# ARITHMETIC SATAKE COMPACTIFICATIONS AND ALGEBRAIC DRINFELD MODULAR FORMS

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Abstract. In this article we construct the arithmetic Satake compactification of the Drinfeld moduli schemes of arbitrary rank over the ring of integers of any global function field away from the level structure, and show that the universal family extends uniquely to a generalized Drinfeld module over the compactification. Using these and functorial properties, we define algebraic Drinfeld modular forms over more general bases and the action of the (prime-to-residue characteristic) Hecke algebra. The construction also furnishes many algebraic Drinfeld modular forms obtained from the coefficients of the universal family which are also Hecke eigenforms. Among them we obtain generalized Hasse invariants which already live on the arithmetic Satake compactification. We use these generalized Hasse invariants to study the geometry of the special fiber and to establish the Jacquet-Langlands correspondence (mod $v$) between Hecke eigensystems of rank $r$ Drinfeld modular forms and those of algebraic modular forms (in the sense of Gross) attached to a compact inner form of $\mathrm{GL}_r$.



## 1. Introduction

Drinfeld modular curves and Drinfeld modular forms of rank 2 are the function field analogues of elliptic modular curves and modular forms and have been intensively studied. Drinfeld modular varieties of higher rank $r$ are the function field $\mathrm{GL}_r$-analogue of Shimura varieties. They have more structure and are even more interesting. Analytic and algebraic Drinfeld modular forms of higher rank with values in the function field were recently introduced and studied by Basson, Breuer and Pink [BBP18a, BBP18b, BBP18c] using the Satake compactification of the Drinfeld modular varieties constructed by Pink and Schieder [Pin13, PiSc14].

Our goal in this article is to define and construct Drinfeld modular forms of higher rank over more general bases, for example, the ring of integers and its reduction modulo a power of a prime ideal. This provides a framework for studying the arithmetic aspect of Drinfeld modular forms. For example, one can look for the congruences between two Drinfeld modular forms of different weights or ranks (the latter for example through morphisms between Drinfeld modular varieties as in Lemma 3.7 or through restriction to the boundary of the Satake compactification), or study the congruences between Drinfeld Hecke eigenforms and related Galois representations. Investigating such congruences has proved to be very useful in the construction of Galois representations of Hecke eigenforms as in the famous article of Deligne and Serre [DeSe74] for modular forms of weight 1, and those of Wiles [Wil88] and of Taylor [Tay89, Tay91, Tay95] for Hilbert modular forms and Siegel modular forms of degree 2 with lower weight. One can also explore the analogous theory for $p$-adic modular forms following Katz [Kat73]. These $\wp$-adic Drinfeld modular forms were studied recently by Hattori [Hat18] for rank 2 and by [NiRo] and [GrHa20] for arbitrary rank.

Before we explain our results, let us explain the (old) strategy of defining elliptic modular forms over rings of integers using algebraic geometry. Suppose $\overline{M}_n$ is the projective elliptic modular curve over $\mathbb{Q}(\zeta_n)$ of level-$n$ structure with $n \geq 3$, where $\zeta_n$ is a primitive $n$-th root of unity. One first realizes the elliptic modular forms of level $n$ and weight $k$, which are a priori

*Date*: September 30, 2020.

2010 *Mathematics Subject Classification.* 11F52, (11G09, 14M27, 20C08).

*Key words and phrases.* Drinfeld modular varieties, arithmetic Satake compactification, Drinfeld modular forms, Hecke eigensystems.

defined analytically, as the elements of $H^0(\overline{M}_n \otimes \mathbb{C}, \omega^{\otimes k} \otimes \mathbb{C})$ for a suitable invertible sheaf $\omega$ on $\overline{M}_n$, which is the dual of the Lie algebra of the universal elliptic curve. Then one constructs an integral model $\overline{\mathbf{M}}_n$ of $\overline{M}_n$ over $\mathbb{Z}[1/n, \zeta_n]$ and extends $\omega$ canonically over $\overline{\mathbf{M}}_n$. Here $\overline{\mathbf{M}}_n$ is the Satake compactification of the moduli space $\mathbf{M}_n$ of elliptic curves with level-$n$ structure over $\mathbb{Z}[1/n, \zeta_n]$. Algebraic modular forms of level $n$ and weight $k$ over a $\mathbb{Z}[1/n, \zeta_n]$-algebra $L$ then are defined as elements of $H^0(\overline{\mathbf{M}}_n \otimes L, \omega^{\otimes k} \otimes L)$.

This strategy has been worked out for Siegel and Hilbert moduli schemes by Chai, Faltings and Rapoport (see [Cha90, FaCh90, Rap78]). Historically, the Satake compactifications of complex Siegel modular varieties were constructed by Satake first analytically. Then Ash, Mumford, Rapoport and Tai [AMRT75] constructed complex smooth toroidal compactifications of locally symmetric varieties. However, the order in the construction of the arithmetic version is reverse: the arithmetic toroidal compactifications were constructed first and were used to construct the arithmetic Satake (minimal) compactification. Chai and Faltings showed that the ample invertible sheaf $\omega$ admits a canonical extension over a smooth toroidal compactification and used them to define arithmetic Siegel modular forms. The arithmetic Satake compactification then is constructed to be the Proj of the graded ring of arithmetic Siegel modular forms.

To explain the results of our article, we describe some background of compactifications of Drinfeld modular varieties. Let $F$ be a global function field with finite constant field $\mathbb{F}_q$ with $q$ elements. Let $\infty$ be a fixed place of $F$, and $A$ the ring of $\infty$-integers of $F$. Denote by $\mathbb{C}_\infty := \widehat{\overline{F_\infty}}$ the completion of an algebraic closure of the completion $F_\infty$ of $F$ at $\infty$. The compactification of Drinfeld moduli schemes of rank 2 over $A$ was constructed by Drinfeld; see [Dri76, Proposition 9.3]. Gekeler [Gek87] gave an outline of the Satake compactification for higher rank over $\mathbb{C}_\infty$. The Satake compactification over $F$ for arbitrary rank was constructed by Kapranov [Kap88] for $F = \mathbb{F}_q(t)$ and by Pink [Pin13] for arbitrary $F$. Pink's method is rather different from previous ones. He showed that the old strategy works fine with Drinfeld modular varieties, namely, the universal family extends to a generalized Drinfeld $A$-module over the Satake compactification. This paves a way to define algebraic Drinfeld modular forms of arbitrary rank over $F$. Arithmetic compactifications for rank 2 were revisited by Lehmkuhl [Leh09] in more details and also by Hattori [Hat17]. Häberli [Häb18] gives an analytic construction of the Satake compactification of Drinfeld modular varieties of arbitrary rank over $\mathbb{C}_\infty$. He also shows the agreement of the Satake compactification by the analytic construction and Pink's Satake compactification by the algebraic method. In particular, the universal Drinfeld $A$-module extends to a generalized Drinfeld $A$-module over the analytic Satake compactification constructed by Kapranov for $A = \mathbb{F}_q[t]$ and by Häberli for an arbitrary global function field $F$. This answers a question of Pink [Pin13, Remark 4.9].

In this article we construct the arithmetic Satake compactification of Drinfeld moduli schemes over the localization $A_{(v)}$ of a prime $v \in \operatorname{Spec} A$. Let $G = \mathrm{GL}_r$ with $r \geq 1$. Denote by $A_v$ the completion of $A$ at $v$, by $\widehat{A}$ the profinite completion of $A$, and by $\mathbb{A}^\infty := \widehat{A} \otimes_A F$ the finite adele ring of $F$.

**Theorem 1.1.** *For every fine open compact subgroup $K = K_v K^v \subset G(\mathbb{A}^\infty)$, where $K_v = G(A_v)$ and $K^v \subset G(\mathbb{A}^{v\infty})$, the Drinfeld moduli scheme $\mathbf{M}^r_K$ over $A_{(v)}$ of rank $r$ and level $K$ possesses a projective arithmetic Satake compactification $\overline{\mathbf{M}}^r_K$. The Satake compactification and its universal family are unique up to unique isomorphism. The dual $\omega_K := \mathrm{Lie}(\overline{E}_K)^\vee$ of the relative Lie algebra of the universal generalized Drinfeld $A$-module $\overline{E}_K$ over $\overline{\mathbf{M}}^r_K$ is ample. Moreover, the Satake compactification is compatible with the transition maps of changing $K$ and the prime-to-$v$ Hecke correspondences, and the universal family $\overline{E}_K$ and $\omega_K$ satisfy the functorial property with respect the transition maps and prime-to-$v$ Hecke correspondences.*

We prove this in Theorem 3.2 even in the classical form, and in the discussions after Definition 4.10 in the adelic form. We follow the approach of Pink [Pin13]. Thus, the generic fiber of our compactification constructed here gives Pink's Satake compactification. Using Theorem 1.1, we define for any positive integer $k$ and any $A_{(v)}$-algebra $L$

$$M_k(r,K,L) := H^0(\overline{\mathbf{M}}^r_K \otimes_{A_{(v)}} L, \omega_K^{\otimes k} \otimes L)$$

the $L$-module of algebraic Drinfeld modular forms of rank $r$, weight $k$ and level $K$. Thanks to work of Basson, Breuer and Pink [BBP18b] on the comparison theorem of analytically and algebraically defined Drinfeld modular forms, we extend the notion of Drinfeld modular forms over the ring of integers away from the level.

In contrast to the case of Siegel moduli schemes, our results may be surprising due to the following reasons:

(i) The construction of arithmetic Satake compactifications for Drinfeld modular varieties does not rely on that of smooth arithmetic toroidal compactifications. Indeed, this is a big advantage (due to Pink's idea) because the smooth arithmetic compactifications for Drinfeld modular varieties have not yet been completely constructed. Some special cases of smooth arithmetic compactifications of arbitrary rank where $A = \mathbb{F}_q[t]$ and the level $K = K(t)$ were constructed by Pink and Schieder [PiSc14, Section 10], and more general level $K = K(\mathfrak{n})$ were constructed very recently by Fukaya, Kato and Sharifi [FKS20].

(ii) The universal family extends to a generalized Drinfeld $A$-module over the arithmetic Satake compactification. It is not known that the universal family over a Siegel moduli scheme extends to its arithmetic Satake compactification.

An advantage in the function field case is that we can construct coefficient modular forms and generalized Hasse invariants on the special fiber over a prime $v \in \operatorname{Spec} A$ of the arithmetic Satake compactification. And we can lift these generalized Hasse invariants to the integral model (but still in positive characteristic). The analogous problem is still unsolved for Shimura varieties although significant progress was made by Boxer [Box15] and Goldring and Koskivirta [GoKo19]. This allows us to prove our main applications:

**Theorem 1.2.** *(Theorem 6.6) Let $\overline{\mathbf{M}}^r_K$ be the Satake compactification as in Theorem 1.1 and put $\overline{\mathscr{M}}^r_K := \overline{\mathbf{M}}^r_K \otimes_{A_{(v)}} \mathbb{F}_v$. Let $h$ be an integer with $1 \le h \le r$, let $a \in \mathfrak{p}$, the prime ideal corresponding to $v$, with $v(a) = 1$, and let $H^a_0, \ldots, H^a_{r-1}$ be the generalized Hasse invariants as in Definition 6.2. For $1 \le h \le r$ let $(\overline{\mathscr{M}}^r_K)^{\ge h}$ (resp. $(\mathscr{M}^r_K)^{\ge h}$) be the vanishing locus of $H^a_0, \ldots, H^a_{h-1}$ in $\overline{\mathscr{M}}^r_K$ (resp. $\mathbf{M}^r_K \otimes_{A_{(v)}} \mathbb{F}_v$), and let $(\overline{\mathscr{M}}^r_K)^{(h)} := (\overline{\mathscr{M}}^r_K)^{\ge h} - (\overline{\mathscr{M}}^r_K)^{\ge h+1}$ and $(\mathscr{M}^r_K)^{(h)} := (\mathscr{M}^r_K)^{\ge h} - (\mathscr{M}^r_K)^{\ge h+1}$.*

(0) *(Lemma 6.4) $(\overline{\mathscr{M}}^r_K)^{\ge h}$ and $(\mathscr{M}^r_K)^{\ge h}$ are independent of $a$ (satisfying $v(a) = 1$).*

(1) *The subschemes $(\overline{\mathscr{M}}^r_K)^{\ge h}$ and $(\overline{\mathscr{M}}^r_K)^{(h)}$ are of pure dimension $r-h$ and $(\mathscr{M}^r_K)^{(h)}$ is Zariski dense in $(\overline{\mathscr{M}}^r_K)^{\ge h}$, in $(\mathscr{M}^r_K)^{\ge h}$ and in $(\overline{\mathscr{M}}^r_K)^{(h)}$.*

(2) *The subschemes $(\overline{\mathscr{M}}^r_K)^{(h)}$ and $(\mathscr{M}^r_K)^{(h)}$ are affine.*

(3) *$(H^a_0, \ldots, H^a_{r-1})$ is a regular sequence on $\overline{\mathbf{M}}^r_K$.*

(4) *For every $1 \le h \le r$ the closed subscheme $X_h := V(H^a_1, \ldots, H^a_h)$ of $\overline{\mathbf{M}}^r_K$ is flat over $A_{(v)}$.*

(5) *For every $h < r$, every irreducible component of $(\overline{\mathscr{M}}^r_K)^{\ge h}$ meets $(\overline{\mathscr{M}}^r_K)^{\ge h+1}$.*

(6) *If $\overline{\mathbf{M}}^r_K$ is Cohen-Macaulay, then so is each subscheme $(\overline{\mathscr{M}}^r_K)^{\ge h}$.*

(7) *If $\overline{\mathbf{M}}^r_K$ is Cohen-Macaulay and $h \ne r$, then the natural map $\pi_0((\overline{\mathscr{M}}^r_K)^{\ge h}) \to \pi_0((\overline{\mathscr{M}}^r_K)^{\ge h-1})$ of connected components is bijective.*

Namely, we study the geometric properties of the vanishing locus of the generalized Hasse invariants inside the special fiber over $v$ of both Drinfeld moduli schemes and their Satake compactifications. These vanishing loci provide the stratification according to the height of the universal generalized Drinfeld $A$-module on the Satake compactification, or equivalently according to the $v$-rank of the Drinfeld $A$-module on the Drinfeld moduli scheme. The closed stratum is the supersingular locus.

We also study the Hecke action on Drinfeld modular forms. Let $\mathcal{A}(G', \overline{\mathbb{F}}_v)$ be the space of all locally constant functions $f : G'(F)\backslash G'(\mathbb{A})/G'(F_\infty) \to \overline{\mathbb{F}}_v$, where $\mathbb{A}$ denotes the adeles of $F$ and $G' = D^\times$ is the group scheme of units in the central division algebra $D$ over $F$ ramified precisely at $\infty$ and $v$, with invariants $\mathrm{inv}_\infty(D) = -1/r$ and $\mathrm{inv}_v(D) = 1/r$. Put $U(v) := \ker(G'(A_v) = O_{D_v}^\times \to \mathbb{F}_{v^r}^\times)$, cf. (5.8).

**Theorem 1.3.** *(Theorem 6.10) Let $\mathfrak{n} \subset A$ be a prime to $v$ non-zero ideal and $K = K_v K^v = K(\mathfrak{n}) := \ker\big(G(\widehat{A}) \to G(A/\mathfrak{n})\big)$. Consider the sets of prime-to-$v\mathfrak{n}$ Hecke eigensystems $\mathcal{H}_{\overline{\mathbb{F}}_v}^{\infty v\mathfrak{n}} \to \overline{\mathbb{F}}_v$ arising from*

1. *algebraic Drinfeld modular forms in $M_k(r, K_v, \overline{\mathbb{F}}_v)^{K^v}$ for all $k \geq 0$, where $M_k(r, K_v, \overline{\mathbb{F}}_v) := \varinjlim_{\widetilde{K}^v} M_k(r, K_v\widetilde{K}^v, \overline{\mathbb{F}}_v)$ and $M_k(r, K_v\widetilde{K}^v, \overline{\mathbb{F}}_v) := H^0(\overline{\mathbf{M}}^r_{K_v\widetilde{K}^v} \otimes_{A_{(v)}} \overline{\mathbb{F}}_v,\ \omega^{\otimes k}_{K_v\widetilde{K}^v} \otimes \overline{\mathbb{F}}_v)$,*
2. *and elements of $\mathcal{A}(G', \overline{\mathbb{F}}_v)^{U(v)K^v}$, respectively,*

*where $\mathcal{H}_{\overline{\mathbb{F}}_v}^{\infty v\mathfrak{n}} = \mathcal{H}_{\overline{\mathbb{F}}_v}(G(\mathbb{A}^{\infty v\mathfrak{n}}), K^{v\mathfrak{n}}) \simeq \mathcal{H}_{\overline{\mathbb{F}}_v}(G'(\mathbb{A}^{\infty v\mathfrak{n}}), K^{v\mathfrak{n}})$ is the prime-to-$v\mathfrak{n}$ spherical Hecke algebra over $\overline{\mathbb{F}}_v$. Then both sets of Hecke eigensystems are equal. In particular, there are only finitely many Hecke eigensystems of algebraic Drinfeld modular forms over $\overline{\mathbb{F}}_v$ of a fixed level and all weights.*

Theorem 1.3 amounts to the Jacquet-Langlands correspondence mod $v$ for Hecke eigensystems. As a corollary we also derive an explicit upper bound and the asymptotic behavior of the size of these Hecke eigensystems. One technical difficulty is that we do not know whether $M_k(r, K_v, \overline{\mathbb{F}}_v)^{K^v} = M_k(r, K_vK^v, \overline{\mathbb{F}}_v)$. This would be true if the special fiber $\overline{\mathscr{M}}^r_{K_vK^v}$ at $v$ of the Satake compactification was normal; see Theorem 4.14. However, we only know that $\overline{\mathscr{M}}^r_{K_vK^v}$ is reduced; see Proposition 3.9. We overcome this difficulty by comparing the Hecke eigensystems in Theorem 1.3 also to those arising on the normalization of $\overline{\mathscr{M}}^r_{K_vK^v}$; see Theorem 6.10. We also prove in Theorem 4.14 that, nevertheless, $M_k(r, K_v, \overline{\mathbb{F}}_v)$ is a smooth admissible $G(\mathbb{A}^{v\infty})$-module.

