, a finitely generated projective, discrete A-submodule) of the same rank as the Drinfeld A-module $\underline{G}$. Note that $\mathbb{C}_\infty$ is infinite dimensional over Q_∞ and therefore contains A-lattices of arbitrarily high rank.

Conversely, let $\Lambda \subset \mathbb{C}_\infty$ be an A-lattice of rank r. Then the function

$$\exp_\Lambda(z) = z \prod_{0\neq\lambda\in\Lambda} (1 - \tfrac{z}{\lambda}) \tag{11.1}$$

converges for every $z \in \mathbb{C}_\infty$ and can be written as an everywhere convergent power series in z. Moreover $\exp_\Lambda\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a surjective $\mathbb{F}_q$-linear map whose zeroes are simple and located at Λ. For more details see [Gos96, §4.2]. For $a \in A \smallsetminus \{0\}$ we can now define the polynomial

$$\varphi_a^\Lambda(x) \;:=\; \gamma(a)\cdot x \cdot \prod_{0\neq\lambda\in\gamma(a)^{-1}\Lambda/\Lambda} \big(1 - \tfrac{x}{\exp_\Lambda(\lambda)}\big) \;\in\; \mathbb{C}_\infty[x]\,. \tag{11.2}$$

It satisfies

$$\exp_\Lambda(\gamma(a)\cdot z) = \varphi_a^\Lambda(\exp_\Lambda(z)) \tag{11.3}$$

and makes the following diagram with exact rows commutative

$$\begin{tikzcd} 0 \arrow[r] & \Lambda \arrow[r] \arrow[d] & \mathbb{C}_\infty \arrow[r, "\exp_\Lambda"] \arrow[d, "\gamma(a)"'] & \mathbb{C}_\infty \arrow[r] \arrow[d, "\varphi_a^\Lambda"] & 0\,. \\ 0 \arrow[r] & \Lambda \arrow[r] & \mathbb{C}_\infty \arrow[r, "\exp_\Lambda"] & \mathbb{C}_\infty \arrow[r] & 0 \end{tikzcd} \tag{11.4}$$

It is easy to see that

(a) $\varphi_a^\Lambda(x)$ is an $\mathbb{F}_q$-linear polynomial, i.e. $\varphi_a^\Lambda \in \mathbb{C}_\infty\{\tau\}$, of τ-degree $\deg_\tau(\varphi_a^\Lambda) = -rd_\infty v_\infty(a)$;

(b) $\varphi^\Lambda : a \mapsto \varphi_a^\Lambda$ defines a ring homomorphism $\varphi^\Lambda : A \to \mathbb{C}_\infty\{\tau\}$.

The additive group $\mathbb{C}_\infty$, considered as the quotient $\mathbb{C}_\infty/\Lambda$ via $\exp_\Lambda$, thus carries a new structure as an A-module given by $z \mapsto \varphi_a^\Lambda(z)$ for $a \in A$. Therefore, for every A-lattice $\Lambda \subset \mathbb{C}_\infty$ of rank r we get a Drinfeld A-module $\underline{G}^\Lambda := (\mathbb{G}_{a,\mathbb{C}_\infty}, \varphi^\Lambda)$ of rank r over $\mathbb{C}_\infty$.

Definition 11.1. Let Λ_1, Λ_2 be two A-lattices of the same rank. A morphism from $\Lambda_1 \to \Lambda_2$ is an element $c \in \mathbb{C}_\infty$, with $c\Lambda_1 \subseteq \Lambda_2$. If the ranks of Λ_1 and Λ_2 are different, then we only allow $0 \in \mathbb{C}_\infty$ to be a morphism.

Theorem 11.2 ([Dri76, Proposition 3.1]). *The functors $\underline{G} \mapsto \Lambda(\underline{G})$ and $\Lambda \mapsto \underline{G}^\Lambda$ give an equivalence of categories between the category of Drinfeld A-modules over $\mathbb{C}_\infty$ and the category of A-lattices in $\mathbb{C}_\infty$.*

Corollary 11.3. *If $\underline{G}$ is a Drinfeld A-module over a field K of generic A-characteristic, then $\mathrm{QEnd}_K(\underline{G})$ is a commutative field whose degree over Q divides $\mathrm{rk}\,\underline{G}$.*

Proof. Since $\underline{G}$ and all elements of $\mathrm{QEnd}_K(\underline{G})$ are defined over a finitely generated subfield K_0 of K, we can choose a Q-embedding $K_0 \hookrightarrow \mathbb{C}_\infty$ and it suffices to prove the corollary when $K = \mathbb{C}_\infty$. In this case $\underline{G} \cong \underline{G}^\Lambda$ for an A-lattice $\Lambda \subset \mathbb{C}_\infty$ of rank equal to $\mathrm{rk}\,\underline{G}$. By Theorem 11.2 we have isomorphisms $\mathrm{End}_K(\underline{G}) \xrightarrow{\sim} \{c \in \mathbb{C}_\infty : c\Lambda \subset \Lambda\}$, $f \mapsto \mathrm{Lie}(f)$ and $\mathrm{QEnd}_K(\underline{G}) \xrightarrow{\sim} \{c \in \mathbb{C}_\infty : c(Q\cdot\Lambda) \subset Q\cdot\Lambda\}$. In particular $\mathrm{QEnd}_K(\underline{G}) \subset \mathbb{C}_\infty$ is a commutative field. Since $Q\cdot\Lambda \subset \mathbb{C}_\infty$ is a Q-vector space of dimension $\mathrm{rk}\,\underline{G}$ and also a $\mathrm{QEnd}_K(\underline{G})$-vector space, the formula $\mathrm{rk}\,\underline{G} = \dim_Q(Q\cdot\Lambda) = [\mathrm{QEnd}_K(\underline{G}) : Q]\cdot \dim_{\mathrm{QEnd}_K(\underline{G})}(Q\cdot\Lambda)$ tells us that $[\mathrm{QEnd}_K(\underline{G}) : Q]$ divides $\mathrm{rk}\,\underline{G}$. □

We regard Drinfeld A-modules and particularly those of rank two as analogs of elliptic curves, where the functional equation (11.3) for $\exp_\Lambda(z)$ corresponds to the group law derived from (2.3). The point is that (2.3) defines a $\mathbb{Z}$-module structure on the elliptic curve $\mathbb{C}/\Lambda \xrightarrow{\sim} E_\Lambda(\mathbb{C})$, while (11.2) and (11.3) define the above A-module structure on the additive group scheme $\mathbb{G}_{a_K}$.

Definition 11.4. Let $\underline{G}$ be a Drinfeld A-module of rank r over $\mathbb{C}_\infty$. The *Betti (co-)homology realization* of $\underline{G}$ is defined by

$$\mathrm{H}^1_{\mathrm{Betti}}(\underline{G}, R) := \Lambda(\underline{G}) \otimes_A R \qquad \text{and} \qquad \mathrm{H}_{1,\mathrm{Betti}}(\underline{G}, R) := \mathrm{Hom}_A(\Lambda(\underline{G}), R)$$

for any A-algebra R. Both are free R-modules of rank r.

12 Torsion Points and v-adic Cohomology of Drinfeld Modules

Definition 12.1. Let $\underline{G} = (G, \varphi)$ be a Drinfeld A-module over an A-field K and let $G(K^{\mathrm{alg}})$ be the set of K^{alg}-valued points of G. For an element $a \in A$, we set

$$\underline{G}[a](K^{\mathrm{alg}}) := \varphi[a](K^{\mathrm{alg}}) := \{P \in G(K^{\mathrm{alg}}) \mid \varphi_a(P) = 0\},$$

and we call $\underline{G}[a](K^{\mathrm{alg}})$ the *module of a-torsion points* of $\underline{G} = (G, \varphi)$. If $\mathfrak{a} \subseteq A$ is an ideal, we set

$$\underline{G}[\mathfrak{a}](K^{\mathrm{alg}}) := \varphi[\mathfrak{a}](K^{\mathrm{alg}}) := \{P \in G(K^{\mathrm{alg}}) \mid \varphi_a(P) = 0 \text{ for all } a \in \mathfrak{a}\}.$$

The latter are the K^{alg}-valued points of a closed subgroup scheme $\underline{G}[\mathfrak{a}]$ of G, which is an $A/\mathfrak{a}$-module scheme via $\overline{a} \mapsto \varphi_a|_{\underline{G}[\mathfrak{a}]}$. If $\mathfrak{a} = (a)$ then $\underline{G}[\mathfrak{a}](K^{\mathrm{alg}}) = \underline{G}[a](K^{\mathrm{alg}})$.

Remark 12.2. We have the following observation, see [Gos96, § 4.5], where we denote the A-characteristic of K by $\mathfrak{p} = A\text{-char}(K) := \ker(\gamma : A \to K)$:

(a) If $a \in A$ is prime to $A\text{-char}(K)$, we see that the polynomial φ_a is separable and $\#\underline{G}[a](K^{\mathrm{alg}}) = (\#A/(a))^{\mathrm{rk}\,\underline{G}}$. Since this holds for every $a \in A$ and $\underline{G}[\mathfrak{a}](K^{\mathrm{alg}})$ is an $A/\mathfrak{a}$-module, one obtains $\underline{G}[\mathfrak{a}](K^{\mathrm{alg}}) \cong (A/\mathfrak{a})^{\mathrm{rk}\,\underline{G}}$ as A-modules.

(b) $\#\underline{G}[\mathfrak{p}](K^{\mathrm{alg}}) = (\#A/(\mathfrak{p}))^{\mathrm{rk}\,\underline{G}-h}$ and $\underline{G}[\mathfrak{p}](K^{\mathrm{alg}}) \cong (A/(\mathfrak{p}))^{\mathrm{rk}\,\underline{G}-h}$, where h is the height of the Drinfeld A-module defined by $h := \frac{w(a)}{v_\mathfrak{p}(a)\cdot[\mathbb{F}_\mathfrak{p}:\mathbb{F}_q]}$ for every $a \in A$, where $w(a)$ is the smallest integer $i \geq 0$ with τ^i occurring in φ_a, with nonzero coefficient.

Example 12.3. The Carlitz module $\underline{G} = (\mathbb{G}_{a,K}, \varphi)$ over $K = \mathbb{F}_q(\theta)$ with $\varphi_t = \theta + \tau$ from Example 9.8 has rank 1. For every $a = \sum_{i=0}^n a_i t^i$ with $a_i \in \mathbb{F}_q$ and $a_n \neq 0$, we have $\varphi_a = \sum_{i=0}^n a_i \varphi_t^n = \sum_{i=0}^n a_i(\theta + \tau)^n = (\sum_{i=0}^n a_i \theta^n) \cdot \tau^0 + \ldots + a_n \tau^n = \gamma(a)\tau^0 + \ldots + a_n\tau^n$. Therefore, the polynomial $\varphi_a(x) = \gamma(a)\, x + \ldots + a_n\, x^{q^n}$ has degree q^n and is separable, because $\gamma(a) \neq 0$. From this it follows that $\#\underline{G}[a](K^{\mathrm{alg}}) = q^n = \#(A/(a))$ and that $\underline{G}[a](K^{\mathrm{alg}}) \cong A/(a)$ for every $a \in A$. This illustrates that the Carlitz module is the function field analog of the multiplicative group $\mathbb{G}_m = \mathbb{G}_{m,\mathbb{Q}}$ from Example 4.3, which for $a \in \mathbb{N}_{>0}$ satisfies $\mathbb{G}_m[a](\mathbb{Q}^{\mathrm{alg}}) := \ker[a](\mathbb{Q}^{\mathrm{alg}}) = \{x \in \mathbb{Q}^{\mathrm{alg}} : x^a = 1\} \cong \mathbb{Z}/(a)$.

Definition 12.4. Let v be a prime ideal of A. Let $\underline{G} = (G, \varphi)$ be a Drinfeld A-module over K of fixed rank r and define the A_v-module $\underline{G}[v^\infty](K^{\mathrm{alg}}) := \cup_{n\geq 1}\underline{G}[v^n](K^{\mathrm{alg}})$. The A_v-module

$$\mathrm{H}_{1,v}(\underline{G}, A_v) := T_v(\underline{G}) = \mathrm{Hom}_{A_v}\big(Q_v/A_v,\, G(K^{\mathrm{alg}})\big) = \mathrm{Hom}_{A_v}\big(Q_v/A_v,\, \underline{G}[v^\infty](K^{\mathrm{alg}})\big). \tag{12.1}$$

is called the *v-adic homology realization* or the *v-adic Tate module* of $\underline{G}$. It carries a continuous $\mathscr{G}_K$-action. Note that when $z = \frac{a}{c} \in Q$ is a uniformizing parameter of A_v then the map $\varphi_z := \varphi_c^{-1} \circ \varphi_a \colon \underline{G}[v^n](K^{\mathrm{alg}}) \to \underline{G}[v^{n-1}](K^{\mathrm{alg}})$ is well defined and

$$T_v(\underline{G}) \cong \varprojlim\big(\underline{G}[v^n](K^{\mathrm{alg}}), \varphi_z\big)\,;$$

see for example [HK20, after Definition 4.8]. A morphism $f : \underline{G} \to \underline{G}'$ of Drinfeld A-modules gives a morphism $T_v(f) : T_v(\underline{G}) \to T_v(\underline{G}')$ of $A_v[\mathscr{G}_K]$-modules. If v is different from the A-characteristic A-char(K) of K, then $T_v(\underline{G})$ is isomorphic to $A_v^{\oplus r}$.

Remark 12.5. The results of this section parallel Remark 2.7 for abelian varieties. Since the ℓ-adic Tate module of an abelian variety X is isomorphic to $(\mathbb{Z}_\ell)^{2\dim X}$, while the v-adic Tate module of a Drinfeld A-module $\underline{G}$ is isomorphic to $A_v^{\mathrm{rk}\,\underline{G}}$ it is natural to call the number $\mathrm{rk}\, X := 2\dim X$ the *rank* of the abelian variety X, compare also Theorems 3.1 and 10.1.

There is a similar theory of Tate modules for A-motives which we will explain in the next section.

13 Cohomology Realizations and Period Maps for A-Motives

13.1 Uniformizability and Betti Cohomology

In this section we discuss the notion of uniformizability, cohomology realizations and period maps for A-motives from [HJ20] and also we generalize the results to the case $d_\infty = [\mathbb{F}_\infty : \mathbb{F}_q] \neq 1$. For a field extension K of $\mathbb{F}_q$ we consider the closed subscheme $\infty_K := \infty \times_{\mathbb{F}_q} \operatorname{Spec} K \subset C_K := C \times_{\mathbb{F}_q} \operatorname{Spec} K$. If K contains $\mathbb{F}_\infty$, then ∞_K is the disjoint union of d_∞-many K-rational points of C_K.

In order to define the notion of uniformizability for A-motives we have to introduce some notation of rigid analytic geometry as in [HP04]. For a general introduction to rigid analytic geometry see [BGR84].

Notation 13.1. With the curve $C_{\mathbb{C}_\infty}$ and its open affine part $C'_{\mathbb{C}_\infty} := C_{\mathbb{C}_\infty} \smallsetminus \infty_{\mathbb{C}_\infty}$ one can associate by [BGR84, §9.3] rigid analytic spaces $\mathfrak{C}_{\mathbb{C}_\infty} := (C_{\mathbb{C}_\infty})^{\mathrm{rig}}$ and $\mathfrak{C}'_{\mathbb{C}_\infty} := (C'_{\mathbb{C}_\infty})^{\mathrm{rig}} = \mathfrak{C}_{\mathbb{C}_\infty} \smallsetminus \infty_{\mathbb{C}_\infty}$. The underlying sets of $\mathfrak{C}_{\mathbb{C}_\infty}$ and $\mathfrak{C}'_{\mathbb{C}_\infty}$ are the sets of $\mathbb{C}_\infty$-valued points of $C_{\mathbb{C}_\infty}$ and $C_{\mathbb{C}_\infty} \smallsetminus \infty_{\mathbb{C}_\infty}$, respectively. The endomorphism σ of $C_{\mathbb{C}_\infty}$ from (9.1) induces endomorphisms of $\mathfrak{C}_{\mathbb{C}_\infty}$ and $\mathfrak{C}'_{\mathbb{C}_\infty}$ which we denote by the same symbol σ.

Let $\mathcal{O}_{\mathbb{C}_\infty}$ be the valuation ring of $\mathbb{C}_\infty$ and let $\kappa_{\mathbb{C}_\infty}$ be its residue field. By the valuative criterion of properness every point of $\mathfrak{C}_{\mathbb{C}_\infty} = C_{\mathbb{C}_\infty}(\mathbb{C}_\infty) = C(\mathbb{C}_\infty)$ extends uniquely to an $\mathcal{O}_{\mathbb{C}_\infty}$-valued point of C and in the reduction gives rise to a $\kappa_{\mathbb{C}_\infty}$-valued point of C. This gives us a reduction map

$$red : \mathfrak{C}_{\mathbb{C}_\infty} = C(\mathbb{C}_\infty) \longrightarrow C(\kappa_{\mathbb{C}_\infty})\,. \tag{13.1}$$

The subscheme $\infty_{\kappa_{\mathbb{C}_\infty}} \subset C_{\kappa_{\mathbb{C}_\infty}}$ contains d_∞ points. We denote them by $\{\infty_i \text{ for } i \in \mathbb{Z}/d_\infty\mathbb{Z}\}$ in such a way that the map σ from (9.1) transports ∞_i to ∞_{i+1} and $(\sigma^{d_\infty})^*$ stabilizes each ∞_i. Since the curve $C_{\kappa_{\mathbb{C}_\infty}}$ is non-singular, [BL85, Proposition 2.2] implies for each i that the preimage $\mathfrak{D}_i$ of $\infty_i \in \infty_{\kappa_{\mathbb{C}_\infty}}$ under *red* is an open rigid analytic unit disc in $\mathfrak{C}_{\mathbb{C}_\infty}$ around ∞_i. Let $\mathfrak{D}'_i := \mathfrak{D}_i \smallsetminus \infty_i$ be the punctured open unit disc around ∞_i in $\mathfrak{C}_{\mathbb{C}_\infty}$. Then σ maps $\mathfrak{D}_i$ isomorphically onto $\mathfrak{D}_{i+1}$. We let $\mathcal{O}(\mathfrak{D}_i)$ and $\mathcal{O}(\mathfrak{C}_{\mathbb{C}_\infty} \smallsetminus \cup_i \mathfrak{D}_i)$ be the coordinate rings of rigid analytic functions on the spaces $\mathfrak{D}_i$ and $\mathfrak{C}_{\mathbb{C}_\infty} \smallsetminus \cup_i\mathfrak{D}_i$, respectively. The uniformizer $z \in \mathcal{O}(\mathfrak{D}_i)$ is a coordinate function on the disc $\mathfrak{D}_i$ for every i.

Example 13.2. If $C = \mathbb{P}^1_{\mathbb{F}_q}$, $A = \mathbb{F}_q[t]$, and $[\mathbb{F}_\infty : \mathbb{F}_q] = 1$, we can give the following explicit description. $\mathfrak{D}_0 \subset \mathbb{P}^1(\mathbb{C}_\infty)$ is the open unit disc around ∞.

$$\mathcal{O}(\mathfrak{C}_{\mathbb{C}_\infty} \smallsetminus \mathfrak{D}_0) := \mathbb{C}_\infty\langle t\rangle := \Big\{ \sum_{i=0}^{\infty} a_i t^i,\ a_i \in \mathbb{C}_\infty,\ a_i \to 0 \text{ as } i \to \infty \Big\}$$

and $\mathfrak{C}_{\mathbb{C}_\infty} \smallsetminus \mathfrak{D}_0$ is the closed unit disc inside $C(\mathbb{C}_\infty) \smallsetminus \infty_{\mathbb{C}_\infty} = \mathbb{C}_\infty$ on which the coordinate t has absolute value less or equal to 1. Also we can take $z = 1/t$ as the coordinate on the disc $\mathfrak{D}_0$, and suggestively write $\mathfrak{D}_0 = \{|z| < 1\}$.

Definition 13.3. For an A-motive $\underline{M}$ over $\mathbb{C}_\infty$, we define the *τ-invariants*

$$\Lambda(\underline{M}) := (M \otimes_{A_{\mathbb{C}_\infty}} \mathcal{O}(\mathfrak{C}_{\mathbb{C}_\infty} \smallsetminus \cup_i \mathfrak{D}_i))^\tau := \{m \in M \otimes_{A_{\mathbb{C}_\infty}} \mathcal{O}(\mathfrak{C}_{\mathbb{C}_\infty} \smallsetminus \cup_i \mathfrak{D}_i) :\ \tau_M(\sigma_M^* m) = m\}.$$

Since the ring of σ^*-invariants in $\mathcal{O}(\mathfrak{C}_{\mathbb{C}_\infty} \smallsetminus \cup_i \mathfrak{D}_i)$ equals A, the set $\Lambda(\underline{M})$ is an A-module. It was shown implicitly by Anderson [And86, Proof of Lemma 2.10.6] that $\Lambda(\underline{M})$ is finite projective of rank at most equal to $\operatorname{rk} \underline{M}$.

Definition 13.4. An A-motive $\underline{M}$ is called *uniformizable* (or *rigid analytically trivial*) if the natural homomorphism

$$h_{\underline{M}} \colon \Lambda(\underline{M}) \otimes_A \mathcal{O}(\mathfrak{C}_{\mathbb{C}_\infty} \smallsetminus \cup_i \mathfrak{D}_i) \longrightarrow M \otimes_{A_{\mathbb{C}_\infty}} \mathcal{O}(\mathfrak{C}_{\mathbb{C}_\infty} \smallsetminus \cup_i \mathfrak{D}_i), \quad \lambda \otimes f \longmapsto f \cdot \lambda$$

is an isomorphism.

Example 13.5. We keep the notation from Example 9.8. We recall that the Carlitz motive over $\mathbb{C}_\infty$ is given by $\underline{\mathcal{C}} = \big(\mathcal{C} = \mathbb{C}_\infty[t], \tau_{\mathcal{C}} = t - \theta\big)$. We set $\ell^- := \prod_{i=0}^\infty \big(1 - \frac{t}{\theta^{q^i}}\big) \in \mathcal{O}(\mathfrak{C}'_{\mathbb{C}_\infty}) \subset \mathcal{O}(\mathfrak{C}_{\mathbb{C}_\infty} \smallsetminus \mathfrak{D}_0)$ and choose an $\eta \in \mathbb{C}_\infty$ with $\eta^{1-q} = -\theta$. Then we see that $\eta\ell^- \in \Lambda(\underline{\mathcal{C}})$, because

$$\tau_{\mathcal{C}}(\sigma_{\mathcal{C}}^*(\eta\ell^-)) \;=\; (t-\theta)\cdot\eta^q\cdot\sigma^*\prod_{i=0}^{\infty}\big(1 - \frac{t}{\theta^{q^i}}\big) \;=\; \eta\cdot\frac{t-\theta}{-\theta}\cdot\prod_{i=1}^{\infty}\big(1 - \frac{t}{\theta^{q^i}}\big) \;=\; \eta\ell^-\,.$$

Since $\eta\ell^-$ has no zeroes outside $\mathfrak{D}_0$ it generates the $\mathcal{O}(\mathfrak{C}_{\mathbb{C}_\infty} \smallsetminus \mathfrak{D}_0)$-module $\mathcal{C} \otimes_{A_{\mathbb{C}_\infty}} \mathcal{O}(\mathfrak{C}_{\mathbb{C}_\infty} \smallsetminus \mathfrak{D}_0) = \mathcal{O}(\mathfrak{C}_{\mathbb{C}_\infty} \smallsetminus \mathfrak{D}_0)$ and so $h_{\underline{\mathcal{C}}}$ is an isomorphism and $\underline{\mathcal{C}}$ is uniformizable.

Anderson [And86] proved the following criterion for uniformizability.

Lemma 13.6. *Let $\underline{M}$ be an A-motive of rank r.*

(a) The homomorphism $h_{\underline{M}}$ is injective and it satisfies $h_{\underline{M}} \circ (\mathrm{id}_{\Lambda(\underline{M})} \otimes \mathrm{id}) = (\tau_M \otimes \mathrm{id}) \circ \sigma^* h_{\underline{M}}$.

(b) $\underline{M}$ is uniformizable if and only if $\operatorname{rk}_A \Lambda(\underline{M}) = r$.

Proof. (b) was proved by Anderson [And86, Lemma 2.10.6].

(a) is implicitly proved by Anderson [And86]. It is explicitly stated for example in [BH07, Lemma 4.2]. □

Next we state the generalization of [HJ20, Proposition 3.25], which we will need to define period maps. The point $\mathrm{V}(\mathcal{J}) \in C_{\mathbb{C}_\infty}(\mathbb{C}_\infty)$ lies in one of the discs $\mathfrak{D}_i$, because $|\gamma(a)|_\infty > 1$ for all $a \in A \smallsetminus \mathbb{F}_q$. We normalize the indexing of the $\mathfrak{D}_i$ in such a way that $\mathrm{V}(\mathcal{J}) \in \mathfrak{D}_0$. Then for any $i \in \mathbb{N}_0$, we consider the pullbacks $\sigma^{i*}\mathcal{J} = (a \otimes 1 - 1 \otimes \gamma(a)^{q^i} :\ a \in A) \subset A_{\mathbb{C}_\infty}$ and the points $\mathrm{V}(\sigma^{i*}\mathcal{J})$ of $C'_{\mathbb{C}_\infty}$ and $\mathfrak{C}'_{\mathbb{C}_\infty}$. They correspond to the point $\mathrm{V}(z - \zeta^{q^i}) \in \mathfrak{D}_i$ and have $\infty_{\mathbb{C}_\infty} = \{\infty_0, \cdots, \infty_{d_\infty - 1}\}$ as accumulation points. More precisely, for each $k = 0, 1, \cdots, d_\infty - 1$ the point ∞_k is the limit of the sequence $\mathrm{V}(\sigma^{(k+d_\infty i)*}\mathcal{J}) = \mathrm{V}(z - \zeta^{q^{k+d_\infty i}})$ for $i \in \mathbb{N}_0$. Therefore, $\mathfrak{C}'_{\mathbb{C}} \smallsetminus \cup_{i\in\mathbb{N}_0} \mathrm{V}(\sigma^{i*}\mathcal{J})$ is an admissible open rigid analytic subspace of $\mathfrak{C}'_{\mathbb{C}_\infty}$.

Proposition 13.7. *[HJ20, Proposition 3.25] Let $\underline{M}$ be a uniformizable effective A-motive over $\mathbb{C}_\infty$. Then $\Lambda(\underline{M})$ equals $\{m \in M \otimes_{A_{\mathbb{C}_\infty}} \mathcal{O}(\mathfrak{C}'_{\mathbb{C}_\infty}) :\ \tau_M(\sigma_M^* m) = m\}$ and the isomorphism $h_{\underline{M}}$ extends to an injective homomorphism*

$$h_{\underline{M}} \colon \Lambda(\underline{M}) \otimes_A \mathcal{O}(\mathfrak{C}'_{\mathbb{C}_\infty}) \longrightarrow M \otimes_{A_{\mathbb{C}_\infty}} \mathcal{O}(\mathfrak{C}'_{\mathbb{C}_\infty}),\ \lambda \otimes f \mapsto f \cdot \lambda$$

with $h_{\underline{M}} \circ (\mathrm{id}_{\Lambda(\underline{M})} \otimes \mathrm{id}) = (\tau_M \otimes \mathrm{id}) \circ \sigma^* h_{\underline{M}}$. At the point $\mathrm{V}(\mathcal{J})$ its cokernel satisfies $\operatorname{coker} h_{\underline{M}} \otimes \mathbb{C}_\infty[\![z - \zeta]\!] = M/\tau_M(\sigma^* M)$*. The morphism $h_{\underline{M}}$ is a local isomorphism away from $\cup_{i\in\mathbb{N}_0} \mathrm{V}(\sigma^{i*}\mathcal{J})$, and $\sigma^* h_{\underline{M}}$ is a local isomorphism away from $\cup_{i\in\mathbb{N}_{>0}} \mathrm{V}(\sigma^{i*}\mathcal{J})$.*

Proof. This follows in the same way as [HJ20, Proposition 3.25]. $\square$

Definition 13.8. Let $\underline{M}$ be an A-motive of rank r over $\mathbb{C}_\infty$. Anderson defined the *Betti cohomology realization* of $\underline{M}$ by setting

$$\mathrm{H}^1_{\mathrm{Betti}}(\underline{M}, R) := \Lambda(\underline{M}) \otimes_A R \qquad \text{and} \qquad \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, R) := \mathrm{Hom}_A(\Lambda(\underline{M}), R)$$

for any A-algebra R. This is most useful when $\underline{M}$ is uniformizable, in which case both are locally free R-modules of rank equal to $\operatorname{rk} \underline{M}$.

Example 13.9. We keep the notation from Example 13.5. There we have calculated $\Lambda(\underline{\mathcal{C}})$ as the A-module generated by $\eta\ell^-$, so

$$\mathrm{H}^1_{\mathrm{Betti}}(\underline{\mathcal{C}}, A) = \eta\ell^- \cdot A \qquad \text{and} \qquad \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, A) = (\eta\ell^-)^{-1} \cdot A.$$

Remark 13.10. To explain the compatibility with Definition 11.4 let $\Omega^1_{A/\mathbb{F}_q}$ be the module of Kähler differentials of A over $\mathbb{F}_q$. Then $\Omega^1_{A/\mathbb{F}_q} \otimes_A Q = \Omega^1_{Q/\mathbb{F}_q} = Q\,dz$ because the field extension $Q/\mathbb{F}_q(z)$ is separable as it is unramified at ∞.

Proposition 13.11 ([And86, Corollary 2.12.1])**.** *Let $\underline{G} = (G, \varphi)$ be a Drinfeld A-module over $\mathbb{C}_\infty$ and let $\underline{M} = \underline{M}(\underline{G})$ be the associated A-motive. Then $\underline{M}$ is uniformizable and there is a perfect pairing of A-modules*

$$\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}, A) \times \mathrm{H}^1_{\mathrm{Betti}}(\underline{M}, A) \longrightarrow \Omega^1_{A/\mathbb{F}_q}, \quad (\lambda, m) \longmapsto \omega_{A,\lambda,m}$$

where $\omega_{A,\lambda,m}$ is determined by the residues $\mathrm{Res}_\infty(a \cdot \omega_{A,\lambda,m}) = -m\big(\exp_{\underline{G}}(\mathrm{Lie}\,\varphi_a(\lambda))\big) \in \mathbb{F}_q$ for all $a \in Q$. The pairing yields a canonical isomorphism

$$\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, A) \otimes_A \Omega^1_{A/\mathbb{F}_q} \xrightarrow{\sim} \mathrm{H}_{1,\mathrm{Betti}}(\underline{G}, A)\,,$$

which is functorial in $\underline{G}$.

13.2 v-adic Cohomology

Definition 13.12. For an A-field K consider the v-adic completion $A_{v,K} := \varprojlim A_K/v^n A_K$ of A_K. Let $\underline{M}$ be an A-motive over K and let $v \subset A$ be a maximal ideal with $v \neq A\text{-char}(K)$. Since $(A_{v,K^{\mathrm{sep}}})^{\tau=\mathrm{id}} = A_v$, we can define the *$v$-adic cohomology realizations* of $\underline{M}$ as the A_v-modules

$$\begin{aligned} \mathrm{H}^1_v(\underline{M}, A_v) &:= (M \otimes_{A_K} A_{v,K^{\mathrm{sep}}})^\tau := \{ m \in M \otimes_{A_K} A_{v,K^{\mathrm{sep}}} \mid \tau_M(\sigma_M^* m) = m \} \qquad \text{and} \\ \mathrm{H}_{1,v}(\underline{M}, A_v) &:= \mathrm{Hom}_{A_v}(\mathrm{H}^1_v(\underline{M}, A_v), A_v). \end{aligned} \tag{13.2}$$

They are free A_v-modules of rank equal to $\operatorname{rk} M$, carrying a continuous action of the Galois group $\mathscr{G}_K$ by [TW96, Proposition 6.1], and the inclusion $\mathrm{H}^1_v(\underline{M}, A_v) \subset M \otimes_{A_K} A_{v,K^{\mathrm{sep}}}$ induces a canonical isomorphism of $A_{v,K^{\mathrm{sep}}}$-modules

$$\mathrm{H}^1_v(\underline{M}, A_v) \otimes_{A_v} A_{v,K^{\mathrm{sep}}} \xrightarrow{\sim} M \otimes_{A_K} A_{v,K^{\mathrm{sep}}}$$

which is both $\mathscr{G}_K$ and τ-equivariant, where on the left module $\mathscr{G}_K$ acts on both factors and τ is $\mathrm{id} \otimes \sigma^*$ and on the right module $\mathscr{G}_K$ acts only on $A_{v,K^{\mathrm{sep}}}$ and τ is $(\tau_M \circ \sigma_M^*) \otimes \sigma^*$. One also sometimes denotes $\mathrm{H}^1_v(\underline{M}, A_v)$ by $\check{T}_v(\underline{M})$ and calls this the *v-adic dual Tate module* associated with $\underline{M}$ at v. We also define the Q_v-vector spaces with continuous $\mathscr{G}_K$-action

$$\begin{aligned} \mathrm{H}^1_v(\underline{M}, Q_v) &:= \mathrm{H}^1_v(\underline{M}, A_v) \otimes_{A_v} Q_v \qquad \text{and} \\ \mathrm{H}_{1,v}(\underline{M}, Q_v) &:= \mathrm{Hom}_{A_v}(\mathrm{H}^1_v(\underline{M}, A_v), Q_v) = \mathrm{H}_{1,v}(\underline{M}, A_v) \otimes_{A_v} Q_v\,. \end{aligned}$$

The association $\underline{M} \mapsto \mathrm{H}^1_v(\underline{M}, A_v)$ or $\underline{M} \mapsto \mathrm{H}^1_v(\underline{M}, Q_v)$ is a covariant functor which is exact and faithful.

The analog of the Tate conjecture is the following theorem which was proved by Taguchi [Tag95] and Tamagawa [Tam94, § 2].

Theorem 13.13 (Tate conjecture for A-motives)**.** *If K is a finitely generated A-field and $v \neq A\text{-char}(K)$ then*

$$\mathrm{Hom}(\underline{M}, \underline{M}') \otimes_A A_v \xrightarrow{\sim} \mathrm{Hom}_{A_v[\mathscr{G}_K]}(\mathrm{H}^1_v(\underline{M}, A_v), \mathrm{H}^1_v(\underline{M}', A_v))$$

is an isomorphism of A_v-modules for A-motives $\underline{M}$ and $\underline{M}'$.

Let us explain the relation between $T_v\underline{G}$ and $\check{T}_v\underline{M}(\underline{G}) := \mathrm{H}^1_v(\underline{M}(\underline{G}), A_v)$ for a Drinfeld A-module $\underline{G}$. The A_v-module $\mathrm{Hom}_{\mathbb{F}_v}(Q_v/A_v, \mathbb{F}_v)$ is canonically isomorphic to the A_v-module $\widehat{\Omega}^1_{A_v/\mathbb{F}_v} = A_v\, dz_v$ of continuous differential forms; see [HK20, Equation (4.5)], and therefore, it is a free A_v-module of rank 1. If $\underline{G}$ is a Drinfeld A-module over K and $\underline{M} = \underline{M}(\underline{G})$ is its associated A-motive, then there is a natural $\mathscr{G}_K$-equivariant perfect pairing of A_v-modules

$$\langle\,.\,,\,.\,\rangle\colon\; T_v\underline{G} \times \check{T}_v\underline{M} \;\longrightarrow\; \mathrm{Hom}_{\mathbb{F}_v}(Q_v/A_v, \mathbb{F}_v) \;\cong\; \widehat{\Omega}^1_{A_v/\mathbb{F}_v}\,, \quad \langle f, m\rangle := m \circ f\,, \tag{13.3}$$

which identifies $T_v\underline{G}$ with the contragredient $\mathscr{G}_K$-representation $\mathrm{Hom}_{A_v}(\check{T}_v\underline{M},\, \widehat{\Omega}^1_{A_v/\mathbb{F}_v})$ of $\check{T}_v\underline{M}$; see [HK20, Proposition 4.9]. Together with Theorems 9.7 and 13.13 this implies the following

Corollary 13.14 (Tate conjecture for Drinfeld A-modules)**.** *Let $\underline{G}$ and $\underline{G}'$ be two Drinfeld A-modules over a finitely generated field K. Then the natural map*

$$\mathrm{Hom}_K(\underline{G}, \underline{G}') \otimes_A A_v \to \mathrm{Hom}_{A_v[\mathscr{G}_K]}(T_v\underline{G}, T_v\underline{G}'), \quad f \otimes a \mapsto a \cdot T_v(f)$$

is an isomorphism of A_v-modules.

13.3 De Rham Cohomology and Period Isomorphisms

In this subsection let (K, γ) be an A-field of generic A-characteristic. Then K is a field extension of Q via γ and we set $\zeta := \gamma(z)$. There is an identification $\varprojlim A_K/\mathcal{J}^n = K[\![z-\zeta]\!]$ from [HJ20, Lemma 1.3].

Definition 13.15. Let $\underline{M}$ be an A-motive over an A-field K of generic A-characteristic. The *de Rham realization of $\underline{M}$* is defined as

$$\begin{aligned}
\mathrm{H}^1_{\mathrm{dR}}\big(\underline{M}, K[\![z-\zeta]\!]\big) \;&:=\; \sigma^*M \otimes_{A_K} \varprojlim A_K/\mathcal{J}^n\,, \\
\mathrm{H}^1_{\mathrm{dR}}\big(\underline{M}, K(\!(z-\zeta)\!)\big) \;&:=\; \mathrm{H}^1_{\mathrm{dR}}\big(\underline{M}, K[\![z-\zeta]\!]\big) \otimes_{K[\![z-\zeta]\!]} K(\!(z-\zeta)\!) \qquad \text{and} \\
\mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K) \;&:=\; \sigma^*M \otimes_{A_K} A_K/\mathcal{J} \\
&=\; \mathrm{H}^1_{\mathrm{dR}}\big(\underline{M}, K[\![z-\zeta]\!]\big) \otimes_{K[\![z-\zeta]\!]} K[\![z-\zeta]\!]/(z-\zeta)\,.
\end{aligned}$$

The *Hodge-Pink lattice of $\underline{M}$* is defined as $\mathfrak{q}^{\underline{M}} := \tau_M^{-1}(M \otimes_{A_K} \varprojlim A_K/\mathcal{J}^n) \subset \mathrm{H}^1_{\mathrm{dR}}\big(\underline{M}, K(\!(z-\zeta)\!)\big)$, and the descending *Hodge-Pink filtration of $\underline{M}$* is defined via $\mathfrak{p}^{\underline{M}} := \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K[\![z-\zeta]\!])$ and

$$\begin{aligned}
F^i\,\mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K) \;&:=\; \big(\mathfrak{p}^{\underline{M}} \cap (z-\zeta)^i\mathfrak{q}^{\underline{M}}\big)\big/\big((z-\zeta)\mathfrak{p}^{\underline{M}} \cap (z-\zeta)^i\mathfrak{q}^{\underline{M}}\big) \\
&=\; \text{image of } \big(\sigma^*M \cap \tau_M^{-1}(\mathcal{J}^iM)\big) \otimes_R K \;\text{ in }\; \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K)\,;
\end{aligned}$$

compare also with [Gos96, § 2.6]. Since $\underline{M}$ is effective, we have $\mathfrak{p}^{\underline{M}} \subset \mathfrak{q}^{\underline{M}}$ with $\tau_M\colon \mathfrak{q}^{\underline{M}}/\mathfrak{p}^{\underline{M}} \xrightarrow{\sim} \mathrm{coker}\,\tau_M$ and $F^0\,\mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K) = \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K)$. Note that the de Rham realization with Hodge-Pink lattice and filtration is a covariant functor on the category of A-motives over K with quasi-morphisms.

Definition 13.16. If $\underline{G}$ is a Drinfeld A-module over an A-field K of generic characteristic, let $\underline{M} = (M, \tau_M) = \underline{M}(\underline{G})$ be the associated A-motive. Then the *de Rham cohomology realization* of $\underline{G}$ is defined to be

$$\begin{aligned}
\mathrm{H}^1_{\mathrm{dR}}(\underline{G}, K) \;&:=\; \mathrm{Hom}_A(\Omega^1_{A/\mathbb{F}_q},\, \sigma^*M/\mathcal{J}\cdot\sigma^*M)\,, \\
\mathrm{H}^1_{\mathrm{dR}}(\underline{G}, K[\![z-\zeta]\!]) \;&:=\; \mathrm{Hom}_A\big(\Omega^1_{A/\mathbb{F}_q},\, \sigma^*M \otimes_{A_K} K[\![z-\zeta]\!]\big)\,, \\
\mathrm{H}_{1,\mathrm{dR}}(\underline{G}, K[\![z-\zeta]\!]) \;&:=\; \mathrm{Hom}_{A_K}(\sigma^*M,\, \widehat{\Omega}^1_{K[\![z-\zeta]\!]/K}) \quad \text{and} \\
\mathrm{H}_{1,\mathrm{dR}}(\underline{G}, K) \;&:=\; \mathrm{Hom}_{A_K}(\sigma^*M,\, \widehat{\Omega}^1_{K[\![z-\zeta]\!]/K}) \otimes_{K[\![z-\zeta]\!]} K[\![z-\zeta]\!]/(z-\zeta)\,,
\end{aligned}$$

where $\Omega^1_{A/\mathbb{F}_q}$ is the module of Kähler differentials of A over $\mathbb{F}_q$ and $\widehat{\Omega}^1_{K[\![z-\zeta]\!]/K} = K[\![z-\zeta]\!]dz$ is the $K[\![z-\zeta]\!]$-module of continuous differentials. We define the *Hodge-Pink lattices* of $\underline{G}$ as the $K[\![z-\zeta]\!]$-submodules

$$\begin{aligned}
\mathfrak{q}^{\underline{G}} \;&:=\; \mathrm{Hom}_A\big(\Omega^1_{A/\mathbb{F}_q},\, \tau_M^{-1}(M) \otimes_{A_K} K[\![z-\zeta]\!]\big) \;\subset\; \mathrm{H}^1_{\mathrm{dR}}\big(\underline{G}, K(\!(z-\zeta)\!)\big) \quad \text{and} \\
\mathfrak{q}_{\underline{G}} \;&:=\; (\tau_M^\vee \otimes \mathrm{id}_{K(\!(z-\zeta)\!)})\big(\mathrm{Hom}_{A_K}(M,\, \widehat{\Omega}^1_{K[\![z-\zeta]\!]/K})\big) \;\subset\; \mathrm{H}_{1,\mathrm{dR}}\big(\underline{G}, K(\!(z-\zeta)\!)\big)\,.
\end{aligned}$$

In both cases the Hodge-Pink filtrations $F^i\,\mathrm{H}^1_{\mathrm{dR}}(\underline{G},K)$ and $F^i\,\mathrm{H}_{1,\mathrm{dR}}(\underline{G},K)$ of $\underline{G}$ are recovered as the images of $\mathrm{H}^1_{\mathrm{dR}}\big(\underline{G},K[\![z-\zeta]\!]\big)\cap(z-\zeta)^i\mathfrak{q}^{\underline{G}}$ in $\mathrm{H}^1_{\mathrm{dR}}(\underline{G},K)$ and of $\mathrm{H}_{1,\mathrm{dR}}\big(\underline{G},K[\![z-\zeta]\!]\big)\cap(z-\zeta)^i\mathfrak{q}_{\underline{G}}$ in $\mathrm{H}_{1,\mathrm{dR}}(\underline{G},K)$ like in Definition 13.15. All these structures are compatible with the natural duality between $\mathrm{H}^1_{\mathrm{dR}}$ and $\mathrm{H}_{1,\mathrm{dR}}$.

Remark 13.17. It was shown in [HJ20, Remark 4.45 and Lemma 5.46] that this definition coincides with the definitions given by Deligne, Anderson, Gekeler and Jing Yu, see [Gos94, Definition 2.6.1], [Gek89, § 2] and [Yu90]. Moreover, it was shown in [HJ20, Diagram (5.36) in the Proof of Theorem 5.40] that the dual of the sequence of $K[\![z-\zeta]\!]$-modules $0\to\mathfrak{p}^{\underline{M}}\to\mathfrak{q}^{\underline{M}}\to\operatorname{coker}\tau_M\to 0$ is isomorphic to the sequence

$$0\longrightarrow\mathfrak{q}_{\underline{G}}\longrightarrow\mathrm{H}_{1,\mathrm{dR}}(\underline{G},K[\![z-\zeta]\!])\longrightarrow\operatorname{Lie}\underline{G}\longrightarrow 0\,.$$

Since $z-\zeta=0$ on $\operatorname{Lie}\underline{G}$ we obtain modulo $(z-\zeta)\,\mathrm{H}_{1,\mathrm{dR}}(\underline{G},K[\![z-\zeta]\!])$ the exact sequence of K-vector spaces

$$0\longrightarrow F^0\,\mathrm{H}_{1,\mathrm{dR}}(\underline{G},K)\longrightarrow\mathrm{H}_{1,\mathrm{dR}}(\underline{G},K)\longrightarrow\operatorname{Lie}\underline{G}\longrightarrow 0\,,\tag{13.4}$$

which is the analog of the decomposition (6.2).

For a uniformizable A-motive $\underline{M}$ over $\mathbb{C}_\infty$ the morphism $h_{\underline{M}}$ from Proposition 13.7 induces comparison isomorphisms between the Betti and the v-adic, respectively the de Rham realizations as follows.

Since $v\neq\infty$ the points in the closed subscheme $\{v\}\times_{\mathbb{F}_q}\operatorname{Spec}\mathbb{C}_\infty\subset C_{\mathbb{C}_\infty}$ do not specialize to $\infty_{\kappa_{\mathbb{C}}}\in C_{\kappa_{\mathbb{C}}}$ and so this closed subscheme lies in $C_{\mathbb{C}_\infty}\smallsetminus\cup_i\mathfrak{D}_i$. This gives us isomorphisms $\mathcal{O}(C_{\mathbb{C}_\infty}\smallsetminus\cup_i\mathfrak{D}_i)/v^n\mathcal{O}(C_{\mathbb{C}_\infty}\smallsetminus\cup_i\mathfrak{D}_i)\xrightarrow{\sim}A_{\mathbb{C}_\infty}/v^nA_{\mathbb{C}_\infty}$ for all $n\in\mathbb{N}$ and $\varprojlim\mathcal{O}(C_{\mathbb{C}_\infty}\smallsetminus\cup_i\mathfrak{D}_i)/v^n\mathcal{O}(C_{\mathbb{C}_\infty}\smallsetminus\cup_i\mathfrak{D}_i)\xrightarrow{\sim}\varprojlim A_{\mathbb{C}_\infty}/v^nA_{\mathbb{C}_\infty}=A_{v,\mathbb{C}_\infty}$. The isomorphism $h_{\underline{M}}$ from Proposition 13.7 induces a τ-equivariant isomorphism

$$\mathrm{H}^1_{\mathrm{Betti}}(\underline{M},A)\otimes_A\varprojlim\mathcal{O}(C_{\mathbb{C}_\infty}\smallsetminus\cup_i\mathfrak{D}_i)/v^n\mathcal{O}(C_{\mathbb{C}_\infty}\smallsetminus\cup_i\mathfrak{D}_i)\xrightarrow{\sim}M\otimes_{A_{\mathbb{C}_\infty}}A_{v,\mathbb{C}_\infty}\,.$$

Taking τ-invariant on both sides provides us with the isomorphism between the Betti and the v-adic realization

$$h_{\mathrm{Betti},v}:\mathrm{H}^1_{\mathrm{Betti}}(\underline{M},A_v)=\mathrm{H}^1_{\mathrm{Betti}}(\underline{M},A)\otimes_A A_v\xrightarrow{\sim}\mathrm{H}^1_v(\underline{M},A_v),\ \ \lambda\otimes f\mapsto(f\cdot\lambda\bmod v^n)_{n\in\mathbb{N}}.$$

On the other hand, Proposition 13.7 implies that $\sigma^*h_{\underline{M}}$ is an isomorphism locally at $\mathrm{V}(\mathcal{J})$ that is

$$\sigma^*h_{\underline{M}}\otimes\mathrm{id}_{\mathbb{C}_\infty[\![z-\zeta]\!]}:\mathrm{H}^1_{\mathrm{Betti}}(\underline{M},A)\otimes_A\mathbb{C}_\infty[\![z-\zeta]\!]\xrightarrow{\sim}\sigma^*M\otimes_{A_{\mathbb{C}_\infty}}\mathbb{C}_\infty[\![z-\zeta]\!].$$

This induces an isomorphism between the Betti and the de Rham realization

$$\begin{aligned}
h_{\mathrm{Betti,dR}}&:=\sigma^*h_{\underline{M}}\otimes\mathrm{id}_{\mathbb{C}_\infty[\![z-\zeta]\!]}: && \mathrm{H}^1_{\mathrm{Betti}}(\underline{M},\mathbb{C}_\infty[\![z-\zeta]\!])\xrightarrow{\sim}\mathrm{H}^1_{\mathrm{dR}}(\underline{M},\mathbb{C}_\infty[\![z-\zeta]\!]),\\
h_{\mathrm{Betti,dR}}&:=\sigma^*h_{\underline{M}}\bmod\mathcal{J}: && \mathrm{H}^1_{\mathrm{Betti}}(\underline{M},\mathbb{C}_\infty)\xrightarrow{\sim}\mathrm{H}^1_{\mathrm{dR}}(\underline{M},\mathbb{C}_\infty).
\end{aligned}$$

We summarize the above result as follows, compare [HJ20, Theorem 3.39].

Theorem 13.18. *If $\underline{M}$ is a uniformizable A-motive over $\mathbb{C}_\infty$ there are canonical* comparison isomorphisms, *sometimes also called* period isomorphisms

$$h_{\mathrm{Betti},v}:\mathrm{H}^1_{\mathrm{Betti}}(\underline{M},A_v)=\mathrm{H}^1_{\mathrm{Betti}}(\underline{M},A)\otimes_A A_v\xrightarrow{\sim}\mathrm{H}^1_v(\underline{M},A_v),\ \ \lambda\otimes f\mapsto(f\cdot\lambda\bmod v^n)_{n\in\mathbb{N}}\tag{13.5}$$

and

$$\begin{aligned}
h_{\mathrm{Betti,dR}}&:=\sigma^*h_{\underline{M}}\otimes\mathrm{id}_{\mathbb{C}_\infty[\![z-\zeta]\!]}: && \mathrm{H}^1_{\mathrm{Betti}}(\underline{M},\mathbb{C}_\infty[\![z-\zeta]\!])\xrightarrow{\sim}\mathrm{H}^1_{\mathrm{dR}}(\underline{M},\mathbb{C}_\infty[\![z-\zeta]\!]),\\
h_{\mathrm{Betti,dR}}&:=\sigma^*h_{\underline{M}}\bmod\mathcal{J}: && \mathrm{H}^1_{\mathrm{Betti}}(\underline{M},\mathbb{C}_\infty)\xrightarrow{\sim}\mathrm{H}^1_{\mathrm{dR}}(\underline{M},\mathbb{C}_\infty).
\end{aligned}\tag{13.6}$$

The latter yields a pairing

$$\begin{aligned}
\langle\,.\,,\,.\,\rangle_\infty:\quad\mathrm{H}_{1,\mathrm{Betti}}(\underline{M},\mathbb{C}_\infty)\times\mathrm{H}^1_{\mathrm{dR}}(\underline{M},\mathbb{C}_\infty)&\longrightarrow\ \mathbb{C}_\infty\,,\\
(u\,,\,\omega)\qquad\qquad&\longmapsto\ \langle u,\omega\rangle_\infty\ :=\ u\otimes\mathrm{id}_{\mathbb{C}_\infty}\big(h^{-1}_{\mathrm{Betti,dR}}(\omega)\big)\,.
\end{aligned}\tag{13.7}$$

All these cohomology realizations and period isomorphisms are functorial in $\underline{M}$ and by [HJ20, Theorem 5.49] compatible with the functor from Drinfeld A-modules to A-motives, Proposition 13.11 and the pairing (13.3).

Example 13.19. For the Carlitz motive $\underline{\mathcal{C}} = (\mathcal{C} = \mathbb{F}_q(\theta)[t], \tau_{\mathcal{C}} = t-\theta)$ from Example 9.8 the period isomorphism $h_{\mathrm{Betti,dR}}$ is given as follows. By Example 13.5 the generator $\eta\ell^-$ of $\mathrm{H}^1_{\mathrm{Betti}}(\underline{\mathcal{C}}, \mathbb{C}_\infty) = A\cdot\eta\ell^-$ is sent under $h_{\mathrm{Betti,dR}}$ to the element $\sigma^*(\eta\ell^-)|_{t=\theta} = \eta^q\prod_{i=1}^\infty(1-\theta^{1-q^i}) \in \mathbb{C}_\infty$ which has absolute value $\left|\eta^q\prod_{i=1}^\infty(1-\theta^{1-q^i})\right|_\infty = |\eta^q|_\infty = |\theta|_\infty^{q/(1-q)} = q^{-q/(q-1)}$. This element is the analog of the period $(2\pi i)^{-1}$ from Example 4.5, because the Carlitz module and Carlitz motive are the analogs of the multiplicative group $\mathbb{G}_m$, see Example 12.3.

Theorem 13.20. *Let $\underline{G}$ be a Drinfeld A-module over $\mathbb{C}_\infty$ and let $\underline{M} = (M,\tau_M) = \underline{M}(\underline{G}) := \mathrm{Hom}_{\mathbb{C}_\infty,\mathbb{F}_q}(G, \mathbb{G}_{a,\mathbb{C}_\infty})$ be the associated A-motive. Let $\mathfrak{q}^{\underline{M}}$ and $\mathfrak{p}^{\underline{M}} := \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, \mathbb{C}_\infty[\![z-\zeta]\!])$ be as in Definition 13.15. Let $m \in \mathfrak{q}^{\underline{M}}$ be such that its image $\overline{m}$ under the isomorphism $\tau_M \otimes \mathrm{id}_{\mathbb{C}_\infty[\![z-\zeta]\!]}\colon \mathfrak{q}^{\underline{M}}/\mathfrak{p}^{\underline{M}} \xrightarrow{\sim} \operatorname{coker}\tau_M$ generates the one dimensional $\mathbb{C}_\infty$-vector space $\operatorname{coker}\tau_M$. Let $\omega := -(z-\zeta)\cdot m \in (z-\zeta)\mathfrak{q}^{\underline{M}} \subset \mathfrak{p}^{\underline{M}}$. Consider the pairing*

$$\operatorname{coker}\tau_M \times \operatorname{Lie}\underline{G} \longrightarrow \operatorname{Lie}\mathbb{G}_{a,\mathbb{C}_\infty} = \mathbb{C}_\infty\,, \qquad (\overline{m},\lambda) \longmapsto \overline{m}(\lambda) \tag{13.8}$$

induced from (9.5) *and the isomorphism*

$$\beta_A\colon \mathrm{H}_{1,\mathrm{Betti}}(\underline{G}, Q) \xrightarrow{\sim} \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, Q)\otimes_Q \Omega^1_{Q/\mathbb{F}_q} = \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, Q)\cdot dz$$

from Proposition 13.11 using Remark 13.10. Let $\lambda \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{G}, Q) \subset \operatorname{Lie}\underline{G}$ and let $u \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, Q)$ be such that $\beta_A(\lambda) = u\,dz$. Then the pairing (13.7) *can be computed as*

$$\langle u,\omega\rangle_\infty = \overline{m}(\lambda)\,. \tag{13.9}$$

Proof. As in [HJ20, Diagram (5.36) in the proof of Theorem 5.39] the isomorphism β_A fits into a commutative diagram

$$\begin{tikzcd} \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, Q)\otimes_Q \Omega^1_{Q/\mathbb{F}_q} \arrow[r, "\tilde{\gamma}_A"] & \mathrm{Hom}_{\mathbb{C}_\infty}(\operatorname{coker}\tau_M, \mathbb{C}_\infty) \\ \mathrm{H}_{1,\mathrm{Betti}}(\underline{G}, Q) \arrow[u, "\beta_A", "\cong"'] \arrow[r, hook] & \operatorname{Lie} G \arrow[u, "\alpha"', "\cong"] \end{tikzcd} \tag{13.10}$$

where the isomorphism α is induced from the pairing (13.8), and the map $\tilde{\gamma}_A$ is given by

$$\begin{aligned} \tilde{\gamma}_A\colon \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, Q)\otimes_Q \Omega^1_{Q/\mathbb{F}_q} = \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, Q)\cdot dz &\longrightarrow \mathrm{Hom}_{\mathbb{C}_\infty}(\operatorname{coker}\tau_M, \mathbb{C}_\infty)\,, \\ u\,dz &\longmapsto \big[\overline{m} \mapsto -\operatorname{Res}_{z=\zeta} u(\overline{m})dz\big]\,. \end{aligned}$$

Here $u(\overline{m}) \in \mathbb{C}_\infty(\!(z-\zeta)\!)$ is defined as

$$\begin{aligned} u(\overline{m}) &:= \big(u\otimes \mathrm{id}_{\mathbb{C}_\infty(\!(z-\zeta)\!)}\big)\circ\big(h_{\underline{M}}\otimes \mathrm{id}_{\mathbb{C}_\infty(\!(z-\zeta)\!)}\big)^{-1}\circ\big(\tau_M\otimes \mathrm{id}_{\mathbb{C}_\infty(\!(z-\zeta)\!)}\big)(m) \\ &= \big(u\otimes \mathrm{id}_{\mathbb{C}_\infty(\!(z-\zeta)\!)}\big)\circ\big(h^{-1}_{\mathrm{Betti,dR}}\otimes \mathrm{id}_{\mathbb{C}_\infty(\!(z-\zeta)\!)}\big)(m) \end{aligned}$$

where

$$h_{\underline{M}}\otimes \mathrm{id}_{\mathbb{C}_\infty(\!(z-\zeta)\!)}\colon \mathrm{H}^1_{\mathrm{Betti}}(\underline{M}, Q)\otimes_Q \mathbb{C}_\infty(\!(z-\zeta)\!) \xrightarrow{\sim} M\otimes_{A_{\mathbb{C}_\infty}} \mathbb{C}_\infty(\!(z-\zeta)\!)$$

is the isomorphism from Proposition 13.7 with $h_{\underline{M}} = \tau_M\circ\sigma^*h_{\underline{M}}$ and $h_{\mathrm{Betti,dR}} = \sigma^*h_{\underline{M}}\otimes\mathrm{id}_{\mathbb{C}_\infty[\![z-\zeta]\!]}$. Note that $u(\overline{m})$ is only well defined up to adding elements of $\mathbb{C}_\infty[\![z-\zeta]\!]$, because the preimage m of $\overline{m}$ is only well defined up to $\mathfrak{p}^{\underline{M}}$ and $(u\circ h^{-1}_{\mathrm{Betti,dR}})(\mathfrak{p}^{\underline{M}}) = u\big(\mathrm{H}^1_{\mathrm{Betti}}(\underline{M}, \mathbb{C}_\infty[\![z-\zeta]\!])\big) \subset \mathbb{C}_\infty[\![z-\zeta]\!]$. This shows that, nevertheless, the residue $-\operatorname{Res}_{z=\zeta} u(\overline{m})dz$ is well defined and independent of the preimage m of $\overline{m}$. We may thus compute

$$\overline{m}(\lambda) = \alpha(\lambda)(\overline{m}) = (\tilde{\gamma}_A\circ\beta_A)(\lambda)(\overline{m}) = \tilde{\gamma}_A(u\,dz)(\overline{m}) = -\operatorname{Res}_{z=\zeta} u(\overline{m})dz\,.$$

Now $m = -(z-\zeta)^{-1}\cdot\omega$ and $u(\overline{m}) = (u\circ h^{-1}_{\mathrm{Betti,dR}})(m) = -(z-\zeta)^{-1}\cdot\langle u,\omega\rangle_\infty$ in $\mathbb{C}_\infty(\!(z-\zeta)\!)/\mathbb{C}_\infty[\![z-\zeta]\!]$. This yields

$$\overline{m}(\lambda) = -\operatorname{Res}_{z=\zeta} u(\overline{m})dz = \operatorname{Res}_{z=\zeta}\Big(\langle u,\omega\rangle_\infty \tfrac{d(z-\zeta)}{z-\zeta}\Big) = \langle u,\omega\rangle_\infty\,.$$

□

14 Local Shtukas and the v-adic Period Isomorphism

We next describe the function field analog of p-divisible groups.

Notation 14.1. We fix a place $v \neq \infty$ of Q. Let $K \subset Q^{\mathrm{alg}}$ be an A-field which is a finite extension of Q via γ. Under the fixed embedding $Q^{\mathrm{alg}} \hookrightarrow \mathbb{C}_v$ let L be the v-adic completion of $K \subset \mathbb{C}_v$. Let R be the valuation ring of L, let π_L be a uniformizing parameter of R and let κ be the residue field of R. Then $R = \kappa[\![\pi_L]\!]$ and $L = \kappa(\!(\pi_L)\!)$. The homomorphism $\gamma\colon A \to K$ extends by continuity to $\gamma\colon A_v \to L$ and factors through $\gamma\colon A_v \to R$ with $\zeta_v = \gamma(z_v) \in \pi_L R \smallsetminus \{0\}$. Let $R[\![z_v]\!]$ be the power series ring in the variable z_v over R and $\hat\sigma_v^*$ the endomorphism of $R[\![z_v]\!]$ with $\hat\sigma_v^*(z_v) = z_v$ and $\hat\sigma_v^*(b) = b^{q_v}$ for $b \in R$, where $q_v = \#\mathbb{F}_v$. For an $R[\![z_v]\!]$-module $\hat M$ we let $\hat\sigma_v^*\hat M := \hat M \otimes_{R[\![z_v]\!],\hat\sigma_v^*} R[\![z_v]\!]$ as well as $\hat M[\frac{1}{z_v-\zeta_v}] := \hat M \otimes_{R[\![z_v]\!]} R[\![z_v]\!][\frac{1}{z_v-\zeta_v}]$ and $\hat M[\frac{1}{z_v}] := \hat M \otimes_{R[\![z_v]\!]} R[\![z_v]\!][\frac{1}{z_v}]$. We obtain a canonical embedding $A_R := A \otimes_{\mathbb{F}_q} R \hookrightarrow R[\![z_v]\!]$ by mapping $z_v \otimes 1 \mapsto z_v$ and $1 \otimes \zeta_v \mapsto \zeta_v$.