This article is organized as follows. After reviewing (generalized) Drinfeld modules and the construction of Drinfeld modular varieties in Section 2, we construct their arithmetic Satake compactification in Section 3 following Pink's approach [Pin13] over the function field. Algebraic Drinfeld modular forms are defined and their preliminary properties are studied in Section 4. The next Section 5 deals with the supersingular locus and its relation with algebraic modular forms in the sense of Gross. In the final Section 6 we introduce the generalized Hasse invariants and prove the two Theorems 6.6 and 6.10 explained above.

## 2. Drinfeld modules and moduli spaces

Let $q$ be a power of a prime number $p$, and $\mathcal{C}$ a geometrically connected smooth projective algebraic curve over a finite finite $\mathbb{F}_q$ of $q$ elements. Let $F$ be the function field of $\mathcal{C}$, which is a global function field of characteristic $p > 0$ with field of constants $\mathbb{F}_q$. Fix a closed point $\infty$ of $\mathcal{C}$, referred as the place of $F$ at infinity. Let $A := \Gamma(\mathcal{C} - \{\infty\}, \mathcal{O}_\mathcal{C})$ be the ring of functions in $F$ regular away from $\infty$. $A$ is a Dedekind domain with finite unit group $A^\times = \mathbb{F}_q^\times$. For any place $v$ of $F$, denote by $F_v$ the completion of $F$ at $v$, $\mathcal{O}_v$ the valuation ring, $\mathbb{F}_v$ the residue field and $|\ |_v$ the normalized valuation of $F$ at $v$. If $v$ is a finite place, we also write $A_v$ for $\mathcal{O}_v$. For any nonzero element $a \in A$, define $\deg(a) := \dim_{\mathbb{F}_q} A/(a)$.

Denote by $\widehat{A}$ the pro-finite completion of $A$, and $\mathbb{A}$ (resp. $\mathbb{A}^\infty$) the (resp. finite) adele ring of $F$. Let $\overline{F} \subset \overline{F}_\infty$ be fixed algebraic closures of $F \subset F_\infty$, respectively. Let $\mathbb{C}_\infty$ be the completion of $\overline{F}_\infty$ with respect to the unique extension of $|\ |_\infty$ .

Let $\tau$ denote the endomorphism $x \mapsto x^q$ of the additive group $\mathbb{G}_{a,\mathbb{F}_p}$. For any field $L \supset \mathbb{F}_p$, denote by $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,L})$ the ring of $\mathbb{F}_q$-linear endomorphisms of $\mathbb{G}_{a,L} = \mathbb{G}_a \otimes_{\mathbb{F}_p} L$ over $L$. It is known that

$$\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,L}) = L\{\tau\} := \left\{ \sum_{i=0}^{n} \varphi_i \tau^i, \text{for some } n \in \mathbb{N},\ \varphi_i \in L \right\}, \quad \tau\varphi_i = \varphi_i^q \tau.$$

Denote by $\partial : L\{\tau\} \to L$, $\sum_i \varphi_i \tau^i \mapsto \varphi_0$, the derivative map.

An *$A$-field* is a field $L$ together with a ring homomorphism $\gamma : A \to L$. We say that $(L, \gamma)$ is of *generic $A$-characteristic* if A-char $(L, \gamma) := \ker \gamma$ is zero, otherwise that $(L, \gamma)$ is of *$A$-characteristic* $v$, where $v$ is the place corresponding to the non-zero prime ideal A-char $(L, \gamma)$. If there is no confusion, we write $L$ for $(L, \gamma)$. More generally, for an $A$-scheme $S$ we let $\gamma\colon A \to \Gamma(S, \mathcal{O}_S)$ be the ring homomorphism induced from the structure morphism $S \to \operatorname{Spec} A$. We write $(S, \gamma)$ for such an $A$-scheme.

Recall that a *Drinfeld $A$-module* over an $A$-field $L$ is a ring homomorphism

$$\varphi : A \to L\{\tau\}, \quad a \mapsto \varphi_a = \sum_i \varphi_{a,i}\tau^i, \tag{2.1}$$

such that $\partial \circ \varphi = \gamma$ and $\varphi$ does not factor through the inclusion $L \subset L\{\tau\}$. There is a unique positive integer $r$ such that for any non-zero element $a \in A$, one has $\varphi_{a,i} = 0$ for all $i > r\deg(a)$ and $\varphi_{a,r\deg(a)} \neq 0$ [Dri76, Prop. 2.1]. The integer $r$ is called the *rank* of $\varphi$.

To introduce families of Drinfeld $A$-modules we mainly follow [Pin13], besides the standard references [Dri76] and [Lau96]. By definition the *trivial line bundle* over a scheme $S$ is the additive group scheme $\mathbb{G}_{a,S}$ over $S$ together with the multiplication $\mathbb{G}_{m,S} \times_S \mathbb{G}_{a,S} \to \mathbb{G}_{a,S}, (x, y) \mapsto xy$. An arbitrary *line bundle over* $S$ is a commutative group scheme $E$ over $S$ together with a scalar multiplication $\mathbb{G}_{m,S} \times_S E \to E$ which as a pair, is Zariski locally on $S$ isomorphic to the trivial line bundle. A *homomorphism* of line bundles is a morphism of group schemes that is compatible with the $\mathbb{G}_m$-actions. By working on a local trivialization the following facts are easy to prove. The sections of $E$ over an open subset $U \subset S$ form a module over $\mathcal{O}_S(U)$ such that the scalar multiplication with elements in $\mathcal{O}_S(U)^\times = \mathbb{G}_{m,S}(U)$ coincides with the $\mathbb{G}_{m,S}$-action. On sections over an open $U \subset S$ a homomorphism of line bundles is automatically $\mathcal{O}_S(U)$-linear. Indeed, it is additive by definition and compatible with scalar multiplication by elements in $\mathcal{O}_S(U)^\times$. The compatibility with all $a \in \mathcal{O}_S(U)$ follows because at every point $s$ of $U$ either $a$ or $1 + a$ is invertible in a neighborhood of $s$. We let $\operatorname{Hom}_{\mathcal{O}_S}(E_1, E_2)$ denote the group of homomorphisms between the line bundles $E_1$ and $E_2$. We also let $\operatorname{Hom}(E_1, E_2)$ denote the group of homomorphisms of the commutative group schemes underlying $E_1$ and $E_2$ forgetting the $\mathbb{G}_m$-actions. As usual we write $\operatorname{End}_{\mathcal{O}_S}(E) := \operatorname{Hom}_{\mathcal{O}_S}(E, E)$ and $\operatorname{End}(E) := \operatorname{Hom}(E, E)$ for the endomorphism rings. Note that the homomorphism

$$\tau\colon E \to E^{\otimes q} = \sigma^* E\,, \qquad x \mapsto x^q$$

is only additive and not compatible with the $\mathbb{G}_m$-actions on $E$ and $E^{\otimes q}$. So it lies in $\operatorname{Hom}(E, E^{\otimes q})$ but not in $\operatorname{Hom}_{\mathcal{O}_S}(E, E^{\otimes q})$.

According to the original definition [Dri76, Sect. 5, p. 575], a *Drinfeld $A$-module of rank $r$* ($r$ being a positive integer) over an $A$-scheme $(S, \gamma)$ is a pair $(E, \varphi)$, where $E$ is a line bundle over $S$ and $\varphi : A \to \operatorname{End}(E)$ is a ring homomorphism such that $\partial \circ \varphi = \gamma$, and for any $L$-valued point $s : \operatorname{Spec} L \to S$ over $\operatorname{Spec} A$, where $L$ is an $A$-field, the pull-back $\varphi_s$ is a Drinfeld $A$-module of rank $r$ over $L$. A *homomorphism* of Drinfeld $A$-modules between $(E, \varphi)$ and $(E', \varphi')$ is a homomorphism

$$u : E \to E' \tag{2.2}$$

of commutative group schemes over $S$ such that $\varphi'_a \circ u = u \circ \varphi_a$ for all $a \in A$. Compatibility with the structure of line bundles, i.e. with the $\mathbb{G}_{\mathrm{m}}$-actions, is not required. Let $\sigma : S \to S$ be the $q$-th power Frobenius map. The endomorphism $\varphi_a$ for $a \in A$ can be expressed uniquely as a locally finite sum [Dri76, Sect. 5]

$$\varphi_a = \sum_{i \geq 0} \varphi_{a,i}\tau^i, \quad \varphi_{a,i} \in \Gamma(S, E^{\otimes(1-q^i)}), \quad \tau^i : E \to E^{\otimes q^i} = \sigma^{i*}E, \ x \mapsto x^{q^i}. \tag{2.3}$$

By this we mean that the expression in (2.3) is a finite sum on any quasi-compact open subset.

A Drinfeld $A$-module $(E,\varphi)$ of rank $r$ over $S$ is called *standard* if for any $a\in A$ and any $i>r\deg(a)$, the term $\varphi_{a,i}$ in (2.3) is zero. It is shown [Dri76, Prop. 5.2] that every Drinfeld $A$-module is isomorphic to a standard Drinfeld $A$-module, and that every isomorphism of standard Drinfeld $A$-modules is $\mathcal{O}_S$-linear, i.e. in $\mathrm{Hom}_{\mathcal{O}_S}(E_1,E_2)$ as opposed to in $\mathrm{Hom}(E_1,E_2)$, see also [Har19, Lemma 3.8].

In [Pin13], R. Pink worked on the notion of generalized Drinfeld modules. These modules play a similar role as what generalized elliptic curves do for compactifying elliptic modular curves.

**Definition 2.1** ([Pin13, Sect. 3])**.** (1) A *generalized Drinfeld $A$-module* over an $A$-scheme $S$ is a pair $(E,\varphi)$ consisting of a line bundle $E$ over $S$ and a ring homomorphism $A\to\mathrm{End}(E)$ satisfying the following properties

(a) The composition $\partial\circ\varphi$ is the structure morphism $\gamma: A\to\Gamma(S,\mathcal{O}_S)$.
(b) Over any point $s\in S$, the fiber $\varphi_s$ at $s$ is a Drinfeld $A$-module of rank $r_s\ge 1$.

(2) A generalized Drinfeld $A$-module $(E,\varphi)$ is said to be *of rank* $\le r$, where $r$ is a positive integer, if

(c) for any $a\in A$, the endomorphism $\varphi_a$ has the form $\sum_{i=0}^{r\deg(a)}\varphi_{a,i}\tau^i$ with sections $\varphi_{a,i}\in\Gamma(S,E^{\otimes(1-q^i)})$.

(3) An isomorphism of generalized Drinfeld $A$-modules is an isomorphism of line bundles that commutes with the actions of $A$.

Note that the property that $(E,\varphi)$ is of rank $\le r$ is preserved under isomorphisms of line bundles, but in general not under (non-linear) isomorphisms of the underlying group schemes. The definition of generalized Drinfeld $A$-modules $(E,\varphi)$ of rank $\le r$ is slightly stronger than the notion of that with the property $r_s\le r$ everywhere. Following from the definitions, these two notions are equivalent if the scheme $S$ is reduced. Indeed, if $r_s\le r$ everywhere, then $\varphi_{a,i}$ is zero in the residue fields of all points of $S$, hence is locally nilpotent on $S$ for all $i>r\cdot\deg(a)$. But for general $S$ and a generalized Drinfeld $A$-module $(E,\varphi)$ of rank $\le r$ over $S$, the smallest integer $r_1$ such that $(E,\varphi)$ is of rank $\le r_1$ can be bigger than the maximal point-wise rank $r_2:=\max\{r_s|s\in S\}$. According to [Pin13], the non-zero nilpotent components $\varphi_{a,i}$ for $i>r_2\deg(a)$ should be regarded as deformations "towards higher rank".

**Definition 2.2** ([Pin13, Sect. 3])**.** A generalized Drinfeld $A$-module of rank $\le r$ with $r_s=r$ everywhere is called a *Drinfeld $A$-module of rank $r$* over an $A$-scheme $S$.

This definition corresponds to that of a standard Drinfeld $A$-module of rank $r$ in [Dri76]. By [Dri76, Prop. 5.2] or [Lau96, Lemma 1.1.2] there is a natural bijection between

$$\{\text{Drinfeld } A\text{-modules of rank } r \text{ over } S \text{ in Definition 2.2}\}/\simeq \quad\text{and}$$

$$\{\text{Drinfeld } A\text{-modules of rank } r \text{ over } S \text{ in the original definition}\}/\simeq\ .$$

Here the $\simeq$ in the second line means up to isomorphisms of group schemes with $A$-action in the sense of (2.2) and not isomorphisms in the sense of Definition 2.1. We shall adopt Definition 2.2 for Drinfeld $A$-modules of constant rank in this article. In particular, every isomorphism between Drinfeld modules is $\mathcal{O}_S$-linear.

Let $r$ be a positive integer, and $\mathfrak{n}\subset A$ a nonzero proper ideal. Denote by $A[\mathfrak{n}^{-1}]\subset F$ the $A$-subalgebra generated by elements of the fractional ideal $\mathfrak{n}^{-1}\subset F$. For each finite place $v$, one easily calculates that $A[\mathfrak{n}^{-1}]\otimes_A A_v=A_v$ if $v\notin V(\mathfrak{n})$, and $A[\mathfrak{n}^{-1}]\otimes_A A_v=F_v$ otherwise. Therefore, $A[\mathfrak{n}^{-1}]$ agrees with the ring $A[1/\mathfrak{n}]:=\Gamma(\mathrm{Spec}\,A-V(\mathfrak{n}),\mathcal{O}_C)$. A *(full) level-$\mathfrak{n}$ structure* on a Drinfeld $A$-module of rank $r$ over an $A[\mathfrak{n}^{-1}]$-scheme $S$ is an isomorphism of finite flat schemes of $A$-modules

$$\lambda:(\mathfrak{n}^{-1}/A)^r_S\xrightarrow{\sim}\varphi[\mathfrak{n}],\quad \varphi[\mathfrak{n}]:=\bigcap_{a\in\mathfrak{n}}\ker(\varphi_a)\subset E, \tag{2.4}$$

where $\bigcap$ is the scheme theoretic intersection. When $S$ is connected, $\lambda$ is simply given by an isomorphism $(\mathfrak{n}^{-1}/A)^r \xrightarrow{\sim} \varphi[\mathfrak{n}](S)$ of finite $A$-modules. Let

$$K(\mathfrak{n}) := \ker(\mathrm{GL}_r(\widehat{A}) \to \mathrm{GL}_r(A/\mathfrak{n})) \subset \mathrm{GL}_r(\widehat{A})$$

denote the principal open compact subgroup of level $\mathfrak{n}$.

Let $\mathbf{M}^r(\mathfrak{n})$ denote the moduli scheme over $A[\mathfrak{n}^{-1}]$ of (isomorphism classes of) Drinfeld $A$-modules of rank $r$ with level-$\mathfrak{n}$ structure. Let $M^r(\mathfrak{n}) := \mathbf{M}^r(\mathfrak{n}) \otimes_{A[\mathfrak{n}^{-1}]} F$ denote the generic fiber of $\mathbf{M}^r(\mathfrak{n})$.

**Theorem 2.3** ([Dri76, Section 5]). *If $\mathfrak{n} \subset A$ a nonzero proper ideal, then the moduli scheme $\mathbf{M}^r(\mathfrak{n})$ is an affine smooth scheme of finite type over* $\operatorname{Spec} A[\mathfrak{n}^{-1}]$ *of relative dimension $r-1$.*

The finite group $\mathrm{GL}_r(A/\mathfrak{n})$ acts on level-$\mathfrak{n}$ structures, and hence gives a right action on $\mathbf{M}^r(\mathfrak{n})$ by $(E, \varphi, \lambda) \mapsto (E, \varphi, \lambda g)$, for $g \in \mathrm{GL}_r(A/\mathfrak{n})$. By inflation, $\mathrm{GL}_r(\widehat{A})$ acts on $\mathbf{M}^r(\mathfrak{n})$ on the right.

**Proposition 2.4.** (1) *There is a bijection between the set $\pi_0(M^r(\mathfrak{n})_{\overline{F}})$ of geometrically connected components and the ray class group $(\mathbb{A}^\infty)^\times/F^\times(1+\mathfrak{n}\widehat{A})^\times$. The set of orbits of the $\mathrm{GL}_r(A/\mathfrak{n})$-action is in bijection with the ideal class group $\mathrm{Cl}(A) = (\mathbb{A}^\infty)^\times/F^\times\widehat{A}^\times$. In particular, the action of $\mathrm{GL}_r(A/\mathfrak{n})$ on $\pi_0(M^r(\mathfrak{n})_{\overline{F}})$ is transitive if and only if $A$ is a principal ideal domain.*

(2) *The moduli schemes $\mathbf{M}^r(\mathfrak{n})$ and $M^r(\mathfrak{n})$ are connected as schemes.*

PROOF. (1) Using the modular interpretation and the analytic theory, there is a natural isomorphism of rigid analytic spaces

$$M^r(\mathfrak{n})(\mathbb{C}_\infty) \simeq \mathrm{GL}_r(F)\backslash\Omega^r(\mathbb{C}_\infty) \times \mathrm{GL}_r(\mathbb{A}^\infty)/K(\mathfrak{n}),$$

where $\Omega^r$ is the Drinfeld period domain of rank $r$ over $\mathbb{C}_\infty$, see [DeHu87, Theorem 5.6]. Recall that $\Omega^r(\mathbb{C}_\infty)$ is the complement of the union of all $F_\infty$-rational hyperplanes in $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$. Since $\Omega^r$ is connected as a rigid analytic space, one has a bijection

$$\pi_0(M^r(\mathfrak{n})_{\mathbb{C}_\infty}) \simeq \mathrm{GL}_r(F)\backslash \mathrm{GL}_r(\mathbb{A}^\infty)/K(\mathfrak{n}).$$

Through the determinant map, strong approximation for $\mathrm{SL}_r$ yields a bijection

$$\pi_0(M^r(\mathfrak{n})_{\overline{F}}) = \pi_0(M^r(\mathfrak{n})_{\mathbb{C}_\infty}) \simeq (\mathbb{A}^\infty)^\times/F^\times \det(K(\mathfrak{n})).$$

Note that $\det K(\mathfrak{n}) = (1+\mathfrak{n}\widehat{A})^\times$. The action of $\mathrm{GL}_r(\widehat{A})$ on $M^r(\mathfrak{n})(\mathbb{C}_\infty)$ is simply the right translation. Through the determinant map, this action factors through the right translation of $\widehat{A}^\times$ on $(\mathbb{A}^\infty)^\times/F^\times\det(K(\mathfrak{n}))$. Then the set of orbits is isomorphic to $\mathrm{Cl}(A)$.

(2) It suffices to show the connectedness of $M^r(\mathfrak{n})$ as $M^r(\mathfrak{n}) \subset \mathbf{M}^r(\mathfrak{n})$ is open and dense. Indeed, $\mathbf{M}^r(\mathfrak{n})$ is smooth over $A[\mathfrak{n}^{-1}]$, and so all its irreducible components meet $M^r(\mathfrak{n})$ by [Har77, Proposition III.9.7]. The connectedness of $M^r(\mathfrak{n})$ is mentioned in [Pin13, p. 335], but this does not seem to be stated explicitly in standard references [Dri76, Lau96]. So we give a proof for the reader's convenience. It suffices to show that the $\mathrm{Gal}(\overline{F}/F)$-action on $\pi_0(M^r(\mathfrak{n})_{\overline{F}})$ is transitive. Using the Weil-pairing map $w_\mathfrak{n} : M^r(\mathfrak{n}) \to M^1(\mathfrak{n})$ constructed by van der Heiden [vdH06, Theorem 4.1], one obtains a surjective morphism

$$\pi_0(M^r(\mathfrak{n})_{\overline{F}}) \to \pi_0(M^1(\mathfrak{n})_{\overline{F}}) = M^1(\mathfrak{n})(\overline{F}) \simeq (\mathbb{A}^\infty)^\times/F^\times(1+\mathfrak{n}\widehat{A})^\times. \tag{2.5}$$

As shown in (1), $\pi_0(M^r(\mathfrak{n})_{\overline{F}})$ and $\pi_0(M^1(\mathfrak{n})_{\overline{F}})$ have the same cardinality, and hence the map (2.5) is bijective. By Drinfeld's description of Drinfeld modules of rank one [Dri76, Section 8], the group $\mathrm{Gal}(\overline{F}/F)$ acts transitively on the set $M^1(\mathfrak{n})(\overline{F})$ and hence on the set $\pi_0(M^r(\mathfrak{n})_{\overline{F}})$. This completes the proof. ∎

For any element $g \in K(1) := \mathrm{GL}_r(\widehat{A})$, denote by

$$J_g : \mathbf{M}^r(\mathfrak{n}) \to \mathbf{M}^r(\mathfrak{n}), \quad (E, \varphi, \lambda) \mapsto (E, \varphi, \lambda g),$$

the isomorphism translating the level structures. Observe that if $(E, \varphi, \lambda)$ is the fiber of the universal family $(\widetilde{E}, \widetilde{\varphi}, \widetilde{\lambda}) \to \mathbf{M}^r(\mathfrak{n})$ at a point $x$, then the fiber of $(\widetilde{E}, \widetilde{\varphi}, \widetilde{\lambda})$ at its image $J_g(x)$ is

$(E,\varphi,\lambda g)$. Consider the family $(\widetilde{E},\widetilde{\varphi},\widetilde{\lambda}g)$ over $\mathbf{M}^r(\mathfrak{n})$. By the universal property of fine moduli schemes, there is a unique morphism $\alpha_g : \mathbf{M}^r(\mathfrak{n}) \to \mathbf{M}^r(\mathfrak{n})$ such that $\beta_g : (\widetilde{E},\widetilde{\varphi},\widetilde{\lambda}g) \simeq \alpha_g^*(\widetilde{E},\widetilde{\varphi},\widetilde{\lambda})$ over $\mathbf{M}^r(\mathfrak{n})$. The above observation and modular interpretation say that $\alpha_g = J_g$. With level structures ignored, the composition of $\beta_g$ with the base change isomorphism $I_g : (\widetilde{E},\widetilde{\varphi}) \xrightarrow{\sim} J_g^*(\widetilde{E},\widetilde{\varphi})$ gives a commutative diagram:

$$
\begin{array}{ccc}
(\widetilde{E},\widetilde{\varphi}) & \xrightarrow{I_g} & (\widetilde{E},\widetilde{\varphi}) \\
\downarrow & & \downarrow \\
\mathbf{M}^r(\mathfrak{n}) & \xrightarrow{J_g} & \mathbf{M}^r(\mathfrak{n}).
\end{array}
\tag{2.6}
$$

One easily checks that for $g_1, g_2 \in K(1)$, the relations $J_{g_1g_2} = J_{g_2} \circ J_{g_1}$ and $I_{g_1g_2} = I_{g_2} \circ I_{g_1}$ hold on the moduli scheme $\mathbf{M}^r(\mathfrak{n})$ and the universal family $(\widetilde{E},\widetilde{\varphi})$, respectively. Therefore, there is a right action of $K(1)$ on the scheme (geometric line bundle) $\widetilde{E}$ which is equivalent with the Hecke action $J_g$ on $\mathbf{M}^r(\mathfrak{n})$.

Let $\tilde{\mathfrak{n}} \subset \mathfrak{n}$ be two ideals of $A$. Then $\mathfrak{n}^{-1}/A \subset \tilde{\mathfrak{n}}^{-1}/A$ and $\mathfrak{n}^{-1}/A$ is the $\mathfrak{n}$-torsion in $\tilde{\mathfrak{n}}^{-1}/A$. Likewise $\varphi[\mathfrak{n}]$ equals the $\mathfrak{n}$-torsion in $\varphi[\tilde{\mathfrak{n}}]$. If $\widetilde{\lambda}\colon (\tilde{\mathfrak{n}}^{-1}/A)^r_S \xrightarrow{\sim} \varphi[\tilde{\mathfrak{n}}]$ is a level-$\tilde{\mathfrak{n}}$ structure on $\varphi$, then its restriction to $\mathfrak{n}$-torsion defines an isomorphism $\lambda\colon (\mathfrak{n}^{-1}/A)^r_S \xrightarrow{\sim} \varphi[\mathfrak{n}]$. Since $\mathrm{GL}_r(A/\tilde{\mathfrak{n}}) \to \mathrm{GL}_r(A/\mathfrak{n})$ is surjective with kernel isomorphic to $K(\mathfrak{n})/K(\tilde{\mathfrak{n}})$, the map

$$
\mathrm{Isom}_S\big((\tilde{\mathfrak{n}}^{-1}/A)^r_S, \varphi[\tilde{\mathfrak{n}}]\big) \to \mathrm{Isom}_S\big((\mathfrak{n}^{-1}/A)^r_S, \varphi[\mathfrak{n}]\big), \quad \widetilde{\lambda} \mapsto \lambda := \widetilde{\lambda}|_{(\mathfrak{n}^{-1}/A)^r_S}
\tag{2.7}
$$

is surjective with fibers being $\big(K(\mathfrak{n})/K(\tilde{\mathfrak{n}})\big)_S$-orbits, if the source is non-empty. Thus, the natural map

$$
\pi_{\mathfrak{n},\tilde{\mathfrak{n}}} : \mathbf{M}^r(\tilde{\mathfrak{n}}) \to \mathbf{M}^r(\mathfrak{n}), \quad (E,\varphi,\widetilde{\lambda}) \mapsto (E,\varphi,\lambda)
\tag{2.8}
$$

induces an isomorphism

$$
\mathbf{M}^r(\tilde{\mathfrak{n}})/(K(\mathfrak{n})/K(\tilde{\mathfrak{n}})) \to \mathbf{M}^r(\mathfrak{n})[\tilde{\mathfrak{n}}^{-1}] := \mathbf{M}^r(\mathfrak{n}) \otimes_{A[\mathfrak{n}^{-1}]} A[\tilde{\mathfrak{n}}^{-1}].
\tag{2.9}
$$

Note that $\mathbf{M}^r(\tilde{\mathfrak{n}})$ is the finite étale Galois cover of $\mathbf{M}^r(\mathfrak{n})[\tilde{\mathfrak{n}}^{-1}]$ parameterizing the level-$\tilde{\mathfrak{n}}$ structures on the universal family $(E,\varphi,\lambda)$ on $\mathbf{M}^r(\mathfrak{n})$ extending $\lambda$. Using this description, the pull-back of $(E,\varphi)$ on $\mathbf{M}^r(\mathfrak{n})$ by $\pi_{\mathfrak{n},\tilde{\mathfrak{n}}}$ is the universal family on $\mathbf{M}^r(\tilde{\mathfrak{n}})$.

Let $K \subset \mathrm{GL}_r(\widehat{A}) =: K(1)$ be an open compact subgroup. Choose any proper nonzero ideal $\mathfrak{n}$ with $K(\mathfrak{n}) \subset K$. Define the moduli scheme of level-$K$ structure by

$$
\mathbf{M}^r_K[\mathfrak{n}^{-1}] := \mathbf{M}^r(\mathfrak{n})/(K/K(\mathfrak{n})),
\tag{2.10}
$$

which is an affine scheme of finite type over $A[\mathfrak{n}^{-1}]$ by [SGA1, Exp. V. Prop. 1.8] and [SGA 3, Exp. V, Thm. 7.1]. Let $\mathfrak{n}_K$ denote the largest ideal satisfying the property $K(\mathfrak{n}_K) \subset K$. One can extend $\mathbf{M}^r_K[\mathfrak{n}^{-1}]$ uniquely to a moduli scheme which is faithfully flat over $A[\mathfrak{n}_K^{-1}]$. When $K \neq K(1)$, one can simply take $\mathfrak{n} := \mathfrak{n}_K$. When $K = K(1)$, the scheme $\mathbf{M}^r_{K(1)}$ is obtained by gluing of affine schemes $\mathbf{M}^r_{K(1)}[\mathfrak{n}_1^{-1}]$ and $\mathbf{M}^r_{K(1)}[\mathfrak{n}_2^{-1}]$ along the open subset $\mathbf{M}^r_{K(1)}[(\mathfrak{n}_1\mathfrak{n}_2)^{-1}]$, where $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are any two coprime proper ideals.

**Definition 2.5.** Let $K \subset \mathrm{GL}_r(\widehat{A}) =: K(1)$ be an open compact subgroup, and $\mathfrak{n}_K$ denote the largest ideal satisfying the property $K(\mathfrak{n}_K) \subset K$. Let $\mathbf{M}^r_K$ denote the unique moduli scheme over $A[\mathfrak{n}_K^{-1}]$ such that $\mathbf{M}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}^{-1}] = \mathbf{M}^r_K[\mathfrak{n}^{-1}]$ for any ideal $\mathfrak{n} \subset \mathfrak{n}_K$. With this notation, one has $\mathbf{M}^r(\mathfrak{n}) = \mathbf{M}^r_{K(\mathfrak{n})}$ for any proper nonzero ideal $\mathfrak{n} \subset A$.

A level-$K$ structure on a Drinfeld $A$-module $(E,\varphi)$ over an $A[\mathfrak{n}^{-1}]$-scheme $S$ with $K(\mathfrak{n}) \subset K$ is a $K$-orbit $\lambda K$ of level-$\mathfrak{n}$ structures that is defined over $S$. By (2.7), this does not depend on the choice of $\mathfrak{n}$, provided that the support of $\mathfrak{n}$ is unchanged. Define the contravariant functor

$$
F^r_K : (A[\mathfrak{n}_K^{-1}]\text{-sch}) \to (\mathrm{Set})
$$

as follows. For $K \neq K(1)$, $F^r_K(S)$ is the set of isomorphism classes of Drinfeld $A$-modules of rank $r$ with level-$K$ structure over $S$. For $K = K(1)$, $F^r_{K(1)}(S)$ is the set of isomorphism classes of Drinfeld $A$-modules of rank $r$ over $S$.

**Definition 2.6.** An open compact subgroup $K \subset K(1)$ is said to be *fine* if there is a prime ideal $\mathfrak{p} \subset A$ such that the image of $K$ in $\mathrm{GL}_r(A/\mathfrak{p})$ is unipotent.

**Lemma 2.7.** *If $K$ is fine, then the automorphism group of any Drinfeld $A$-module of rank $r$ with level-$K$ structure $(E, \varphi, \lambda K)$ over $S$ is trivial.*

PROOF. One can assume that $S = \operatorname{Spec} k$ with an algebraically closed field $k$. As $K$ is fine, the image of $K$ in $\mathrm{GL}_r(A/\mathfrak{p})$ is unipotent for some prime $\mathfrak{p}$. Let $g \in \mathrm{Aut}(\varphi)$ be an automorphism fixing $\lambda K$. Observe that $g$ fixes $\lambda K'$ for any subgroup $K' \supset K$. Thus, after replacing $K$ by $K(\mathfrak{p})K$, one can assume that $K(\mathfrak{p}) \subset K$. It is known that the automorphic group $\mathrm{Aut}(\varphi)$ is isomorphic to $\mathbb{F}^\times_{q^m}$ for some integer $m|r$. This follows from the fact that the endomorphism algebra $D = \mathrm{End}^0(\varphi)$ of $\varphi$ is totally ramified at $\infty$, i.e. $D_\infty$ is a central division $F_\infty$-algebra. See an argument in [Gek92, p.322] or [WeYu15, Lemma 2.5]. Thus, we have an $\mathbb{F}_q$-algebra homomorphism

$$\mathbb{F}_{q^m} \subset \mathrm{End}(\varphi) \to \mathrm{End}(\varphi[\mathfrak{p}]) \simeq \mathrm{Mat}_r(A/\mathfrak{p}).$$

Since $\mathbb{F}_{q^m}$ is a field, the map $\mathbb{F}^\times_{q^m} = \mathrm{Aut}(\varphi) \to \mathrm{GL}_r(A/\mathfrak{p})$ is injective. Now $g\lambda K = \lambda K$ implies that $g \in \mathbb{F}^\times_{q^m} \cap \lambda K \lambda^{-1}$. Therefore, the image of $g$ in $\mathrm{GL}_r(A/\mathfrak{p})$ is both semi-simple and unipotent and is trivial by the Jordan decomposition. This shows that $\mathrm{Aut}(\varphi, \lambda K) = 1$. ∎

**Proposition 2.8.** (1) *For any open compact subgroup $K$, $\mathbf{M}^r_K$ is the coarse moduli scheme for the functor $F^r_K$. That is, there is a natural transformation $\tau : F^r_K \to h_{\mathbf{M}^r_K} := \mathrm{Hom}(-, \mathbf{M}^r_K)$ and $\tau$ is universal among such natural transformations.*

(2) *If $K$ is fine, then $\mathbf{M}^r_K$ represents the functor $F^r_K$ and is smooth over $A[\mathfrak{n}_K^{-1}]$ of relative dimension $r-1$. If $K' \lhd K$ and $K$ is fine, then the natural morphism*

$$\pi_{K,K'} : \mathbf{M}^r_{K'} \to \mathbf{M}^r_K[\mathfrak{n}_{K'}^{-1}]$$

*is finite Galois étale with group $K/K'$. The pull-back of the universal family $(\widetilde{E}, \widetilde{\varphi})$ on $\mathbf{M}^r_K$ by $\pi_{K,K'}$ is the universal family on $\mathbf{M}^r_{K'}$.*

PROOF. (1) This follows from the construction of $\mathbf{M}^r_K$ and the fine moduli scheme $\mathbf{M}^r_{K(\mathfrak{n})} = \mathbf{M}^r(\mathfrak{n})$. Indeed, suppose we have an object $(E, \varphi, \lambda K)$ over $S$. Then one finds a finite Galois étale $K/K(\mathfrak{n})$-cover $S_\mathfrak{n} \to S[\mathfrak{n}^{-1}]$ and a family $(E, \varphi, \lambda)$ with level-$\mathfrak{n}$ structure over $S_\mathfrak{n}$. By the universal property, there is a unique morphism $f : S_\mathfrak{n} \to \mathbf{M}^r_{K(\mathfrak{n})}$ such that $(E, \varphi, \lambda)$ is isomorphic to the pull-back of the universal family. The composition $\pi_{K,K(\mathfrak{n})} \circ f : S_\mathfrak{n} \to \mathbf{M}^r_K[\mathfrak{n}^{-1}]$ is $K/K(\mathfrak{n})$-invariant, and hence it induces a unique morphism $f' : S[\mathfrak{n}^{-1}] = S_\mathfrak{n}/(K/K(\mathfrak{n})) \to \mathbf{M}^r_K[\mathfrak{n}^{-1}]$. It is straightforward to check this transformation satisfies the universal property.