The function field analog of p-divisible groups is given by the following

Definition 14.2. A *z_v-divisible local Anderson module over R* is a sheaf of $\mathbb{F}_q[\![z_v]\!]$-modules G on the big *fppf*-site of $\operatorname{Spec} R$ such that

(a) G is *z_v-torsion*, that is $G = \varinjlim G[z_v^n]$,

(b) G is *z_v-divisible*, that is $z_v\colon G \to G$ is an epimorphism,

(c) for every n the $\mathbb{F}_q$-module $G[z_v^n]$ is representable by a finite locally free strict $\mathbb{F}_q$-module scheme over R in the sense of Faltings (see [Fal02] or [HS20, Definition 4.7]), and

(d) locally on $\operatorname{Spec} R$ there exists an integer $d \in \mathbb{Z}_{\geq 0}$, such that $(z_v - \zeta_v)^d = 0$ on ω_G where $\omega_G := \varprojlim \omega_{G[z_v^n]}$ and $\omega_{G[z_v^n]} := \varepsilon^*\Omega^1_{G[z_v^n]/\operatorname{Spec} R}$ for the unit section ε of $G[z_v^n]$ over R.

Example 14.3. Let $\underline{G} = (G, \varphi)$ be a Drinfeld A-module over R which is defined as in Definition 9.2 by replacing K by R. By [Har17, Theorem 6.6] the torsion module $\underline{G}[v^n]$ is a finite locally free strict $\mathbb{F}_v$-module scheme and the inductive limit $\underline{G}[v^\infty] := \varinjlim \underline{G}[v^n]$ is a z_v-divisible local Anderson module over R for which one can take $d = 1$ in Definition 14.2(d).

Similarly to Remark 5.3, divisible local Anderson modules have a description by semi-linear algebra. It is given by local $\hat\sigma_v^*$-shtukas.

Definition 14.4. A *local $\hat\sigma_v^*$-shtuka of rank r* over R is a pair $\underline{\hat M} = (\hat M, \tau_{\hat M})$ consisting of a free $R[\![z_v]\!]$-module $\hat M$ of rank r, and an isomorphism $\tau_{\hat M}: \hat\sigma_v^*\hat M[\frac{1}{z_v-\zeta_v}] \xrightarrow{\sim} \hat M[\frac{1}{z_v-\zeta_v}]$. It is *effective* if $\tau_{\hat M}(\hat\sigma_v^*\hat M) \subset \hat M$ and *étale* if $\tau_{\hat M}(\hat\sigma_v^*\hat M) = \hat M$. We write $\operatorname{rk}\underline{\hat M}$ for the rank of $\underline{\hat M}$.

A *morphism* of local shtukas $f: \underline{\hat M} = (\hat M, \tau_{\hat M}) \to \underline{\hat N} = (\hat N, \tau_{\hat N})$ over R is a morphism of the underlying modules $f: \hat M \to \hat N$ which satisfies $\tau_{\hat N} \circ \hat\sigma_v^* f = f \circ \tau_{\hat M}$. We denote the A_v-module of homomorphisms $f\colon \underline{\hat M} \to \underline{\hat N}$ by $\operatorname{Hom}_R(\underline{\hat M}, \underline{\hat N})$ and write $\operatorname{End}_R(\underline{\hat M}) = \operatorname{Hom}_R(\underline{\hat M}, \underline{\hat M})$.

A *quasi-morphism* between local shtukas $f\colon (\hat M, \tau_{\hat M}) \to (\hat N, \tau_{\hat N})$ over R is a morphism of $R[\![z_v]\!][\frac{1}{z_v}]$-modules $f\colon M[\frac{1}{z_v}] \xrightarrow{\sim} N[\frac{1}{z_v}]$ with $\tau_{\hat N} \circ \hat\sigma_v^* f = f \circ \tau_{\hat M}$. It is called a *quasi-isogeny* if it is an isomorphism of $R[\![z_v]\!][\frac{1}{z_v}]$-modules. We denote the Q_v-vector space of quasi-morphisms from $\underline{\hat M}$ to $\underline{\hat N}$ by $\operatorname{QHom}_R(\underline{\hat M}, \underline{\hat N})$ and write $\operatorname{QEnd}_R(\underline{\hat M}) = \operatorname{QHom}_R(\underline{\hat M}, \underline{\hat M})$.

Note that $\operatorname{Hom}_R(\underline{\hat M}, \underline{\hat N})$ is a finite free A_v-module of rank at most $\operatorname{rk}\underline{\hat M} \cdot \operatorname{rk}\underline{\hat N}$ by [HK20, Corollary 4.5] and $\operatorname{QHom}_R(\underline{\hat M}, \underline{\hat N}) = \operatorname{Hom}_R(\underline{\hat M}, \underline{\hat N}) \otimes_{A_v} Q_v$. Also every quasi-isogeny $f\colon \underline{\hat M} \to \underline{\hat N}$ induces an isomorphism of Q_v-algebras $\operatorname{QEnd}_R(\underline{\hat M}) \xrightarrow{\sim} \operatorname{QEnd}_R(\underline{\hat N})$, $g \mapsto fgf^{-1}$, similarly to Remark 2.5(a).

The analog of the ("local") Dieudonné functor from Remark 5.3 is given by the following

Theorem 14.5 ([HS20, Theorem 8.3]). *There is an anti-equivalence between the category of z_v-divisible local Anderson modules over R and the category of effective local $\hat\sigma_v^*$-shtukas over R given by the contravariant functor $\underline{\hat M}_{q_v}$ defined by $\underline{\hat M}_{q_v}(G) := \varprojlim_n \underline{\hat M}_{q_v}\big(G[z_v^n]\big)$, where*

$$\underline{\hat M}_{q_v}(G[z_v^n]) \;:=\; \big(\operatorname{Hom}_{R\text{-groups},\mathbb{F}_q\text{-lin}}(G[z_v^n], \mathbb{G}_{a,R}), \hat\tau_{M_q(G[z_v^n])}\big)$$

and $\hat\tau_{M_q(G[z_v^n])}$ is provided by the relative q_v-Frobenius of the additive group scheme $\mathbb{G}_{a,R}$ over R like in (9.4).

It turns out that like with abelian Anderson A-modules, one can dispense with the notions of z_v-divisible local Anderson modules, because their equivalent description by local $\hat{\sigma}_v^*$-shtukas can be obtained purely from A-motives as in the following

Example 14.6. Let $\underline{M} = (M, \tau_M)$ be an A-motive over K and assume that it has *good reduction*, that is, there exist a pair $\underline{\mathcal{M}} = (\mathcal{M}, \tau_{\mathcal{M}})$ consisting of a locally free module $\mathcal{M}$ over $A_R := A \otimes_{\mathbb{F}_q} R$ of finite rank and a morphism $\tau_{\mathcal{M}} : \sigma^*\mathcal{M} \to \mathcal{M}$ of A_R-modules whose cokernel is annihilated by a power of the ideal $\mathcal{J} := (a \otimes 1 - 1 \otimes \gamma(a) : a \in A) \subset A_R$, such that $\underline{\mathcal{M}} \otimes_R L \cong \underline{M} \otimes_K L$. The reduction $\underline{\mathcal{M}} \otimes_R \kappa$ is an A-motive over κ of A-characteristic $v = \ker(\gamma : A \to \kappa)$. The pair $\underline{\mathcal{M}}$ is called an *A-motive over R* and a *good model* of $\underline{M}$.

We consider the v-adic completions $A_{v,R}$ of A_R and $\underline{\mathcal{M}} \otimes_{A_R} A_{v,R} := (\mathcal{M} \otimes_{A_R} A_{v,R}, \tau_{\mathcal{M}} \otimes \mathrm{id})$ of $\underline{\mathcal{M}}$. We let $d_v := [\mathbb{F}_v : \mathbb{F}_q]$ and discuss the two cases $d_v = 1$ and $d_v > 1$ separately. If $d_v = 1$, and hence $q_v = q$ and $\hat{\sigma}_v^* = \sigma^*$, we have $A_{v,R} = R[\![z_v]\!]$, and $\underline{\mathcal{M}} \otimes_{A_R} A_{v,R}$ is an effective local $\hat{\sigma}_v^*$-shtuka over $\operatorname{Spec} R$ which we denote by $\underline{\hat{M}}_v(\underline{\mathcal{M}})$ and call the *local $\hat{\sigma}_v^*$-shtuka at v associated with $\underline{\mathcal{M}}$*.

If $d_v > 1$, the situation is more complicated, because $\mathbb{F}_v \otimes_{\mathbb{F}_q} R$ and $A_{v,R}$ fail to be integral domains. Namely,

$$\mathbb{F}_v \otimes_{\mathbb{F}_q} R = \prod_{\mathrm{Gal}(\mathbb{F}_v/\mathbb{F}_q)} \mathbb{F}_v \otimes_{\mathbb{F}_v} R = \prod_{i \in \mathbb{Z}/d_v\mathbb{Z}} \mathbb{F}_v \otimes_{\mathbb{F}_q} R \,/\, (a \otimes 1 - 1 \otimes \gamma(a)^{q^i} : a \in \mathbb{F}_v)$$

and σ^* transports the i-th factor to the $(i+1)$-th factor. In particular $\hat{\sigma}_v^*$ stabilizes each factor. Denote by $\mathfrak{a}_i$ the ideal of $A_{v,R}$ generated by $\{a \otimes 1 - 1 \otimes \gamma(a)^{q^i} : a \in \mathbb{F}_v\}$. Then

$$A_{v,R} = \prod_{\mathrm{Gal}(\mathbb{F}_v/\mathbb{F}_q)} A_v \widehat{\otimes}_{\mathbb{F}_v} R = \prod_{i \in \mathbb{Z}/d_v\mathbb{Z}} A_{v,R} \,/\, \mathfrak{a}_i.$$

Note that each factor is isomorphic to $R[\![z_v]\!]$ and the ideals $\mathfrak{a}_i$ correspond precisely to the places v_i of $C_{\mathbb{F}_v}$ lying above v. The ideal $\mathcal{J}$ decomposes as follows $\mathcal{J} \cdot A_{v,R}/\mathfrak{a}_0 = (z_v - \zeta_v)$ and $\mathcal{J} \cdot A_{v,R}/\mathfrak{a}_i = (1)$ for $i \neq 0$. We define the *local $\hat{\sigma}_v^*$-shtuka at v associated with $\underline{\mathcal{M}}$* as $\underline{\hat{M}}_v(\underline{\mathcal{M}}) := (\hat{M}, \tau_{\hat{M}}) := \big(\mathcal{M} \otimes_{A_R} A_{v,R}/\mathfrak{a}_0\,,\, (\tau_{\mathcal{M}} \otimes 1)^{d_v}\big)$, where $\tau_{\mathcal{M}}^{d_v} := \tau_{\mathcal{M}} \circ \sigma^*\tau_{\mathcal{M}} \circ \ldots \circ \sigma^{(d_v-1)*}\tau_{\mathcal{M}}$. Of course if $d_v = 1$ we get back the definition of $\underline{\hat{M}}_v(\underline{\mathcal{M}})$ given above. Also note that $\mathcal{M}/\tau_{\mathcal{M}}(\sigma^*\mathcal{M}) = \hat{\mathcal{M}}/\tau_{\hat{\mathcal{M}}}(\hat{\sigma}_v^*\hat{\mathcal{M}})$.

The local shtuka $\underline{\hat{M}}_v(\underline{\mathcal{M}})$ allows to recover $\underline{\mathcal{M}} \otimes_{A_R} A_{v,R}$ via the isomorphism

$$\bigoplus_{i=0}^{d_v-1} (\tau_{\mathcal{M}} \otimes 1)^i \bmod \mathfrak{a}_i \colon \Big(\bigoplus_{i=0}^{d_v-1} \sigma^{i*}(\mathcal{M} \otimes_{A_R} A_{v,R}/\mathfrak{a}_0),\ (\tau_{\mathcal{M}} \otimes 1)^{d_v} \oplus \bigoplus_{i \neq 0} \mathrm{id}\Big) \xrightarrow{\ \sim\ } \underline{\mathcal{M}} \otimes_{A_R} A_{v,R}\,,$$

because for $i \neq 0$ the equality $\mathcal{J} \cdot A_{v,R}/\mathfrak{a}_i = (1)$ implies that $\tau_{\mathcal{M}} \otimes 1$ is an isomorphism modulo $\mathfrak{a}_i$; see [BH11, Propositions 8.8 and 8.5] for more details.

Proposition 14.7 ([Har17, Theorem 7.6]). *Let $\underline{G} = (G, \varphi)$ be a Drinfeld A-module over R and let $\underline{G}[v^\infty] := \varinjlim \underline{G}[v^n]$ be its z_v-divisible local Anderson module over R from Example 14.3. Let $\underline{\mathcal{M}}(\underline{G})$ be the associated A-motive over R and let $\underline{\hat{M}}_{q_v}(\underline{G}[v^\infty])$ be the associated local $\hat{\sigma}_v^*$-shtuka over R. Then $\underline{\hat{M}}_{q_v}(\underline{G}[v^\infty])$ is canonically isomorphic to the local $\hat{\sigma}_v^*$-shtuka $\underline{\hat{M}}_v(\underline{\mathcal{M}})$ from Example 14.6.*

Example 14.8. It was shown in [HK20, Example 2.7] that the local $\hat{\sigma}_v^*$-shtuka at v associated with the Carlitz motive $\underline{\mathcal{C}} = (\mathcal{C} = \mathbb{F}_q(\theta)[t], \tau_{\mathcal{C}} = t - \theta)$ from Example 9.8 equals $\underline{\hat{M}}_v(\underline{\mathcal{C}}) = \big(\mathbb{F}_v[\![\zeta_v]\!][\![z]\!], \tau_{\hat{M}} = (z_v - \zeta_v)\big)$. Here $L = \mathbb{F}_v(\!(\zeta_v)\!)$ and $R = \mathcal{O}_L = \mathbb{F}_v[\![\zeta_v]\!]$.

Next we define the v-adic realization and the de Rham realization of a local shtuka $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ over R. Since $\tau_{\hat{M}}$ induces an isomorphism $\tau_{\hat{M}} : \hat{\sigma}_v^*\hat{M} \otimes_{R[\![z_v]\!]} L[\![z_v]\!] \xrightarrow{\sim} \hat{M} \otimes_{R[\![z_v]\!]} L[\![z_v]\!]$, we can think of $\underline{\hat{M}} \otimes_{R[\![z_v]\!]} L[\![z_v]\!]$ as an étale local shtuka over L.

Definition 14.9. The *v-adic realization* $\mathrm{H}^1_v(\underline{\hat{M}}, A_v)$ of a local $\hat{\sigma}_v^*$-shtuka $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ is the $\mathscr{G}_L$-module of τ-invariants

$$\mathrm{H}^1_v(\underline{\hat{M}}, A_v) \ :=\ (\hat{M} \otimes_{R[\![z_v]\!]} L^{\mathrm{sep}}[\![z_v]\!])^\tau := \{m \in \hat{M} \otimes_{R[\![z_v]\!]} L^{\mathrm{sep}}[\![z_v]\!] : \tau_{\hat{M}}(\hat{\sigma}_{\hat{M}}^* m) = m\},$$

where we set $\hat{\sigma}_{\hat{M}}^* m := m \otimes 1 \in \hat{M} \otimes_{R[\![z_v]\!], \hat{\sigma}_v^*} R[\![z_v]\!] =: \sigma^* M$ for $m \in M$. One also writes sometimes $\check{T}_v\underline{\hat{M}} = \mathrm{H}^1_v(\underline{\hat{M}}, A_v)$ and calls this the *dual Tate module of $\underline{\hat{M}}$*. By [HK20, Proposition 4.2] it is a free A_v-module of the same rank as $\hat{M}$. We also write $\mathrm{H}^1_v(\underline{\hat{M}}, B) := \mathrm{H}^1_v(\underline{\hat{M}}, A_v) \otimes_{A_v} B$ for an A_v-algebra B.

If $\underline{M} = (M, \tau_M)$ is an A-motive over L with good model $\underline{\mathcal{M}}$ and $\underline{\hat{M}} = \underline{\hat{M}}_v(\underline{\mathcal{M}})$ is the local shtuka at v associated with $\underline{\mathcal{M}}$, then $\mathrm{H}^1_v(\underline{\hat{M}}, A_v)$ is by [HK20, Proposition 4.6] canonically isomorphic as a representation of $\mathscr{G}_L$ to the v-adic realization $\mathrm{H}^1_v(\underline{M}, A_v)$ of $\underline{M}$.

Example 14.10. We describe the v-adic realization $\mathrm{H}^1_v(\underline{\mathcal{C}}, A_v) = \mathrm{H}^1_v(\underline{\hat{M}}_v(\underline{\mathcal{C}}), A_v)$ of the Carlitz module from Example 14.8 by using its local shtuka $\underline{\hat{M}}_v(\underline{\mathcal{C}}) = \big(\mathbb{F}_v[\![\zeta_v]\!][\![z_v]\!], \tau_{\hat{M}} = (z_v - \zeta_v)\big)$ at v computed there. For all $i \in \mathbb{N}_0$ let $\ell_i \in L^{\mathrm{sep}}$ be solutions of the equations $\ell_0^{q_v-1} = -\zeta_v$ and $\ell_i^{q_v} + \zeta_v \ell_i = \ell_{i-1}$. This implies $|\ell_i| = |\zeta_v|^{q_v^{-i}/(q_v-1)} < 1$. Define the power series $\ell_v^+ = \sum_{i=0}^\infty \ell_i z_v^i \in \mathcal{O}_{L^{\mathrm{sep}}}[\![z_v]\!]$. It satisfies $\hat{\sigma}_v^*(\ell_v^+) = (z_v - \zeta_v)\cdot \ell_v^+$, but depends on the choice of the ℓ_i. A different choice yields a different power series $\tilde{\ell}_v^+$ which satisfies $\tilde{\ell}_v^+ = u\ell_v^+$ for a unit $u \in (L^{\mathrm{sep}}[\![z_v]\!]^\times)^{\hat{\sigma}_v^* = \mathrm{id}} = \mathbb{F}_v[\![z_v]\!]^\times = A_v^\times$, because $\hat{\sigma}_v^*(u) = \frac{\hat{\sigma}_v^*(\tilde{\ell}_v^+)}{\hat{\sigma}_v^*(\ell_v^+)} = \frac{\tilde{\ell}_v^+}{\ell_v^+} = u$. The field extension $\mathbb{F}_v(\!(\zeta_v)\!)(\ell_i \colon i \in \mathbb{N}_0)$ of $\mathbb{F}_v(\!(\zeta_v)\!)$ is the function field analog of the cyclotomic tower $\mathbb{Q}_p(\sqrt[p^i]{1} \colon i \in \mathbb{N}_0)$; see [Har09, § 1.3 and § 3.4]. There is an isomorphism of topological groups called the *v-adic cyclotomic character*

$$\chi_v \colon \mathrm{Gal}\big(\mathbb{F}_v(\!(\zeta_v)\!)(\ell_i \colon i \in \mathbb{N}_0)/\mathbb{F}_v(\!(\zeta_v)\!)\big) \xrightarrow{\sim} A_v^\times,$$

which satisfies $g(\ell_v^+) := \sum_{i=0}^\infty g(\ell_i) z_v^i = \chi_v(g)\cdot \ell_v^+$ in $L^{\mathrm{sep}}[\![z_v]\!]$ for g in the Galois group. It is independent of the choice of the ℓ_i. The v-adic (dual) Tate module $\check{T}_v \underline{\hat{M}} = \mathrm{H}^1_v(\underline{\hat{M}}_v(\underline{\mathcal{C}}), A_v)$ of $\underline{\hat{M}}_v(\underline{\mathcal{C}})$ and $\underline{\mathcal{C}}$ is generated by $(\ell_v^+)^{-1}$ on which the Galois group acts by the inverse of the v-adic cyclotomic character. The reader should compare this to Example 5.6.

Definition 14.11. Let $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ be a local $\hat{\sigma}_v^*$-shtuka over R. We define the *de Rham realizations* of $\underline{\hat{M}}$ as

$$\begin{aligned}
\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, R) &:= \hat{\sigma}_v^* \hat{M}/(z_v - \zeta_v)\hat{M} \;=\; \hat{\sigma}_v^*\hat{M} \otimes_{R[\![z_v]\!], z_v \mapsto \zeta_v} R, \quad \text{as well as} \\
\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L[\![z_v - \zeta_v]\!]) &:= \hat{\sigma}_v^*\hat{M} \otimes_{R[\![z_v]\!]} L[\![z_v - \zeta_v]\!] \quad \text{and} \\
\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L) &:= \hat{\sigma}_v^*\hat{M} \otimes_{R[\![z_v]\!], z_v \mapsto \zeta_v} L \;=\; \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L[\![z_v - \zeta_v]\!]) \otimes_{L[\![z_v - \zeta_v]\!]} L[\![z_v - \zeta_v]\!]/(z_v - \zeta_v) \\
&= \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, R) \otimes_R L\,.
\end{aligned}$$

It carries the *Hodge-Pink lattice* $\mathfrak{q}^{\underline{\hat{M}}} := \tau_{\hat{M}}^{-1}(\hat{M} \otimes_{R[\![z_v]\!]} L[\![z_v - \zeta_v]\!]) \subset \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L[\![z_v - \zeta_v]\!])[\tfrac{1}{z_v - \zeta_v}]$.

If $\underline{M} = (M, \tau_M)$ is an A-motive over L with good model $\underline{\mathcal{M}}$ and $\underline{\hat{M}} = \underline{\hat{M}}_v(\underline{\mathcal{M}})$ is the local shtuka at v associated with $\underline{\mathcal{M}}$ and $d_v = [\mathbb{F}_v : \mathbb{F}_q]$ is as in Example 14.6, the map

$$\sigma^*\tau_M^{d_v-1} = \sigma^*\tau_M \circ \sigma^{2*}\tau_M \circ \cdots \circ \sigma^{(d_v-1)*}\tau_M \colon \sigma^{d_v *} M \otimes_{A_R} A_{v,R}/\mathfrak{a}_0 \xrightarrow{\sim} \sigma^* M \otimes_{A_R} A_{v,R}/\mathfrak{a}_0$$

is an isomorphism, because τ_M is an isomorphism over $A_{v,R}/\mathfrak{a}_i$ for all $i \neq 0$. Therefore, it defines canonical isomorphisms of the de Rham realizations

$$\begin{aligned}
\sigma^*\tau_M^{d_v-1} \colon \mathrm{H}^1_{\mathrm{dR}}\big(\underline{\hat{M}}, L[\![z_v - \zeta_v]\!]\big) &\xrightarrow{\sim} \mathrm{H}^1_{\mathrm{dR}}\big(\underline{M}, L[\![z_v - \zeta_v]\!]\big) \quad \text{and} \\
\sigma^*\tau_M^{d_v-1} \colon \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L) &\xrightarrow{\sim} \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, L)\,,
\end{aligned}$$

which are compatible with the Hodge-Pink lattices and the Hodge-Pink filtrations.

The *v-adic period isomorphism* for an A-motive $\underline{M}$ over a field $K \subset Q_v^{\mathrm{alg}}$ is provided by the following theorem by using the local $\hat{\sigma}_v^*$-shtuka $\underline{\hat{M}} := \underline{\hat{M}}_v(\underline{M})$.

Theorem 14.12 ([HK20, Theorem 4.14]). *If $\underline{\hat{M}}$ is a local $\hat{\sigma}_v^*$-shtuka over R then there is a canonical comparison isomorphism*

$$h_{v,\mathrm{dR}} \colon \; \mathrm{H}^1_v\big(\underline{\hat{M}}, Q_v\big) \otimes_{Q_v} \mathbb{C}_v(\!(z_v - \zeta_v)\!) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{dR}}\big(\underline{\hat{M}}, L(\!(z_v - \zeta_v)\!)\big) \otimes_{L(\!(z_v - \zeta_v)\!)} \mathbb{C}_v(\!(z_v - \zeta_v)\!)$$

If $\underline{M}$ is an A-motive over L (which does not need to have good reduction) then there is a canonical comparison isomorphism

$$h_{v,\mathrm{dR}} \colon \; \mathrm{H}^1_v\big(\underline{M}, Q_v\big) \otimes_{Q_v} \mathbb{C}_v(\!(z_v - \zeta_v)\!) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{dR}}\big(\underline{M}, L(\!(z_v - \zeta_v)\!)\big) \otimes_{L(\!(z_v - \zeta_v)\!)} \mathbb{C}_v(\!(z_v - \zeta_v)\!) \tag{14.1}$$

Both isomorphisms are equivariant for the action of $\mathscr{G}_L$, where on the source this group acts on both factors of the tensor product and on the target it acts only on $\mathbb{C}_v$.

In comparison with the p-adic comparison isomorphism for an abelian variety over a finite extension of $\mathbb{Q}_p$ from Theorem 5.5, the ring $\mathbb{C}_v((z_v - \zeta_v))$ is the function field analog of $\mathbb{B}_{p,\mathrm{dR}}$.

Example 14.13. For the Carlitz motive $\underline{\mathcal{C}} = (\mathcal{C} = \mathbb{F}_q(\theta)[t], \tau_{\mathcal{C}} = t - \theta)$ from Example 9.8 we have $\mathrm{H}^1_v(\underline{\mathcal{C}}, Q_v) = Q_v \cdot (\ell_v^+)^{-1} \cong Q_v$ and $\mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{C}}, \mathbb{F}_q(\theta)[\![z_v - \zeta_v]\!]) = \mathbb{F}_q(\theta)[\![z_v - \zeta_v]\!] =: \mathfrak{p}$, see Example 14.10. The Hodge-Pink lattice is $\mathfrak{q} = (z_v - \zeta_v)^{-1}\mathfrak{p}$ and the Hodge filtration satisfies $F^1 = \mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{C}}, \mathbb{F}_q(\theta)) \supset F^2 = (0)$. With respect to the bases $(\ell_v^+)^{-1}$ of $\mathrm{H}^1_v(\underline{\mathcal{C}}, Q_v)$ and 1 of $\mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{C}}, \mathbb{F}_q(\theta)[\![z_v - \zeta_v]\!])$ the comparison isomorphism $h_{v,\mathrm{dR}}$ from Theorem 14.12 is given by the *v-adic Carlitz period* $(z_v - \zeta_v)^{-1}(\ell_v^+)^{-1} = \hat{\sigma}_v^*(\ell_v^+)^{-1}$. It has a pole of order one at $z_v = \zeta_v$ because $\ell_v^+ \in \mathbb{F}_v((\zeta_v))^{\mathrm{sep}}\langle\frac{z_v}{\zeta_v}\rangle^\times \subset \mathbb{C}_v[\![z_v - \zeta_v]\!]^\times$. So $h_{v,\mathrm{dR}}\big(\mathrm{H}^1_v(\underline{\mathcal{C}}, Q_v) \otimes_{Q_v} \mathbb{C}_v[\![z_v - \zeta_v]\!]\big) = (z_v - \zeta_v)^{-1}\mathbb{C}_v[\![z_v - \zeta_v]\!] = \mathfrak{q} \otimes_{K[\![z_v - \zeta_v]\!]} \mathbb{C}_v[\![z_v - \zeta_v]\!]$.

Definition 14.14. On the power series ring $\mathcal{O}_{\mathbb{C}_v}[\![z_v]\!]$ we consider the $\mathcal{O}_{\mathbb{C}_v}$-embedding $\mathcal{O}_{\mathbb{C}_v}[\![z_v]\!] \hookrightarrow \mathbb{C}_v[\![z_v - \zeta_v]\!]$ given by $z_v \mapsto z_v = \zeta_v + (z_v - \zeta_v)$. Let $\Theta : \mathbb{C}_v[\![z_v - \zeta_v]\!] \to \mathbb{C}_v,\ z_v \mapsto \zeta_v$ be the residue map. Then $\mathcal{O}_{\mathbb{C}_v}[\![z_v]\!] \cap \ker\Theta$ is a principal ideal of $\mathcal{O}_v[\![z_v]\!]$ generated by $z_v - \zeta_v$. Any other generator is of the form $(z_v - \zeta_v)\cdot u$ with $u \in \mathcal{O}_{\mathbb{C}_v}[\![z_v]\!]^\times$. On $\mathbb{C}_v((z_v - \zeta_v))$ we define a valuation $\hat{v}$ by

$$\hat{v}\left(\sum_{i=-N}^{\infty} b_i (z_v - \zeta_v)^i\right) := \min\{i :\ b_i \neq 0\}.$$

and we extend the valuation v on $\mathbb{C}_v$ to $\mathbb{C}_v((z_v - \zeta_v))$ by

$$v(f) := v\big(\Theta(f \cdot (z_v - \zeta_v)^{-\hat{v}(f)})\big). \tag{14.2}$$

If f and g are two elements of $\mathbb{C}_v((z_v - \zeta_v))$, then $\hat{v}(fg) = \hat{v}(f) + \hat{v}(g)$, and hence $v(fg) = v(f) + v(g)$. But note that v does not satisfy the triangle inequality. The valuation $v(f)$ is unchanged, if we replace the generator $z_v - \zeta_v$ of $\mathcal{O}_{\mathbb{C}_v}[\![z_v]\!] \cap \ker\Theta$ by another generator $(z_v - \zeta_v)\cdot u$ with $u \in \mathcal{O}_{\mathbb{C}_v}[\![z_v]\!]^\times$, because then $v\big(\Theta(f\cdot((z_v - \zeta_v)\cdot u)^{-\hat{v}(f)}\big) = v\big(\Theta(f \cdot (z_v - \zeta_v)^{-\hat{v}(f)}\big) + v(\Theta(u))^{-\hat{v}(f)} = v\big(\Theta(f \cdot (z_v - \zeta_v)^{-\hat{v}(f)}\big)$ as $\Theta(u) \in \mathcal{O}_v^\times$.

Example 14.15. The inverse $(z_v - \zeta_v)(\ell_v^+) = \hat{\sigma}_v^*(\ell_v^+)$ of the v-adic Carlitz period $\sigma_v^*(\ell_v^+)^{-1}$ from Example 14.13 satisfies $\hat{v}\big((z_v - \zeta_v)(\ell_v^+)\big) = 1$ and $v_p(\hat{\sigma}_v^*(\ell_v^+)) = v_p\big((z_v - \zeta_v)(\ell_v^+)\big) = v_p(\Theta(\ell_v^+)) = v_p(\sum_{i=0}^\infty \ell_i \zeta_v^i) = v_p(\ell_0) = \frac{1}{q_v - 1}$, see Example 14.10. The reader should compare this to Example 5.6.

15 Complex Multiplication

Definition 15.1. Let $\underline{M}$ be an A-motive over an A-field K. If $\mathrm{QEnd}_K(\underline{M})$ contains a commutative semi-simple Q-algebra E of dimension $\dim_Q E = \mathrm{rk}\,\underline{M}$, then we call $\underline{M}$ a *CM A-motive over K* and we say that $\underline{M}$ *has complex multiplication by E over K.*

Here semi-simple means that E is a product of fields. Note that we do not assume that E is itself a field. By [Sch09, Theorem 4.2.5] any CM A-motive $\underline{M}$ is semi-simple. We know from [Sch09, Theorem 4.4.7] if $\underline{M}$ is simple, uniformizable then $\dim_Q \mathrm{QEnd}_K(\underline{M}) \leq \mathrm{rk}\,\underline{M}$ and if in addition $\underline{M}$ has complex multiplication by E, then $E = \mathrm{QEnd}_K(\underline{M})$ is a field.