(2) It follows from Lemma 2.7 that the right action of $K/K(\mathfrak{n})$ on $\mathbf{M}^r(\mathfrak{n})$ is free. Thus, the universal family $(\widetilde{E}, \widetilde{\varphi}, \widetilde{\lambda})$ on $\mathbf{M}^r(\mathfrak{n})$ descends uniquely to a family $(E_K, \varphi_K)$ on $\mathbf{M}^r_K$. One can show that the $K$-orbit $\widetilde{\lambda}K$ is defined over $\mathbf{M}^r_K$. Thus, one obtains a family $(E_K, \varphi_K, \widetilde{\lambda}K)$ in $F^r_K(\mathbf{M}^r_K)$. Using the same argument as in (1), we show that for any object $(E, \varphi, \lambda K) \in F^r_K(S)$, there is a unique map $f : S \to \mathbf{M}^r_K$ such that $(E, \varphi, \lambda K) \simeq f^*(E_K, \varphi_K, \widetilde{\lambda}K)$. This shows that $\mathbf{M}^r_K$ represents $F^r_K$. The remaining assertions follow from the same reason as in (2.9). ∎

**Definition 2.9** (cf. [Pin13, Sect. 3])**.** A generalized Drinfeld $A$-module $(E, \varphi)$ over $S$ is called *weakly separating*, if for any Drinfeld $A$-module $(E', \varphi')$ over any $A$-field $L$, at most finitely many fibers of $(E, \varphi)$ over $L$-valued points of $S$ are isomorphic to $(E', \varphi')$.

Note that our "test" objects $(E', \varphi')$ can be in finite characteristic, in contrast to [Pin13].

**Proposition 2.10.** *Let $(E,\varphi)$ be a weakly separating generalized Drinfeld $A$-module over an $A$-scheme $S$ of finite type. Then for any positive integer $r$, there is a unique closed subscheme $S_{\le r}$ of $S$ such that any morphism $f : T \to S$ with the property that $f^*(E,\varphi)$ is of rank $\le r$ over $T$ factors through the inclusion $S_{\le r} \to S$. Moreover, $S_{\le r}$ has relative dimension $\le r-1$ over* $\operatorname{Spec} A$.

PROOF. The proof is the same as [Pin13, Prop. 3.10] and is included merely for the reader's convenience. The first statement is local on $S$. Thus, we may assume that $E = \mathbb{G}_{a,S}$ and $S = \operatorname{Spec} R$ is affine. Suppose $A$ is generated by $a_1, \dots, a_s$ as an $\mathbb{F}_q$-algebra. Let $S_{\le r}$ be the closed subscheme defined by the ideal generated by the elements $\varphi_{a_j,i}$ for $j = 1, \dots, s$ and $i > r\deg(a_j)$. Then it is easy to verify that $S_{\le r}$ satisfies the universal property in the proposition. The second statement is local for $\operatorname{Spec} A$. Thus, we may further assume that $S$ is an $A[\mathfrak{n}^{-1}]$-scheme for a nonzero proper ideal $\mathfrak{n} \subset A$. For any integer $1 \le r' \le r$, let $S_{r'} := S_{\le r'} - S_{\le r'-1}$ and $S_1 := S_{\le 1}$. It suffices to show that each $S_{r'}$ has relative dimension $\le r'-1$.

Adding level-$\mathfrak{n}$ structures to $(E,\varphi)$ over $S_{r'}$, one obtains a finite étale cover $\widetilde{S}_{r'}$ of $S_{r'}$ and a morphism $f : \widetilde{S}_{r'} \to \mathbf{M}^{r'}(\mathfrak{n})$, by the universal property of fine moduli schemes. As $\widetilde{S}_{r'}$ is finite over $S_{r'}$, it suffices to show that $\widetilde{S}_{r'}$ has relative dimension $\le r'-1$, which is $\le r-1$. By the property that $(E,\varphi)$ is weakly separating, the morphism $f$ is quasi-finite. Therefore, $\widetilde{S}_{r'}$ has relative dimension $\le r'-1$ over $\operatorname{Spec} A$ and the proposition is proved. ∎

Note that in general the locally closed stratum $S_r := S_{\le r} - S_{\le r-1}$ may not be dense in $S_{\le r}$ even if $S_r$ is non-empty. For example suppose we have a family $S$ with both $S_r$ and $S_{\le r-1}$ non-empty. Define a new family $T := S_{\le r-1} \coprod S_{\le r}$ as the topologically disjoint union of $S_{\le r-1}$ and $S_{\le r}$. Then $T_r$ is not dense in $T$.

We may view generalized Drinfeld modules of higher rank as the function field analogue of semi-abelian varieties. The following is the analogous result for the semistable reduction theorem for abelian varieties due to Grothendieck, Deligne and Mumford [DeMu69].

**Proposition 2.11.** *Let $R$ be a discrete valuation ring with fraction field $L$, $\gamma : A \to R$ a ring homomorphism, and let $(E,\varphi)$ be a Drinfeld $A$-module of rank $r$ over $L$. Then there is a finite tamely ramified extension $L'/L$, a generalized Drinfeld $A$-module of rank $\le r$ over $R'$, and an isomorphism $\alpha : (E,\varphi) \otimes_L L' \simeq (E',\varphi') \otimes_{R'} L'$, where $R'$ is the integral closure of $R$ in $L'$.*

PROOF. This is proved by Drinfeld [Dri76, Prop. 7.1] when $R$ is complete in a terse style. We provide more details for the reader's convenience. Let $\pi$ be a uniformizer of $R$ and let $v$ be the valuation on $L$ with $v(\pi) = 1$. We first prove the case where $A = \mathbb{F}_q[t]$. Suppose $\varphi_t = a_0 + a_1\tau + \cdots + a_r\tau^r$, $a_i \in L$ and $a_r \ne 0$. Over a field extension $L'/L$, $\varphi_t$ is isomorphic to $\varphi_t^c = a'_0 + \cdots + a'_r\tau^r$ with $a'_i = c^{q^i-1}a_i$ $(0 \le i \le r)$ for some $0 \ne c \in L'$. The isomorphism is given by $c \circ \varphi_t^c = \varphi_t \circ c$. Let

$$\nu := \min_{1 \le i \le r} \left\{ \frac{v(a_i)}{(q^i-1)} \right\}.$$

Then there is an integer $0 < i_0 \le r$ such that $v(a_{i_0}) = (q^{i_0}-1)\nu$ and $v(a_i) \ge (q^i-1)\nu$ for all $0 \le i \le r$.

Now take $L' = L(\pi')$, $\pi' = \pi^{1/(q^{i_0}-1)}$ and $c := \pi'^{-v(a_{i_0})}$. Clearly, $v(c) = -v(a_{i_0})/(q^{i_0}-1) = -\nu$. Thus, one has

$$v(a_i c^{q^i-1}) = v(a_i) - (q^i-1)\nu \ge 0, \quad 0 \le i \le r, \quad \text{and} \quad v(a_{i_0}c^{q^{i_0}-1}) = 0.$$

The morphism $\varphi'_t := \varphi_t^c$ defines a generalized Drinfeld $A$-module $(E',\varphi')$ of rank $\le r$ over $R'$, which has the desired property.

Now $F$ is arbitrary. Choose $t \in A - \mathbb{F}_q$ and put $A_0 := \mathbb{F}_q[t]$ with fraction field $F_0$. Then the restriction of $\varphi$ to $A_0$ gives rise to a Drinfeld $A_0$-module $(E_0,\varphi_0)$ over $L$ of rank $r_0 = nr$ , where $n = [F : F_0]$. We have shown that there is a tamely ramified extension $L'/L$, a generalized Drinfeld $A_0$-module $(E'_0,\varphi'_0)$ of rank $\le r_0$ over $R'$, and an isomorphism $\alpha : (E,\varphi) \otimes_L L' \simeq (E',\varphi')$

of Drinfeld $A_0$-modules. Since $A$ commutes with $A_0$, each element $\varphi_a$ for $a \in A$, can be viewed as an element $\varphi'_a$ in $\mathrm{Hom}((E'_0, \varphi'_0)|_{L'}, (E'_0, \varphi'_0)|_{L'})$. By [Pin13, Prop. 3.7], this homomorphism extends uniquely to a homomorphism $\varphi'_a$ over $R'$, i.e. $\varphi'_a \in R'\{\tau\}$. Thus, we have a generalized Drinfeld $A$-module $(E', \varphi')$ of rank $\le r$ over $R'$ satisfying the desired property. ∎

## 3. The arithmetic Satake compactification

We keep the notation in the previous section. In this section we construct the arithmetic Satake compactification of the Drinfeld moduli scheme $\mathbf{M}^r(\mathfrak{n})$ over $A[\mathfrak{n}^{-1}]$. The Satake compactification of the generic fiber $M^r(\mathfrak{n}) = \mathbf{M}^r(\mathfrak{n}) \otimes_{A[\mathfrak{n}^{-1}]} F$ has been constructed by Kapranov [Kap88] for $F = \mathbb{F}_q(t)$ and $A = \mathbb{F}_q[t]$ using the analytic construction. Pink [Pin13] gave a different construction for the Satake compactification of the generic fiber $M^r_K := \mathbf{M}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} F$ for any global function field $F$ and any fine subgroup $K \subset \mathrm{GL}_r(\widehat{A})$ using generalized Drinfeld modules. The arithmetic construction for $\mathbf{M}^r_K$ over $A[\mathfrak{n}_K^{-1}]$ presented here follows directly along the line of Pink's construction.

**Definition 3.1.** For any fine open compact subgroup $K$, a dominant open immersion $\mathbf{M}^r_K \hookrightarrow \overline{\mathbf{M}}^r_K$ over $A[\mathfrak{n}^{-1}]$, where $\mathfrak{n} = \mathfrak{n}_K$, with the properties

(a) $\overline{\mathbf{M}}^r_K$ is a normal integral scheme which is proper flat over $\mathrm{Spec}\, A[\mathfrak{n}^{-1}]$, and
(b) the universal family over $\mathbf{M}^r_K$ extends to a weakly separating generalized Drinfeld $A$-module $(\overline{E}, \overline{\varphi})$ over $\overline{\mathbf{M}}^r_K$.

is called an (*arithmetic*) *Satake-Pink*, or *Satake compactification* of $\mathbf{M}^r_K$. By abuse of terminology we call $(\overline{E}, \overline{\varphi})$ the universal family on $\overline{\mathbf{M}}^r_K$.

As far as we know, it was expected that the universal Drinfeld module extends to a generalized Drinfeld module over a certain compactification of $\mathbf{M}^r_K$. For rank $r = 2$, where $\mathbf{M}^r_K$ is a relative curve, this was proved by Drinfeld [Dri76, Section 9]. However, for arbitrary ranks the precise description (as Definition 3.1) first appeared in the work of Pink [Pin13]. As shown in loc. cit., Pink's formulation and approach give a rather simple way for compactifying Drinfeld modular varieties. One advantage is that one does not need to construct the boundary components and analyze how to glue them, which was done for $A = \mathbb{F}_q[t]$ by Kapranov [Kap88] using the analytic construction. In [Pin13, Remark 4.9], Pink suggested to compare Kapranov's compactification and the Satake-Pink compactification. For example, whether or not the universal family over the generic fiber $M^r(\mathfrak{n})$ extends to a weakly separating generalized Drinfeld module over Kapranov's compactification.

**Theorem 3.2.** *(1) For every fine open compact subgroup $K$, the moduli scheme $\mathbf{M}^r_K$ possesses a projective arithmetic Satake compactification $\overline{\mathbf{M}}^r_K$. The Satake compactification and its universal family are unique up to unique isomorphism. The dual $\omega_K := \mathrm{Lie}(\overline{E}_K)^\vee$ of the relative Lie algebra of the universal family $\overline{E}_K$ over $\overline{\mathbf{M}}^r_K$ is ample.*

*(2) If $\widetilde{K} \subset K$ are fine open compact subgroups, then "forgetting the level" induces a finite surjective and open morphism*

$$\pi_{\widetilde{K},K} \colon \overline{\mathbf{M}}^r_{\widetilde{K}} \longrightarrow \overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}]$$

*over $\mathrm{Spec}\, A[\mathfrak{n}_{\widetilde{K}}^{-1}]$ which satisfies $\pi^*_{\widetilde{K},K}(\overline{E}_K, \overline{\varphi}_K) = (\overline{E}_{\widetilde{K}}, \overline{\varphi}_{\widetilde{K}})$ and $\pi^*_{\widetilde{K},K}(\omega_K) = \omega_{\widetilde{K}}$. Moreover, if $\widetilde{K} \lhd K$ is normal, then $\pi_{\widetilde{K},K}$ identifies $\overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}]$ with the quotient $(\overline{\mathbf{M}}^r_{\widetilde{K}})^{(K/\widetilde{K})}$ of $\overline{\mathbf{M}}^r_{\widetilde{K}}$ by the action of the finite group $K/\widetilde{K}$.*

We begin the proof of the theorem with the following lemma.

**Lemma 3.3.** *If a Satake compactification $\overline{\mathbf{M}}^r_K$ and the universal family $(\overline{E}, \overline{\varphi})$ exist, then they are unique up to unique isomorphism.*

PROOF. The proof is the same as that of [Pin13, Lemma 4.3]. We include it for the sake of completeness. Abbreviate $\overline{\mathbf{M}} := \overline{\mathbf{M}}^r_K$, and let $\overline{\mathbf{M}}'$ be another Satake compactification of $\mathbf{M} := \mathbf{M}^r_K$ with universal family $(\overline{E}', \overline{\varphi}')$. Let $\widetilde{\mathbf{M}}$ be the normalization of the Zariski closure $\mathbf{M}^{\mathrm{zar}}$ of the diagonal embedding $\mathbf{M} \hookrightarrow \overline{\mathbf{M}} \times_A \overline{\mathbf{M}}'$. Then we have two projections

$$\overline{\mathbf{M}} \xleftarrow{\ \pi\ } \widetilde{\mathbf{M}} \xrightarrow{\ \pi'\ } \overline{\mathbf{M}}'$$

which are proper and are the identity when restricted to $\mathbf{M}$. The morphism $(\pi, \pi') : \widetilde{\mathbf{M}} \to \overline{\mathbf{M}} \times_A \overline{\mathbf{M}}'$ is finite, because $\mathbf{M}^{\mathrm{zar}}$ is excellent, see [EGA, IV$_2$, Scholie 7.8.3 (ii), (v)]. One obtains two generalized Drinfeld modules $\pi^*(\overline{E}, \overline{\varphi})$ and $\pi'^*(\overline{E}', \overline{\varphi}')$ over $\widetilde{\mathbf{M}}$. By [Pin13, Prop. 3.7 and 3.8], the identity on the universal family over $\mathbf{M} \subset \widetilde{\mathbf{M}}$ extends to an isomorphism $\pi^*(\overline{E}, \overline{\varphi}) \xrightarrow{\sim} \pi'^*(\overline{E}', \overline{\varphi}')$ over $\widetilde{\mathbf{M}}$. For any geometric point $x \in \overline{\mathbf{M}}(\overline{L})$ where $\overline{L}$ is an algebraically closed field, we restrict the map $\pi' : \widetilde{\mathbf{M}}(\overline{L}) \to \overline{\mathbf{M}}'(\overline{L})$ to the subset $\pi^{-1}(x) \subset \widetilde{\mathbf{M}}(\overline{L})$. One obtains a finite-to-one map

$$\pi' : \pi^{-1}(x) \xrightarrow{(\pi, \pi')} \{x\} \times \pi'(\pi^{-1}(x)) \underset{\sim}{\xrightarrow{\mathrm{pr}_2}} \pi'(\pi^{-1}(x)). \tag{3.1}$$

Since the fibers $(\overline{E}', \overline{\varphi}')$ over points $y \in \pi'(\pi^{-1}(x))$ are isomorphic to $(\overline{E}_x, \overline{\varphi}_x)$ and $(\overline{E}', \overline{\varphi}')$ is weakly separating, the set $\pi'(\pi^{-1}(x))$ is finite. By (3.1), $\pi^{-1}(x)$ is finite and hence the morphism $\pi$ is finite. As $\pi$ is birational and both $\widetilde{\mathbf{M}}$ and $\overline{\mathbf{M}}$ are normal, the morphism $\pi$ is an isomorphism by Zariski's Main Theorem. Similarly, one proves that $\pi'$ is also an isomorphism. Thus one obtains unique isomorphisms $\xi := \pi' \circ \pi^{-1} : \overline{\mathbf{M}} \xrightarrow{\sim} \overline{\mathbf{M}}'$ and $(\overline{E}, \overline{\varphi}) \xrightarrow{\sim} \xi^*(\overline{E}', \overline{\varphi}')$. ∎

**Proposition 3.4.** *Let $f : X \to S = \operatorname{Spec} R$ be a quasi-projective scheme over a Noetherian ring $R$, together with a right action by a finite group $G$ over $R$. Then*

*(1) The quotient scheme $Y = X/G$ exists and the canonical morphism $\pi : X \to Y$ is finite surjective and open.*

*(2) The quotient scheme $Y$ is quasi-projective over $R$. It is projective over $R$ if and only if $X$ is so.*

*(3) If the group $G$ acts freely in the sense that the morphism $X \times G \to X \times_S X, (x, g) \mapsto (x, g.x)$ is a monomorphism, then $\pi : X \to Y$ is faithfully flat.*

*(4) Assume that $Y$ is regular. Then $X$ is Cohen-Macaulay if and only if $\pi : X \to Y$ is flat.*

PROOF. (1) and (3) By [SGA 3, Exp. V, Thm. 7.1] the quotient scheme exists, is of finite type over $R$ and the morphism $\pi\colon X \to Y$ is surjective, open, proper and quasi-finite, hence finite. Moreover, $\pi$ is flat if $G$ acts freely.

(2) Since $f$ is quasi-projective, we can choose an $f$-very ample invertible sheaf $\mathcal{L}$ on $X$. Replacing $\mathcal{L}$ by $\otimes_{g \in G} g^*\mathcal{L}$, one may assume that $\mathcal{L}$ is $G$-equivariant and still $f$-ample by [EGA, II, Corollaire 4.6.10]. Some power $\mathcal{L}^{\otimes n}$ with $n > 0$ will be $f$-very ample again by [EGA, II, Proposition 4.6.11]. Then $H^0(X, \mathcal{L}^{\otimes n})$ gives rise to a $G$-equivariant embedding $X \hookrightarrow \mathbb{P}^{N-1}$ where $G$ acts on $\mathbb{P}^{N-1}$ linearly. The closure $\overline{X}$ of $X$ is equal to $\operatorname{Proj} T$ of a graded $T = \oplus_{n \geq 0} T_n$ that is generated by $T_1$ over $T_0$. $T_0$ is an $R$-algebra which is finite as an $R$-module, and therefore it is Noetherian. By the construction of $\overline{X}/G$ and $X/G$, we know that $\overline{Y} := \overline{X}/G$ is isomorphic to $\operatorname{Proj} T^G$ and $Y$ is an open subscheme of $\overline{Y}$. So it suffices to show that $\overline{Y}$ is projective. Since $T$ is of finite type over a Noetherian ring $T_0$ and $G$ is finite, the invariant subring $T^G$ is again of finite type over $T_0$; see [Has04, Theorem 1.2]. Suppose that $T^G$ is generated by elements of degree $\geq d$ for some $d$. Then the graded $T' := \oplus_{n \geq 0} T^G_{d!n}$ is generated by elements in $T_{d!n}$, which are of degree 1 in the new graded ring. By [EGA] II, Prop. 2.4.7, the induced morphism $\operatorname{Proj} T' \to \operatorname{Proj} T^G$ is an isomorphism. This shows that $\overline{Y}$ is projective over $T_0$ (and hence over $R$).

(4) This follows from [Eis95, Thm. 18.16 and Cor. 18.17]. ∎

*Remark* 3.5. (1) The finite surjective morphism $X \to Y$ in Proposition 3.4 may not be flat in general. Singularities of $X$ and $Y$ play a crucial role. Here is a counter-example where $X$ is regular and $Y$ is normal.

Take $X = \operatorname{Spec} B$ with $B = \mathbb{C}[x_1, x_2]$ and let the finite group $G = \{\pm 1\}$ act on $B$ by $-1 : (x_1, x_2) \mapsto (-x_1, -x_2)$. Then we have $C := B^G = \mathbb{C}[x_1^2, x_1x_2, x_2^2]$ and $Y = \operatorname{Spec} C$. We show that $B$ is not $C$-flat. To see this, consider the maximal ideal $\mathfrak{m} = (x_1^2, x_1x_2, x_2^2)$ of $C$ and put $L := \operatorname{Frac}(C)$. One computes $\dim_{\mathbb{C}} B \otimes_C C/\mathfrak{m} = \dim_{\mathbb{C}} B/\mathfrak{m}B = 3$ and $\dim_L B \otimes_C L = 2$. Thus, $B$ can not be $C$-flat.

(2) We thank David Rydh for explaining the proof of Prop. 3.4(2) to us and pointing out the reference [Knu71, IV Prop. 1.5, p. 180]. Rydh proves a more general result for algebraic spaces with finite flat groupoids; see [Ryd13, Prop. 4.7 (B')] for more details. Prop. 3.4(2) also follows from [AlKl80, Theorem 2.9], because a strongly quasi-projective scheme over an affine base is quasi-projective.

**Lemma 3.6.** *Suppose that $K$ is fine and $K(\mathfrak{n}) \subset K$. Suppose that $\mathbf{M}^r_{K(\mathfrak{n})}$ has a projective Satake compactification $\overline{\mathbf{M}}^r_{K(\mathfrak{n})}$ with universal family $(\overline{E}, \overline{\varphi})$ for which the dual $\omega_{K(\mathfrak{n})}$ of the relative Lie algebra of $\overline{E}$ is ample. Then*

*(1) The action of $K/K(\mathfrak{n})$ on $\mathbf{M}^r_{K(\mathfrak{n})}$ and $(E, \varphi)$ as in (2.6) extends uniquely to an action on $\overline{\mathbf{M}}^r_{K(\mathfrak{n})}$ and $(\overline{E}, \overline{\varphi})$, respectively.*

*(2) The quotient $\overline{\mathbf{M}}^r_{K(\mathfrak{n})}/(K/K(\mathfrak{n}))$ furnishes the projective Satake compactification $\overline{\mathbf{M}}^r_K[\mathfrak{n}^{-1}]$ of $\mathbf{M}^r_K[\mathfrak{n}^{-1}]$ with universal family $(\overline{E}_K, \overline{\varphi}_K)$, where $\overline{E}_K := \overline{E}/(K/K(\mathfrak{n}))$ is the quotient and $\overline{\varphi}_K$ is the Drinfeld $A$-module structure on $\overline{E}_K$ descended from $\overline{\varphi}$.*

*(3) The dual $\omega_K := \operatorname{Lie}(\overline{E}_K)^\vee$ of the relative Lie algebra of $\overline{E}_K$ over $\overline{\mathbf{M}}^r_K[\mathfrak{n}^{-1}]$ is ample.*

Proof. (1) Write $\mathbf{M} = \mathbf{M}^r_{K(\mathfrak{n})}$ and $\overline{\mathbf{M}} = \overline{\mathbf{M}}^r_{K(\mathfrak{n})}$. For any $g \in K$, the action of $g$ gives the following commutative morphism:

$$\begin{array}{ccccccc} (E,\varphi) & \longrightarrow & \mathbf{M} & \qquad & (\overline{E},\overline{\varphi}) & \longrightarrow & \overline{\mathbf{M}} \\ {\scriptstyle I_g}\big\downarrow & & {\scriptstyle J_g}\big\downarrow & & {\scriptstyle ?\,\overline{I}_g}\big\downarrow & & {\scriptstyle ?\,\overline{J}_g}\big\downarrow \\ (E,\varphi) & \longrightarrow & \mathbf{M} & & (\overline{E},\overline{\varphi}) & \longrightarrow & \overline{\mathbf{M}}. \end{array} \tag{3.2}$$

Let $\widetilde{\mathbf{M}}$ be the normalization of the Zariski closure of the graph of $J_g$ in $\overline{\mathbf{M}} \times_{A[\mathfrak{n}^{-1}]} \overline{\mathbf{M}}$. It is equipped with two projections $\pi : \widetilde{\mathbf{M}} \to \overline{\mathbf{M}}$ and $\pi' : \widetilde{\mathbf{M}} \to \overline{\mathbf{M}}$. The argument of Lemma 3.3 shows that $\pi$ and $\pi'$ are isomorphisms and the isomorphism $I_g$ extends to an isomorphism $\pi^*(\overline{E},\overline{\varphi}) \xrightarrow{\sim} \pi'^*(\overline{E},\overline{\varphi})$. Since $\pi$ and $\pi'$ are isomorphisms, the morphisms $J_g$ and $I_g$ extend to morphisms $\overline{J}_g$ and $\overline{I}_g$ on $\overline{\mathbf{M}}$ and $(\overline{E},\overline{\varphi})$, respectively.

(2) By Proposition 3.4, the quotients $\overline{\mathbf{M}}/(K/K(\mathfrak{n}))$ and $\overline{E}/(K/K(\mathfrak{n}))$ exist as schemes and the quotient $\mathbf{M}^r_K[\mathfrak{n}^{-1}]$ of $\mathbf{M}$ is open in $\overline{\mathbf{M}}/(K/K(\mathfrak{n}))$. Moreover, $\overline{\mathbf{M}}/(K/K(\mathfrak{n}))$ is projective over $A[\mathfrak{n}^{-1}]$. We next show that $\overline{\mathbf{M}}/(K/K(\mathfrak{n}))$ is a normal integral scheme proper flat over $A[\mathfrak{n}^{-1}]$. Note that if $R$ is a normal domain with quotient field $Q$ with an action by a finite group $G$, then $R^G$ is again a normal domain. To see this, suppose that $x \in Q^G$ is integral over $R^G$. Then $x \in R \cap Q^G = R^G$. The normality of $\overline{\mathbf{M}}/(K/K(\mathfrak{n}))$ follows. To show the flatness of $\overline{\mathbf{M}}/(K/K(\mathfrak{n}))$, one must show that the generic point of $\overline{\mathbf{M}}/(K/K(\mathfrak{n}))$ dominates $\operatorname{Spec} A[\mathfrak{n}^{-1}]$; see [Har77, Proposition III.9.7]. This follows from that the generic point of $\overline{\mathbf{M}} \otimes_{A[\mathfrak{n}^{-1}]} F$ maps to that of $\overline{\mathbf{M}}/(K/K(\mathfrak{n}))$ and it dominates $\operatorname{Spec} A[\mathfrak{n}^{-1}]$.

As $K$ is fine, it is proved in [Pin13, Lemma 4.4] that $\overline{E}_K := \overline{E}/(K/K(\mathfrak{n}))$ is again a line bundle over $\overline{\mathbf{M}}$. Since $\overline{\varphi}$ is invariant under the action of $K$, it descends to a Drinfeld module structure $\overline{\varphi}_K$ on $\overline{E}_K$. Since $(\overline{E},\overline{\varphi})$ is weakly separating, also $(\overline{E}_K, \overline{\varphi}_K)$ trivially is weakly separating.

(3) Observe that $\omega_{K(\mathfrak{n})} = g^*\omega_K$ under the finite surjective quotient morphism $g : X' := \overline{\mathbf{M}}^r_{K(\mathfrak{n})} \longrightarrow X := \overline{\mathbf{M}}^r_{K(\mathfrak{n})}/(K/K(\mathfrak{n}))$, because $g^*\overline{E}_K = \overline{E}$. Since the base $Y := \operatorname{Spec} A[\mathfrak{n}^{-1}]$ is

affine, $\omega_K$ is ample on $X$, if and only if it is ample relative to $Y$, by [EGA] II, Cor. 4.6.6. Since $\omega_{K(\mathfrak{n})}$ is ample on $\overline{\mathbf{M}}^r_{K(\mathfrak{n})}$ we will now apply [EGA] II, Cor. 6.6.3 to conclude that $\omega_K$ is ample on $X$. We may apply loc. cit. because condition (II bis) in [EGA] II, Prop. 6.6.1 is satisfied for $(X, \mathcal{O}_X)$ and $g_*\mathcal{O}_{X'}$. Namely, let $\eta \in X$ and $qf(X) := \mathcal{O}_{X,\eta}$ be the generic point and the function field of $X$ and similarly for $X'$. Then condition (II bis), stated on page 126 of loc. cit., requires that for any affine open subscheme $U \subset X$ and any section $f \in (g_*\mathcal{O}_{X'})(U)$ the characteristic polynomial $T^n - \sigma_1(f)T^{n-1} + \ldots + (-1)^n\sigma_n(f) \in qf(X)[T]$ of the multiplication by $f$ on the $qf(X)$-vector space $g_*\mathcal{O}_{X'} \otimes_{\mathcal{O}_X} qf(X) = qf(X')$ has all its coefficients $\sigma_i(f)$ in $\mathcal{O}_X(U)$. Since $X$ is normal, $\mathcal{O}_X(U)$ equals the intersection in $qf(X)$ of the local rings $\mathcal{O}_{X,x}$ for all points $x \in X$ of codimension one. All these local rings are discrete valuation rings. Since $g_*\mathcal{O}_{X'} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}$ equals the normalization of $\mathcal{O}_{X,x}$ in $qf(X')$, it is a free $\mathcal{O}_{X,x}$-module and this implies that all the coefficients $\sigma_i(f)$ lie in $\mathcal{O}_{X,x}$. So condition (II bis) is indeed satisfied. ∎

Let $F'$ be a finite field extension of $F$ with only one place $\infty'$ over $\infty$, and let $A'$ be the integral closure of $A$ in $F'$. Then $A'$ consists of all elements in $F'$ regular away from $\infty'$, and $A'$ is a projective finite $A$-module of rank $[F' : F]$. Let $r$ and $r'$ be positive integers with $r = r'[F' : F]$. Let $\mathfrak{n} \subset A$ be a non-zero proper ideal, and put $\mathfrak{n}' := \mathfrak{n}A'$. Note that $\mathfrak{n}'^{-1}/A' = A' \otimes_A (\mathfrak{n}^{-1}/A)$ which is non-canonically isomorphic to $(\mathfrak{n}^{-1}/A)^{\oplus[F':F]}$ as a module over the principal ring $A/\mathfrak{n}$. Thus, we can fix an isomorphism $(\mathfrak{n}'^{-1}/A')^{r'} \simeq (\mathfrak{n}^{-1}/A)^r$ of $A$-modules. Then we have a natural finite morphism

$$I_b : \mathbf{M}^{r'}_{A'}(\mathfrak{n}') \to \mathbf{M}^r_A(\mathfrak{n}) \tag{3.3}$$

sending each $(E', \varphi', \lambda')$ to $(E', \varphi'|_A, \lambda')$, which fits into the commutative diagram

$$\begin{array}{ccc} \mathbf{M}^{r'}_{A'}(\mathfrak{n}') & \xrightarrow{I_b} & \mathbf{M}^r_A(\mathfrak{n}) \\ \downarrow & & \downarrow \\ \operatorname{Spec} A'[\mathfrak{n}'^{-1}] & \longrightarrow & \operatorname{Spec} A[\mathfrak{n}^{-1}]. \end{array} \tag{3.4}$$

**Lemma 3.7.** *Suppose that the Satake compactification $\overline{\mathbf{M}}$ of $\mathbf{M} := \mathbf{M}^r_A(\mathfrak{n})$ over $A[\mathfrak{n}^{-1}]$ exists. Then the Satake compactification $\overline{\mathbf{M}'}$ of $\mathbf{M}' := \mathbf{M}^{r'}_{A'}(\mathfrak{n}')$ over $A[\mathfrak{n}^{-1}]$ exists and the morphism $I_b$ extends uniquely to a finite morphism $\overline{I}_b : \overline{\mathbf{M}'} \to \overline{\mathbf{M}}$. Moreover, if the dual $\omega_A$ of the Lie algebra of the universal Drinfeld module over $\overline{\mathbf{M}}$ is ample, then also the dual $\omega_{A'}$ of the Lie algebra of the universal Drinfeld module over $\overline{\mathbf{M}'}$ is ample.*

Proof. The statement of Lemma 3.7 for the generic fiber is proved in [Pin13, Lemma 4.5], and the proof also works in the present situation. We sketch the construction for the reader's convenience. Let $I_b(\mathbf{M}')^{\mathrm{zar}}$ be the Zariski closure of $I_b(\mathbf{M}')$ in $\overline{\mathbf{M}}$, and let $\overline{\mathbf{M}'}$ be the normalization of $I_b(\mathbf{M}')^{\mathrm{zar}}$ in the function field of $\mathbf{M}'$. $\overline{\mathbf{M}'}$ is a normal integral scheme flat over $A'[\mathfrak{n}'^{-1}]$, using [Har77, Proposition III.9.7] again. Since $I_b(\mathbf{M}')^{\mathrm{zar}}$ is excellent we have a natural finite morphism $\overline{I}_b : \overline{\mathbf{M}'} \to \overline{\mathbf{M}}$ extending $I_b$ by [EGA, IV$_2$, Scholie 7.8.3 (ii), (v)]. The pull-back of the universal family on $\overline{\mathbf{M}}$ gives a generalized Drinfeld $A$-module $(\overline{E'}, \widetilde{\varphi})$ over $\overline{\mathbf{M}'}$ where the $A$-action $\widetilde{\varphi}$ extends to the $A'$-action $\varphi'$ on the open subscheme $\mathbf{M}'$. We can view $\varphi'_{a'}$, for $a' \in A'$, as an endomorphism of $(\overline{E'}, \widetilde{\varphi})$ over $\mathbf{M}'$, which extends to an endomorphism $\overline{\varphi}'_{a'}$ over $\overline{\mathbf{M}'}$ by [Pin13, Prop. 3.7]. Since the morphism $\overline{\mathbf{M}'} \to \overline{\mathbf{M}}$ is finite, it follows that $(\overline{E'}, \overline{\varphi}')$ is a weakly separating family on $\overline{\mathbf{M}'}$. That $\omega_{A'}$ is ample on $\overline{\mathbf{M}'}$ follows from the equality $\omega_{A'} = (\overline{I}_b)^*\omega_A$ by [EGA] II, Prop. 5.1.12 and Cor. 4.6.6. ∎

**Lemma 3.8.** *If $A = \mathbb{F}_q[t]$, then a projective Satake compactification $\overline{\mathbf{M}}^r_{K(t)}$ for $\mathbf{M}^r_{K(t)}$ exists for any $r \geq 1$ and the dual $\omega$ of the Lie algebra of the universal Drinfeld module over $\overline{\mathbf{M}}^r_{K(t)}$ is ample.*

Proof. See Proposition 3.11 below for a more detailed statement and the proof. ∎

**Proof of Theorem 3.2.** (1) Choose $\mathfrak{n} := \mathfrak{n}_K^m = tA$ for some $m \in \mathbb{N}$. Put $A_0 := \mathbb{F}_q[t] \subset A$ and $F_0 := \mathrm{Frac}(A_0)$. The moduli scheme $\mathbf{M}^r_{K(\mathfrak{n})}$ is defined over $A[\mathfrak{n}_K^{-1}]$ and one has a morphism $I_b : \mathbf{M}^r_{K(\mathfrak{n})} \to \mathbf{M}^{r[F:F_0]}_{A_0,K(t)}$ as in (3.3). By Lemma 3.8, the moduli scheme $\mathbf{M}^{r[F:F_0]}_{A_0,K(t)}$ admits a projective Satake compactification over $A_0[1/t]$. It follows from Lemma 3.7 that the moduli scheme $\mathbf{M}^r_{K(\mathfrak{n})}$ admits a projective Satake compactification over $A[\mathfrak{n}_K^{-1}]$. As $K(\mathfrak{n}) \subset K$, it follows from Lemma 3.6 that the moduli scheme $\mathbf{M}^r_K$ admits a projective Satake compactification over $A[\mathfrak{n}_K^{-1}]$. This proves part (1) of Theorem 3.2.