Let $\underline{M}$ be an A-motive over K with complex multiplication through E and let $\mathcal{O}_E$ be the integral closure of A in E. If $E = \prod_i E_i$ is a product of finite field extensions of Q, then $\mathcal{O}_E = \prod_i \mathcal{O}_{E_i}$, where $\mathcal{O}_{E_i}$ is the integral closure of A in E_i. By [Sch09, Theorem 3.3.3] there exists an A-motive $\underline{M}'$ isogenous to $\underline{M}$ such that $\mathcal{O}_E \subseteq \mathrm{End}_K(\underline{M}')$. So for all aspects which only depend on the isogeny class of $\underline{M}$ we can assume that $\mathcal{O}_E \subseteq \mathrm{End}_K(\underline{M})$. Then M is a locally free module over the ring $\mathcal{O}_E \otimes_{\mathbb{F}_q} K$ and

$$M = \bigoplus_i (M \otimes_{\mathcal{O}_E} \mathcal{O}_{E_i}).$$

Since $\mathcal{O}_E \hookrightarrow \mathrm{End}_K(\underline{M})$ is injective, $M \otimes_{\mathcal{O}_E} \mathcal{O}_{E_i}$ is a locally free module over the ring $\mathcal{O}_{E_i} \otimes_{\mathbb{F}_q} K$ of rank ≥ 1, because otherwise $\mathcal{O}_{E_i}$ acts as 0 on $\underline{M}$, which is a contradiction. Now the estimate

$$\begin{aligned}\mathrm{rk}_{A_K} M = \sum_i \mathrm{rk}_{A_K}(M \otimes_{\mathcal{O}_E} \mathcal{O}_{E_i}) &= \sum_i \mathrm{rk}_{(\mathcal{O}_{E_i} \otimes_{\mathbb{F}_q} K)}(M \otimes_{\mathcal{O}_E} \mathcal{O}_{E_i}) \cdot [E_i : Q] \\ &\geq \sum_i [E_i : Q] = [E : Q] = \mathrm{rk}_{A_K} M\end{aligned}$$

shows that $\mathrm{rk}_{(\mathcal{O}_{E_i} \otimes_{\mathbb{F}_q} L)}(M \otimes_{\mathcal{O}_E} \mathcal{O}_{E_i}) = 1$ for all i. Therefore, M is a locally free module over $\mathcal{O}_E \otimes_{\mathbb{F}_q} K$ of rank 1. Thus we have the following proposition.

Proposition 15.2. *[Sch09, Proposition 3.3.5] Let $\underline{M} = (M, \tau_M)$ be an A-motive over K with complex multiplication E such that $\mathcal{O}_E \subseteq \mathrm{End}_K(\underline{M})$, then*

(a) M is a locally free $\mathcal{O}_E \otimes_{\mathbb{F}_q} K$-module of rank 1.

(b) $\tau_M : \sigma^* M \to M$ is an $\mathcal{O}_E \otimes_{\mathbb{F}_q} K$-linear injection.

Theorem 15.3 ([Sch09, Theorem 6.3.6]). *Let $\underline{M}$ be an A-motive over an A-field K with complex multiplication E such that $\mathcal{O}_E \subseteq \mathrm{End}_K(\underline{M})$ and E is separable over Q. Then $\underline{M}$ is already defined over a finite separable extension L of the A-field $\mathrm{Quot}(A/A\text{-char}(K))$ which is Q or a finite field, i.e. $\underline{M} \cong \underline{M}_L \otimes_L K$ for an A-motive $\underline{M}_L$ over L.*

Theorem 15.4 ([Pel09, Section 3.6]). *If $\underline{M}$ is an A-motive defined over a finite extension K/Q with complex multiplication by a separable Q-algebra E, then there exists a finite separable extension L/K such that $\underline{M}$ has good reduction at every prime of $\mathcal{O}_L$.*

Remark 15.5. If $\underline{M} = \underline{M}(\underline{G})$ is the A-motive of a Drinfeld A-module $\underline{G}$ then both theorems are well known. Namely, in this case there is exactly one place of E above ∞ by [Gos96, Proposition 4.7.17]. Then $\underline{G}$ can be viewed as a Drinfeld $\mathcal{O}_E$-module of rank 1. All these are defined over the Hilbert class field of E and have everywhere good reduction by [Hay79], see [Tha04, Theorems 2.6.4 and 3.4.2].

Definition 15.6. A *CM-type* is a pair $(E, (d_\psi)_{\psi \in H_E})$ consisting of a finite dimensional, semi-simple, commutative Q-algebra E and a tuple of integers $(d_\psi)_{\psi \in H_E}$ indexed by $H_E := \mathrm{Hom}_Q(E, Q^{\mathrm{alg}})$.

An *isomorphism* $f\colon (E, (d_\psi)_{\psi \in H_E}) \xrightarrow{\sim} (E', (d'_{\psi'})_{\psi' \in H_{E'}})$ of CM-types is an isomorphism $f\colon E \xrightarrow{\sim} E'$ of Q-algebras with $d_{\psi' \circ f} = d'_{\psi'}$ for all $\psi' \in H_{E'}$.

Remark 15.7. The analog of a classical CM-type (E, Φ) as in Definition 6.2 would be a tuple $(d_\psi)_{\psi \in H_E}$ for which $d_\psi \in \{0, 1\}$. Then one can set $\Phi := \{\psi \in H_E\colon d_\psi = 1\}$ and has $d_\psi = 1$ for all $\psi \in \Phi$ and $d_\psi = 0$ for all $\psi \in H_E \smallsetminus \Phi$. But note, that we need a more flexible definition of CM-type here, due to the construction of the CM-type of a CM A-motive in Definition 15.8 below.

To prepare for this construction let $z \in Q$ be a uniformizer at ∞ and denote by ζ the image of z in Q^{alg} under the natural inclusion $Q \subset Q^{\mathrm{alg}}$. We consider the power series ring $Q^{\mathrm{alg}}[\![z - \zeta]\!]$ over Q^{alg} in the "variable" $z - \zeta$ as a Q-algebra via $z \mapsto \zeta + (z - \zeta)$. Let E be a finite dimensional, semi-simple, commutative Q-algebra. Then by [HS20, Lemma A.3] there is a decomposition

$$E \otimes_Q Q^{\mathrm{alg}}[\![z - \zeta]\!] \;=\; \prod_{\psi \in H_E} Q^{\mathrm{alg}}[\![y_\psi - \psi(y_\psi)]\!], \tag{15.1}$$

where y_ψ is a uniformizer at a place of E such that $\psi(y_\psi) \neq 0$. By [HJ20, Lemma 1.5] the factors are obtained as the completion of $\mathcal{O}_E \otimes_A A_{Q^{\mathrm{alg}}} = \mathcal{O}_E \otimes_{\mathbb{F}_q} Q^{\mathrm{alg}}$ along the kernels $(a \otimes 1 - 1 \otimes \psi(a)\colon a \in \mathcal{O}_E)$ of the homomorphisms $\psi \otimes \mathrm{id}_{Q^{\mathrm{alg}}}\colon \mathcal{O}_E \otimes_{\mathbb{F}_q} Q^{\mathrm{alg}} \to Q^{\mathrm{alg}}$ for $\psi \in H_E$. If $(E, (d_\psi)_{\psi \in H_E})$ is a CM-type, then there is a finite free $Q^{\mathrm{alg}}[\![z - \zeta]\!]$-submodule

$$\mathfrak{q} \;:=\; \prod_{\psi \in H_E} \big(y_\psi - \psi(y_\psi)\big)^{-d_\psi} \cdot Q^{\mathrm{alg}}[\![y_\psi - \psi(y_\psi)]\!] \subseteq E \otimes_Q Q^{\mathrm{alg}}(\!(z - \zeta)\!) \tag{15.2}$$

with $\mathfrak{q} \cdot Q^{\mathrm{alg}}(\!(z - \zeta)\!) = E \otimes_Q Q^{\mathrm{alg}}(\!(z - \zeta)\!)$. Conversely, every such $Q^{\mathrm{alg}}[\![z - \zeta]\!]$-submodule $\mathfrak{q}$ uniquely determines a tuple $(d_\psi)_{\psi \in H_E}$ of integers satisfying (15.2). So we could equivalently call $(E, \mathfrak{q})$ a "CM-type". In this description, an isomorphism $f\colon (E, \mathfrak{q}) \xrightarrow{\sim} (E', \mathfrak{q}')$ of CM-types is an isomorphism $f : E \xrightarrow{\sim} E'$ of Q-algebras which satisfies $(f \otimes \mathrm{id}_{Q^{\mathrm{alg}}(\!(z-\zeta)\!)})(\mathfrak{q}) = \mathfrak{q}'$.

Definition 15.8. Let $\underline{M}$ be an A-motive over a finite field extension $K \subset Q^{\mathrm{alg}}$ of Q with complex multiplication through E. We assume that K contains $\psi(E)$ for all $\psi \in H_E$. Then the decomposition (15.1) exists already with Q^{alg} replaced by K. The $E \otimes_Q K[\![z - \zeta]\!]$-module $\mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K[\![z - \zeta]\!])$ is finite free of rank one, and correspondingly decomposes into eigenspaces

$$\mathrm{H}^\psi(\underline{M}, K[\![y_\psi - \psi(y_\psi)]\!]) \;:=\; \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K[\![z - \zeta]\!]) \otimes_{E \otimes_Q K[\![z - \zeta]\!]} K[\![y_\psi - \psi(y_\psi)]\!]$$

each of which is free of rank one over $K[\![y_\psi - \psi(y_\psi)]\!]$, that is

$$\mathfrak{p}^{\underline{M}} \;:=\; \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K[\![z - \zeta]\!]) \;=\; \prod_{\psi \in H_E} \mathrm{H}^\psi(\underline{M}, K[\![y_\psi - \psi(y_\psi)]\!])\,.$$

Since the Hodge-Pink lattice $\mathfrak{q}^{\underline{M}}$ from Definition 14.11 is also an $E \otimes_Q K[\![z-\zeta]\!]$-module and contains $\mathfrak{p}^{\underline{M}}$, there are non-negative integers $d_\psi \in \mathbb{Z}_{\geq 0}$ such that

$$\mathfrak{q}^{\underline{M}} \;=\; \prod_{\psi \in H_E} (y_\psi - \psi(y_\psi))^{-d_\psi}\, \mathrm{H}^\psi(\underline{M}, K[\![y_\psi - \psi(y_\psi)]\!]) .$$

The tuple $(d_\psi)_{\psi \in H_E}$ is the *CM-type* of $\underline{M}$. Since $\operatorname{coker} \tau_M = \mathfrak{q}^{\underline{M}} / \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K[\![z-\zeta]\!])$ we see that d_ψ is the dimension over K of the generalized ψ-eigenspace of the action of E on $\operatorname{coker} \tau_M$.

If we fix an isomorphism $\alpha\colon \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K[\![z-\zeta]\!]) \xrightarrow{\sim} E \otimes_Q K[\![z-\zeta]\!]$, then the CM-type of $\underline{M}$ can equivalently be described as $(E, \alpha(\mathfrak{q}^{\underline{M}}))$.

Example 15.9. Let $\underline{G}$ be a Drinfeld A-module over an A-field K of generic A-characteristic, such that $\underline{M} := \underline{M}(\underline{G})$ has CM by $\mathcal{O}_E$ for a field extension E of Q with $[E:Q] = \operatorname{rk} \underline{M} = \operatorname{rk} \underline{G}$. By Remark 15.5 we may assume that K is a finite extension of Q, and we can fix an embedding $K \subset Q^{\mathrm{alg}}$. Theorem 9.7 and Corollary 11.3 imply that $\mathrm{QEnd}_K(\underline{M}) = \mathrm{QEnd}_K(\underline{G})^{\mathrm{op}}$ is a (commutative) field extension of Q of degree dividing $\operatorname{rk} \underline{G}$ and containing E. Thus, $E = \mathrm{QEnd}_K(\underline{M}) = \mathrm{QEnd}_K(\underline{G})$. The field E acts K-linearly on the one dimensional K-vector space $\operatorname{Lie} G$. Therefore, there is a Q-homomorphism $\psi_0\colon E \to \operatorname{End}_K(\operatorname{Lie} G) = K$, that is, an element $\psi_0 \in H_E$ such that every $a \in E$ acts on $\operatorname{Lie} G$ via multiplication with $\psi_0(a)$. If K contains $\psi(E)$ for all $\psi \in H_E$, then as E-modules, sequence (13.4) takes the form

$$0 \longrightarrow \bigoplus_{\psi \neq \psi_0} K_\psi \longrightarrow \mathrm{H}_{1,\mathrm{dR}}(\underline{G}, K) \longrightarrow K_{\psi_0} \longrightarrow 0$$

where K_ψ denotes the 1-dimensional K-vector space on which E acts via ψ. In particular $\operatorname{Lie} \underline{G} = K_{\psi_0}$, and hence (13.4) is analogous to the decomposition (6.3). Since $\operatorname{coker} \tau_M \cong (\operatorname{Lie} \underline{G})^\vee$ is 1-dimensional with the induced E-action also given by ψ_0, the CM-type of $\underline{G}$ is $(E, (d_\psi)_{\psi \in H_E})$ with $d_{\psi_0} = 1$ and $d_\psi = 0$ for all $\psi \neq \psi_0$. This yields an isomorphism

$$\tau_M\colon \big(y_{\psi_0} - \psi_0(y_{\psi_0})\big)^{-1} \mathrm{H}^{\psi_0}\big(\underline{M}, K[\![y_{\psi_0} - \psi_0(y_{\psi_0})]\!]\big) / \mathrm{H}^{\psi_0}\big(\underline{M}, K[\![y_{\psi_0} - \psi_0(y_{\psi_0})]\!]\big) \;=\; \mathfrak{q}^{\underline{M}} / \mathfrak{p}^{\underline{M}} \xrightarrow{\;\sim\;} \operatorname{coker} \tau_M .$$

Let $\omega_{\psi_0} \in \mathrm{H}^{\psi_0}(\underline{M}, K[\![y_{\psi_0} - \psi_0(y_{\psi_0})]\!])$ be a $K[\![y_{\psi_0} - \psi_0(y_{\psi_0})]\!]$-generator. Then $m := (y_{\psi_0} - \psi_0(y_{\psi_0}))^{-1} \cdot \omega_{\psi_0} \in \mathfrak{q}^{\underline{M}}$ and the image of m in $\operatorname{coker} \tau_M \cong \mathfrak{q}^{\underline{M}} / \mathfrak{p}^{\underline{M}}$ generates the one dimensional K-vector space $\operatorname{coker} \tau_M$. In particular, if E/Q is separable, we can take $y_{\psi_0} = z$ and $\psi_0(y_{\psi_0}) = \zeta$ by [HJ20, Lemma 1.3]. Then $y_{\psi_0} - \psi_0(y_{\psi_0}) = z - \zeta$ and $K[\![y_{\psi_0} - \psi_0(y_{\psi_0})]\!] = K[\![z-\zeta]\!]$.

16 The Taguchi height of a Drinfeld module

Pushing the analogy between abelian varieties and Drinfeld modules forward, Taguchi [Tag93, Section 5] defined the analog of the Faltings height for Drinfeld modules. It is today called the *Taguchi height*. Taguchi used it to prove the Tate Conjecture 13.14 for Drinfeld modules. We follow the exposition of Wei [Wei20, § 5.1].

Definition 16.1. For an A-lattice $\Lambda \subset \mathbb{C}_\infty$ of rank r, a Q_∞-basis $\{\lambda_i\}_{1 \leq i \leq r}$ of $Q_\infty \cdot \Lambda$ is called *orthogonal* if $\lambda_1, \ldots, \lambda_r$ satisfy that

(a) $\lambda_i \in \Lambda$ for $1 \leq i \leq r$,

(b) $|a_1\lambda_1 + \ldots + a_r\lambda_r|_\infty = \max\{\, |a_i\lambda_i|_\infty ; 1 \leq i \leq r \,\}$ for all $a_1, \ldots, a_r \in Q_\infty$,

(c) $Q_\infty \cdot \Lambda = \Lambda + (A_\infty\lambda_1 + \ldots + A_\infty\lambda_r)$.

Note that if $\lambda_i \in Q \cdot \Lambda$ for $1 \leq i \leq r$ such that $\oplus_{i=1}^r Q\lambda_i = Q \cdot \Lambda$ and (b) holds, then (a) and (c) can be achieved by multiplying all λ_i with some $a \in A$ that has $v_\infty(a) \ll 0$. Then we define the *A-volume* $D_A(\Lambda)$ of Λ by

$$D_A(\Lambda) \;:=\; \left(\frac{\prod_{1 \leq i \leq r} |\lambda_i|_\infty}{\#\big(\Lambda / (A\lambda_1 + \cdots + A\lambda_r)\big)} \right)^{1/r} \;=\; q^{1-g_Q} \cdot \left(\frac{\prod_{1 \leq i \leq r} |\lambda_i|_\infty}{\#\big(\Lambda \cap (A_\infty\lambda_1 + \cdots + A_\infty\lambda_r)\big)} \right)^{1/r} , \tag{16.3}$$

where g_Q is the genus of Q

Example 16.2. Let E be a finite *imaginary* field extension of Q, that is, $E_\infty := E \otimes_Q Q_\infty$ is still a field. Then the absolute value $|\,.\,|_\infty$ on Q_∞ extends in a unique way to an absolute value on E_∞. The latter equals the restriction of the absolute value $|\,.\,|_\infty$ on $\mathbb{C}_\infty$ for any Q_∞-embedding $E_\infty \hookrightarrow \mathbb{C}_\infty$. Under any such embedding $\mathcal{O}_E$ is an A-lattice in $\mathbb{C}_\infty$ of rank $[E:Q]$, and we can define $D_A(\mathcal{O}_E)$, which is independent of the chosen embedding. If the ramification of ∞ in E/Q is tame then

$$\log D_A(\mathcal{O}_E) \;=\; \frac{1}{2[E:Q]} \cdot \log \#(A/\mathfrak{d}_{\mathcal{O}_E/A})$$

by [Wei20, Remark 5.6] where $\mathfrak{d}_{\mathcal{O}_E/A}$ is the (relative) discriminant of $\mathcal{O}_E$ over A.

For the Taguchi height [Tag93, § 5] of a Drinfeld module the following alternative, equivalent definition was given by Wei [Wei20, § 5.1].

Definition 16.3 ([Tag93, § 5], [Wei20, § 5.1])**.** Let $\underline{G} = (G,\varphi)$ be a Drinfeld A-module of rank r over a finite field extension $K \subset Q^{\mathrm{alg}}$ of Q. For every $\eta \in H_K := \mathrm{Hom}_Q(K, Q^{\mathrm{alg}})$ the embedding $\eta\colon K \hookrightarrow Q^{\mathrm{alg}} \subset Q_v^{\mathrm{alg}}$ allows to restrict the valuation v on Q_v^{alg} to a valuation, that is, a place $\tilde{v}_\eta$ of K, such that the completion $K_{\tilde{v}_\eta}$ equals the closure of $\eta(K)$ in Q_v^{alg}. Conversely, for each place $\tilde{v}$ of K with $\tilde{v}|v$, we let $K_{\tilde{v}}$ be the completion of K at $\tilde{v}$. We choose a Q_v-embedding $\eta\colon K_{\tilde{v}} \hookrightarrow Q_v^{\mathrm{alg}}$ and the induced Q-embedding $\eta\colon K \hookrightarrow Q^{\mathrm{alg}}$. Then $\tilde{v} = \tilde{v}_\eta$. In this way the place $\tilde{v}$ is obtained $[K_{\tilde{v}} : Q_v]$-many times. We let $\underline{G}^\eta = (G^\eta, \varphi^\eta)$ be the base change of $\underline{G}$ to Q^{alg} via $\eta\colon K \hookrightarrow Q^{\mathrm{alg}}$ and also to $\mathbb{C}_\infty$ via the fixed inclusion $Q^{\mathrm{alg}} \subset \mathbb{C}_\infty$.

We choose an isomorphism $m\colon G \xrightarrow{\sim} \mathbb{G}_{a,K}$ and consider the induced isomorphisms $m^\eta\colon G^\eta \xrightarrow{\sim} \mathbb{G}_{a,Q^{\mathrm{alg}}}$ and $\mathrm{Lie}\, m^\eta\colon \mathrm{Lie}\, G^\eta \xrightarrow{\sim} Q^{\mathrm{alg}}$ for every $\eta \in H_K$. The *local height of $\underline{G}$ at $\widetilde{\infty}_\eta$ with respect to m* is given by

$$ht_{\mathrm{Tag},\widetilde{\infty}_\eta}(\underline{G}/K) \;:=\; -[K_{\widetilde{\infty}_\eta} : Q_\infty] \cdot \log_q D_A\big(\mathrm{Lie}\, m^\eta\big(\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta, A)\big)\big). \tag{16.4}$$

To define the local height of $\underline{G}$ at a finite place $\tilde{v}_\eta$ of K with $\tilde{v}_\eta | v \neq \infty$ we write

$$m^\eta \circ \varphi_a^\eta \circ (m^\eta)^{-1} \;=\; \gamma(a) + \sum_{i=1}^{r \deg a} \varphi_{a,i}^\eta\, \tau^i \;\in\; \mathrm{End}_{Q^{\mathrm{alg}},\mathbb{F}_q}(\mathbb{G}_{a,Q^{\mathrm{alg}}}) \;=\; Q^{\mathrm{alg}}\{\tau\} \qquad \text{with} \quad \varphi_{a,i}^\eta \;\in\; Q^{\mathrm{alg}}.$$

for each $a \in A$. We put $\mathrm{ord}_{\tilde{v}_\eta}(\underline{G}) := \min\left\{ \dfrac{e(\tilde{v}_\eta|v)\cdot v(\varphi_{a,i}^\eta)}{q^i - 1} : a \in A \smallsetminus \mathbb{F}_q,\ 1 \le i \le r \deg a \right\}$, where $e(\tilde{v}_\eta|v)$ is the ramification index of $\tilde{v}_\eta$ in K/Q. The *local height of $\underline{G}$ at $\tilde{v}_\eta$ with respect to m* is given by

$$ht_{\mathrm{Tag},\tilde{v}_\eta}(\underline{G}/K) \;:=\; -[\mathbb{F}_{\tilde{v}_\eta} : \mathbb{F}_q] \cdot \lfloor \mathrm{ord}_{\tilde{v}_\eta}(\underline{G}) \rfloor\,, \tag{16.5}$$

where $\lfloor x \rfloor$ denotes the largest integer $n \le x$, and $\mathbb{F}_{\tilde{v}_\eta}$ is the residue field of $\tilde{v}_\eta$.

Then the *Taguchi height* $ht_{\mathrm{Tag}}(\underline{G}/K)$ of $\underline{G}$ is defined by taking the sum over all places of K

$$ht_{\mathrm{Tag}}(\underline{G}/K) \;:=\; \frac{1}{[K:Q]} \cdot \Big(\sum_{\tilde{v} \nmid \infty} ht_{\mathrm{Tag},\tilde{v}}(\underline{G}/K) + \sum_{\widetilde{\infty}|\infty} ht_{\mathrm{Tag},\widetilde{\infty}}(\underline{G}/K) \Big). \tag{16.6}$$

It does not depend on the isomorphism m.

Remark 16.4. (1) Let K' be a finite field extension of K. Let $\eta'\colon K' \hookrightarrow Q^{\mathrm{alg}}$ be a Q-homomorphism and let $\eta\colon K \hookrightarrow Q^{\mathrm{alg}}$ be its restriction to K. Let $\widetilde{\infty}'_{\eta'}$ and $\widetilde{\infty}_\eta$ be the corresponding places of K' and K, respectively. It is clear that $\mathrm{Lie}\, m\big(\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^{\eta'}, A)\big) = \mathrm{Lie}\, m\big(\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta, A)\big) \subset \mathbb{C}_\infty$, and

$$ht_{\mathrm{Tag},\widetilde{\infty}'_{\eta'}}(\underline{G}/K') \;=\; [K'_{\widetilde{\infty}'_{\eta'}} : K_{\widetilde{\infty}_\eta}] \cdot ht_{\mathrm{Tag},\widetilde{\infty}_\eta}(\underline{G}/K).$$

For places $\tilde{v}$ of K and $\tilde{v}'$ of K' with $\tilde{v}' \mid \tilde{v} \nmid \infty$, one has $\mathrm{ord}_{\tilde{v}'}(\underline{G}) = e(\tilde{v}'|\tilde{v}) \cdot \mathrm{ord}_{\tilde{v}}(\underline{G})$, where $e(\tilde{v}'|\tilde{v})$ is the ramification index of $\tilde{v}'/\tilde{v}$. Thus we get

$$ht_{\mathrm{Tag},\tilde{v}'}(\underline{G}/K') \;\le\; [K'_{\tilde{v}'} : K_{\tilde{v}}] \cdot ht_{\mathrm{Tag},\tilde{v}}(\underline{G}/K).$$

In particular, assume that $\underline{G}$ has stable reduction at $\tilde{v}$, that is, there is an $x \in K_{\tilde{v}}$ such that $v(x^{q^i-1}\varphi_{a,i}^\eta) \ge 0$ for all i and a, and for every $a \in A \smallsetminus \mathbb{F}_q$ there is an $i \ge 1$ such that $v(x^{q^i-1}\varphi_{a,i}^\eta) = 0$. Then $\mathrm{ord}_{\tilde{v}}(\underline{G}) = -e(\tilde{v}|v) \cdot v(x) = -\tilde{v}(x)$ is an integer, which implies that $ht_{\mathrm{Tag},\tilde{v}'}(\underline{G}/K') = [K'_{\tilde{v}'} : K_{\tilde{v}}] \cdot ht_{\mathrm{Tag},\tilde{v}}(\underline{G}/K)$. In conclusion, we have $ht_{\mathrm{Tag}}(\underline{G}/K') \le ht_{\mathrm{Tag}}(\underline{G}/K)$, and the equality holds when $\underline{G}$ has stable reduction everywhere.

(2) Note that every Drinfeld A-module $\underline{G}$ over K has potentially stable reduction everywhere by [Dri76, Proposition 7.1]. Define the *stable Taguchi height of* $\underline{G}$ as

$$ht^{\mathrm{st}}_{\mathrm{Tag}}(\underline{G}) \;:=\; \log q \cdot \lim_{K'/K \text{ finite}} ht_{\mathrm{Tag}}(\underline{G}/K'),$$

which is always convergent by (1).

(3) Let $\underline{G}$ and $\underline{G}'$ be two Drinfeld A-modules over Q^{alg} which are isomorphic over Q^{alg}. Then

$$ht^{\mathrm{st}}_{\mathrm{Tag}}(\underline{G}) \;=\; ht^{\mathrm{st}}_{\mathrm{Tag}}(\underline{G}')\,.$$

17 The Analog of Colmez's Conjecture for CM A-Motives

In [HS20] the authors have formulated the analog of Colmez's conjecture (Section 8) for periods of CM A-motives. We consider the following

Situation 17.1. Let $\underline{M}$ be a uniformizable A-motive over a finite extension $K \subset Q^{\mathrm{alg}}$ of Q with complex multiplication of CM-type $(E,(d_\psi)_{\psi\in H_E})$, in the sense of Definition 15.6 such that E is a product of *separable* field extensions of Q and $\underline{M}$ has complex multiplication by the ring of integers $\mathcal{O}_E$ of E. As an abbreviation we denote the CM-Type of $\underline{M}$ by (E,Φ) with $\Phi = (d_\psi)_{\psi\in H_E}$. Let $H_E := \mathrm{Hom}_Q(E, Q^{\mathrm{alg}})$ be the set of all Q-homomorphisms $E \hookrightarrow Q^{\mathrm{alg}}$ and assume that K contains $\psi(E)$ for every $\psi \in H_E$. By Theorems 15.3 and 15.4 we may assume moreover, that K is a finite Galois extension of Q and that $\underline{M}$ has good reduction at every prime of K. For a fixed $\psi \in H_E$ let ω_ψ be a generator of the $K[\![y_\psi - \psi(y_\psi)]\!]$-module $\mathrm{H}^\psi(\underline{M}, K[\![y_\psi - \psi(y_\psi)]\!])$. The image of ω_ψ in $\mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K)$ is non-zero and satisfies $a^*\omega_\psi = \psi(a)\cdot\omega_\psi$ for all $a \in E$. For every embedding $\eta\colon K \hookrightarrow Q^{\mathrm{alg}}$, let $\underline{M}^\eta := \underline{M} \otimes_{K,\eta} K$ and $\omega^\eta_\psi \in \mathrm{H}^{\eta\psi}(\underline{M}^\eta, K[\![y_{\eta\psi} - \eta\psi(y_{\eta\psi})]\!])$ be deduced from $\underline{M}$ and ω_ψ by base extension, and let $u_\eta \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta, Q) = \mathrm{Hom}_A\big(\mathrm{H}^1_{\mathrm{Betti}}(\underline{M}^\eta, A), Q\big)$ be an E-generator. Let v be a place of Q.

If $v = \infty$ the pairing (13.7) from Theorem 13.18 between Betti and de Rham cohomology gives a pairing

$$\langle\,.\,,\,.\,\rangle_\infty\colon\ \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta, Q) \times \mathrm{H}^1_{\mathrm{dR}}(\underline{M}^\eta, K) \;\longrightarrow\; \mathbb{C}_\infty\,, \quad (u_\eta, \omega^\eta_\psi) \;\longmapsto\; \langle u_\eta, \omega^\eta_\psi\rangle_\infty \;=:\; \textstyle\int_{u_\eta}\omega^\eta_\psi\,.$$

We define the absolute value $\big|\int_{u_\eta}\omega^\eta_\psi\big|_\infty := |\langle u_\eta, \omega^\eta_\psi\rangle_\infty|_\infty = q_\infty^{-v_\infty(\langle u_\eta,\omega^\eta_\psi\rangle_\infty)} \in \mathbb{R}$.

If $v \subset A$ is a maximal ideal, the comparison isomorphism $h_{\mathrm{Betti},v}$ from (13.5) in Theorem 13.18 between Betti and v-adic cohomology together with the comparison isomorphism $h_{v,\mathrm{dR}}$ between v-adic and de Rham cohomology from (14.1) in Theorem 14.12 yield a pairing

$$\begin{aligned} \langle\,.\,,\,.\,\rangle_v\colon\quad \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta, Q) \times \mathrm{H}^1_{\mathrm{dR}}(\underline{M}^\eta, K) \;&\longrightarrow\; \mathbb{C}_v(\!(z_v - \zeta_v)\!)\,,\\ (u_\eta, \omega^\eta_\psi) \qquad\qquad &\longmapsto\; \langle u_\eta, \omega^\eta_\psi\rangle_v \;:=\; u_\eta \otimes \mathrm{id}_{\mathbb{C}_v(\!(z_v-\zeta_v)\!)}\big(h^{-1}_{\mathrm{Betti},v} \circ h^{-1}_{v,\mathrm{dR}}(\omega^\eta_\psi)\big)\,. \end{aligned}$$

We define the absolute value $\big|\int_{u_\eta}\omega^\eta_\psi\big|_v := |\langle u_\eta, \omega^\eta_\psi\rangle_v|_v := q_v^{-v(\langle u_\eta,\omega^\eta_\psi\rangle_v)} \in \mathbb{R}$, where the "valuation" v on $\mathbb{C}_v(\!(z_v - \zeta_v)\!)$ was defined in (14.2) in Definition 14.14.

In analogy with Section 8 we now consider the product $\prod\limits_v \prod\limits_{\eta\in H_K} \big|\int_{u_\eta}\omega^\eta_\psi\big|_v$ over all places v of Q, or equivalently $\frac{1}{\#H_K}$ times its logarithm

$$\tfrac{1}{\#H_K}\sum_v \sum_{\eta\in H_K} \log\big|\textstyle\int_{u_\eta}\omega^\eta_\psi\big|_v \;=\; \tfrac{1}{\#H_K}\displaystyle\sum_{\eta\in H_K} \log\big|\textstyle\int_{u_\eta}\omega^\eta_\psi\big|_\infty - \tfrac{1}{\#H_K}\displaystyle\sum_{v\neq\infty}\sum_{\eta\in H_K} v\big(\textstyle\int_{u_\eta}\omega^\eta_\psi\big)\log q_v\,. \tag{17.1}$$

Again the right sum over all $v \neq \infty$ does not converge. Namely, we prove in [HS20, Theorem 1.3] the following Theorem 17.3 below. To formulate the theorem we recall Definition 8.2. For our CM-type (E,Φ) and for every $\psi \in H_E$ we define the functions

$$a_{E,\psi,\Phi}\colon\ \mathscr{G}_Q \to \mathbb{Z}, \quad g \mapsto d_{g\psi} \qquad \text{and} \tag{17.2}$$

$$a^0_{E,\psi,\Phi}\colon\ \mathscr{G}_Q \to \mathbb{Q}, \quad g \mapsto \tfrac{1}{\#H_K}\sum_{\eta\in H_K} a_{E,\eta\psi,\eta\Phi}(g) \;=\; \tfrac{1}{\#H_K}\sum_{\eta\in H_K} d_{\eta^{-1}g\eta\psi} \tag{17.3}$$

which factor through $\mathrm{Gal}(K/Q)$ by our assumption that $\psi(E) \subset K$ for all $\psi \in H_E$. In particular, $a_{E,\psi,\Phi} \in \mathcal{C}(\mathscr{G}_Q, \mathbb{Q})$ and $a^0_{E,\psi,\Phi} \in \mathcal{C}^0(\mathscr{G}_Q, \mathbb{Q})$ is independent of K.

We also define integers $v(\omega_\psi^\eta)$ and $v_{\eta\psi}(u_\eta)$ for all $v \neq \infty$ which are all zero except for finitely many. Let $\mathcal{O}_{E_v} := \mathcal{O}_E \otimes_A A_v$ and let $c \in E_v := E \otimes_Q Q_v$ be such that $c^{-1}u_\eta$ is an $\mathcal{O}_{E_v}$-generator of $\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta, A) \otimes_A A_v = \mathrm{H}_{1,v}(\underline{M}^\eta, A_v)$, which exists because $\mathcal{O}_{E_v}$ is a product of discrete valuation rings. Then c is unique up to multiplication by an element of $\mathcal{O}_{E_v}^\times$ and we set

$$v_{\eta\psi}(u_\eta) \;:=\; v\big(\eta\psi(c)\big) \;\in\; \mathbb{Q}, \tag{17.4}$$

where we extend $\eta\psi \in H_E$ by continuity to $\eta\psi\colon E_v \to Q_v^{\mathrm{alg}}$.

Also let K_v be the v-adic completion of $K \subset Q^{\mathrm{alg}} \subset Q_v^{\mathrm{alg}} \subset \mathbb{C}_v$ and let $\underline{\mathcal{M}}^\eta = (\mathcal{M}^\eta, \tau_{\mathcal{M}^\eta})$ be an A-motive over $\mathcal{O}_{K_v}$ with good reduction and $\underline{\mathcal{M}}^\eta \otimes_{\mathcal{O}_{K_v}} K_v \cong \underline{M}^\eta \otimes_K K_v$; see Example 14.6. On $\mathrm{H}^1_{\mathrm{dR}}(\underline{M}^\eta, K_v)$ we consider the following two integral structures arising from $\mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{M}}^\eta, \mathcal{O}_{K_v}) := \sigma^*\mathcal{M}^\eta \otimes_{A_{\mathcal{O}_{K_v}}, \gamma\otimes\mathrm{id}_{\mathcal{O}_{K_v}}} \mathcal{O}_{K_v}$

$$\widetilde{\mathrm{H}}^{\eta\psi}(\underline{\mathcal{M}}^\eta, \mathcal{O}_{K_v}) \;:=\; \big\{\omega \in \mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{M}}^\eta, \mathcal{O}_{K_v})\colon [a]^*\omega = \eta\psi(a)\cdot\omega \;\;\forall\, a \in \mathcal{O}_E\big\} \qquad \text{and}$$

$$\mathrm{H}^{\eta\psi}(\underline{\mathcal{M}}^\eta, \mathcal{O}_{K_v}) := \;\mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{M}}^\eta, \mathcal{O}_{K_v})\big/([a]^* - \eta\psi(a)\colon a \in \mathcal{O}_E)\cdot \mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{M}}^\eta, \mathcal{O}_{K_v})\,.$$

By [HS21, Lemma 1] (see also the arXiv version of [HS20, Lemma B.1]) these are free $\mathcal{O}_{K_v}$-modules of rank one contained in

$$\mathrm{H}^{\eta\psi}(\underline{M}^\eta, K_v) \;=\; \widetilde{\mathrm{H}}^{\eta\psi}(\underline{\mathcal{M}}^\eta, \mathcal{O}_{K_v}) \otimes_{\mathcal{O}_{K_v}} K_v \;=\; \mathrm{H}^{\eta\psi}(\underline{\mathcal{M}}^\eta, \mathcal{O}_{K_v}) \otimes_{\mathcal{O}_{K_v}} K_v$$

and satisfying $\widetilde{\mathrm{H}}^{\eta\psi}(\underline{\mathcal{M}}^\eta, \mathcal{O}_{K_v}) \subset \mathrm{H}^{\eta\psi}(\underline{\mathcal{M}}^\eta, \mathcal{O}_{K_v})$ with $\mathrm{H}^{\eta\psi}(\underline{\mathcal{M}}^\eta, \mathcal{O}_{K_v})/\widetilde{\mathrm{H}}^{\eta\psi}(\underline{\mathcal{M}}^\eta, \mathcal{O}_{K_v}) \cong \mathcal{O}_{K_v}/\eta\psi(\mathfrak{D}_{\mathcal{O}_E/A})$, where $\mathfrak{D}_{\mathcal{O}_E/A}$ is the different of $\mathcal{O}_E$ over A. Then there are elements $\tilde{x}, x \in K_v^\times$, unique up to multiplication by $\mathcal{O}_{K_v}^\times$, such that

$$\tilde{x}^{-1}\omega_\psi^\eta \bmod y_{\eta\psi} - \eta\psi(y_{\eta\psi}) \text{ is an } \mathcal{O}_{K_v}\text{-generator of } \widetilde{\mathrm{H}}^{\eta\psi}(\underline{\mathcal{M}}^\eta, \mathcal{O}_{K_v}) \text{ and}$$

$$x^{-1}\omega_\psi^\eta \bmod y_{\eta\psi} - \eta\psi(y_{\eta\psi}) \text{ is an } \mathcal{O}_{K_v}\text{-generator of } \mathrm{H}^{\eta\psi}(\underline{\mathcal{M}}^\eta, \mathcal{O}_{K_v})\,.$$

We set

$$v^\sim(\omega_\psi^\eta) \;\;:=\;\; v(\tilde{x}) \;\in\; \mathbb{Q} \quad\text{and} \tag{17.5}$$

$$v(\omega_\psi^\eta) \;\;:=\;\; v(x) \;\in\; \mathbb{Q}. \tag{17.6}$$

Then

$$v(\omega_\psi^\eta) - v^\sim(\omega_\psi^\eta) \;=\; v\big(\eta\psi(\mathfrak{D}_{\mathcal{O}_E/A})\big) \;=\; v(\mathfrak{D}_{\eta\psi(E_v)/Q_v})$$

by [HS21, Corollary 2] (see also the arXiv version of [HS20, Corollary B.2]), and consequently

$$\begin{aligned}\sum_{\eta\in H_K} v(\omega_\psi^\eta) - v^\sim(\omega_\psi^\eta) \;&=\; \sum_{\eta\in H_K} v\big(\eta\psi(\mathfrak{D}_{\mathcal{O}_E/A})\big) \;=\; v\Big(\prod_{\eta\in H_K}\eta\psi(\mathfrak{D}_{\mathcal{O}_E/A})\Big) \;=\; v\big(N_{K/Q}(\mathfrak{D}_{\psi(\mathcal{O}_E)/A})\big)\\ &=\; v\Big(N_{\psi(E)/Q}\big(N_{K/\psi(E)}(\mathfrak{D}_{\psi(\mathcal{O}_E)/A})\big)\Big) \;=\; [K:\psi(E)]\cdot v(\mathfrak{d}_{\psi(\mathcal{O}_E)/A}) \quad\text{and}\end{aligned}$$

$$\sum_{\eta\in H_K}\sum_{v\neq\infty}\big(v(\omega_\psi^\eta) - v^\sim(\omega_\psi^\eta)\big)\log q_v \;=\; [K:\psi(E)]\cdot\log\#(A/\mathfrak{d}_{\psi(\mathcal{O}_E)/A})\,. \tag{17.7}$$

These value only depend on the image of ω_ψ^η in $\mathrm{H}^1_{\mathrm{dR}}(\underline{M}^\eta, K)$. They also do not depend on the choice of the model $\underline{\mathcal{M}}^\eta$ with good reduction, because all such models are isomorphic over $\mathcal{O}_{K_v}$ by [Gar03, Proposition 2.13(ii)].

Remark 17.2. In [HS20, Formula (1.13) and Definition 4.10] there is an error in the definition of $v(\omega_\psi^\eta)$. Namely, there $v(\omega_\psi^\eta)$ is defined to be $v^\sim(\omega_\psi^\eta)$ as in (17.5). However, in the rest of [HS20] the above definition (17.6) for $v(\omega_\psi^\eta)$ is used; see [HS21] or the arXiv version of [HS20, Erratum B].

In [HS20, Theorem 1.3] we computed the terms in (17.1) as follows, where we use (17.7) and the logarithmic derivative Z_v of the Artin L-function from (8.3) in Definition 8.2.

Theorem 17.3. *Let $\mathfrak{d}_{\psi(\mathcal{O}_E)/A}$ denote the discriminant of the extension of Dedekind rings $\psi(\mathcal{O}_E)/A$. Then for every $v \neq \infty$ we have*

$$\begin{aligned}\tfrac{1}{\#H_K}\sum_{\eta\in H_K} v\big(\textstyle\int_{u_\eta}\omega_\psi^\eta\big) \;&=\; Z_v(a^0_{E,\psi,\Phi},1) - \mu_{\mathrm{Art},v}(a^0_{E,\psi,\Phi}) - \dfrac{v(\mathfrak{d}_{\psi(\mathcal{O}_E)/A})}{[\psi(E):Q]} + \tfrac{1}{\#H_K}\sum_{\eta\in H_K}\big(v(\omega_\psi^\eta) + v_{\eta\psi}(u_\eta)\big)\\ &=\; Z_v(a^0_{E,\psi,\Phi},1) - \mu_{\mathrm{Art},v}(a^0_{E,\psi,\Phi}) + \tfrac{1}{\#H_K}\sum_{\eta\in H_K}\big(v^\sim(\omega_\psi^\eta) + v_{\eta\psi}(u_\eta)\big)\,.\end{aligned}$$

This formula holds more generally for all tuples of E_v-generators $u_\eta \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta, Q_v) = \mathrm{H}_{1,v}(\underline{M}^\eta, Q_v)$.

Since $-\mu_{\mathrm{Art},v}(a^0_{E,\psi,\Phi}) - \dfrac{v(\mathfrak{d}_{\psi(\mathcal{O}_E)/A})}{[\psi(E):Q]} + \frac{1}{\#H_K}\sum\limits_{\eta\in H_K}\big(v(\omega^\eta_\psi) + v_{\eta\psi}(u_\eta)\big)$ vanishes for all but finitely many places v and $\sum\limits_{v\neq\infty} Z_v(a^0_{E,\psi,\Phi},1)$ diverges, the sum (17.1) diverges. But as in Section 8 we can assign to this divergent sum a value by the following

Convention 17.4. Let $(x_v)_{v\neq\infty}$ be a tuple of complex numbers indexed by the finite places v of Q. We will give a sense to the (divergent) series $\Sigma \overset{?}{=} \sum_{v\neq\infty} x_v$ in the following situation. We suppose that there exists an element $a\in\mathcal{C}^0(\mathscr{G}_Q,\mathbb{Q})$ such that $x_v = -Z_v(a,1)\log q_v$ for all v except for finitely many. Then we let $a^*\in\mathcal{C}^0(\mathscr{G}_Q,\mathbb{Q})$ be defined by $a^*(g) := a(g^{-1})$. We further assume that $Z^\infty(a^*,s)$ does not have a pole at $s=0$, and we define the limit of the series $\sum_{v\neq\infty} x_v$ as

$$\Sigma \;:=\; -Z^\infty(a^*,0) - \mu^\infty_{\mathrm{Art}}(a) + \sum_{v\neq\infty}\big(x_v + Z_v(a,1)\log q_v\big) \tag{17.8}$$

inspired by Weil's [Wei48, p. 82] functional equation

$$Z(\chi,1-s) \;=\; -Z(\chi^*,s) - (2\cdot \mathit{genus}(C)-2)\chi(1)\log q - \mu_{\mathrm{Art}}(\chi)$$

deprived of the summands at ∞, where the genus term is considered as belonging to ∞.

Convention 17.4, Theorem 17.3 and (17.7) allow us to give to the divergent sum (17.1) the convergent interpretation

$$\begin{aligned}
&-Z^\infty((a^0_{E,\psi,\Phi})^*,0) + \frac{\log\#(A/\mathfrak{d}_{\psi(\mathcal{O}_E)/A})}{[\psi(E):Q]} + \tfrac{1}{\#H_K}\sum_{\eta\in H_K}\Big(\log\big|\textstyle\int_{u_\eta}\omega^\eta_\psi\big|_\infty - \displaystyle\sum_{v\neq\infty}\big(v(\omega^\eta_\psi) + v_{\eta\psi}(u_\eta)\big)\log q_v\Big)\\
=\;\; & -Z^\infty((a^0_{E,\psi,\Phi})^*,0) + \tfrac{1}{\#H_K}\sum_{\eta\in H_K}\Big(\log\big|\textstyle\int_{u_\eta}\omega^\eta_\psi\big|_\infty - \displaystyle\sum_{v\neq\infty}\big(v^\sim(\omega^\eta_\psi) + v_{\eta\psi}(u_\eta)\big)\log q_v\Big).
\end{aligned} \tag{17.9}$$

Remark 17.5. The problem arises that formulas (17.1) and (17.9) depend on the choices of the E-generators u_η of $\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta,Q)$. Namely, multiplying one u_η with an element $a\in E$ changes these sums by the summand $\frac{1}{\#H_K}\sum_{\text{all } v} v\big(\eta\psi(a)\big)\log q_v$, which may be different from zero. On the other hand, if all u_η are simultaneously multiplied with the same $a\in E$ then the term $\frac{1}{\#H_K}\sum_{\eta\in H_K}\sum_{\text{all } v} v\big(\eta\psi(a)\big)\log q_v$ is added, which is zero by (1.2).

Colmez [Col93] faces the same problem and overcomes it by considering the terms (8.10) instead, which are independent of the chosen u_η. This is not possible for general A-motives, because it relies on the existence of a Q-automorphism c of Q^{alg} such that the set of integers $\{d_\psi, d_{c\psi}\}$ does not depend on $\psi\in H_E$. In (8.10), c is complex conjugation and $\{d_\psi,d_{c\psi}\} = \{0,1\}$ for every $\psi\in H_E$. These conditions are not satisfied for the more general CM-types we considered so far for A-motives.

It should also be noted, that it is in general *not* possible to choose all u_η in a compatible way, although this is possible for the generators ω^η_ψ by pulling back ω_ψ under η. However, it is possible for A-motives to pull back the induced E_v-generators $u_\eta\otimes 1\in \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta,Q)\otimes_Q Q_v = \mathrm{H}_{1,v}(\underline{M}^\eta,Q_v)$ under additional automorphisms $\tilde\eta\in\mathscr{G}_Q = \mathrm{Gal}(Q^{\mathrm{sep}}/Q)$. Namely, it follows from the definition in (13.2) that applying $\tilde\eta$ yields an $\mathcal{O}_{E_v}$-isomorphism

$$\tilde\eta\colon\ \mathrm{H}^1_v(\underline{M}^\eta,A_v) \xrightarrow{\;\sim\;} \mathrm{H}^1_v(\underline{M}^{\tilde\eta\eta},A_v)\,,\quad m\mapsto\tilde\eta(m)\,.$$

If $\tilde\eta = \kappa\in\mathrm{Gal}(Q^{\mathrm{sep}}/K)$ then this isomorphism is just $\rho_{\underline{M}^\eta}(\kappa)$ where $\rho_{\underline{M}^\eta}\colon\mathscr{G}_Q\to\mathrm{Aut}_{\mathcal{O}_{E_v}}\mathrm{H}_{1,v}(\underline{M}^\eta,A_v) = \mathcal{O}^\times_{E_v}$ is the Galois representation. Then $\tilde\eta(u_\eta\otimes 1)\in\mathrm{H}_{1,v}(\underline{M}^{\tilde\eta\eta},Q_v) = \mathrm{Hom}_{Q_v}(\mathrm{H}^1_v(\underline{M}^{\tilde\eta\eta},Q_v),Q_v)$ is defined by requiring

$$\tilde\eta(u_\eta\otimes 1)\big(\tilde\eta(m)\big) \;=\; (u_\eta\otimes 1)(m)\qquad\text{for every}\quad m\in\mathrm{H}^1_v(\underline{M}^\eta,Q_v)\,. \tag{17.10}$$

$\tilde\eta(u_\eta\otimes 1)$ is an E_v-generator of $\mathrm{H}_{1,v}(\underline{M}^{\tilde\eta\eta},Q_v)$. If $\tilde\eta$ is replaced by $\tilde\eta' = \tilde\eta\circ\kappa$ with $\kappa\in\mathrm{Gal}(Q^{\mathrm{sep}}/K)$ then $\underline{M}^{\tilde\eta'\eta} = \underline{M}^{\tilde\eta\eta}$ and $\tilde\eta'(m) = \rho_{\underline{M}^\eta}(\kappa)\cdot\tilde\eta(m)$, and hence $\tilde\eta'(u_\eta\otimes 1) = \rho_{\underline{M}^\eta}(\kappa)^{-1}\cdot\tilde\eta(u_\eta\otimes 1) = \rho^\vee_{\underline{M}^\eta}(\kappa)\cdot\tilde\eta(u_\eta\otimes 1)$. In particular, the value $v_{\tilde\eta\eta\psi}\big(\tilde\eta(u_\eta\otimes 1)\big)$ only depends on the image of $\tilde\eta$ in $\mathrm{Gal}(K/Q) = H_K$. We abbreviate $\tilde\eta(u_\eta\otimes 1)$ to $u^{\tilde\eta}_\eta$. Allthough the notation is similar to ω^η_ψ, it is understood, that $u^{\tilde\eta}_\eta$ does not exist in $\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^{\tilde\eta\eta},Q)$, but only in $\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^{\tilde\eta\eta},\mathbb{A}^\infty_Q) = \prod'_{v\neq\infty}\mathrm{H}_{1,v}(\underline{M}^{\tilde\eta\eta},Q_v)$ where $\mathbb{A}^\infty_Q$ is the adèle ring of Q. Then for every fixed $\eta\in H_K$

Convention 17.4, Theorem 17.3 and (17.7) yield

$$\log\big|\textstyle\int_{u_\eta}\omega_\psi^\eta\big|_\infty+\frac{1}{\#H_K}\sum\limits_{\tilde\eta\in H_K}\sum\limits_{v\neq\infty}\log\big|\int_{u_\eta^{\tilde\eta}}\omega_\psi^{\tilde\eta\eta}\big|_v \;=\; \tag{17.11}$$

$$=\;\log\big|\textstyle\int_{u_\eta}\omega_\psi^\eta\big|_\infty-Z^\infty((a^0_{E,\psi,\Phi})^*,0)+\dfrac{\log\#(A/\mathfrak{d}_{\psi(\mathcal{O}_E)/A})}{[\psi(E):Q]}-\frac{1}{\#H_K}\sum\limits_{\tilde\eta\in H_K}\sum\limits_{v\neq\infty}\big(v(\omega_\psi^{\tilde\eta\eta})+v_{\tilde\eta\eta\psi}(u_\eta^{\tilde\eta})\big)\log q_v$$

$$=\;\log\big|\textstyle\int_{u_\eta}\omega_\psi^\eta\big|_\infty-Z^\infty((a^0_{E,\psi,\Phi})^*,0)-\frac{1}{\#H_K}\sum\limits_{\tilde\eta\in H_K}\sum\limits_{v\neq\infty}\big(v^\sim(\omega_\psi^{\tilde\eta\eta})+v_{\tilde\eta\eta\psi}(u_\eta^{\tilde\eta})\big)\log q_v\,.$$

If we restrict to *imaginary* CM-fields E, which means that $E_\infty:=E\otimes_Q Q_\infty$ is still a field and carries a unique extension of the valuation v_∞, then this sum is independent of the choice of the E-generator $u_\eta\in \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta,Q)$. Indeed, if u_η is multiplied with a unit $a\in E^\times$, then in (17.11) the term

$$-v_\infty(\eta\psi(a))\log q_\infty-\tfrac{1}{\#H_K}\sum_{\tilde\eta\in H_K}\sum_{v\neq\infty}v\big(\tilde\eta(\eta\psi(a))\big)\log q_v\;=\;-\tfrac{1}{\#H_K}\sum_{\tilde\eta\in H_K}\sum_{\text{all }v}v\big(\tilde\eta(\eta\psi(a))\big)\log q_v$$

is added, which is zero by (1.2). Imaginary CM-fields are particularly relevant for Drinfeld modules, see Theorem 17.8 below. On the other hand, if E has more than one place above ∞, then only the place induced from the embedding $\eta\psi\colon E\hookrightarrow Q_\infty^{\mathrm{alg}}\subset\mathbb{C}_\infty$ contributes to (17.11), and then this formula is not invariant under changing u_η.

We thus propose to average twice over $\eta,\tilde\eta\in H_K$ and make the following

Conjecture 17.6. *Let E be a finite* imaginary *field extension of Q, which means that $E_\infty:=E\otimes_Q Q_\infty$ is still a field. Then the sum*

$$\sum_{\eta\in H_K}\Big(\log\big|\textstyle\int_{u_\eta}\omega_\psi^\eta\big|_\infty-Z^\infty((a^0_{E,\psi,\Phi})^*,0)+\dfrac{\log\#(A/\mathfrak{d}_{\psi(\mathcal{O}_E)/A})}{[\psi(E):Q]}-\frac{1}{\#H_K}\sum\limits_{\tilde\eta\in H_K}\sum\limits_{v\neq\infty}\big(v(\omega_\psi^{\tilde\eta\eta})+v_{\tilde\eta\eta\psi}(u_\eta^{\tilde\eta})\big)\log q_v\Big)$$

$$=\;\sum_{\eta\in H_K}\Big(\log\big|\textstyle\int_{u_\eta}\omega_\psi^\eta\big|_\infty-Z^\infty((a^0_{E,\psi,\Phi})^*,0)-\frac{1}{\#H_K}\sum\limits_{\tilde\eta\in H_K}\sum\limits_{v\neq\infty}\big(v^\sim(\omega_\psi^{\tilde\eta\eta})+v_{\tilde\eta\eta\psi}(u_\eta^{\tilde\eta})\big)\log q_v\Big) \tag{17.12}$$

is zero, or equivalently the product formula *holds:*

$$\prod_{\tilde\eta,\eta\in H_K}\Big(\big|\textstyle\int_{u_\eta}\omega_\psi^\eta\big|_\infty\cdot\prod\limits_{v\neq\infty}\big|\int_{u_\eta^{\tilde\eta}}\omega_\psi^{\tilde\eta\eta}\big|_v\Big)\;:=\;\prod\limits_{\tilde\eta,\eta\in H_K}\Big(\big|\langle u_\eta,\omega_\psi^\eta\rangle_v\big|_\infty\cdot\prod\limits_{v\neq\infty}\big|\langle u_\eta^{\tilde\eta},\omega_\psi^{\tilde\eta\eta}\rangle_v\big|_v\Big)\;=\;1\,.$$

Example 17.7. Similarly to Example 8.5, the convention allows to prove the product formula for the Carlitz motive $\underline{\mathcal{C}}=(\mathcal{C}=\mathbb{F}_q(\theta)[t],\tau_{\mathcal{C}}=t-\theta)$ from Example 9.8 over the field $K=\mathbb{F}_q(\theta)=Q$ for which $H_K=\{\mathrm{id}_K\}$. We let $u\in\mathrm{H}_{1,\mathrm{Betti}}(\underline{\mathcal{C}},A)$ be the generator which is dual to $\eta\ell^-\in\mathrm{H}^1_{\mathrm{Betti}}(\underline{\mathcal{C}},A)$ and we let $\omega=1\in\mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{C}},\mathbb{C}_\infty)$. Then we have computed in Examples 13.19, 14.13 and 14.15 that

$$\begin{aligned}\langle u,\omega\rangle_\infty\;&=\;\eta^{-q}\prod_{i=1}^\infty(1-\theta^{1-q^i})^{-1}\qquad\text{and}\qquad \log\big|\langle u,\omega\rangle_\infty\big|_\infty\;=\;\log(q^{q/(q-1)})\;=\;\tfrac{q}{q-1}\log q\,,\\ \langle u,\omega\rangle_v\;&=\;\hat\sigma_v^*(\ell_v^+)\qquad\qquad\qquad\text{and}\qquad \log\big|\langle u,\omega\rangle_v\big|_v\;=\;-v\big(\hat\sigma_v^*(\ell_v^+)\big)\log q_v\;=\;-\tfrac{\log q_v}{q_v-1}\;=\;-Z_v(\mathbb{1},1)\log q_v\,,\end{aligned}$$

where $\mathbb{1}(g)=1$ for every $g\in\mathscr{G}_Q$. Here the CM-field is $E=Q$, $H_E=\{\mathrm{id}\}$ and the CM-type is given by $d_{\mathrm{id}}=1$. This implies that $a^0_{E,\mathrm{id},\Phi}=\mathbb{1}$. So Convention 17.4 implies $\sum_{v\neq\infty}\log|\langle u,\omega\rangle_v|_v=-\frac{\zeta_A'(0)}{\zeta_A(0)}=-\frac{q}{q-1}\log q$ for the Riemann Zeta-function

$$\zeta_A(s)\;:=\;\prod_{v\neq\infty}(1-(\#\mathbb{F}_v)^{-s})^{-1}\;=\;\prod_{v\neq\infty}(1-q_v^{-s})^{-1}\;=\;\frac{1}{1-q^{1-s}}\,.$$

We conclude $\sum_v\log|\langle u,\omega\rangle_v|_v=0$ and $\prod\limits_v|\langle u,\omega\rangle_v|_v=1$.

In Section 18 we will discuss an interesting example where C and Q have genus 1. In the remainder of this section we focus on CM A-motives which come from Drinfeld modules. As analog of Colmez's Theorem 8.8 we have the following

Theorem 17.8. *Let $\underline{G}$ be a Drinfeld A-module over a finite separable field extension $K \subset Q^{\mathrm{alg}}$ of Q with complex multiplication of CM-type (E, Φ) as in Example 15.9, where $\Phi = (d_\psi)_{\psi \in H_E}$ with $d_{\psi_0} = 1$ for one $\psi_0 \in H_E$ and $d_\psi = 0$ for all $\psi \neq \psi_0$. Assume that $\underline{G}$ has complex multiplication by $\mathcal{O}_E$ and that E is a separable field extension of Q. Let $\underline{M} = \underline{M}(\underline{G})$ and choose ω_{ψ_0} and u_η as in Situation 17.1. Then the stable Taguchi height $ht^{\mathrm{st}}_{\mathrm{Tag}}(\underline{G})$ of $\underline{G}$ satisfies*

$$\begin{aligned} ht^{\mathrm{st}}_{\mathrm{Tag}}(\underline{G}) &= \tfrac{1}{\#H_K} \sum_{\eta \in H_K} \Big(-\log\big|\textstyle\int_{u_\eta} \omega^\eta_{\psi_0}\big|_\infty + \tfrac{1}{\#H_K} \displaystyle\sum_{\tilde\eta \in H_K} \sum_{v \neq \infty} \big(v(\omega^{\tilde\eta\eta}_{\psi_0}) + v_{\tilde\eta\eta\psi_0}(u^{\tilde\eta}_\eta)\big) \log q_v \Big) \\ &\qquad - \frac{\log \#(A/\mathfrak{d}_{\mathcal{O}_E/A})}{[E:Q]} - \log D_A(\mathcal{O}_E) \qquad (17.13) \\ &= \tfrac{1}{\#H_K} \sum_{\eta \in H_K} \Big(-\log\big|\textstyle\int_{u_\eta} \omega^\eta_{\psi_0}\big|_\infty + \tfrac{1}{\#H_K} \displaystyle\sum_{\tilde\eta \in H_K} \sum_{v \neq \infty} \big(v^\sim(\omega^{\tilde\eta\eta}_{\psi_0}) + v_{\tilde\eta\eta\psi_0}(u^{\tilde\eta}_\eta)\big) \log q_v \Big) - \log D_A(\mathcal{O}_E)\,. \end{aligned}$$

Proof. 1. Since both sides of the claimed equality (17.13) are invariant under extending the field K, we may assume that K is Galois over Q and that $\underline{G}$ has good reduction at every finite place of K. Via the inclusion $K \subset Q^{\mathrm{alg}} \subset Q^{\mathrm{alg}}_\infty$ the restriction of the valuation v_∞ on Q^{alg}_∞ to K corresponds to a place $\widetilde\infty$ of K such that the completion $K_{\widetilde\infty}$ equals the closure of K in Q^{alg}_∞. For every $\eta \in H_K = \mathrm{Gal}(K/Q)$ we denote the image of $\widetilde\infty$ under η by $\widetilde\infty_\eta$. Note that $\widetilde\infty_{\eta'} = \widetilde\infty_\eta$ if and only if $\eta'\eta^{-1} \in \mathrm{Gal}(K_{\widetilde\infty}/Q_\infty)$.