Part (2) follows from Lemma 3.6. Namely, let $\mathfrak{n} := \mathfrak{n}_{\widetilde{K}}$, so that $K(\mathfrak{n}) \subset \widetilde{K}$, and let $\overline{\mathbf{M}}^r_{K(\mathfrak{n})}$ be the Satake compactification of $\mathbf{M}^r_{K(\mathfrak{n})}$. By Lemma 3.6 the quotients $\overline{\mathbf{M}}^r_{\widetilde{K}} := \overline{\mathbf{M}}^r_{K(\mathfrak{n})}/\big(\widetilde{K}/K(\mathfrak{n})\big)$ and $\overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}] := \overline{\mathbf{M}}^r_{K(\mathfrak{n})}/\big(K/K(\mathfrak{n})\big)$ are the Satake compactifications of $\mathbf{M}^r_{\widetilde{K}}$ and $\mathbf{M}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}]$, respectively. The generalized Drinfeld $A$-module $(\overline{E}_{K(\mathfrak{n})}, \overline{\varphi}_{K(\mathfrak{n})})$ descends to generalized Drinfeld $A$-modules $(\overline{E}_{\widetilde{K}}, \overline{\varphi}_{\widetilde{K}})$ and $(\overline{E}_K, \overline{\varphi}_K)$ on $\overline{\mathbf{M}}^r_{\widetilde{K}}$ and $\overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}]$, respectively. The forgetful quotient morphisms $\pi_{K(\mathfrak{n}),\widetilde{K}}$ and $\pi_{K(\mathfrak{n}),K}$ are finite surjective and open by Proposition 3.4(1). Therefore, also $\pi_{\widetilde{K},K}$ is finite surjective an open and satisfies $\pi^*_{\widetilde{K},K}(\overline{E}_K, \overline{\varphi}_K) = (\overline{E}_{\widetilde{K}}, \overline{\varphi}_{\widetilde{K}})$ and $\pi^*_{\widetilde{K},K}(\omega_K) = \omega_{\widetilde{K}}$. ∎

**Proposition 3.9.** *At every place $v \nmid \mathfrak{n}_K$ the fiber $\overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} \mathbb{F}_v$ is geometrically reduced.*

Proof. Write $\mathscr{M}^r_K = \mathbf{M}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} \mathbb{F}_v$ and $\overline{\mathscr{M}}^r_K = \overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} \mathbb{F}_v$. Since the residue field $\mathbb{F}_v$ is perfect, it suffices to show that $\overline{\mathscr{M}}^r_K$ is reduced. Using Serre's criterion [EGA, IV$_2$, Proposition 5.8.5] we prove this by showing that every point $x \in \overline{\mathscr{M}}^r_K$ has an open affine neighborhood $\operatorname{Spec} B \subset \overline{\mathscr{M}}^r_K$ for which the ring $B$ satisfies conditions $(R_0)$ and $(S_1)$:

$(R_0)$ For every minimal prime $\mathfrak{p} \subset B$, the local ring $B_\mathfrak{p}$ is regular.

$(S_1)$ Every prime ideal $\mathfrak{p} \subset B$ with $\operatorname{depth} B_\mathfrak{p} = 0$ has codimension 0.

Let $\operatorname{Spec} \mathbf{B}$ be an open affine neighborhood of $x$ in $\overline{\mathbf{M}}^r_K$. Since $\mathbf{M}^r_K$ is dense in $\overline{\mathbf{M}}^r_K$ the dimension of the integral domain $\mathbf{B}$ is $r$ by Proposition 2.8(2). Also the dimension of $\mathbf{B}_{A_{(v)}} := \mathbf{B} \otimes_{A[\mathfrak{n}_K^{-1}]} A_{(v)}$ is $r$. Let $a \in A$ be an element with $v(a) = 1$ and let $B := \mathbf{B} \otimes_{A[\mathfrak{n}_K^{-1}]} \mathbb{F}_v = \mathbf{B}_{A_{(v)}}/\gamma(a)\mathbf{B}_{A_{(v)}}$. Then $\operatorname{Spec} B$ is an affine open neighborhood of $x$ in $\overline{\mathscr{M}}^r_K$. Let $\mathfrak{p} \subset B$ be a minimal prime ideal and let $\mathfrak{P} \subset \mathbf{B}$ be the preimage of $\mathfrak{p}$ in $\mathbf{B}$. Then $\mathfrak{P}\mathbf{B}_{A_{(v)}}$ is the preimage of $\mathfrak{p}$ in $\mathbf{B}_{A_{(v)}}$ and is minimal among primes containing $\gamma(a)$. So $\mathbf{B}_\mathfrak{P}$, which equals the localization of $\mathbf{B}_{A_{(v)}}$ at $\mathfrak{P}\mathbf{B}_{A_{(v)}}$, has dimension $\leq 1$ by Krull's principal ideal theorem [Eis95, Theorem 10.2]. On the other hand $\dim B = r - 1$, because $\gamma(a)$ is not a unit and not a zero-divisor in $\mathbf{B}_{A_{(v)}}$ due to the flatness of $\overline{\mathbf{M}}^r_K$ over $A[\mathfrak{n}_K^{-1}]$. Therefore, $\dim \mathbf{B}/\mathfrak{P} = \dim B/\mathfrak{p} \leq \dim B = r - 1$. Since $\mathbf{B}$ is a finitely generated $\mathbb{F}_q$-algebra we have $r = \dim \mathbf{B} = \dim \mathbf{B}_\mathfrak{P} + \dim \mathbf{B}/\mathfrak{P}$ by [Eis95, Corollary 13.4], and hence $\dim B/\mathfrak{p} = r - 1$. Since $\dim(\overline{\mathscr{M}}^r_K - \mathscr{M}^r_K) \leq r - 2$ by Proposition 2.10, the point $\mathfrak{p} \in \operatorname{Spec} B \subset \overline{\mathscr{M}}^r_K$ must lie in $\mathscr{M}^r_K$. And since $\mathscr{M}^r_K$ is smooth over $\mathbb{F}_v$ the local ring $B_\mathfrak{p}$ is regular, proving $(R_0)$.

To prove $(S_1)$ let $\mathfrak{p} \subset B$ be a prime ideal with $\operatorname{depth} B_\mathfrak{p} = 0$ and let $\mathfrak{P} \subset \mathbf{B}$ be the preimage of $\mathfrak{p}$ in $\mathbf{B}$. Since $\gamma(a)$ is a non-zero-divisor in $\mathbf{B}_\mathfrak{P}$ and $B_\mathfrak{p} = \mathbf{B}_\mathfrak{P}/(\gamma(a))$, we have $\operatorname{depth} \mathbf{B}_\mathfrak{P} = 1$ by [EGA, $0_{\mathrm{IV}}$, Proposition 16.4.6]. Since $\mathbf{B}$ is normal we conclude $\dim \mathbf{B}_\mathfrak{P} = 1$, and hence $\dim B_\mathfrak{p} = 0$ as desired. This proves $(S_1)$ and the proposition. ∎

In the remainder of this section we let $A = \mathbb{F}_q[t]$, $F = \mathbb{F}_q(t)$ and $\mathfrak{n} = (t)$. Let $r \geq 1$ be a positive integer, and write $\mathbf{M} = \mathbf{M}^r(t)$. The aim is to construct the Satake compactification of $\mathbf{M}$ over $A[1/t]$ and prove Lemma 3.8. It turns out that methods and proofs in the construction for the Satake compactification $\overline{M}$ of the generic fibre $M := \mathbf{M} \otimes_{A[1/t]} F$ already suffice for our purpose.

Set $V_r := \mathbb{F}_q^r$ and identify it with the $\mathbb{F}_q$-vector space $(t^{-1}A/A)^r$. Put $V_r^0 := V_r - \{0\}$. Let $(E, \varphi, \lambda)$ be a Drinfeld $A$-module of rank $r$ with level-$t$ structure over an $A[1/t]$-scheme $S$. Then the level structure $\lambda$ induces an $\mathbb{F}_q$-linear map $\lambda : V_r \to E(S)$ which is *fiber-wise injective*, i.e. for any point $s \in S$ the induced map $V_r \to E(s)$ is injective. In particular, for any $v \in V_r^0$, the section $\lambda(v)$ is nowhere zero.

**Lemma 3.10.** *For any line bundle $E$ over an $A[1/t]$-scheme $S$ and any fiber-wise injective $\mathbb{F}_q$-linear map $\lambda : V_r \to E(S)$, there exists a unique homomorphism $\varphi : A \to \mathrm{End}(E)$ turning $(E, \varphi, \lambda)$ into a Drinfeld $A$-module of rank $r$ with level-$t$ structure over $S$.*

Proof. The assertion is local on $S$. Thus, we may assume that $S = \operatorname{Spec} R$ is connected and $E = \mathbb{G}_{a,S} = \operatorname{Spec} R[X]$. For any $v \in V_r^0 := V_r - \{0\}$, one has $\lambda(v) \in R^\times$, as it is non-zero everywhere. Put $f(X) := \prod_{v \in V_r} (X - \lambda(v))$. By [Gos96, Cor. 1.2.2], $f \in \mathrm{End}(\mathbb{G}_{a,S})$ is an $\mathbb{F}_q$-linear endomorphism of degree $q^r$ in $X$ over $R$. Thus, $\ker f \subset S$ is a finite constant group over $S$ of order $q^r$, which is the union of the image of the sections $\lambda(v)$. Note that $(-1)^{q^r-1} = +1$ if $q$ is odd, and also if $q$ is even when $-1 = +1$. Therefore,

$$\begin{aligned} \varphi_t := \gamma(t) \prod_{v \in V_r^0} \lambda(v)^{-1} f(X) &= \gamma(t) X \prod_{v \in V_r^0} (1 - \lambda(v)^{-1} X) \\ &= \gamma(t)\tau^0 + \cdots + \gamma(t) \prod_{v} \lambda(v)^{-1} \tau^r, \quad (\tau^i = X^{q^i}) \end{aligned} \tag{3.5}$$

defines a Drinfeld $A$-module of rank $r$ over $S$. Note that $\lambda : V_r \xrightarrow{\sim} (\ker f)(S) = \varphi[t](S) \subset R$ is an $\mathbb{F}_q$-linear isomorphism, which is also $A$-linear as $t$ annihilates both sides. Thus, $\lambda$ is a level-$t$ structure on $(\mathbb{G}_{a,S}, \varphi)$. Note that $\varphi_t$ in (3.5) is the unique polynomial such that the coefficient of $X$ is $\gamma(t)$ and $(\ker \varphi_t)(S) = \lambda(V_r)$. Therefore, the homomorphism $\varphi$ is uniquely determined by $\lambda$. ∎

Write $\mathbb{P}^{r-1} := \mathbb{P}^{r-1}_{\mathbb{F}_q} = \operatorname{Proj} S_r$, where $S_r := \mathbb{F}_q[x_0, \ldots, x_{r-1}]$ is the graded polynomial ring over $\mathbb{F}_q$ with degree one on each $x_i$. We identify $S_r$ with the symmetric algebra $\mathrm{Sym}_{\mathbb{F}_q} V_r$ of $V_r = \mathbb{F}_q^r$ by sending $x_0, \ldots, x_{r-1}$ to the standard basis of $V_r$. As is well known [Har77, Prop. II.7.12], $\mathbb{P}^{r-1}$ represents the functor that associates to any $\mathbb{F}_q$-scheme $T$ the set of isomorphism classes of $(E, e_0, e_1, \ldots, e_{r-1})$ consisting of a line bundle $E$ over $T$ and sections $e_0, \ldots, e_{r-1} \in E(T)$ that generate $E$. Given such a tuple $(E, e_0, \ldots, e_{r-1})$, one associates an $\mathbb{F}_q$-linear map $\lambda : V_r \to E(T)$, sending $x_i$ to $e_i$, which induces a surjective map $\mathcal{O}_T^r \to E$. Clearly, the datum $(e_0, \ldots, e_{r-1})$ is determined by $\lambda$. The universal family on $\mathbb{P}^{r-1}$ is $(\mathcal{O}_{\mathbf{P}^{r-1}}(1), x_0, \ldots, x_{r-1})$, or equivalently by $(\mathcal{O}_{\mathbf{P}^{r-1}}(1), \lambda_{\mathbb{P}^{r-1}})$, where $\lambda_{\mathbb{P}^{r-1}} : V_r \to \mathcal{O}_{\mathbb{P}^{r-1}}(1)(\mathbb{P}^{r-1})$ is the identity map.

Let $\Omega_r$ be the open subscheme of $\mathbb{P}^{r-1}$ obtained by removing all $\mathbb{F}_q$-rational hyperplanes. By definition, $\Omega_r$ is the largest open subset $U$ such that $v$ is nowhere zero on $U$ for any $v \in V_r^0$, or equivalently, the restriction on $U$ of $\lambda_{\mathbb{P}^{r-1}} : V_r \to \mathcal{O}_{\mathbb{P}^{r-1}}(1)(U)$ is fiber-wise injective. Thus, $\Omega_r$ represents the functor $\mathcal{F}$ which associates to each $\mathbb{F}_q$-scheme $T$ the pairs $(E, \lambda)$ in $\mathbb{P}^{r-1}(T)$ with the property that the $\mathbb{F}_q$-linear map $\lambda : V_r \to E(T)$ is fiber-wise injective. On the other hand, by Lemma 3.10, the functor $\mathcal{F}$ restricted to the category of $A[1/t]$-schemes is the same as the representable functor associated to $\mathbf{M}$. Thus, one obtains $\mathbf{M} = \Omega_{r,A[1/t]} := \Omega_r \otimes_{\mathbb{F}_q} A[1/t]$.

Let $E = E_{\Omega_{r,A[1/t]}}$ be the line bundle over $\Omega_{r,A[1/t]}$ corresponding to the invertible sheaf $\mathcal{O}_{\mathbb{P}^{r-1}}(1) \otimes_{\mathbb{F}_q} A[1/t]$ on $\Omega_{r,A[1/t]}$. The map $\lambda_{\mathbb{P}^{r-1}}$ induces a fiber-wise injective $\mathbb{F}_q$-linear map $\lambda : V_r \to E(\Omega_{r,A[1/t]})$. Let $\varphi : A \to \mathrm{End}(E)$ be the homomorphism defined by

$$\varphi_t = \gamma(t) X \prod_{v \in V_r^0} \left(1 - \frac{X}{\lambda(v)}\right) = \gamma(t)\tau^0 + \sum_{i=1}^{r} \varphi_{t,i}\tau^i, \tag{3.6}$$

where $\tau^i = X^{q^i}$ and

$$\varphi_{t,i} = \sum_{v_1, \ldots, v_{q^i-1} \in V_r^0, v_i \neq v_j} \frac{\gamma(t)}{\lambda(v_1) \cdots \lambda(v_{q^i-1})} \in \Gamma(\Omega_r, E^{\otimes(1-q^i)}), \tag{3.7}$$

where we use again that $(-1)^{q^i-1} = +1$ if $q$ is odd, and also if $q$ is even when $-1 = +1$. From Lemma 3.10, $(E, \varphi, \lambda)$ is the universal family on $\mathbf{M}$.

Denote the quotient field of $S_r = \mathbb{F}_q[x_0, \ldots, x_{r-1}]$ by $K_r$. Let $R_r$ be the $\mathbb{F}_q$-subalgebra of $K_r$ generated by $1/v$ for all $v \in V_r^0$. Impose a graded structure on $R_r$ by assigning degree one to each $1/v$, and define $Q_r := \operatorname{Proj} R_r$. We change the graded structure on $S_r$ by now assigning degree $-1$ to each $x_i$ and to each $v \in V_r^0$. Let $RS_r$ be the graded $\mathbb{F}_q$-subalgebra of $K_r$ generated by $R_r$ and $S_r$, and $RS_{r,0} \subset RS_r$ be the $\mathbb{F}_q$-subalgebra consisting of homogeneous elements of degree zero. Then $\Omega_r = \operatorname{Spec}(RS_{r,0})$ is the open subscheme in $Q_r$ by removing the hyperplane sections defined by $1/x_i$ for $i = 0, \ldots, r-1$.