For $\eta \in \mathrm{Gal}(K/Q)$, we obtained $\underline{G}^\eta = (G^\eta, \varphi^\eta), \underline{M}^\eta$ and $\omega^\eta_{\psi_0} \in \mathrm{H}^{\eta\psi_0}(\underline{M}^\eta, K[\![z-\zeta]\!])$ from $\underline{G} = (G, \varphi), \underline{M}$ and ω_{ψ_0} in Situation 17.1 by applying η to the coefficients in K. Note that $K[\![y_{\psi_0} - \psi_0(y_{\psi_0})]\!] = K[\![z-\zeta]\!]$ by [HJ20, Lemma 1.3] because E/Q is separable. In addition, we chose E-generators $u_\eta \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta, Q)$ and as in Remark 17.5 we obtain E_v-generators $u^{\tilde\eta}_\eta \in \mathrm{H}_{1,v}(\underline{M}^{\tilde\eta\eta}, Q_v)$ for every $v \neq \infty$ and every $\tilde\eta \in H_K$. As in Example 15.9 let $m^\eta := -(z-\zeta)^{-1} \cdot \omega^\eta_{\psi_0} \in \mathfrak{q}^{\underline{M}^\eta}$. The image $\overline{m}^\eta = \tau_{M^\eta}(m^\eta)$ of m^η in $\operatorname{coker} \tau_{M^\eta} = \mathrm{Hom}_K(\mathrm{Lie}\, G^\eta, K)$ provides an isomorphism

$$\overline{m}^\eta \colon \mathrm{Lie}\, G^\eta \xrightarrow{\;\sim\;} K\,.$$

using (9.5). We can lift $\overline{m}^\eta$ in a unique way to an element $\widetilde{m}^\eta \in M^\eta$ which is an isomorphism $\widetilde{m}^\eta \colon G^\eta \xrightarrow{\sim} \mathbb{G}_{a,K}$. Indeed, if we choose any isomorphism $n \colon G^\eta \xrightarrow{\sim} \mathbb{G}_{a,K}$ with $n \in M^\eta$, then $\overline{m}^\eta = b \cdot \mathrm{Lie}\, n$ for some $b \in K^\times$, and we may take $\widetilde{m}^\eta := b \cdot n$. In particular, $\widetilde{m}^\eta$ is obtained from $\widetilde{m} := \widetilde{m}^{\mathrm{id}} \colon G \xrightarrow{\sim} \mathbb{G}_{a,K}$ by pull back under η. We recall the E-equivariant isomorphism for Betti-homology from Proposition 13.11

$$\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta, A) \otimes_A \Omega^1_{A/\mathbb{F}_q} \xrightarrow{\;\sim\;} \mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta, A)\,. \qquad (17.14)$$

We tensor it to Q and observe that $\Omega^1_{A/\mathbb{F}_q} \otimes_A Q = \Omega^1_{Q/\mathbb{F}_q} = Q\,dz$; see Remark 13.10. Under the isomorphism (17.14) we consider the element $\lambda_\eta := u_\eta\,dz \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta, Q)$. We may multiply u_η by an element $a \in A$ such that we can assume $u_\eta \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta, A)$ and $\lambda_\eta \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta, A)$. Since $c \in E$ acts on $\mathrm{Lie}\, G^\eta$ as multiplication with $\eta\psi_0(c)$, Theorem 13.20 implies for every $c \in E$

$$\big|\textstyle\int_{cu_\eta} \omega^\eta_{\psi_0}\big|_\infty = \big|\langle cu_\eta, \omega^\eta_{\psi_0}\rangle_\infty\big|_\infty = \big|\overline{m}^\eta(c\lambda_\eta)\big|_\infty = \big|\eta\psi_0(c)\big|_\infty \cdot \big|\overline{m}^\eta(\lambda_\eta)\big|_\infty\,.$$

2. We want to compute $ht_{\mathrm{Tag},\widetilde\infty_\eta}(\underline{G}/K)$ as in Equations (16.4) and (16.3). From [Gos96, Proposition 4.7.17] we know that $E_\infty := E \otimes_Q Q_\infty$ is still a field, that is E/Q is imaginary in the sense of Example 16.2 and Remark 17.5. For every $\eta \in H_E$ we consider the Q_∞-homomorphism $\eta\psi_0 \otimes \mathrm{id}_{Q_\infty} \colon E_\infty \to K_{\widetilde\infty} \subset Q^{\mathrm{alg}}_\infty$ which is hence injective. Therefore, the restriction to E_∞ of the valuation v_∞ on Q^{alg}_∞ is the unique valuation on E_∞ extending v_∞ on Q_∞. It is thus independent of η. By [Ser79, § I.4, Proposition 10] and [BGR84, § 3.6.2, Proposition 5] there are elements $c_1, \ldots, c_r \in E$ such that $E = \oplus_{i=1}^r Q \cdot c_i$ and

$$\Big|\sum_i a_i \cdot \eta\psi_0(c_i)\Big|_\infty = \max\big\{\, |a_i \cdot \eta\psi_0(c_i)|_\infty \,\big\} \qquad (17.15)$$

for every tuple $a_1, \ldots, a_r \in Q_\infty$. Under the isomorphism (17.14) we consider the elements $\lambda_{\eta,i} := c_i \cdot \lambda_\eta = c_i \cdot u_\eta\,dz \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta, Q)$. Then $\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta, Q) = \sum_{i=1}^r Q \cdot \lambda_{\eta,i}$, because u_η is an E-generator of $\mathrm{H}_1(\underline{M}^\eta, Q)$. We will check whether the tuple $\overline{m}^\eta(\lambda_{\eta,1}), \ldots, \overline{m}^\eta(\lambda_{\eta,r})$ is orthogonal in the sense of Definition 16.1 for the A-lattice

$$\Lambda(\underline{G}^\eta) := \overline{m}^\eta\big(\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta, A)\big) \subset \overline{m}^\eta(\mathrm{Lie}\, G^\eta \otimes_K \mathbb{C}_\infty) = \mathbb{C}_\infty\,.$$

For $a_1,\ldots,a_r\in Q_\infty$ equation (17.15) implies

$$\begin{aligned}\Big|\sum_i a_i\,\overline{m}^\eta(\lambda_{\eta,i})\Big|_\infty &= \Big|\sum_i a_i\cdot\eta\psi_0(c_i)\cdot\overline{m}^\eta(\lambda_\eta)\Big|_\infty\\ &= \Big|\sum_i a_i\cdot\eta\psi_0(c_i)\Big|_\infty\cdot\big|m^\eta(\lambda_\eta)\big|_\infty \;=\; \max\big\{\,\big|a_i\,\overline{m}^\eta(\lambda_{\eta,i})\big|_\infty\,\big\}\,.\end{aligned}$$

By multiplying all c_i by the same element $a\in A$ with $v_\infty(a)\ll 0$, we may assume that $c_i\in\mathcal{O}_E$ for all i and that conditions (a), (b) and (c) from Definition 16.1 are satisfied for $\overline{m}^\eta(\lambda_{\eta,i})$. We observe that $\big(\sum_{i=1}^r A\,c_i\big)\lambda_\eta\subset\mathcal{O}_E\,\lambda_\eta\subset\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta,A)$, and hence

$$\begin{aligned}\#\Big(\frac{\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta,A)}{\big(\sum_{i=1}^r A\,c_i\big)\lambda_\eta}\Big) &= \#\Big(\frac{\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta,A)}{\mathcal{O}_E\,\lambda_\eta}\Big)\cdot\#\Big(\frac{\mathcal{O}_E\,\lambda_\eta}{\big(\sum_{i=1}^r A\,c_i\big)\lambda_\eta}\Big)\\ &= \#\Big(\frac{\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta,A)}{\mathcal{O}_E\,\lambda_\eta}\Big)\cdot\#\Big(\frac{\mathcal{O}_E}{\sum_{i=1}^r A\,c_i}\Big)\,.\end{aligned}$$

Then

$$\begin{aligned}\frac{ht_{\mathrm{Tag},\widetilde{\infty}_\eta}(\underline{G}/K)}{-[K_{\widetilde{\infty}_\eta}:Q_\infty]} &= \log_q D_A\big(\overline{m}^\eta\big(\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta,A)\big)\big)\\ &= \log_q\left(\frac{\prod_{1\le i\le r}|\overline{m}^\eta(\lambda_{\eta,i})|_\infty}{\#\big(\Lambda(G^\eta)/(A\cdot\overline{m}^\eta(\lambda_{\eta,1})+\cdots+A\cdot\overline{m}^\eta(\lambda_{\eta,r}))\big)}\right)^{1/r}\\ &= \log_q\left(\frac{\prod_{1\le i\le r}\big|\eta\psi_0(c_i)\big|_\infty\cdot\big|\int_{u_\eta}\omega^\eta_{\psi_0}\big|_\infty}{\#\big(\,\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta,A)/(\sum_{i=1}^r A\,c_i)\lambda_\eta\big)}\right)^{1/r}\\ &= \log_q\big|\textstyle\int_{u_\eta}\omega^\eta_{\psi_0}\big|_\infty-\log_q\#\left(\dfrac{\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta,A)}{\mathcal{O}_E\,\lambda_\eta}\right)^{1/r}+\log_q\left(\dfrac{\prod_{1\le i\le r}\big|\eta\psi_0(c_i)\big|_\infty}{\#(\mathcal{O}_E/\sum_{i=1}^r A\,c_i)}\right)^{1/r}\\ &= \log_q\big|\textstyle\int_{u_\eta}\omega^\eta_{\psi_0}\big|_\infty-\dfrac{1}{r}\,\log_q\#\left(\dfrac{\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta,A)}{\mathcal{O}_E\,\lambda_\eta}\right)+\log_q D_A(\mathcal{O}_E)\,,\end{aligned}\tag{17.16}$$

where the last equation is the definition of $D_A(\mathcal{O}_E)$ from Example 16.2. In particular, this formula holds equally for all $\eta'\in H_K$ with $\widetilde{\infty}_{\eta'}=\widetilde{\infty}_\eta$ of K, that is for all $\eta'\in\mathrm{Gal}(K_{\widetilde{\infty}_\eta}/Q_\infty)\cdot\eta$.

3. We compute further

$$\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta,A)/\mathcal{O}_E\,\lambda_\eta \;=\; \prod_{v\ne\infty}\big(\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta,A)/\mathcal{O}_E\,\lambda_\eta\big)\otimes_A A_v \;=\; \prod_{v\ne\infty}\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta,A_v)/\mathcal{O}_{E_v}\,\lambda_\eta\,.$$

Under the isomorphism (17.14), tensored to A_v we have

$$\begin{array}{ccc}\mathcal{O}_{E_v}\,\lambda_\eta & \hookrightarrow & \mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta,A_v)\\ \uparrow\cong & & \uparrow\cong\\ \mathcal{O}_{E_v}\,u_\eta\otimes_{A_v}A_v\,dz \leftarrow - - \rightarrow \mathcal{O}_{E_v}\,u_\eta\otimes_A\Omega^1_{A/\mathbb{F}_q} & \subset & \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta,A_v)\otimes_A\Omega^1_{A/\mathbb{F}_q}\,,\end{array}$$

where the dashed arrow in the lower left corner comes from a comparison of A_v-modules of rank one, which is an inclusion $A_v\,dz\subset\Omega^1_{A/\mathbb{F}_q}\otimes_A A_v$ or $A_v\,dz\supset\Omega^1_{A/\mathbb{F}_q}\otimes_A A_v$ and even an equality for almost all v. Therefore,

$$\log_q\#\big(\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta,A_v)/\mathcal{O}_{E_v}\,\lambda_\eta\big) \;=\; r\,\mathrm{ord}_v(dz)\cdot[\mathbb{F}_v:\mathbb{F}_q]+\log_q\#\big(\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta,A_v)/\mathcal{O}_{E_v}\,u_\eta\big)\,.$$

Here the factor $r=\mathrm{rk}_{A_v}\mathcal{O}_{E_v}$ comes from the tensor product with $\mathcal{O}_{E_v}\,u_\eta$, and $\mathrm{ord}_v(dz)$ is the order at v of the rational section dz of the line bundle $\Omega^1_{C/\mathbb{F}_q}$. That is, if $A_v\,dz\subset\Omega^1_{A/\mathbb{F}_q}\otimes_A A_v$ then $\log_q\#\big(\Omega^1_{A/\mathbb{F}_q}\otimes_A A_v/A_v\,dz\big)=[\mathbb{F}_v:\mathbb{F}_q]\,\mathrm{ord}_v(dz)$. Adding over all places $v\ne\infty$ we obtain

$$\log_q\#\big(\mathrm{H}_{1,\mathrm{Betti}}(\underline{G}^\eta,A)/\mathcal{O}_E\,\lambda_\eta\big) \;=\; \sum_{v\ne\infty}\Big(r\,\mathrm{ord}_v(dz)\cdot[\mathbb{F}_v:\mathbb{F}_q]+\log_q\#\big(\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta,A_v)/\mathcal{O}_{E_v}\,u_\eta\big)\Big)\,.\tag{17.17}$$

4. We now fix a place $v \neq \infty$ and let $e_\eta \in E_v := E \otimes_Q Q_v$ such that $e_\eta^{-1} u_\eta$ is an $\mathcal{O}_{E_v}$-generator of $\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta, A_v) = \mathrm{H}_{1,v}(\underline{M}^\eta, A_v)$. Then $\mathcal{O}_{E_v}/e_\eta \mathcal{O}_{E_v} \xrightarrow{\sim} \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta, A_v)/\mathcal{O}_{E_v}\, u_\eta$ under $a \mapsto a\, e_\eta^{-1} u_\eta$. By the definition of $u_\eta^{\tilde\eta}$ in (17.10) also $e_\eta^{-1} u_\eta^{\tilde\eta}$ is an $\mathcal{O}_{E_v}$-generator of $\mathrm{H}_{1,v}(\underline{M}^{\tilde\eta\eta}, A_v)$. This means

$$v_{\tilde\eta\eta\psi_0}(u_\eta^{\tilde\eta}) \;:=\; v\big(\tilde\eta\eta\psi_0(e_\eta)\big)\,.$$

The Q_v-algebra E_v decomposes into a product of fields $E_v = \prod_i E_{v,i}$. To compute the cardinality of $\mathcal{O}_{E_v}/e_\eta\mathcal{O}_{E_v} = \prod_i \mathcal{O}_{E_{v,i}}/e_\eta \mathcal{O}_{E_{v,i}}$, note that each $\mathcal{O}_{E_{v,i}}/e_\eta \mathcal{O}_{E_{v,i}}$ is an $\mathbb{F}_v$-vector space. We denote its dimension by n_i. Let K_v be the closure in $\mathbb{C}_v$ of $K \subset Q^{\mathrm{alg}} \subset Q_v^{\mathrm{alg}} \subset \mathbb{C}_v$, let $\mathcal{O}_{K_v}$ be its valuation ring and k_v its residue field. For every Q_v-homomorphism $\widetilde\psi_i \in H_{E_{v,i}} := \mathrm{Hom}_{Q_v}(E_{v,i}, Q_v^{\mathrm{alg}})$ the $\mathbb{F}_v$-vector space

$$\big(\mathcal{O}_{E_{v.i}}/e_\eta\mathcal{O}_{E_{v,i}}\big) \otimes_{\mathcal{O}_{E_{v,i}},\widetilde\psi_i} \mathcal{O}_{K_v} \;=\; \mathcal{O}_{K_v}/\widetilde\psi_i(e_\eta)\mathcal{O}_{K_v}$$

has dimension $n_i \cdot [K_v : \widetilde\psi_i(E_{v,i})]$, because $\mathcal{O}_{K_v}$ is free over $\mathcal{O}_{E_{v,i}}$ of rank $[K_v : \widetilde\psi_i(E_{v,i})]$. This dimension is equal to $[k_v : \mathbb{F}_v] \cdot \mathrm{ord}_{K_v}(\widetilde\psi_i(e_\eta)) = [K_v : Q_v] \cdot v(\widetilde\psi_i(e_\eta))$. We conclude that

$$n_i \;:=\; \dim_{\mathbb{F}_v}\big(\mathcal{O}_{E_{v.i}}/e_\eta\mathcal{O}_{E_{v,i}}\big) \;=\; \frac{[K_v : Q_v]}{[K_v : \widetilde\psi_i(E_{v,i})]} \cdot v(\widetilde\psi_i(e_\eta)) \;=\; [E_{v,i} : Q_v] \cdot v(\widetilde\psi_i(e_\eta))$$

$$\text{and} \qquad \log_q \#\Big(\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta, A_v)/\mathcal{O}_{E_v}\, u_\eta\Big) \;=\; \sum_i n_i \cdot [\mathbb{F}_v : \mathbb{F}_q]\,.$$

We now consider the following maps

$$\begin{array}{ccccc} H_K & \twoheadrightarrow & H_E & \xrightarrow{\;\sim\;} & \mathrm{Hom}_{Q_v}(E_v, Q_v^{\mathrm{alg}}) \\ \tilde\eta & \longmapsto & \tilde\eta\eta\psi_0 \;=:\; \widetilde\psi & \longmapsto & \widetilde\psi \otimes \mathrm{id}_{Q_v} \end{array}$$

The set $\mathrm{Hom}_{Q_v}(E_v, Q_v^{\mathrm{alg}})$ is equal to $\coprod_i H_{E_{v,i}}$, because every $\widetilde\psi \otimes \mathrm{id}_{Q_v}$ factors in a unique way

$$\begin{tikzcd} \widetilde\psi \otimes \mathrm{id}_{Q_v} \colon E_v = \prod_i E_{v,i} \arrow[rr] \arrow[dr, two heads] & & Q_v^{\mathrm{alg}} \\ & E_{v,i(\widetilde\psi)} \arrow[ur, hook, "\widetilde\psi_i \;:=\; (\widetilde\psi \otimes \mathrm{id}_{Q_v})|_{E_{v,i(\widetilde\psi)}}"'] & \end{tikzcd} \tag{17.18}$$

for an index $i(\widetilde\psi)$. The number of elements $\tilde\eta \in H_K$ which are mapped to the same $\widetilde\psi := \tilde\eta\eta\psi_0 \in H_E$ equals $\#\,\mathrm{Gal}(K/\eta\psi_0(E)) = [K : \eta\psi_0(E)] = \frac{[K:Q]}{[E:Q]}$, and the number of $\tilde\eta \in H_K$ which are mapped into the set $H_{E_{v,i}}$ equals

$$\#H_{E_{v,i}} \cdot \tfrac{[K:Q]}{[E:Q]} \;=\; [E_{v,i} : Q_v] \cdot \tfrac{[K:Q]}{[E:Q]}\,. \tag{17.19}$$

For each of the latter $\tilde\eta$ the valuation $v(\tilde\eta\eta\psi_0(e_\eta)) = v(\widetilde\psi_i(e_\eta)) = \frac{n_i}{[E_{v,i}:Q_v]}$ is the same. This implies

$$\begin{aligned} \tfrac{1}{\#H_K} \sum_{\tilde\eta \in H_K} v_{\tilde\eta\eta\psi_0}(u_\eta^{\tilde\eta}) \cdot [\mathbb{F}_v : \mathbb{F}_q] \;&=\; \tfrac{1}{\#H_K} \sum_{\tilde\eta \in H_K} v(\tilde\eta\eta\psi_0(e_\eta)) \cdot [\mathbb{F}_v : \mathbb{F}_q] \\ &=\; \tfrac{[K:Q]}{\#H_K} \sum_i \frac{n_i}{[E_{v,i} : Q_v]} \cdot \frac{[E_{v,i} : Q_v]}{[E : Q]} \cdot [\mathbb{F}_v : \mathbb{F}_q] \\ &=\; \frac{1}{r}\, \log_q \#\Big(\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta, A_v)/\mathcal{O}_{E_v}\, u_\eta\Big)\,. \end{aligned} \tag{17.20}$$

Putting equations (17.16), (17.17) and (17.20) together we can compute

$$\frac{ht_{\mathrm{Tag},\widetilde\infty_\eta}(\underline{G}/K)}{[K_{\widetilde\infty_\eta} : Q_\infty]} = -\log_q \big|\textstyle\int_{u_\eta} \omega_{\psi_0}^\eta\big|_\infty + \tfrac{1}{\#H_K} \displaystyle\sum_{\tilde\eta \in H_K} \sum_{v \neq \infty} \big(\mathrm{ord}_v(dz) + v_{\tilde\eta\eta\psi_0}(u_\eta^{\tilde\eta})\big)\,[\mathbb{F}_v : \mathbb{F}_q] - \log_q D_A(\mathcal{O}_E)\,. \tag{17.21}$$

5. Now we take a finite place $\tilde{v}_\eta$ of K and let $v \neq \infty$ be the place of Q with $\tilde{v}_\eta | v$. We choose an $\eta \in H_K$ such that $\tilde{v}_\eta$ is the place induced from v via $\eta\colon K \hookrightarrow Q^{\rm alg} \subset Q_v^{\rm alg} \subset \mathbb{C}_v$ and view $\underline{G}^\eta$ as a Drinfeld module over $\mathbb{C}_v$. We use the isomorphism $\widetilde{m}^\eta$ from Step 1 above to write

$$\widetilde{m}^\eta \circ \varphi_a^\eta \circ (\widetilde{m}^\eta)^{-1} \;=\; \gamma(a) + \sum_{i=1}^{r\deg a} \varphi_{a,i}^\eta \tau^i \;\in\; \mathrm{End}_{\mathbb{C}_v,\mathbb{F}_q}(\mathbb{G}_{a,\mathbb{C}_v}) \;=\; \mathbb{C}_v\{\tau\} \quad \text{with} \quad \varphi_{a,i}^\eta \;\in\; \mathbb{C}_v\,.$$

Since $\underline{G}$ has good reduction at $\tilde{v}_\eta$ there exists an element $x_\eta \in K_{\tilde{v}_\eta}^\times$ such that

$$x_\eta \widetilde{m}^\eta \circ \varphi_a^\eta \circ (\widetilde{m}^\eta)^{-1} x_\eta^{-1} \;=\; \gamma(a) + \sum_{i=1}^{r\deg a} \varphi_{a,i}^\eta \cdot x_\eta^{1-q^i}\, \tau^i \;\in\; \mathcal{O}_{\mathbb{C}_v}\{\tau\} \quad \text{and} \quad \varphi_{a,\,r\deg a}^\eta \cdot x_\eta^{1-q^{r\deg a}} \;\in\; \mathcal{O}_{\mathbb{C}_v}^\times\,.$$

We have $\dfrac{e(\tilde{v}_\eta|v)\cdot v(\varphi_{a,i}^\eta)}{q^i-1} = \dfrac{e(\tilde{v}_\eta|v)\cdot v(\varphi_{a,i}^\eta\cdot x_\eta^{1-q^i})}{q^i-1} + e(\tilde{v}_\eta|v)\cdot v(x_\eta)$. Note that $\dfrac{e(\tilde{v}_\eta|v)\cdot v(\varphi_{a,i}^\eta\cdot x_\eta^{1-q^i})}{q^i-1} \geq 0$ for all i and equal to 0 for $i = r\deg a$. So

$$\mathrm{ord}_{\tilde{v}_\eta}(\underline{G}) \;:=\; \min\left\{\frac{e(\tilde{v}_\eta|v)\cdot v(\varphi_{a,i}^\eta)}{q^i-1} : a\in A\smallsetminus\mathbb{F}_q,\; 1\le i\le r\deg a\right\} \;=\; e(\tilde{v}_\eta|v)\cdot v(x_\eta) \;\in\; \mathbb{Z}\,.$$

Then

$$ht_{\mathrm{Tag},\tilde{v}_\eta}(\underline{G}/K) \;:=\; -[\mathbb{F}_{\tilde{v}_\eta}:\mathbb{F}_q]\cdot e(\tilde{v}_\eta|v)\cdot v(x_\eta) \;=\; -[K_{\tilde{v}_\eta}:Q_v]\cdot v(x_\eta)\cdot[\mathbb{F}_v:\mathbb{F}_q]\,. \tag{17.22}$$

It remains to relate $v(x_\eta)$ to $v(\omega_{\psi_0}^\eta)$. For this let $\underline{\mathcal{G}}^\eta$ be the good model of $\underline{G}^\eta$ over $\mathcal{O}_{\mathbb{C}_v}$ and let $\underline{\mathcal{M}}^\eta$ be the A-motive of $\underline{\mathcal{G}}^\eta$. The latter is the good model of $\underline{M}^\eta$ over $\mathcal{O}_{\mathbb{C}_v}$. Then $x_\eta\widetilde{m}^\eta$ extends to a coordinate system $x_\eta\widetilde{m}^\eta\colon \underline{\mathcal{G}}^\eta \xrightarrow{\sim} \mathbb{G}_{a,\mathcal{O}_{\mathbb{C}_v}}$ over $\mathcal{O}_{\mathbb{C}_v}$ of $\underline{\mathcal{G}}^\eta$ and induces an isomorphism

$$\mathrm{End}_{\mathcal{O}_{\mathbb{C}_v},\mathbb{F}_q}(\mathbb{G}_{a,\mathcal{O}_{\mathbb{C}_v}}) \;=\; \mathcal{O}_{\mathbb{C}_v}\{\tau\} \;\xrightarrow{\sim}\; \underline{\mathcal{M}}^\eta \;:=\; \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_v},\mathbb{F}_q}(\underline{\mathcal{G}}^\eta,\mathbb{G}_{a,\mathcal{O}_{\mathbb{C}_v}})\,,\quad f \;\longmapsto\; f\circ x_\eta\widetilde{m}^\eta\,.$$

This implies that $x_\eta\overline{m}^\eta$ generates the $\mathcal{O}_{\mathbb{C}_v}$-module $\operatorname{coker}\tau_{\mathcal{M}^\eta}$. Next let $w = w_\eta$ be the place of E which is induced from the place $\tilde{v}_\eta$ of K under the embedding $\psi_0\colon E \hookrightarrow K$. Then w_η is induced from the valuation v on $\mathbb{C}_v$ under the embedding $\eta\psi_0\colon E\hookrightarrow\mathbb{C}_v$ and lies above the place v of Q. Let $y_w\in\mathcal{O}_E$ be an element which is a uniformizing parameter at w, that is, which satisfies $w(y_w)=1$. Set $\theta_w := \eta\psi_0(y_w)\in\mathcal{O}_{\mathbb{C}_v}$. We use the isomorphism induced from τ_{M^η}

$$(y_w-\theta_w)^{-1}\,\mathrm{H}^{\eta\psi_0}(\underline{M}^\eta,\mathbb{C}_v[\![y_w-\theta_w]\!])/\,\mathrm{H}^{\eta\psi_0}(\underline{M}^\eta,\mathbb{C}_v[\![y_w-\theta_w]\!]) \;\xrightarrow{\sim}\; \mathfrak{q}^{\underline{M}^\eta}/\mathfrak{p}^{\underline{M}^\eta} \;\xrightarrow[\tau_{M^\eta}]{\sim}\; \operatorname{coker}\tau_{M^\eta}\,.$$

In the source of this isomorphism the elements $x_\eta m^\eta$ and $\tau_{M^\eta}^{-1}(x_\eta\widetilde{m}^\eta)$ are equal, because both have the same image $x_\eta\overline{m}^\eta$ in the target $\operatorname{coker}\tau_{M^\eta}$. Therefore, $x_\eta m^\eta$ is a generator of the canonical $\mathcal{O}_{\mathbb{C}_v}$-module structure on the source induced from $\underline{\mathcal{M}}^\eta$. Multiplication with $y_w-\theta_w$ maps this $\mathcal{O}_{\mathbb{C}_v}$-structure isomorphically onto the $\mathcal{O}_{\mathbb{C}_v}$-module $\mathrm{H}^{\eta\psi_0}(\underline{\mathcal{M}}^\eta,\mathcal{O}_{\mathbb{C}_v})$, which is hence generated by $(y_w-\theta_w)x_\eta m^\eta$. On the other hand, after multiplication with $-(z-\zeta)$ mod $(z-\zeta)^2$ we obtain $x_\eta\omega_{\psi_0}^\eta = -(z-\zeta)x_\eta m^\eta$ in

$$\mathrm{H}^{\eta\psi_0}(\underline{M}^\eta,\mathbb{C}_v) \;=\; \mathrm{H}^{\eta\psi_0}(\underline{M}^\eta,\mathbb{C}_v[\![y_w-\theta_w]\!])/(y_w-\theta_w)\,\mathrm{H}^{\eta\psi_0}(\underline{M}^\eta,\mathbb{C}_v[\![y_w-\theta_w]\!])\,.$$

All these are one dimensional $\mathbb{C}_v$-vector spaces. Note that $y_w-\theta_w$ and $z-\zeta$ are not equal. Namely, if we write $I := \ker\big(\mathcal{O}_E\otimes_{\mathbb{F}_q}\mathcal{O}_E\to\mathcal{O}_E,\ a\otimes a'\mapsto aa'\big) = (a\otimes 1 - 1\otimes a\colon a\in\mathcal{O}_E)$, the element $(z-\zeta)$ mod $(z-\zeta)^2$ of $\mathbb{C}_v$ is the image of $dz := (z\otimes 1 - 1\otimes z)$ mod $I^2 \in \Omega^1_{\mathcal{O}_E/\mathbb{F}_q} := I/I^2$ under the $\mathcal{O}_E$-homomorphism

$$\Omega^1_{\mathcal{O}_E/\mathbb{F}_q} \;\longrightarrow\; \Omega^1_{\mathcal{O}_E/\mathbb{F}_q} \underset{\mathcal{O}_E\otimes\mathcal{O}_E/I,\,\mathrm{id}_{\mathcal{O}_E}\otimes\eta\psi_0}{\otimes} (\mathcal{O}_E\otimes_{\mathbb{F}_q}\mathbb{C}_v)/(a\otimes 1 - 1\otimes\eta\psi_0(a)\colon a\in\mathcal{O}_E) \;=\; \Omega^1_{\mathcal{O}_E/\mathbb{F}_q} \underset{\mathcal{O}_E,\,\eta\psi_0}{\otimes} \mathbb{C}_v\,.$$

On the other hand, $y_w-\theta_w$ is the image of $dy_w := (y_w\otimes 1 - 1\otimes y_w)$ mod I^2 and is a generator of the $\mathcal{O}_{\mathbb{C}_v}$-module $\Omega^1_{\mathcal{O}_E/\mathbb{F}_q}\otimes_{\mathcal{O}_E,\eta\psi_0}\mathcal{O}_{\mathbb{C}_v}$. Therefore, $x_\eta\frac{y_w-\theta_w}{z-\zeta}\cdot\omega_{\psi_0}^\eta$ is an $\mathcal{O}_{\mathbb{C}_v}$-generator of $\mathrm{H}^{\eta\psi_0}(\underline{\mathcal{M}}^\eta,\mathcal{O}_{\mathbb{C}_v})$, and hence

$$v(\omega_{\psi_0}^\eta) \;=\; v\big(x_\eta^{-1}\tfrac{z-\zeta}{y_w-\theta_w}\big) \;=\; v\big(x_\eta^{-1}\cdot\eta\psi_0(\tfrac{dz}{dy_w})\big) \;=\; -v(x_\eta) + \frac{\mathrm{ord}_{w_\eta}(dz)}{e(w_\eta|v)}\,,$$

where again $\mathrm{ord}_{w_\eta}(dz) \in \mathbb{Z}$ is the order at w_η of the rational section dz of the line bundle $\Omega^1_{\mathcal{O}_E/\mathbb{F}_q}$. From (17.22) we obtain for the local Taguchi height at $\tilde{v}_\eta$

$$\frac{ht_{\mathrm{Tag},\tilde{v}_\eta}(\underline{G}/K)}{[K_{\tilde{v}_\eta} : Q_v]} \;=\; -v(x_\eta)\cdot[\mathbb{F}_v : \mathbb{F}_q] \;=\; v(\omega^\eta_{\psi_0})\cdot[\mathbb{F}_v : \mathbb{F}_q] - \frac{\mathrm{ord}_{w_\eta}(dz)\cdot[\mathbb{F}_{w_\eta} : \mathbb{F}_q]}{[E_{w_\eta} : Q_v]}\,. \tag{17.23}$$

6. The summand on the right is related to the different $\mathfrak{D}_{\mathcal{O}_E/A}$. Namely, by [Ser79, § III.7, Proposition 14] the $\mathcal{O}_E$-module of relative differentials $\Omega^1_{\mathcal{O}_E/A}$ is generated by one element and is isomorphic to $\mathcal{O}_E/\mathfrak{D}_{\mathcal{O}_E/A}$. This gives rise to the exact sequence [EGA, 0_{IV}, Théorème 0.20.5.7]

$$0 \longrightarrow \Omega^1_{A/\mathbb{F}_q} \otimes_A \mathcal{O}_E \longrightarrow \Omega^1_{\mathcal{O}_E/\mathbb{F}_q} \longrightarrow \mathcal{O}_E/\mathfrak{D}_{\mathcal{O}_E/A} \longrightarrow 0\,.$$

There is an element $0 \neq a \in A$ with $a\,dz \in \Omega^1_{A/\mathbb{F}_q}$. Dividing out $\mathcal{O}_E \cdot a\,dz$ yields the exact sequence

$$0 \longrightarrow \big(\Omega^1_{A/\mathbb{F}_q} \otimes_A \mathcal{O}_E\big)/\mathcal{O}_E\cdot a\,dz \longrightarrow \Omega^1_{\mathcal{O}_E/\mathbb{F}_q}/\mathcal{O}_E\cdot a\,dz \longrightarrow \mathcal{O}_E/\mathfrak{D}_{\mathcal{O}_E/A} \longrightarrow 0\,.$$

Counting elements, and denoting the places of E by w and their residue fields by $\mathbb{F}_w$, we obtain

$$\begin{aligned}
\prod_{w\nmid\infty}(\#\mathbb{F}_w)^{\mathrm{ord}_w(a\,dz)} &= \#\big(\Omega^1_{\mathcal{O}_E/\mathbb{F}_q}/\mathcal{O}_E\cdot a\,dz\big)\\
&= \#\big(\mathcal{O}_E/\mathfrak{D}_{\mathcal{O}_E/A}\big)\cdot\#\big((\Omega^1_{A/\mathbb{F}_q}/A\cdot a\,dz)\otimes_A\mathcal{O}_E\big)\\
&= \#\big(\mathcal{O}_E/\mathfrak{D}_{\mathcal{O}_E/A}\big)\cdot\#\big(\Omega^1_{A/\mathbb{F}_q}/A\cdot a\,dz\big)^{[\mathcal{O}_E:A]}\\
&= \#\big(\mathcal{O}_E/\mathfrak{D}_{\mathcal{O}_E/A}\big)\cdot\Big(\prod_{v\neq\infty}(\#\mathbb{F}_v)^{\mathrm{ord}_v(a\,dz)}\Big)^r.
\end{aligned}$$

We observe $\mathrm{ord}_w(a\,dz) = w(a) + \mathrm{ord}_w(dz)$ and that for every place $v \neq \infty$ of Q

$$\prod_{w|v}(\#\mathbb{F}_w)^{w(a)} \;=\; \prod_{w|v}(\#\mathbb{F}_v)^{[\mathbb{F}_w:\mathbb{F}_v]\cdot e(w|v)\cdot v(a)} \;=\; (\#\mathbb{F}_v)^{\sum_{w|v}[\mathbb{F}_w:\mathbb{F}_v]\cdot e(w|v)\cdot v(a)} \;=\; (\#\mathbb{F}_v)^{r\cdot v(a)}.$$

Taking $\log_q$ this yields

$$\sum_{w\nmid\infty}[\mathbb{F}_w : \mathbb{F}_q]\cdot\mathrm{ord}_w(dz) - r\cdot\sum_{v\neq\infty}[\mathbb{F}_v : \mathbb{F}_q]\cdot\mathrm{ord}_v(dz) \;=\; \log_q\#\big(\mathcal{O}_E/\mathfrak{D}_{\mathcal{O}_E/A}\big) \;=\; \log_q\#\big(A/\mathfrak{d}_{\mathcal{O}_E/A}\big)\,, \tag{17.24}$$

where $\mathfrak{d}_{\mathcal{O}_E/A} = N_{E/Q}(\mathfrak{D}_{\mathcal{O}_E/A})$ is the discriminant of $\mathcal{O}_E$ over A, and the last equality comes from the fact that for all maximal ideals $\mathfrak{P} \subset \mathcal{O}_E$ and $\mathfrak{p} := A \cap \mathfrak{P} \subset A$ with residue fields $\mathbb{F}_\mathfrak{P}$, respectively $\mathbb{F}_\mathfrak{p}$, and for every $n \in \mathbb{N}$ we have $N_{E/Q}(\mathfrak{P}^n) = \mathfrak{p}^{[\mathbb{F}_\mathfrak{P}:\mathbb{F}_\mathfrak{p}]n}$ and $\#(\mathcal{O}_E/\mathfrak{P}^n) = \#(\mathbb{F}_\mathfrak{P})^n = (\#\mathbb{F}_\mathfrak{p})^{[\mathbb{F}_\mathfrak{P}:\mathbb{F}_\mathfrak{p}]n} = \#\big(A/N_{E/Q}(\mathfrak{P}^n)\big)$.

7. Fix a place w of E above v. In terms of the decomposition $E_v := E \otimes_Q Q_v = \prod_i E_{v,i}$ from diagram (17.18) the completion E_w of E at w equals $E_{v,i}$ for some i and the number of $\eta \in H_K$ which give rise to the same $w_\eta = w$

equals $[E_w : Q_v]\cdot\frac{[K:Q]}{[E:Q]}$ by (17.19). This together with (17.23), (17.21) and (17.24) finally implies

$$\begin{aligned}
ht^{\mathrm{st}}_{\mathrm{Tag}}(\underline{G}) &= \frac{\log q}{[K:Q]}\cdot\Big(\sum_{\tilde v\nmid\infty} ht_{\mathrm{Tag},\tilde v}(\underline{G}/K) + \sum_{\widetilde\infty|\infty} ht_{\mathrm{Tag},\widetilde\infty}(\underline{G}/K)\Big)\\
&= \frac{\log q}{[K:Q]}\cdot\sum_{\eta\in H_K}\Big(\sum_{v\neq\infty}\frac{ht_{\mathrm{Tag},\tilde v_\eta}(\underline{G}/K)}{[K_{\tilde v_\eta}:Q_v]} + \frac{ht_{\mathrm{Tag},\widetilde\infty_\eta}(\underline{G}/K)}{[K_{\widetilde\infty_\eta}:Q_\infty]}\Big)\\
&= \frac{\log q}{[K:Q]}\cdot\sum_{\eta\in H_K}\Big(\sum_{v\neq\infty}\Big(v(\omega^\eta_{\psi_0})\cdot[\mathbb{F}_v:\mathbb{F}_q] - \frac{\mathrm{ord}_{w_\eta}(dz)\cdot[\mathbb{F}_{w_\eta}:\mathbb{F}_q]}{[E_{w_\eta}:Q_v]}\Big)\\
&\qquad -\log_q\big|\textstyle\int_{u_\eta}\omega^\eta_{\psi_0}\big|_\infty + \frac{1}{\#H_K}\sum_{\tilde\eta\in H_K}\sum_{v\neq\infty}\big(\mathrm{ord}_v(dz)+v_{\tilde\eta\eta\psi_0}(u^{\tilde\eta}_\eta)\big)[\mathbb{F}_v:\mathbb{F}_q] - \log_q D_A(\mathcal{O}_E)\Big)\\
&= \tfrac{1}{\#H_K}\sum_{\eta\in H_K}\Big(-\log\big|\textstyle\int_{u_\eta}\omega^\eta_{\psi_0}\big|_\infty + \frac{1}{\#H_K}\sum_{\tilde\eta\in H_K}\sum_{v\neq\infty}\big(v(\omega^{\tilde\eta\eta}_{\psi_0})+v_{\tilde\eta\eta\psi_0}(u^{\tilde\eta}_\eta)\big)\log q_v\Big) - \log D_A(\mathcal{O}_E)\\
&\qquad + \frac{\log q}{[K:Q]}\cdot\Big(-\frac{[K:Q]}{[E:Q]}\sum_{w\nmid\infty}[\mathbb{F}_w:\mathbb{F}_q]\cdot\mathrm{ord}_w(dz) + [K:Q]\sum_{v\neq\infty}[\mathbb{F}_v:\mathbb{F}_q]\cdot\mathrm{ord}_v(dz)\Big)\\
&= \tfrac{1}{\#H_K}\sum_{\eta\in H_K}\Big(-\log\big|\textstyle\int_{u_\eta}\omega^\eta_{\psi_0}\big|_\infty + \frac{1}{\#H_K}\sum_{\tilde\eta\in H_K}\sum_{v\neq\infty}\big(v(\omega^{\tilde\eta\eta}_{\psi_0})+v_{\tilde\eta\eta\psi_0}(u^{\tilde\eta}_\eta)\big)\log q_v\Big)\\
&\qquad - \frac{\log\#(A/\mathfrak{d}_{\mathcal{O}_E/A})}{[E:Q]} - \log D_A(\mathcal{O}_E)
\end{aligned}$$

which finishes the proof. $\square$

Remark 17.9. For a Drinfeld module $\underline{G}$ of rank r over a finite Galois extension K/Q with CM by $\mathcal{O}_E$ for a separable field extension E/Q with CM type as in Theorem 17.8, the functions from (17.2) and (17.3) are

$$a_{E,\psi_0,\Phi}(g) \;=\; \left\{\begin{array}{ll}1 & \text{if } g\in\mathrm{Gal}(K/\psi_0(E))\\ 0 & \text{else}\end{array}\right\} \;=\; \mathbb{1}_{\mathrm{Gal}(K/\psi_0E)}(g) \qquad \text{and}$$

$$a^0_{E,\psi_0,\Phi}(g) \;=\; \tfrac{1}{\#H_K}\sum_{\eta\in H_K}\mathbb{1}_{\mathrm{Gal}(K/\eta\psi_0E)}(g) \;=\; \Big(\tfrac{1}{r}\cdot\mathrm{Ind}^{\mathrm{Gal}(K/Q)}_{\mathrm{Gal}(K/\psi_0E)}\,\mathbb{1}_{\mathrm{Gal}(K/\psi_0E)}\Big)(g)\,,$$

where $\mathbb{1}_{\mathrm{Gal}(K/\eta\psi_0E)}$ is the characteristic function of the subset $\mathrm{Gal}(K/\eta\psi_0(E))\subset\mathrm{Gal}(K/Q)$ and Ind denotes the induction of characters; see [Cas67, Chapter VIII, § 3, Property (V), page 222]. Then $(a^0_{E,\psi_0,\Phi})^* = a^0_{E,\psi_0,\Phi}$ and [Cas67, loc. cit.] implies

$$\begin{aligned}
L^\infty\big((a^0_{E,\psi_0,\Phi})^*,s,K/Q\big)^r &= L^\infty(\mathrm{Ind}^{\mathrm{Gal}(K/Q)}_{\mathrm{Gal}(K/\psi_0E)}\,\mathbb{1}_{\mathrm{Gal}(K/\psi_0E)},s,K/Q)\\
&= L^\infty(\mathbb{1}_{\mathrm{Gal}(K/\psi_0E)},s,K/\psi_0E)\\
&= \zeta_{\mathcal{O}_E}(s)\,,
\end{aligned}$$

and hence

$$r\cdot Z^\infty\big((a^0_{E,\psi_0,\Phi})^*,0\big) \;=\; \frac{\zeta'_{\mathcal{O}_E}(0)}{\zeta_{\mathcal{O}_E}(0)}\,.$$

If ∞ is tamely ramified in E/Q then Example 16.2 and [HS20, Lemma 5.17 and Proposition 5.18] imply that

$$\log D_A(\mathcal{O}_E) \;=\; \frac{\log\#(A/\mathfrak{d}_{\mathcal{O}_E/A})}{2r} \;=\; \frac{1}{2}\cdot\mu^\infty_{\mathrm{Art}}(a^0_{E,\psi_0,\Phi})\,,$$

where $\mu^\infty_{\mathrm{Art}}$ was defined in (8.5). This puts Theorem 17.8 in a form analogous to Colmez's Theorem 8.8.

Thus to establish the product formula in Conjecture 17.6 for a CM Drinfeld A-module $\underline{G}$ it suffices to relate the Taguchi height of $\underline{G}$ to the logarithmic derivative of the Zeta-function $\zeta_{\mathcal{O}_E}$. This was achieved by Fu-Tsun Wei [Wei20]:

Theorem 17.10 ([Wei20, Theorem 1.6])**.** *In Situation 17.1 let $\underline{M}=\underline{M}(\underline{G})$ for a Drinfeld A-module $\underline{G}$ of rank r with complex multiplication by $\mathcal{O}_E$ over K which has everywhere good reduction. Then the stable Taguchi height (Definition 16.3) satisfies*

$$ht^{\mathrm{st}}_{\mathrm{Tag}}(\underline{G}) \;=\; -\frac{1}{r}\cdot\frac{\zeta'_{\mathcal{O}_E}(0)}{\zeta_{\mathcal{O}_E}(0)} - \log D_A(\mathcal{O}_E)$$

Theorems 17.10 and 17.8 and Remark 17.9 imply the following

Corollary 17.11. *The product formula from Conjecture 17.6 holds for CM Drinfeld A-modules.* □

In [Wei20] Theorem 17.10 follows from the function field analogs of Kronecker's limit theorem and Lerch's formula (1.3). In that sense, Wei's theorem can be viewed as the analog of Colmez's Theorem 8.10 in the abelian case. Analogously to Remark 8.12, it would be interesting to describe, also in the function field case, the relation on the one hand between the Kronecker limit and the Lerch-type formulas in [Wei20], and on the other hand Gross-Zagier formulas like the ones proved by Yun, Wei Zhang, Howard and Shnidman [YZ17, YZ19, HS19] for the intersection numbers of Heegner cycles on moduli spaces of global PGL_2-shtukas.

In the direction of the André-Oort conjecture over function fields there is the following analog of Theorem 8.14 by Breuer and Hubschmid.

Theorem 17.12. *The André-Oort-Conjecture holds for irreducible closed subvarieties X in Drinfeld modular varieties M in the following cases:*

(a) [Bre07] M is a product of Drinfeld modular curves which parameterize Drinfeld A-modules of rank 2.

(b) [Bre12] M is a Drinfeld modular variety parameterizing Drinfeld A-modules of rank r and X is a curve.

(c) [Hub13] M is a Drinfeld modular variety parameterizing Drinfeld A-modules of rank r such that $(q,r)=1$.

That is, in both cases $X \subset M$ is a special subvariety if and only if it contains a dense set of CM points.

Like in Theorem 8.14 one crucial ingredient is to show that the Galois orbit of a special point, that is a CM Drinfeld module, is large. This is done by following the strategy of Edixhoven [EMO01, Edi05], who proved cases of the original André-Oort-Conjecture for Shimura varieties conditionally under assuming the generalized Riemann Hypothesis. Over function fields various zeta functions are known to satisfy the Riemann Hypothesis by Deligne [Del74]. So this approach to the André-Oort-Conjecture over function fields can become unconditional. One the other hand, Conjecture 17.6 might also imply lower bounds for Galois orbits once it is related to heights of A-motives.

18 Example

We give an example for Conjecture 17.6 in case of an A-motive $\underline{M}$ of rank 1 where the curve C has genus 1. In this case, Conjecture 17.6 follows from Theorem 17.10. This example was studied in detail by Green and Papanikolas [GP16]. It is a beautiful exercise in computing with elliptic curves.

18.1. Let C be an elliptic curve over $\mathbb{F}_q$, given by the (non-homogeneous) Weierstraß equation

$$F \;:=\; F(t,y) \;:=\; y^2 + a_1 ty + a_3 y - t^3 - a_2 t^2 - a_4 t - a_6, \qquad \text{with} \quad a_i \in \mathbb{F}_q,$$

in the variables $t = \frac{X}{Z}$ and $y = \frac{Y}{Z}$, compare (2.1). Let $\infty \in \mathrm{V}(Z^3 \cdot F) \subset \mathbb{P}^2_{\mathbb{F}_q}$ be the $\mathbb{F}_q$-rational point with $(X:Y:Z) = (0:1:0)$ at which t and y have pole order given by

$$v_\infty(t) = -2, \quad v_\infty(y) = -3\,.$$

We have $A = \Gamma(C \smallsetminus \{\infty\}, \mathcal{O}_C) = \mathbb{F}_q[t,y]/(F(t,y))$. For any field extension L of $\mathbb{F}_q$ there is exactly one point ∞_L on C_L above ∞, because ∞ is $\mathbb{F}_q$-rational. To shorten the notation we sometimes denote the point ∞_L again by ∞.

We consider a second copy of the ring A given by $\mathbb{F}_q[\theta,\varepsilon]/(F(\theta,\varepsilon))$ in the variables θ and ε, and its fraction field $\mathbb{F}_q(\theta,\varepsilon)$. This is the function field of a second copy of the elliptic curve C, which we denote by X_0 and which has coordinates θ and ε. That is $\mathbb{F}_q(\theta,\varepsilon) = \mathbb{F}_q(X_0)$. Let $\gamma : A \to \mathbb{F}_q(\theta,\varepsilon)$ be given by $\gamma(t) = \theta$ and $\gamma(y) = \varepsilon$. This makes $\mathbb{F}_q(\theta,\varepsilon)$ into an A-field. We use the isomorphism $\gamma : Q \xrightarrow{\sim} \mathbb{F}_q(\theta,\varepsilon)$ to embed $\mathbb{F}_q(\theta,\varepsilon)$ canonically into $\mathbb{C}_v$ for all places v of Q. We note that

$$\Xi \;=\; \mathrm{V}(t-\theta, y-\varepsilon) \;=\; \mathrm{V}(\mathcal{J}) \qquad \text{for the ideal} \quad \mathcal{J} \;:=\; (a\otimes 1 - 1\otimes\gamma(a) \colon a \in A) \;=\; (t-\theta, y-\varepsilon)$$

is an $\mathbb{F}_q(\theta,\varepsilon)$-rational point of C. Furthermore, $\Xi \in C\big(\mathbb{F}_q(\theta,\varepsilon)\big) \subset C(\mathbb{C}_\infty)$ specializes to $\infty \in C(\kappa_\infty)$ under the reduction map $red : C(\mathbb{C}_\infty) \to C(\kappa_\infty)$ from (13.1). Recall the rigid analytic space $\mathfrak{C} := \mathfrak{C}_{\mathbb{C}_\infty} = (C_{\mathbb{C}_\infty})^{\mathrm{rig}}$ and the disc $\mathfrak{D} \subset \mathfrak{C}$, which is defined in Notation 13.1 as the preimage in $\mathfrak{C} = C(\mathbb{C}_\infty)$ of $\infty \in C(\kappa_\infty)$. This disc $\mathfrak{D}$ is the formal group of the elliptic curve $C_{\mathbb{C}_\infty}$ over $\mathbb{C}_\infty$, see [Sil86, Example IV.3.1.3], where this formal group is denoted $\hat{C}(\mathfrak{m}_\infty)$ for the maximal ideal $\mathfrak{m}_\infty \subset \mathcal{O}_{\mathbb{C}_\infty}$.

For any field extension L of $\mathbb{F}_q$ the relative q-Frobenius isogeny $\mathrm{Fr}_{q,C_L/L}: C_L \to C_L$ of C_L over L is given on $\operatorname{Spec} A_L \subset C_L$ by the L-homomorphism $\mathrm{Fr}^*_{q,C_L/L}\colon A_L \to A_L$, $t \mapsto t^q$, $y \mapsto y^q$. For any point $P \in C_L(L)$ we denote by $P^{(1)} := \mathrm{Fr}_{q,C_L/L}(P) \in C_L(L)$ the image of P. The composition $\sigma \circ \mathrm{Fr}_{q,C_L/L} = \mathrm{Fr}_{q,C_L/L} \circ \sigma$ with the morphism $\sigma\colon C_L \to C_L$ from (9.1) equals the absolute q-Frobenius on C_L, which is the identity on points and the q-power map on the structure sheaf. For example, the morphism $\mathrm{Fr}_{q,C/\mathbb{F}_q}$ sends Ξ to $\Xi^{(1)} = \mathrm{Fr}_{q,C/\mathbb{F}_q}(\Xi) = \mathrm{V}(t-\theta^q, y-\varepsilon^q)$.

The isogeny $1 - \mathrm{Fr}_{q,C/\mathbb{F}_q}: C \to C$ is separable by [Sil86, Corollary III.5.5] and it induces an isomorphism of formal groups $1 - \mathrm{Fr}_{q,C/\mathbb{F}_q}: \hat{C}(\mathfrak{m}_\infty) \to \hat{C}(\mathfrak{m}_\infty)$ by [Sil86, Corollary IV.4.3 and Lemma IV.2.4]. Therefore, we can pick a unique point $V \in \hat{C}(\mathfrak{m}_\infty) = \mathfrak{D} \subset C(\mathbb{C}_\infty)$ so that under the group law of C

$$(1 - \mathrm{Fr}_{q,C/\mathbb{F}_q})(V) = V - V^{(1)} = \Xi, \tag{18.1}$$

and moreover, $(1 - \mathrm{Fr}_{q,C/\mathbb{F}_q})^{-1}(\Xi) = \{V + P \mid P \in C(\mathbb{F}_q)\}$.

If we set $V = \mathrm{V}(t-\alpha, y-\beta)$ with $\alpha, \beta \in \mathbb{C}_\infty$ then $K := \mathbb{F}_q(\theta,\varepsilon)(\alpha,\beta) = \mathbb{F}_q(\alpha,\beta) \subset \mathbb{C}_\infty$ is the Hilbert class field of $\mathbb{F}_q(\theta,\varepsilon)$ by [GP16, Proposition 3.3]. We view K as the function field of a third copy of the elliptic curve C, which we denote by X_1 and which has coordinates α and β. The inclusion of fields $\mathbb{F}_q(\theta,\varepsilon) \subset K$ corresponds to a morphism $X_1 \to X_0$ which is equal to the morphism $1 - \mathrm{Fr}_{q,C/\mathbb{F}_q}: C \to C$ under the identifications $X_1 = C = X_0$. In particular, the set $X_1(\mathbb{F}_q)$ equals the preimage of $\infty = (0:1:0) \in X_0$ under this map. This set consists of the points with $\alpha, \beta \in \mathbb{F}_q$ together with the point $P = \infty_1 \in X_1$ where α and β have poles of order 2 and 3 respectively. It follows that $X_1 \smallsetminus X_1(\mathbb{F}_q) = \operatorname{Spec} \mathcal{O}_K$ for the integral closure $\mathcal{O}_K$ of A in K.

18.2. Now by (18.1) and the definition of the group law on C, see [Sil86, § III.2], the K-valued points $V^{(1)} = \mathrm{V}(t-\alpha^q, y-\beta^q)$ and $-V = \mathrm{V}(t-\alpha, y+\beta+a_1\alpha+a_3)$ and Ξ in $C(K)$ are collinear. We take m to be the slope of the line connecting them:

$$m \;=\; \frac{\varepsilon-\beta^q}{\theta-\alpha^q} \;=\; \frac{\varepsilon+\beta+a_1\alpha+a_3}{\theta-\alpha} \;=\; \frac{\beta^q+\beta+a_1\alpha+a_3}{\alpha^q-\alpha} \;\in\; K. \tag{18.2}$$

With respect to the valuation v_∞ on $K \subset \mathbb{C}_\infty$ we compute $v_\infty(\theta) = v_\infty(\alpha) = -2$ and $v_\infty(\varepsilon) = v_\infty(\beta) = -3$, and hence obtain $v_\infty(m) = v_\infty(\frac{\varepsilon-\beta^q}{\theta-\alpha^q}) = -q$. We extend this to the following

Lemma 18.3. *Let $P \in X_1$ be a closed point. Then the element $m \in K$ has a pole at P if and only if $P \in X_1(\mathbb{F}_q) = X_1 \smallsetminus \operatorname{Spec} \mathcal{O}_K$. In particular, $m \in \mathcal{O}_K$. Moreover, for the normalized valuation v_P corresponding to P we have*

$$v_P(m) = \begin{cases} -1 & \text{when} \quad P \in X_1(\mathbb{F}_q), P \neq \infty_1\,, \\ -q & \text{when} \quad P = \infty_1\,. \end{cases}$$

Proof. This can be proved by computing a uniformizing parameter at P, but we use the following different strategy. The element $m \in K$ was defined as the slope of the line through $V^{(1)}$, $-V$ and Ξ. This also holds over X_1 for the canonical extensions of $V^{(1)}$, $-V$ and Ξ to X_1-valued points of $C \times_{\mathbb{F}_q} X_1$. We now specialize to the residue field $L := \kappa(P)$ of P. If $m(P) = \infty$, that is $\frac{1}{m}(P) = 0$ then on the elliptic curve $C_L := C \times_{\mathbb{F}_q} \operatorname{Spec} L$ the line through $V^{(1)}$, $-V$ and Ξ contains the neutral element ∞_L, so $V^{(1)} = \infty_L$ or $-V = \infty_L$ or $\Xi = \infty_L$. If $V^{(1)} = \infty_L$ or $-V = \infty_L$ then $V = \infty_L$, because $\infty_L = -\infty_L$ and this is the only point in $\mathrm{Fr}^{-1}_{q,C_L/L}(\infty_L)$. From $V = \mathrm{V}(t-\alpha, y-\beta)$ it follows that $P = \infty_1 \in X_1(\mathbb{F}_q)$. In this case $v_P(\theta) = v_P(\alpha) = -2$ and $v_P(\varepsilon) = v_P(\beta) = -3$, and we obtain $v_P(m) = v_P(\frac{\varepsilon-\beta^q}{\theta-\alpha^q}) = -q$ as above. If $\infty_L = \Xi = V - V^{(1)}$ and $V \neq \infty_L$, then $V^{(1)} = V = \mathrm{V}(t-\alpha, y-\beta)$ lies in $C(\mathbb{F}_q)$. Thus $\alpha, \beta \in \mathbb{F}_q$ and $P \in X_1(\mathbb{F}_q)$. In this case $v_P(\alpha), v_P(\beta) \geq 0$, and $\Xi = \mathrm{V}(t-\theta, y-\varepsilon) = \infty_L$ implies $v_P(\theta) = -2$ and $v_P(\varepsilon) = -3$. We obtain $v_P(m) = v_P(\frac{\varepsilon-\beta^q}{\theta-\alpha^q}) = -1$. Conversely, if $P \in X_1(\mathbb{F}_q)$, then $V = \mathrm{V}(t-\alpha, y-\beta) \in C(\mathbb{F}_q)$ and $\Xi = V - V^{(1)} = V - V = \infty_L$ and so the line through $V^{(1)}$, $-V$ and Ξ has slope $m = \infty$. □

18.4. By (18.1) and [Sil86, Corollary III.3.5] the divisor $[V^{(1)}] - [V] + [\Xi] - [\infty]$ on C_K is principal. So there is a function $f \in K(t,y) = \operatorname{Quot}(A_K)$, called the *shtuka function* for A with

$$\operatorname{div}(f) = [V^{(1)}] - [V] + [\Xi] - [\infty]\,. \tag{18.3}$$

The shtuka function f can be written as

$$f \;=\; \frac{\nu(t,y)}{\delta(t)} \;=\; \frac{y-\varepsilon-m(t-\theta)}{t-\alpha} \;=\; \frac{y+\beta+a_1\alpha+a_3-m(t-\alpha)}{t-\alpha} \;=\; \frac{y+\beta+a_1\alpha+a_3}{t-\alpha} - m, \tag{18.4}$$

for

$$\nu \;:=\; \nu(t,y) \;:=\; y-\varepsilon-m\cdot(t-\theta) \;\in\; \mathcal{O}_K[t,y] \qquad \text{and} \qquad \delta \;:=\; \delta(t) \;:=\; t-\alpha \;\in\; \mathcal{O}_K[t,y],$$

with divisors on C_K given by

$$\operatorname{div}(\nu) \;=\; [V^{(1)}]+[-V]+[\Xi]-3[\infty] \qquad \text{and} \qquad \operatorname{div}(\delta) \;=\; [V]+[-V]-2[\infty]. \tag{18.5}$$

The formulas (18.3) and (18.5) also hold for the Cartier divisors of f, ν and δ on the two dimensional scheme $C_{\mathcal{O}_K} := C\times_{\mathbb{F}_q}\operatorname{Spec}\mathcal{O}_K$, because ν and δ do not vanish on an entire fiber of $C_{\mathcal{O}_K}$ over a closed point of $\operatorname{Spec}\mathcal{O}_K$. Here we consider the $\mathcal{O}_K$-valued points $\infty := \mathrm{V}(\frac{1}{t},\frac{t}{y}) = \{\infty\}\times_{\mathbb{F}_q}\operatorname{Spec}\mathcal{O}_K$ and $V = \mathrm{V}(t-\alpha, y-\beta)$ and $\Xi = \mathrm{V}(t-\theta, y-\varepsilon)$, etc. as Cartier divisors on $C_{\mathcal{O}_K}$.