**Proposition 3.11.** *The scheme $Q_{r,A[1/t]} = Q_r \otimes_{\mathbb{F}_q} A[1/t]$ is a projective Satake compactification of $\mathbf{M}$. The dual $\omega$ of the relative Lie algebra of the universal family on $Q_{r,A[1/t]}$ is $\mathcal{O}_{Q_{r,A[1/r]}}(1)$, in particular, $\omega$ is very ample relative to $A[1/t]$.*

Proof. Let $\overline{E}$ be the line bundle whose sheaf of sections is $\mathcal{O}_{r,A[1/t]}(-1)$. Define the homomorphism $\overline{\varphi} : A \to \operatorname{End}(\overline{E})$ by setting

$$\overline{\varphi}_t = \gamma(t) + \sum_{i=1}^{r} \overline{\varphi}_{t,i}\tau^i, \quad \overline{\varphi}_{t,i} = \varphi_{t,i} \in \mathcal{O}_{Q_{r,A[1/t]}}(q^i - 1) = \overline{E}^{\otimes(1-q^i)}(Q_{r,A[1/t]}), \tag{3.8}$$

where $\varphi_{t,i}$ is defined in (3.7). By (3.7), if $\varphi_{t,i}(x) = 0$ at some point $x$ for every $i = 1, \ldots, r$ then $\frac{1}{v}(x) = 0$ for every $v \in V_r^0$, which is not possible. Thus, $(\overline{E}, \overline{\varphi})$ has rank $r \geq 1$ everywhere and it is a generalized Drinfeld $A$-module of rank $\leq r$ on $Q_{r,A[1/t]}$ which extends the universal family $(E, \varphi)$ over $\mathbf{M}$. The proof of [Pin13, Prop. 7.2] shows that $(\overline{E}, \overline{\varphi})$ is weakly separating. Thus, $Q_{r,A[1/t]}$ is a projective Satake compactification of $\mathbf{M}$. Note that the relative Lie algebra $\operatorname{Lie}(\overline{E})$ is $\mathcal{O}_{Q_{r,A[1/r]}}(-1)$ and its dual $\omega := \operatorname{Lie}(\overline{E})^\vee$ is $\mathcal{O}_{Q_{r,A[1/r]}}(1)$, particularly $\omega$ is very ample relative to $A[1/t]$. ∎

*Remark* 3.12. Note that the generic fiber $\overline{M}^r_K := \overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} F$ satisfies the characterizing properties of the Satake compactification [Pin13, Definition 4.1 (a) and (b)]. Thus, the generic fiber $\overline{M}^r_K$ of $\overline{\mathbf{M}}^r_K$ is the Satake compactification of $M^r_K$ constructed by Pink.

## 4. Drinfeld modular forms and Hecke operators

4.1. **Drinfeld modular forms over** $A[\mathfrak{n}^{-1}]$**.** Let $A, F, \infty$ be as in previous sections, and let $G := \mathrm{GL}_r$. For a fine subgroup $K$, let $(\overline{E}_K, \overline{\varphi}_K)$ be the universal family over the Satake compactification $\overline{\mathbf{M}}^r_K$ over $A[\mathfrak{n}_K^{-1}]$. By Theorem 3.2,

$$\omega_K := \operatorname{Lie}(\overline{E}_K)^\vee$$

is an ample invertible sheaf on $\overline{\mathbf{M}}^r_K$. We also write $\omega_K$ for its restriction on $\mathbf{M}^r_K$.

**Definition 4.1.** (1) For any integer $k \geq 0$, fine open compact subgroup $K \subset K(1) = \mathrm{GL}_r(\widehat{A})$ and $A[\mathfrak{n}_K^{-1}]$-algebra $L$, denote by

$$M_k(r, K, L) := H^0(\overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} L, \omega_K^{\otimes k} \otimes L) \tag{4.1}$$

the $L$-module of *algebraic Drinfeld modular forms of rank $r$, weight $k$, level $K$ over $L$.* The terminology "algebraic" is meant to distinguish them from the modular forms which are defined using the analytic theory for $L = \mathbb{C}_\infty$.

(2) Denote by

$$M(r, K, L) := \bigoplus_{k \geq 0} M_k(r, K, L) \tag{4.2}$$

the *graded ring of algebraic Drinfeld modular forms of rank $r$, level $K$ over $L$.*

Note that for $k<0$ the analogously defined $L$-module $M_k(r,K,L)$ is zero by the following well known lemma and Lemma 3.9.

**Lemma 4.2.** *Let $\pi\colon X\to Y$ be a proper and flat morphism of noetherian schemes and let $\mathcal{L}$ be an invertible sheaf on $X$ which is relatively ample over $Y$. Assume that for every $y\in Y$ the fiber $X_y$ is reduced and all irreducible components of $X_y$ have dimension at least one. Then $\pi_*\mathcal{L}^{\otimes k}=(0)$ for every $k<0$.*

We include a proof, because we could not find a reference. Note that the condition that the fibers are reduced is crucial, as one sees from Example 4.3 below.

Proof. Fix a $k<0$. By cohomology and base change it suffices to treat the case when $Y$ is the spectrum of a field. More precisely, by [Mum08, Chapter II, § 5, Theorem on page 44] there is a complex of finite locally free sheaves $K^\bullet\colon 0\longrightarrow K^0\xrightarrow{d^0}K^1\xrightarrow{d^1}\dots$ on $Y$ such that for every $Y$-scheme $Y'$

$$(\pi\times\mathrm{id}_{Y'})_*(\mathcal{L}^{\otimes k}\otimes_{\mathcal{O}_Y}\mathcal{O}_{Y'})\simeq H^0(K^\bullet\otimes_{\mathcal{O}_Y}\mathcal{O}_{Y'}):=\ker(d^0\otimes_{\mathcal{O}_Y}\mathcal{O}_{Y'})\,.$$

Locally on $Y$ we can choose bases of $K^0$ and $K^1$ and write $d^0$ as an $n_1\times n_0$-matrix, where $n_i$ is the rank of $K^i$. If at some point $y\in Y$

$$(0)=H^0(X_y,\mathcal{L}^{\otimes k}\otimes_{\mathcal{O}_Y}\kappa(y))=(\pi\times\mathrm{id}_y)_*(\mathcal{L}^{\otimes k}\otimes_{\mathcal{O}_Y}\kappa(y))\simeq H^0(K^\bullet\otimes_{\mathcal{O}_Y}\kappa(y)),$$

then $d^0\otimes\kappa(y)$ is injective. This means that $n_0\le n_1$ and there is an $n_0\times n_0$-minor in the matrix $d^0$ whose image in $d^0\otimes\kappa(y)$ has invertible determinant. Then the determinant of this minor is already invertible in $\mathcal{O}_{Y,y}$ and so $d^0\otimes\mathcal{O}_{Y,y}$ is injective. If this holds at every point $y\in Y$, then $d^0$ is injective and $\pi_*\mathcal{L}^{\otimes k}=(0)$.

Note that $\mathcal{L}^{\otimes k}\otimes_{\mathcal{O}_Y}\kappa(y)$ on $X_y$ is relatively ample over $\kappa(y)$ by [EGA, II, Proposititon 4.6.13 (iii)]. So we may replace $Y$ by $\operatorname{Spec}\kappa(y)$ for a point $y\in Y$, and thus assume that $Y$ is the spectrum of a field. Then we must show that $H^0(X,\mathcal{L}^{\otimes k})=(0)$. Assume that there is a non-zero global section $0\ne s\in H^0(X,\mathcal{L}^{\otimes k})$. Let $U\subset X$ be an open subset with $0\ne s|_U$; use [EGA, $\mathrm{I_{new}}$, Lemma 9.7.9.1]. By shrinking $U$ we may assume that it is contained in exactly one irreducible component of $X$. Since $X$ is reduced, the scheme theoretic closure $\overline{U}$ of $U$ in $X$ is still reduced and also irreducible. It contains $U$ as an open subscheme. Let $i\colon\overline{U}\hookrightarrow X$ be the corresponding closed immersion. Then $0\ne i^*s\in H^0(\overline{U},i^*\mathcal{L}^{\otimes k})$ is a regular section and defines an effective Cartier divisor $D$ on $\overline{U}$ by [GöWe10, Proposition 11.32]. Since $\dim\overline{U}\ge1$ and the support

$$\operatorname{Supp}(D):=\{x\in\overline{U}\mid D_x\ne1\}=\{x\in\overline{U}\mid\mathcal{O}_{\overline{U},x}\cdot(i^*s)_x\subsetneq(i^*\mathcal{L})_x^{\otimes k}\}$$

of $D$ is strictly contained in $\overline{U}$ by [GöWe10, Lemma 11.33], we may choose a proper curve $C\subset\overline{U}$ (that is, an irreducible and reduced closed subscheme of dimension one) which is not contained in $\operatorname{Supp}(D)$. Let $\widetilde{C}$ be the normalization of $C$ and let $f\colon\widetilde{C}\to X$ be the induced map. Then $f$ is finite and $f^*\mathcal{L}$ is ample on $\widetilde{C}$ by [EGA, II, Corollaire 4.6.6 and Proposition 5.1.12]. Therefore $\deg(f^*\mathcal{L})>0$ by [Har77, Corollary IV.3.3] and $k\cdot\deg(f^*\mathcal{L})<0$, because $k<0$. On the other hand, since $C\not\subset\operatorname{Supp}(D)$ and $i^*s$ generates $i^*\mathcal{L}^{\otimes k}$ on $\overline{U}-\operatorname{Supp}(D)$, we have $0\ne f^*s\in H^0(\widetilde{C},f^*\mathcal{L}^{\otimes k})$. Therefore, $k\cdot\deg(f^*\mathcal{L})=\deg(f^*\mathcal{L}^{\otimes k})\ge0$ by [Har77, Lemma IV.1.2]. This is a contradiction and proves the lemma. ∎

**Example 4.3.** We show that the condition on the reduced fibers in Lemma 4.2 cannot be dropped. Let $k$ be a field and let $X=\operatorname{Proj}k[S,T,U]/(TU,U^2)=V(TU,U^2)\subset\operatorname{Proj}k[S,T,U]=\mathbb{P}^2_k$. Then $X$ is non-reduced at the point $V(T,U)$. The line bundle $\mathcal{O}(1)$ is ample on $X$. But its dual $\mathcal{O}(-1)$ has the non-zero global section $\frac{U}{S^2}\in H^0(X,\mathcal{O}(-1))$, which vanishes outside $V(T,U)$.

Since $\omega_K$ is ample and $\overline{\mathbf{M}}^r_K$ is proper over $A[\mathfrak{n}_K^{-1}]$, by [Stack, Tags 01CV and 01Q1] there is a canonical isomorphism

$$\overline{\mathbf{M}}^r_K \xrightarrow{\sim} \operatorname{Proj} M(r,K,A[\mathfrak{n}_K^{-1}]). \tag{4.3}$$

Applying the base change theorem for cohomology groups to $\operatorname{Spec} L \to \operatorname{Spec} A[\mathfrak{n}_K^{-1}]$, we obtain canonical maps

$$M_k(r,K,A[\mathfrak{n}_K^{-1}]) \otimes L \to M_k(r,K,L), \quad k = 0,1,\dots. \tag{4.4}$$

and these are isomorphisms when $L$ is flat over $A[\mathfrak{n}_K^{-1}]$; see [Har77, Proposition III.9.3]. We will need the following well known

**Proposition 4.4.** *Let $Y$ be a Noetherian scheme, $f\colon X \to Y$ a projective morphism, $\mathcal{F}$ a coherent $\mathcal{O}_X$-module which is flat over $Y$. Assume that for some $i$ the cohomology in the fiber*

$$H^i(X_y, \mathcal{F} \otimes k(y)) = 0 \tag{4.5}$$

*for all points $y \in Y$. Then we have*

$$(R^{i-1}f_*\mathcal{F}) \otimes_{\mathcal{O}_Y} B \simeq H^{i-1}(X \times_Y \operatorname{Spec} B, \mathcal{F} \otimes B) \tag{4.6}$$

*for any $Y$-scheme* $\operatorname{Spec} B$.

Proof. This is proved in the same way as [Har77, Theorem III.12.11] by combining [Har77, Chapter III, Propositions 12.4, 12.5, 12.7 and 12.10]. ∎

**Corollary 4.5.** *There is a positive integer $k_0$ such that for all $k \geq k_0$, the canonical map in (4.4) is an isomorphism and $M_k(r,K,A[\mathfrak{n}_K^{-1}])$ is a finite projective $A[\mathfrak{n}_K^{-1}]$-module.*

Proof. Let $f : X := \overline{\mathbf{M}}^r_K \to Y := \operatorname{Spec} A[\mathfrak{n}_K^{-1}]$. Since $\omega_K$ is ample, by [Har77, Proposition III.5.3] there exists an integer $k_0$ such that $R^i f_*(\omega_K^{\otimes k}) = 0$ on $Y$ for all $i > 0$ and $k \geq k_0$. We now prove that for each point $y \in Y$ the natural map

$$\theta^i(y) : R^i f_*(\omega_K^{\otimes k}) \otimes_{\mathcal{O}_Y} k(y) \to H^i(X_y, \omega_K^{\otimes k} \otimes k(y))$$

is surjective for all $i > 0$. This is true for $i > \dim X + 1$ as the target is zero. Since $R^i f_*(\omega_K^{\otimes k}) = 0$ is locally free everywhere for all $i > 0$, by [Har77, Theorem III.12.11] and by induction on $i$ decreasingly, the map $\theta^i(y)$ is surjective for $i = \dim X + 1, \dots, 1$. Using $R^1 f_*(\omega_K^{\otimes k}) = 0$ and that $\theta^1(y)$ is surjective again, we show $H^1(X_y, \omega_K^{\otimes k} \otimes k(y)) = 0$ for all $y \in Y$. Then (4.4) is an isomorphism by Proposition 4.4, and the projectivity of $M_k(r,K,A[\mathfrak{n}_K^{-1}])$ follows from [Har77, Theorem III.12.11]. ∎

**Lemma 4.6.** *Let $\widetilde{K} \subset K \subset K(1)$ be two fine open compact subgroups and let $\pi := \pi_{\widetilde{K},K} : \overline{\mathbf{M}}^r_{\widetilde{K}} \to \overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}]$ be the finite cover from Theorem 3.2(2).*

*(1) Let $L$ be an $A[\mathfrak{n}_{\widetilde{K}}^{-1}]$-module. Then the map of quasi-coherent sheaves*

$$\omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L \longrightarrow \pi_{\widetilde{K},K\,*}\big(\omega_{\widetilde{K}}^{\otimes k} \otimes_{A[\mathfrak{n}_{\widetilde{K}}^{-1}]} L\big) = \pi_{\widetilde{K},K\,*}\pi^*_{\widetilde{K},K}\big(\omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L\big) \tag{4.7}$$

*on $\overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}]$ is injective.*

*(2) For any $A[\mathfrak{n}_{\widetilde{K}}^{-1}]$-algebra $L$ the pullback map $\pi^*_{\widetilde{K},K} : M_k(r,K,L) \to M_k(r,\widetilde{K},L)$ is injective.*

*(3) Moreover, if $\widetilde{K} \lhd K$ is normal and $L$ is a flat $A[\mathfrak{n}_{\widetilde{K}}^{-1}]$-algebra, then $\pi_{\widetilde{K},K}$ induces an isomorphism of $L$-modules*

$$M_k(r,K,L) \xrightarrow{\sim} M_k(r,\widetilde{K},L)^{K/\widetilde{K}}. \tag{4.8}$$

Proof. Statement (2) follows from (1) by taking global sections on $\overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}]$, because $M_k(r,K,L) = H^0\big(\overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}], \omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L\big)$.

(1) Note that $\omega_K^{\otimes k}$ and $\omega_{\widetilde{K}}^{\otimes k}$ are flat over $A[\mathfrak{n}_{\widetilde{K}}^{-1}]$ and the direct image functor $\pi_{\widetilde{K},K\,*}$ is exact by [EGA, III$_1$, Corollaire 1.3.2] as $\pi_{\widetilde{K},K}$ is finite. Therefore, any exact sequence $0 \to L' \to L \to L'' \to 0$ of $A[\mathfrak{n}_{\widetilde{K}}^{-1}]$-modules induces a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L' & \longrightarrow & \omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L & \longrightarrow & \omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L'' & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \pi_*\big(\omega_{\widetilde{K}}^{\otimes k} \otimes_{A[\mathfrak{n}_{\widetilde{K}}^{-1}]} L'\big) & \longrightarrow & \pi_*\big(\omega_{\widetilde{K}}^{\otimes k} \otimes_{A[\mathfrak{n}_{\widetilde{K}}^{-1}]} L\big) & \longrightarrow & \pi_*\big(\omega_{\widetilde{K}}^{\otimes k} \otimes_{A[\mathfrak{n}_{\widetilde{K}}^{-1}]} L''\big) & \longrightarrow & 0\,.
\end{array}$$

Assume that there is a section $w := \sum_i w_i \otimes \ell_i \in H^0(U, \omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L)$ over an open affine subset $U \subset \overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}]$ in the kernel of (4.7), where $w_i \in H^0(U, \omega_K^{\otimes k})$ and $\ell_i \in L$ for all $i$. Let $L'$ be the $A[\mathfrak{n}_{\widetilde{K}}^{-1}]$-submodule of $L$ generated by the finitely many $\ell_i$. Then $w \in H^0(U, \omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L')$. Replacing $L$ by $L'$ we reduce to the case that $L$ is a finitely generated module over the Dedekind domain $A[\mathfrak{n}_{\widetilde{K}}^{-1}]$. By the structure theory of such modules we have $L = P \oplus \bigoplus_j A[\mathfrak{n}_{\widetilde{K}}^{-1}]/\mathfrak{p}_j^{n_j}$ for a finite projective $A[\mathfrak{n}_{\widetilde{K}}^{-1}]$-module $P$ and maximal ideals $\mathfrak{p}_i \subset A[\mathfrak{n}_{\widetilde{K}}^{-1}]$ and integers $n_j > 0$. Using the commutative diagram above, we thus reduce to the cases where $L = A[\mathfrak{n}_{\widetilde{K}}^{-1}]/\mathfrak{p}^n$ or where $L = P$ is finite projective over $A[\mathfrak{n}_{\widetilde{K}}^{-1}]$.