18.5. We consider the invertible sheaf $\mathcal{O}_{C_K}([V])$ on C_K with

$$\begin{aligned}\Gamma(\operatorname{Spec}A_K,\mathcal{O}_{C_K}([V])) &= \big\{\, x\in\operatorname{Quot}(A_K)\colon\ \operatorname{ord}_P(x)\ge 0\ \forall\, P\in C_K\smallsetminus\{V,\infty\} \text{ and } \operatorname{ord}_V(x)\ge -1\,\big\}\\ &= \big\{\, x\in\operatorname{Quot}(A_K)\colon\ \operatorname{ord}_P(x)\ge 0\ \forall\, P\neq V,\infty \text{ and } (t-\alpha)x,(y-\beta)x\in A_K\,\big\}\,.\end{aligned}$$

Then we compute $\Gamma(\operatorname{Spec}A_K,\sigma^*\mathcal{O}_{C_K}([V]))$ as the A_K-module

$$\begin{aligned}&\big\{\, x\otimes b\in\operatorname{Quot}(A_K)\otimes_{A_K,\sigma^*}A_K\colon\ \operatorname{ord}_P(x)\ge 0\ \forall\, P\neq V,\infty \text{ and } (t-\alpha)x,(y-\beta)x\in A_K\,\big\}\\ =&\ \big\{\, x\otimes b\in\operatorname{Quot}(A_K)\otimes_{A_K,\sigma^*}A_K\colon\ \operatorname{ord}_P(x)\ge 0\ \forall\, P\neq V,\infty \text{ and } x\otimes b(t-\alpha^q),x\otimes b(y-\beta^q)\in A_K\,\big\}\\ =&\ \Gamma(\operatorname{Spec}A_K,\mathcal{O}_{C_K}([V^{(1)}]))\,.\end{aligned}\tag{18.6}$$

We define an A-motive $\underline{M}=(M,\tau_M)$ over K of rank 1 and dimension 1 as follows.

$$\begin{aligned}M &= \Gamma(\operatorname{Spec}A_K,\mathcal{O}_{C_K}([V]))\\ \sigma^*M &= \Gamma(\operatorname{Spec}A_K,\mathcal{O}_{C_K}([V^{(1)}]))\\ \tau_M &:= f\colon \sigma^*M \xrightarrow{\sim} M\otimes\mathcal{O}_{C_K}(-[\Xi]) \;\subset\; M\\ \operatorname{coker}\tau_M &\cong \mathcal{O}_{C_K}/\mathcal{O}_{C_K}(-[\Xi]) \;\cong\; K\,.\end{aligned}$$

This A-motive corresponds to a Drinfeld A-module of rank 1 over K, which is described more explicitly in [GP16, § 3]. In particular, $\underline{M}$ is uniformizable. Moreover, $\underline{M}$ has CM through $\mathcal{O}_E := A$. We set $E=Q$ and then $H_E=\operatorname{Hom}_Q(E,Q^{\mathrm{alg}})=\{\mathrm{id}_E\}$ consists of one single element $\psi=\mathrm{id}_E$. Correspondingly we drop all occurrences of ψ from the notation used in Section 17. The de Rham cohomology of $\underline{M}$ is

$$\mathrm{H}^1_{\mathrm{dR}}(\underline{M},K[\![t-\theta]\!]) = \sigma^*M\otimes_{\mathcal{O}_{C_K}}\varprojlim A_K/\mathcal{J}^n = \Gamma(\operatorname{Spec}A_K,\mathcal{O}_{C_K}(V^{(1)}))\otimes_{\mathcal{O}_{C_K}}K[\![t-\theta]\!] = K[\![t-\theta]\!],$$

because $\varprojlim A_K/\mathcal{J}^n = K[\![t-\theta]\!]$, and $\mathcal{O}_{C_K}(V^{(1)})$ equals $\mathcal{O}_{C_K}$ on the neighborhood $C_K\smallsetminus\{V^{(1)}\}$ of Ξ. For the unique element $\psi=\mathrm{id}_E$ in H_E we have $\mathrm{H}^\psi(\underline{M},K[\![y_\psi-\psi(y_\psi)]\!])=\mathrm{H}^1_{\mathrm{dR}}(\underline{M},K[\![t-\theta]\!])$ and the Hodge-Pink lattice $\mathfrak{q}^{\underline{M}} := \tau_M^{-1}(M\otimes_{A_R}\varprojlim A_K/\mathcal{J}^n)\subset\mathrm{H}^1_{\mathrm{dR}}\big(\underline{M},K(\!(t-\theta)\!)\big)$ of $\underline{M}$ satisfies

$$\mathfrak{q}^{\underline{M}} = f^{-1}\cdot\mathrm{H}^1_{\mathrm{dR}}\big(\underline{M},K[\![t-\theta]\!]\big) = (t-\theta)^{-1}\cdot\mathrm{H}^1_{\mathrm{dR}}\big(\underline{M},K[\![t-\theta]\!]\big)$$

by (18.3). So according to Definition 15.8 the CM-type of $\underline{M}$ is $\Phi=(d_{\mathrm{id}_E})$ with $d_{\mathrm{id}_E}=1$.

18.6. We will next see that $\underline{M}$ has a good integral model $\underline{\mathcal{M}}$ over $\mathcal{O}_K$. Namely, by a similar computation as in (18.6) the invertible sheaf $\mathcal{O}_{C_{\mathcal{O}_K}}([V])$ on $C_{\mathcal{O}_K}$ satisfies

$$\sigma^*\mathcal{O}_{C_{\mathcal{O}_K}}([V]) \;=\; \mathcal{O}_{C_{\mathcal{O}_K}}([V^{(1)}]).$$

Then the good model $\underline{\mathcal{M}}=(\mathcal{M},\tau_{\mathcal{M}})$ of $\underline{M}$ over $\mathcal{O}_K$ is given by

$$\begin{aligned}\mathcal{M} &= \Gamma(\operatorname{Spec}A_{\mathcal{O}_K},\mathcal{O}_{C_{\mathcal{O}_K}}([V]))\\ \sigma^*\mathcal{M} &= \Gamma(\operatorname{Spec}A_{\mathcal{O}_K},\mathcal{O}_{C_{\mathcal{O}_K}}([V^{(1)}]))\\ \tau_{\mathcal{M}} &:= f\colon \sigma^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}\otimes\mathcal{O}_{C_{\mathcal{O}_K}}(-[\Xi]) \;\subset\; \mathcal{M}\\ \operatorname{coker}\tau_{\mathcal{M}} &\cong A_{\mathcal{O}_K}/A_{\mathcal{O}_K}(-[\Xi]) \;\cong\; \mathcal{O}_K\,.\end{aligned}$$

18.7. With respect to the inclusion $K \subset \mathbb{C}_\infty$ Papanikolas and Green [GP16, § 4] calculate $\mathrm{H}^1_{\mathrm{Betti}}(\underline{M}, A)$ as follows. They fix $(q-1)$-st roots of $-\alpha$ and $m\theta - \varepsilon$, and set

$$\begin{aligned} \nu_\varphi &:= (m\theta-\varepsilon)^{1/(1-q)} \prod_{i=0}^{\infty} \left(1 - \left(\frac{m}{m\theta-\varepsilon}\right)^{q^i} t + \left(\frac{1}{m\theta-\varepsilon}\right)^{q^i} y\right), \\ \delta_\varphi &:= (-\alpha)^{1/(1-q)} \prod_{i=0}^{\infty} \left(1 - \frac{t}{\alpha^{q^i}}\right). \end{aligned}$$

Since $v_\infty(\alpha) = -2$ in $\mathbb{C}_\infty$, it follows that the product for δ_φ converges in $\Gamma(\mathfrak{C} \smallsetminus \{\infty\}, \mathcal{O}_{\mathfrak{C}})$, is invertible on $\mathfrak{C} \smallsetminus \mathfrak{D}$ and has zeroes of order 1 at $V^{(i)}$ and $-V^{(i)}$ for all $i \in \mathbb{N}_0$. Since $v_\infty(m) = -q$, and so $v_\infty(m\theta - \varepsilon) = -q-2$ and $v_\infty(\frac{m}{m\theta-\varepsilon}) = 2$ it similarly follows that ν_φ converges in $\Gamma(\mathfrak{C} \smallsetminus \{\infty\}, \mathcal{O}_{\mathfrak{C}})$ and is invertible on $\mathfrak{C} \smallsetminus \mathfrak{D}$. Moreover, ν_φ has zeroes of order 1 at $\Xi^{(i)}$ and $-V^{(i)}$ and $V^{(i+1)}$ for all $i \in \mathbb{N}_0$, because $1 - \frac{m}{m\theta-\varepsilon}\theta + \frac{1}{m\theta-\varepsilon}\varepsilon = 0$ and $1 - \frac{m}{m\theta-\varepsilon}\alpha - \frac{1}{m\theta-\varepsilon}(\beta + a_1\alpha + a_3) = 0$ and $1 - \frac{m}{m\theta-\varepsilon}\alpha^q + \frac{1}{m\theta-\varepsilon}\beta^q = 0$. These functions satisfy the equations

$$\nu_\varphi \;=\; \nu \cdot \sigma^*\nu_\varphi \;=\; (y - \varepsilon - m\cdot(t-\theta)) \cdot \sigma^*\nu_\varphi \qquad \text{and} \qquad \delta_\varphi \;=\; \delta \cdot \sigma^*\delta_\varphi \;=\; (t-\alpha)\cdot\sigma^*\delta_\varphi\,.$$

Thus with the corresponding $(q-1)$-st root $\xi^{1/(q-1)}$ of $\xi = -\frac{m\theta-\varepsilon}{\alpha} = -(m + \frac{\beta+a_1\alpha+a_3}{\alpha})$ we set

$$\lambda_{\underline{M}} \;:=\; \frac{\nu_\varphi}{\delta_\varphi} \;=\; \xi^{1/(1-q)} \prod_{i=0}^{\infty} \frac{\sigma^{i*} f}{\xi^{q^i}} \;\in\; \Gamma(\mathfrak{C} \smallsetminus \mathfrak{D}, \mathcal{O}_{\mathfrak{C}})^\times. \tag{18.7}$$

Then $\tau_M(\sigma^*\lambda_{\underline{M}}) = f\cdot\sigma^*\lambda_{\underline{M}} = \lambda_{\underline{M}}$, and $\lambda_{\underline{M}}$ is a meromorphic function on $\mathfrak{C} \smallsetminus \{\infty\}$ without poles or zeroes on $\mathfrak{C} \smallsetminus \mathfrak{D}$. (By looking at the product decomposition of $\lambda_{\underline{M}}$ one even sees that it has a simple pole at V and simple zeroes at $\Xi^{(i)}$ for all $i \in \mathbb{N}_0$.) So we obtain

$$\mathrm{H}^1_{\mathrm{Betti}}(\underline{M}, A) \;=\; \lambda_{\underline{M}} \cdot A. \tag{18.8}$$

Let $u \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, A)$ be the generator such that $\langle u, \lambda_{\underline{M}}\rangle = 1$. We also write $u_{\mathrm{id}_K} := u$.

18.8. We can take $\omega := \omega_\psi := \sigma^*\delta^{-1} = (t-\alpha^q)^{-1}$ as a generator of $\mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K[\![t-\theta]\!])$. Then the comparison isomorphism $h_{\mathrm{Betti,dR}} = \sigma^* h_{\underline{M}}$ from Theorem 13.18 sends the generator $\lambda_{\underline{M}}$ of $\mathrm{H}^1_{\mathrm{Betti}}(\underline{M}, A)$ to $\sigma^*\lambda_{\underline{M}} = \sigma^*(\lambda_{\underline{M}}\delta)\cdot \omega \in \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K[\![t-\theta]\!])$ and the comparison isomorphism $h_{\mathrm{Betti,dR}} = \sigma^* h_{\underline{M}} \bmod \mathcal{J}$ from (13.6) sends the generator $\lambda_{\underline{M}}$ of $\mathrm{H}^1_{\mathrm{Betti}}(\underline{M}, A)$ to $\sigma^*(\lambda_{\underline{M}}\delta)(\Xi)\cdot\omega \in \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K)$. Therefore,

$$\langle u, h^{-1}_{\mathrm{Betti,dR}}(\omega)\rangle_\infty \;=\; \langle u, \sigma^*(\lambda_{\underline{M}}\delta)(\Xi)^{-1}\cdot\lambda_{\underline{M}}\rangle_\infty \;=\; \frac{\xi^{q/(q-1)}}{(\sigma^*\delta)(\Xi)} \prod_{i=1}^{\infty} \frac{\xi^{q^i}}{(\sigma^{i*}f)(\Xi)}\,.$$

To compute the absolute value of $\langle u, h^{-1}_{\mathrm{Betti,dR}}(\omega)\rangle_\infty$ we observe that for every $i \in \mathbb{N}_{>0}$

$$\left|\frac{\xi^{q^i}}{(\sigma^{i*}f)(\Xi)}\right|_\infty \;=\; \left|\frac{1 - \frac{\theta}{\alpha^{q^i}}}{1 - (\frac{m}{m\theta-\varepsilon})^{q^i}\theta + (\frac{1}{m\theta-\varepsilon})^{q^i}\varepsilon}\right|_\infty \;=\; 1\,,$$

as well as $v_\infty(\xi) = -q$, whence $|\xi^{q/(q-1)}|_\infty = q^{q^2/(q-1)}$, and $|(\sigma^*\delta)(\Xi)|_\infty = |(t-\alpha^q)(\Xi)|_\infty = |\theta - \alpha^q|_\infty = |\alpha^q|_\infty = q^{2q}$. Thus we obtain

$$\Big|\int_u \omega\Big|_\infty := \Big|\,\langle u, h^{-1}_{\mathrm{Betti,dR}}(\omega)\rangle_\infty\Big|_\infty \;=\; q^{\frac{q^2}{q-1}-2q} \;=\; q^{\frac{q}{q-1}-q} \qquad \text{and}$$

$$\boxed{\log\Big|\int_u\omega\Big|_\infty \;=\; \big(\tfrac{q}{q-1} - q\big)\log q\,.} \tag{18.9}$$

18.9. We consider the set $H_K := \mathrm{Hom}_Q(K, Q^{\mathrm{alg}}) = \mathrm{Gal}\big(K/\mathbb{F}_q(\theta,\varepsilon)\big)$ which actually is a group, because K is Galois over $\mathbb{F}_q(\theta,\varepsilon)$. It is isomorphic to the group $C(\mathbb{F}_q)$ under the map $\eta \mapsto P_\eta := V - \eta(V)$. Indeed, since $\eta(\Xi) = \Xi \in C(K)$ is fixed by η we see that $\eta(V)$ still satisfies $\eta(V) - \eta(V)^{(1)} = \eta(V) - \eta(V^{(1)}) = \eta(\Xi) = \Xi = V - V^{(1)}$. Therefore, the point $P_\eta = V - \eta(V)$ satisfies $P_\eta^{(1)} = P_\eta$, and hence $P_\eta \in C(\mathbb{F}_q)$. Since the coordinates (α, β) of V generate the field extension $K/\mathbb{F}_q(\theta,\varepsilon)$, the map $\eta \mapsto P_\eta$ is bijective. It is a group homomorphism, because $P_{\tilde\eta\eta} = V - \tilde\eta\eta(V) = V - \tilde\eta(V) + \tilde\eta(V) - \tilde\eta\eta(V) = P_{\tilde\eta} + \tilde\eta(P_\eta) = P_{\tilde\eta} + P_\eta$, as $P_\eta \in C(\mathbb{F}_q)$ is fixed by $\tilde\eta$. In particular, $\#H_K = \#C(\mathbb{F}_q)$.

We now fix an element $\eta \in H_K$ with $\eta \neq \mathrm{id}_K$ and let the A-motive $\underline{\mathcal{M}}^\eta$ over $\mathcal{O}_K$ and $\omega^\eta \in \mathrm{H}^1_{\mathrm{dR}}(\underline{M}^\eta, K[\![t-\theta]\!])$ be deduced from $\underline{\mathcal{M}}$ and ω by base extension. Then $\underline{\mathcal{M}}^\eta$ is isogenous to $\underline{\mathcal{M}}$ by the theory of complex multiplication, which was developed for Drinfeld modules by Hayes [Hay79] and for general A-motives by Pelzer [Pel09]. We give an elementary and explicit treatment for our $\underline{\mathcal{M}}$. We claim that there is an isomorphism

$$g_\eta \colon \underline{\mathcal{M}}^\eta \xrightarrow{\;\sim\;} \underline{\mathcal{M}} \otimes \mathcal{O}(-[P_\eta]) \;=:\; \underline{\mathcal{M}}(-[P_\eta])\,, \tag{18.10}$$

where $\mathcal{O}(-[P_\eta])$ denotes the invertible sheaf on $\operatorname{Spec} A_{\mathcal{O}_K}$ associated to the divisor $-[P_\eta] \times_{\mathbb{F}_q} \operatorname{Spec} \mathcal{O}_K$. Namely, the A-motives $\underline{\mathcal{M}}^\eta$ and $\underline{\mathcal{M}}(-[P_\eta])$ correspond to the invertible sheaves $\mathcal{O}_{C_K}([\eta(V)]) = \mathcal{O}_{C_K}([V - P_\eta])$ and $\mathcal{O}_{C_K}([V]) \otimes \mathcal{O}_{C_K}(-[P_\eta]) = \mathcal{O}_{C_K}([V] - [P_\eta])$ on C_K, respectively.

By (18.1) and [Sil86, Corollary III.3.5] the divisor $[V - P_\eta] - [V] + [P_\eta] - [\infty]$ on C_K is principal and there is a function $g_\eta \in K(t,y) = \operatorname{Quot}(A_K)$ with

$$\operatorname{div}(g_\eta) \;=\; [V - P_\eta] - [V] + [P_\eta] - [\infty] \;=\; [V - P_\eta] + [-V] + [P_\eta] - [V] - [-V] - [\infty]\,. \tag{18.11}$$

It can be written explicitly as follows. By construction of the group law on C, the three points $V - P_\eta = \eta(V)$ and $-V$ and P_η lie on a single line whose slope is

$$\frac{\eta(\beta) - y(P_\eta)}{\eta(\alpha) - t(P_\eta)} \;=\; \frac{\eta(\beta) + \beta + a_1\alpha + a_3}{\eta(\alpha) - \alpha} \;=\; \frac{y(P_\eta) + \beta + a_1\alpha + a_3}{t(P_\eta) - \alpha} \;\in\; \mathcal{O}_K\,.$$

This slope is a priory an element of K, but we see that it lies in $\mathcal{O}_K$ by reasoning like in Lemma 18.3. Indeed, the slope has a pole if and only if one of the points P_η or $-V$ or $V - P_\eta = \eta(V)$ equals ∞. If $P_\eta = \infty$, then the bijectivity of the map $\eta \mapsto P_\eta$ implies $\eta = \mathrm{id}_K$ which was excluded. If $V - P_\eta = \infty$, and hence $V = P_\eta \in C(\mathbb{F}_q)$, or if $-V = \infty$, then $\Xi = \infty$, and so the poles of the slope do not lie in $\operatorname{Spec} \mathcal{O}_K$. That is, the slope lies in $\mathcal{O}_K$ as claimed. Then we can take

$$g_\eta \;=\; \frac{y - \eta(\beta) - \frac{\eta(\beta)+\beta+a_1\alpha+a_3}{\eta(\alpha)-\alpha}(t - \eta(\alpha))}{t - \alpha} \tag{18.12}$$

as an isomorphism $\mathcal{M}^\eta \xrightarrow{\sim} \mathcal{M} \otimes \mathcal{O}(-[P_\eta])$. Here we use that formula (18.11) for the divisor of g_η also holds on $C_{\mathcal{O}_K}$, because both numerator and denominator of g_η lie in $\mathcal{O}_K[t,y]$ and do not vanish on an entire fiber of $C_{\mathcal{O}_K}$ over a closed point of $\operatorname{Spec} \mathcal{O}_K$.

In order to see that g_η is an isomorphism of A-motives, it remains to prove that $g_\eta \circ \eta(f) = f \circ \sigma^* g_\eta$. Since the divisor on both sides equals $[\eta(V)^{(1)}] + [P_\eta] - [V] + [\Xi] - 2[\infty]$, both sides differ by multiplication with an element of $K^\times$. Multiplying both sides with the common denominator and comparing the coefficients of $t^2 y$ shows that both sides are equal as desired.

18.10. The isomorphism $g_\eta \colon \underline{\mathcal{M}}^\eta \xrightarrow{\sim} \underline{\mathcal{M}}(-[P_\eta])$ induces isomorphisms on (co-)homology

$$\begin{aligned} g_\eta \colon\; &\mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{M}}^\eta, \mathcal{O}_K) &&\xrightarrow{\;\sim\;} \mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{M}}(-[P_\eta]), \mathcal{O}_K)\,, \\ g_\eta \colon\; &\mathrm{H}^1_{\mathrm{Betti}}(\underline{M}^\eta, A) &&\xrightarrow{\;\sim\;} \mathrm{H}^1_{\mathrm{Betti}}(\underline{M}(-[P_\eta]), A)\,, \qquad \text{and} \\ g_\eta \colon\; &\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta, A) &&\xrightarrow{\;\sim\;} \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}(-[P_\eta]), A)\,. \end{aligned}$$

These are compatible with the period isomorphisms $h_{\mathrm{Betti,dR}}$ and the pairing between $\mathrm{H}^1_{\mathrm{Betti}}$ and $\mathrm{H}_{1,\mathrm{Betti}}$. So we may replace $\underline{M}^\eta$ by $\underline{M}(-[P_\eta])$ in the rest of our computation.

Since $\omega = (t - \alpha^q)^{-1}$ and $\omega \bmod (t-\theta) = (\theta - \alpha^q)^{-1} \in \mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{M}}, \mathcal{O}_K)$ we obtain $\omega^\eta = (t - \eta(\alpha)^q)^{-1}$ and $\omega^\eta \bmod (t-\theta) = (\theta - \eta(\alpha)^q)^{-1}$, and we set $\widetilde{\omega}^\eta := g_\eta(\omega^\eta) \in \mathrm{H}^1_{\mathrm{dR}}(\underline{M}(-[P_\eta]), K[\![t-\theta]\!])$ and $\widetilde{\omega}^\eta \bmod (t-\theta) = g_\eta(\omega^\eta) \bmod (t-\theta) \in \mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{M}}(-[P_\eta]), \mathcal{O}_K)$. By definition, $\mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{M}}, \mathcal{O}_K) := \sigma^*\mathcal{M}/\mathcal{J}\sigma^*\mathcal{M} = \sigma^*\mathcal{M}|_\Xi$, with

$\mathcal{J} = (t-\theta, y-\varepsilon)$ being the vanishing ideal of the $\mathcal{O}_K$-valued point $\Xi \in C(\mathcal{O}_K)$. We compute

$$\begin{aligned}
\widetilde{\omega}^\eta &= \sigma^*(g_\eta)\cdot(t-\eta(\alpha)^q)^{-1} \\
&= \frac{y-\eta(\beta)^q - \frac{\eta(\beta)^q+\beta^q+a_1\alpha^q+a_3}{\eta(\alpha)^q-\alpha^q}(t-\eta(\alpha)^q)}{t-\alpha^q}\cdot(t-\eta(\alpha)^q)^{-1} \\
&= \frac{y-\eta(\beta)^q - \frac{\eta(\beta)^q+\beta^q+a_1\alpha^q+a_3}{\eta(\alpha)^q-\alpha^q}(t-\eta(\alpha)^q)}{t-\eta(\alpha)^q}\cdot(t-\alpha^q)^{-1} \qquad \text{and} \\
\widetilde{\omega}^\eta \bmod (t-\theta) &= \frac{\varepsilon-\eta(\beta)^q - \frac{\eta(\beta)^q+\beta^q+a_1\alpha^q+a_3}{\eta(\alpha)^q-\alpha^q}(\theta-\eta(\alpha)^q)}{\theta-\eta(\alpha)^q}\cdot(\theta-\alpha^q)^{-1} \\
&= \Big(\frac{\varepsilon-\eta(\beta)^q}{\theta-\eta(\alpha)^q} - \frac{\eta(\beta)^q+\beta^q+a_1\alpha^q+a_3}{\eta(\alpha)^q-\alpha^q}\Big)\cdot\omega \bmod (t-\theta)\,.
\end{aligned}$$

The element $\sigma^* g_\eta|_\Xi := \frac{\varepsilon-\eta(\beta)^q}{\theta-\eta(\alpha)^q} - \frac{\eta(\beta)^q+\beta^q+a_1\alpha^q+a_3}{\eta(\alpha)^q-\alpha^q}$ has absolute value

$$\big|\sigma^* g_\eta|_\Xi\big|_\infty = q^q\,, \qquad \text{and hence} \qquad \log\big|\sigma^* g_\eta|_\Xi\big|_\infty = q\log q\,, \tag{18.13}$$

because the first summand has absolute value q and is dominated by the second summand which has absolute value q^q.

18.11. We now compute $v(\omega^\eta)$ for all places $v \neq \infty$ of Q and for all $\eta \in H_K$. Observe that by (18.5) the multiplication with $t-\alpha$ induces an isomorphism $\mathcal{O}_{C_{\mathcal{O}_K}}([V]) \xrightarrow{\sim} \mathcal{O}_{C_{\mathcal{O}_K}}(2[\infty]-[-V])$ and the multiplication with $t-\alpha^q$ induces an isomorphism $\mathcal{O}_{C_{\mathcal{O}_K}}([V^{(1)}]) \xrightarrow{\sim} \mathcal{O}_{C_{\mathcal{O}_K}}(2[\infty]-[-V^{(1)}])$. We restrict this morphism to the $\mathcal{O}_K$-valued point Ξ, that is, we pull it back under the corresponding morphism $h_\Xi\colon \operatorname{Spec}\mathcal{O}_K \to C_{\mathcal{O}_K}$. To do so we first claim that h_Ξ factors through the open subscheme of $C_{\mathcal{O}_K}$ which is the complement of $\{\infty\}\cup\{-V^{(1)}\}$. Indeed, the locus on $C_{\mathcal{O}_K}$ where $\Xi = -V^{(1)}$ is equal to the locus where $V=\infty$, and the latter locus does not lie above $\operatorname{Spec}\mathcal{O}_K$. The same is true for the locus where $\Xi=\infty$. We conclude that multiplication with $\theta-\alpha^q$ induces an isomorphism

$$\begin{aligned}
\theta-\alpha^q\colon\ \mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{M}},\mathcal{O}_K) \;=\; h_\Xi^*\,\mathcal{O}_{C_{\mathcal{O}_K}}([V]) &\xrightarrow{\ \sim\ } h_\Xi^*\,\mathcal{O}_{C_{\mathcal{O}_K}}(2[\infty]-[-V]) \;=\; h_\Xi^*\,\mathcal{O}_{C_{\mathcal{O}_K}} \;=\; \mathcal{O}_K \\
\omega \bmod (t-\theta) \;=\; (\theta-\alpha^q)^{-1} &\longmapsto 1\,.
\end{aligned}$$

This shows that $\mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{M}},\mathcal{O}_K) = \mathcal{O}_K\cdot\omega \bmod (t-\theta)$, and by base extension under η, also $\mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{M}}^\eta,\mathcal{O}_K) = \mathcal{O}_K\cdot\omega^\eta \bmod (t-\theta)$. This yields

$$\boxed{v(\omega^\eta) \;=\; 0 \qquad \text{for every place } v\neq\infty \text{ and every } \eta\in H_K\,.} \tag{18.14}$$

18.12. We next compute $\mathrm{H}^1_{\mathrm{Betti}}(\underline{M}(-[P_\eta]),A)$ for the A-motive $\underline{M}(-[P_\eta]) = \big(\mathcal{O}_{C_K}([V]-[P_\eta]),\tau=f\big)$. The function $\lambda_{\underline{M}}$ from (18.7) satisfies $\tau(\sigma^*\lambda_{\underline{M}}) = f\cdot\sigma^*\lambda_{\underline{M}} = \lambda_{\underline{M}}$, but it does not have a zero at P_η, and hence does not lie in $\underline{M}(-[P_\eta])\otimes_{A_K}\mathcal{O}_{\mathfrak{C}\smallsetminus\mathfrak{D}}$ and not in $\mathrm{H}^1_{\mathrm{Betti}}(\underline{M}(-[P_\eta]),A)$. Instead,

$$\mathrm{H}^1_{\mathrm{Betti}}(\underline{M}(-[P_\eta]),A) \;=\; \lambda_{\underline{M}}\cdot\Gamma(\operatorname{Spec}A,\mathcal{O}_C(-[P_\eta])) \;=\; \lambda_{\underline{M}}\cdot\mathfrak{p}_\eta\,,$$

where $\mathfrak{p}_\eta\subset A$ is the maximal ideal defining the $\mathbb{F}_q$-valued point $P_\eta\in C$. Correspondingly, when we take $\tilde{u}_\eta := u \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}(-[P_\eta]),Q) = \mathrm{H}_{1,\mathrm{Betti}}(\underline{M},Q)$, which pairs with $\lambda_{\underline{M}}$ to $\langle\tilde{u}_\eta,\lambda_{\underline{M}}\rangle = \langle u,\lambda_{\underline{M}}\rangle = 1$, we obtain

$$\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}(-[P_\eta]),A) \;=\; \tilde{u}_\eta\cdot\Gamma(\operatorname{Spec}A,\mathcal{O}_C([P_\eta])) \;=\; \tilde{u}_\eta\cdot\mathfrak{p}_\eta^{-1}\,.$$

This yields

$$\boxed{v_\eta(\tilde{u}_\eta)\cdot\log q_v \;=\; \begin{cases} 0 & \text{if } v\neq\mathfrak{p}_\eta \text{ or } \eta=\mathrm{id}_K\,, \\ \log q & \text{if } v=\mathfrak{p}_\eta \text{ and } \eta\neq\mathrm{id}_K\,. \end{cases}} \tag{18.15}$$

Also from (18.9) and (18.13) we compute the absolute value

$$\boxed{\log\Big|\int_{\tilde{u}_\eta}\widetilde{\omega}^\eta\Big|_\infty = \log\Big|\langle u,\sigma^*g_\eta|_\Xi\cdot\omega\rangle_\infty\Big|_\infty = \log\big|\sigma^*g_\eta|_\Xi\big|_\infty + \log\big|\langle u,\omega\rangle_\infty\big|_\infty = \tfrac{q}{q-1}\log q\,.} \tag{18.16}$$

18.13. Finally, we recall the zeta functions for the elliptic curve C, which are defined as the following products which converge for $s \in \mathbb{C}$ with $\mathcal{R}e(s) > 1$

$$\zeta_C(s) := \prod_{\text{all } v} (1-(\#\mathbb{F}_v)^{-s})^{-1} = \prod_{\text{all } v}(1-q_v^{-s})^{-1} = \frac{1-(q+1-\#C(\mathbb{F}_q))q^{-s}+q^{1-2s}}{(1-q^{-s})(1-q^{1-s})} \quad \text{and}$$

$$\zeta_A(s) := \prod_{v\neq\infty} (1-(\#\mathbb{F}_v)^{-s})^{-1} = \prod_{v\neq\infty}(1-q_v^{-s})^{-1} = \frac{1-(q+1-\#C(\mathbb{F}_q))q^{-s}+q^{1-2s}}{1-q^{1-s}}.$$

Since the CM-field is $E = Q$, $H_E = \{\mathrm{id}\}$ and the CM-type is given by $d_{\mathrm{id}} = 1$, we have $a^0_{E,\mathrm{id},\Phi} = \mathbb{1}$. Since $L^\infty(\mathbb{1}, s) = \zeta_A(s)$ we obtain

$$\boxed{Z^\infty(\mathbb{1},0) = \frac{\zeta_A'(0)}{\zeta_A(0)} = \Big(\frac{q+1-\#C(\mathbb{F}_q)-2q}{1-(q+1-\#C(\mathbb{F}_q))+q} - \frac{q}{1-q}\Big)\log q = \Big(\frac{1-\#C(\mathbb{F}_q)-q}{\#C(\mathbb{F}_q)} + \frac{q}{q-1}\Big)\log q\,.} \tag{18.17}$$

We now put everything together using Theorem 17.3 and formula (17.9) to compute

$$\begin{aligned}
\tfrac{1}{\#H_K}\sum_v \sum_{\eta\in H_K} \log\big|\textstyle\int_{\tilde u_\eta}\omega_\psi^\eta\big|_v &= \Big(\frac{q+\#C(\mathbb{F}_q)-1}{\#C(\mathbb{F}_q)} - \frac{q}{q-1}\Big)\cdot\log q && \text{from (18.17)}\\
&\quad + \frac{1}{\#C(\mathbb{F}_q)}\Big(\frac{q}{q-1}-q\Big)\cdot\log q && \text{from (18.9)}\\
&\quad + \frac{\#C(\mathbb{F}_q)-1}{\#C(\mathbb{F}_q)}\,\frac{q}{q-1}\cdot\log q && \text{from (18.16)}\\
&\quad - \frac{\#C(\mathbb{F}_q)-1}{\#C(\mathbb{F}_q)}\cdot\log q && \text{from (18.14) and (18.15)}\\
&= 0\,.
\end{aligned}$$

Miraculously, all terms cancel and this shows that in the present example our Conjecture 17.6 holds true.

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