In the first case we choose a uniformizer $z$ of $\mathfrak{p}$ and consider the exact sequence $0 \to A[\mathfrak{n}_{\widetilde{K}}^{-1}]/\mathfrak{p}^{n-1} \xrightarrow{z\cdot} A[\mathfrak{n}_{\widetilde{K}}^{-1}]/\mathfrak{p}^n \longrightarrow A[\mathfrak{n}_{\widetilde{K}}^{-1}]/\mathfrak{p} \to 0$. Using the diagram again we reduce to the case where $L = A[\mathfrak{n}_{\widetilde{K}}^{-1}]/\mathfrak{p} = \mathbb{F}_v$ for the place $v$ of $A$ corresponding to $\mathfrak{p}$. For $L = \mathbb{F}_v$ the sheaf $\omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L$ is (the pushforward to $\overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}]$ of) an invertible sheaf on $\overline{\mathscr{M}}^r_K := \overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} \mathbb{F}_v$. We consider the kernel sheaf

$$\mathcal{I} := \ker\big(\mathcal{O}_{\overline{\mathscr{M}}^r_K} \longrightarrow \pi_{\widetilde{K},K\,*}\pi^*_{\widetilde{K},K}\mathcal{O}_{\overline{\mathscr{M}}^r_K}\big)\,.$$

The vanishing locus of $\mathcal{I}$ in $\overline{\mathscr{M}}^r_K$ is the scheme theoretic image of $\pi_{\widetilde{K},K} \colon \overline{\mathscr{M}}^r_{\widetilde{K}} \to \overline{\mathscr{M}}^r_K$. Since the set theoretic image equals the entire space $\overline{\mathscr{M}}^r_K$, which is the vanishing locus of the zero ideal, the radicals coincide $\sqrt{\mathcal{I}} = \sqrt{(0)}$. But the latter equals $(0)$, because $\overline{\mathscr{M}}^r_K$ is reduced by Proposition 3.9. This shows that $\mathcal{I} = (0)$. Tensoring with the flat $\mathcal{O}_{\overline{\mathscr{M}}^r_K}$-module $\omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L$ proves the injectivity of (4.7) for $L = \mathbb{F}_v$.

We treat the second case, where $L$ is a finite projective $A[\mathfrak{n}_{\widetilde{K}}^{-1}]$-module simultaneously to assertion (3). By the construction of $\overline{\mathbf{M}}^r_{\widetilde{K}}$ as $\overline{\mathbf{M}}^r_{K(\mathfrak{n}_{\widetilde{K}})}/\big(\widetilde{K}/K(\mathfrak{n}_{\widetilde{K}})\big)$ in Lemma 3.6(2) we have an exact sequence of sheaves on $\overline{\mathbf{M}}^r_{\widetilde{K}}$

$$0 \longrightarrow \mathcal{O}_{\overline{\mathbf{M}}^r_{\widetilde{K}}} \longrightarrow \pi_{K(\mathfrak{n}_{\widetilde{K}}),\widetilde{K}\,*}\mathcal{O}_{\overline{\mathbf{M}}^r_{K(\mathfrak{n}_{\widetilde{K}})}} \rightrightarrows \prod_{\widetilde{K}/K(\mathfrak{n}_{\widetilde{K}})} \pi_{K(\mathfrak{n}_{\widetilde{K}}),\widetilde{K}\,*}\mathcal{O}_{\overline{\mathbf{M}}^r_{K(\mathfrak{n}_{\widetilde{K}})}}\,,$$

where the two maps on the right are the diagonal inclusion, and the action of $\widetilde{K}/K(\mathfrak{n}_{\widetilde{K}})$, respectively. Using the analogous reasoning for $K$ and replacing $\widetilde{K}/K(\mathfrak{n}_{\widetilde{K}})$ by $K/K(\mathfrak{n}_{\widetilde{K}})$ we get a similar description of $\mathcal{O}_{\overline{\mathbf{M}}^r_K} \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}]$. Putting both together we obtain the exact sequence of sheaves on $\overline{\mathbf{M}}^r_{\widetilde{K}} \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}]$

$$0 \longrightarrow \mathcal{O}_{\overline{\mathbf{M}}^r_K} \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}] \longrightarrow \pi_{\widetilde{K},K\,*}\mathcal{O}_{\overline{\mathbf{M}}^r_{\widetilde{K}}} \rightrightarrows \prod_{K/\widetilde{K}} \pi_{\widetilde{K},K\,*}\mathcal{O}_{\overline{\mathbf{M}}^r_{\widetilde{K}}}\,.$$

If $L$ is flat over $A[\mathfrak{n}_{\widetilde{K}}^{-1}]$ we may tensor the latter sequence with the flat $\mathcal{O}_{\overline{\mathbf{M}}^r_K} \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}]$-module $\omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L$ to obtain the exact sequence

$$0 \longrightarrow \omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L \longrightarrow \pi_{\widetilde{K},K\,*}(\omega_{\widetilde{K}}^{\otimes k} \otimes_{A[\mathfrak{n}_{\widetilde{K}}^{-1}]} L) \rightrightarrows \prod_{K/\widetilde{K}} \pi_{\widetilde{K},K\,*}(\omega_{\widetilde{K}}^{\otimes k} \otimes_{A[\mathfrak{n}_{\widetilde{K}}^{-1}]} L)\,.$$

This proves the injectivity of (4.7) for flat $L$ and finishes the proof of (2). Here we use the projection formula $(\pi_{\widetilde{K},K\,*}\mathcal{O}_{\overline{\mathbf{M}}^r_{\widetilde{K}}}) \otimes_{\mathcal{O}_{\overline{\mathbf{M}}^r_K}} (\omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L) = \pi_{\widetilde{K},K\,*}(\pi^*_{\widetilde{K},K}\omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L)$ from [EGA, $\mathrm{I}_{\mathrm{new}}$, Corollaire 9.3.9]

To prove (3) we use that $L$ is a flat $A[\mathfrak{n}_{\widetilde{K}}^{-1}]$-algebra. Taking global sections on $\overline{\mathbf{M}}^r_{\widetilde{K}} \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}]$, which is a left exact functor yields the exact sequence

$$0 \longrightarrow M_k(r,K,L) \longrightarrow M_k(r,\widetilde{K},L) \rightrightarrows \prod_{K/\widetilde{K}} M_k(r,\widetilde{K},L)\,,$$

and hence the isomorphism (4.8). ∎

If the $A[\mathfrak{n}_{\widetilde{K}}^{-1}]$-algebra $L$ is not flat we do not know whether assertion (3) of the previous lemma still holds true. However, if the special fibers $\overline{\mathscr{M}}^r_K := \overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} \mathbb{F}_v$ and $\overline{\mathscr{M}}^r_{\widetilde{K}} := \overline{\mathbf{M}}^r_{\widetilde{K}} \otimes_{A[\mathfrak{n}_{\widetilde{K}}^{-1}]} \mathbb{F}_v$ are *normal* at all places $v \nmid \mathfrak{n}_{\widetilde{K}}$ then (3) can be proved along the same lines as (2) by using the following

**Proposition 4.7.** *Let $\widetilde{K} \lhd K \subset K(1)$ be two fine open compact subgroups such that $\widetilde{K}$ is normal in $K$, and let $L$ be an $\mathbb{F}_v$-algebra for a place $v \nmid \mathfrak{n}_{\widetilde{K}}$. Let $\overline{\mathscr{M}}^{r,\mathrm{nor}}_K$ and $\overline{\mathscr{M}}^{r,\mathrm{nor}}_{\widetilde{K}}$ be the normalizations of $\overline{\mathscr{M}}^r_K := \overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} \mathbb{F}_v$ and $\overline{\mathscr{M}}^r_{\widetilde{K}} := \overline{\mathbf{M}}^r_{\widetilde{K}} \otimes_{A[\mathfrak{n}_{\widetilde{K}}^{-1}]} \mathbb{F}_v$, respectively. Then for any $k$, the canonical morphism $\pi_{\widetilde{K},K} \colon \overline{\mathscr{M}}^{r,\mathrm{nor}}_{\widetilde{K}} \to \overline{\mathscr{M}}^{r,\mathrm{nor}}_K$ induces an isomorphism of $L$-modules*

$$H^0\big(\overline{\mathscr{M}}^{r,\mathrm{nor}}_K, \omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L\big) \xrightarrow{\ \sim\ } H^0\big(\overline{\mathscr{M}}^{r,\mathrm{nor}}_{\widetilde{K}}, \omega_{\widetilde{K}}^{\otimes k} \otimes_{A[\mathfrak{n}_{\widetilde{K}}^{-1}]} L\big)^{K/\widetilde{K}} \tag{4.9}$$

*for the natural action of the group $K/\widetilde{K}$. Moreover, the natural map*

$$M_k(r,K,L) = H^0\big(\overline{\mathscr{M}}^r_K, \omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L\big) \longrightarrow H^0\big(\overline{\mathscr{M}}^{r,\mathrm{nor}}_K, \omega_K^{\otimes k} \otimes_{A[\mathfrak{n}_K^{-1}]} L\big)$$

*is injective.*

Proof. The final statement follows as in the previous lemma, because the scheme theoretic image of $\overline{\mathscr{M}}^{r,\mathrm{nor}}_K \to \overline{\mathscr{M}}^r_K$ is the entire space $\overline{\mathscr{M}}^r_K$ which is reduced.

Like in the previous lemma, the proof of the isomorphism (4.9) follows by considering the sequence of coherent sheaves on $\overline{\mathscr{M}}^{r,\mathrm{nor}}_K$

$$0 \longrightarrow \mathcal{O}_{\overline{\mathscr{M}}^{r,\mathrm{nor}}_K} \longrightarrow \pi_{\widetilde{K},K\,*}\mathcal{O}_{\overline{\mathscr{M}}^{r,\mathrm{nor}}_{\widetilde{K}}} \rightrightarrows \prod_{K/\widetilde{K}} \pi_{\widetilde{K},K\,*}\mathcal{O}_{\overline{\mathscr{M}}^{r,\mathrm{nor}}_{\widetilde{K}}}\,, \tag{4.10}$$

tensoring it with the flat $\mathcal{O}_{\overline{\mathscr{M}}^{r,\mathrm{nor}}_K}$-module $\omega_K^{\otimes k} \otimes_{\mathbb{F}_v} L$ and taking global sections. It thus remains to prove that the sequence (4.10) is exact.

Exactness on the left follows as in the previous lemma, because the scheme theoretic image of $\pi_{\widetilde{K},K} \colon \overline{\mathscr{M}}^{r,\mathrm{nor}}_K \to \overline{\mathscr{M}}^{r,\mathrm{nor}}_{\widetilde{K}}$ is the entire space $\overline{\mathscr{M}}^{r,\mathrm{nor}}_K$ which is reduced.

To prove exactness in the middle let $\overline{U} = \operatorname{Spec} R \subset \overline{\mathscr{M}}^{r,\mathrm{nor}}_K$ be an affine open subset and let $U := \overline{U} \cap \mathscr{M}^r_K$, where we observe that $\mathscr{M}^r_K := \mathbf{M}^r_{\widetilde{K}} \otimes_{A[\mathfrak{n}_{\widetilde{K}}^{-1}]} \mathbb{F}_v$ is smooth over $\mathbb{F}_v$, hence normal, and hence an (affine) open subset of $\overline{\mathscr{M}}^{r,\mathrm{nor}}_K$. By Proposition 2.8(2) the scheme $U \times_{\overline{\mathscr{M}}^{r,\mathrm{nor}}_K} \overline{\mathscr{M}}^{r,\mathrm{nor}}_{\widetilde{K}}$ is a finite étale Galois cover of $U$ with Galois group $K/\widetilde{K}$. Thus the restriction of the sequence (4.10) to the dense open $U$ is exact. Let $f \in H^0(\overline{U}, \pi_{\widetilde{K},K\,*}\mathcal{O}_{\overline{\mathscr{M}}^{r,\mathrm{nor}}_{\widetilde{K}}}) = H^0(\overline{U} \times_{\overline{\mathscr{M}}^{r,\mathrm{nor}}_K} \overline{\mathscr{M}}^{r,\mathrm{nor}}_{\widetilde{K}}, \mathcal{O}_{\overline{\mathscr{M}}^{r,\mathrm{nor}}_{\widetilde{K}}})$ lie in the equalizer of the two morphisms on the right. By the exactness of (4.10) on $U$ the restriction $f|_U$ of $f$ to $U$ lies in $H^0(U, \mathcal{O}_{\overline{U}})$. Since $\overline{U}$ is normal the ring $R = H^0(\overline{U}, \mathcal{O}_{\overline{U}})$ equals the intersection $R_\mathfrak{p}$ of its local rings at height one primes $\mathfrak{p} \subset R$ by [Mat89, Theorem 11.5(ii)]. Thus it suffices to prove that $f \in R_\mathfrak{p}$ for all such $\mathfrak{p}$, or equivalently that $v_\mathfrak{p}(f) \geq 0$ where $v_\mathfrak{p}$ is the valuation of the discrete valuation ring $R_\mathfrak{p}$. Let $\mathfrak{q}$ be a point in $\overline{U} \times_{\overline{\mathscr{M}}_K^{r,\mathrm{nor}}} \overline{\mathscr{M}}_{\tilde{K}}^{r,\mathrm{nor}}$ lying above $\mathfrak{p}$. The local ring at $\mathfrak{q}$ is also a discrete valuation ring with valuation $v_\mathfrak{q}$ extending $v_\mathfrak{p}$. Since $f \in H^0(\overline{U} \times_{\overline{\mathscr{M}}_K^{r,\mathrm{nor}}} \overline{\mathscr{M}}_{\tilde{K}}^{r,\mathrm{nor}}, \mathcal{O}_{\overline{\mathscr{M}}_{\tilde{K}}^{r,\mathrm{nor}}})$, we have $v_\mathfrak{q}(f) \geq 0$, and hence $v_\mathfrak{p}(f) \geq 0$. This proves that $f \in H^0(\overline{U}, \mathcal{O}_{\overline{U}})$, whence the exactness of (4.10). ∎

For any arithmetic subgroup $\Gamma \subset G(F) = \mathrm{GL}_r(F)$, Basson, Breuer and Pink [BBP18a] have defined the space $M_k(\Gamma)$ of Drinfeld modular forms of weight $k$ and level $\Gamma$ over $\mathbb{C}_\infty := \widehat{\overline{F_\infty}}$ the completion of an algebraic closure of the completion $F_\infty$ of $F$ at $\infty$. These are $\mathbb{C}_\infty$-valued (rigid analytic) holomorphic functions on the Drinfeld period domain $\Omega^r$ that satisfy the usual conditions defined by automorphy factors (i.e. are weakly modular forms) and are required to be holomorphic "at infinity"; see [BBP18a, Definition 6.1]. They proved the following comparison theorem.

**Theorem 4.8** ([BBP18b, Theorem 10.9])**.** *There is an isomorphism*

$$\mathbf{M}_K^r \otimes_{A[\mathfrak{n}_K^{-1}]} \mathbb{C}_\infty \simeq \coprod_{i=1}^{h} \Gamma_{g_i} \backslash \Omega^r, \quad \Gamma_{g_i} := G(F) \cap g_i K {g_i}^{-1}, \tag{4.11}$$

*where $g_1, \dots, g_h$ are complete representatives for the double coset space $G(F) \backslash G(\mathbb{A}^\infty)/K$. When $K \subset K(1)$ is fine, there is a natural isomorphism of $\mathbb{C}_\infty$-vector spaces*

$$M_k(r, K, \mathbb{C}_\infty) \xrightarrow{\sim} \bigoplus_{i=1}^{h} M_k(\Gamma_{g_i}). \tag{4.12}$$

4.2. **Drinfeld modular forms over $A_{(v)}$.** Let $v$ be a finite place of $F$. We say that an open compact subgroup $K \subset G(\mathbb{A}^\infty)$ is *fine* if it is conjugate to a fine subgroup of $K(1)$. Let $A_{(v)}$ denote the localization of $A$ at the place $v$, and let $\widehat{A}^{(v)} := \prod_{v' \neq v} A_{v'}$. In this subsection we shall define algebraic Drinfeld modular forms of rank $r$ over an $A_{(v)}$-algebra $L$ and prime-to-$v$ Hecke operators on these modular forms. To do this, we need to define moduli schemes $\mathbf{M}_K^r$ over $A_{(v)}$ and construct their Satake compactification for fine level subgroups $K \subset G(\mathbb{A}^\infty)$ of the form $K = K_v K^v$, where $K_v = K_v(1) = G(A_v)$ is a fixed maximal open compact subgroup, and $K^v \subset G(\mathbb{A}^{\infty v})$ is an open compact subgroup not necessarily contained in $G(\widehat{A}^{(v)})$ which can vary. For the remainder of this article unless stated otherwise $K_v$ and $K^v$ are as above and $K = K_v K^v$.

For a Drinfeld $A$-module $(E, \varphi)$ over an $A_{(v)}$-field $L$, the *prime-to-$v$ Tate module* and *Tate space* of $(E, \varphi)$ are defined as

$$T^{(v)}(\varphi) := \mathrm{Hom}_A(\mathbb{A}^{\infty v}/\widehat{A}^{(v)}, \varphi(L^{\mathrm{sep}})), \quad V^{(v)}(\varphi) := \mathbb{A}^{\infty v} \otimes_{\widehat{A}^{(v)}} T^{(v)}(\varphi),$$

where $L^{\mathrm{sep}}$ denotes a separable closure of $L$.

**Definition 4.9.** Let $S$ be a *connected* locally Noetherian $A_{(v)}$-scheme, and let $(E, \varphi)$ be a Drinfeld $A$-module of rank $r$ over $S$. A *level-$K^v$ structure* on $(E, \varphi)$ is a $K^v$-orbit $\overline{\eta} = \eta K^v$ of $\mathbb{A}^{\infty v}$-linear isomorphisms

$$\eta : (\mathbb{A}^{\infty v})^r \xrightarrow{\sim} V^{(v)}(\varphi_{\bar{s}})$$

which is $\pi_1(S, \bar{s})$-invariant, where $\bar{s}$ is a geometric point of $S$. Here $K^v \subset \mathrm{GL}_r(\mathbb{A}^{\infty v})$ acts on $(\mathbb{A}^{\infty v})^r$ and $\pi_1(S, \bar{s})$ on $V^{(v)}(\varphi_{\bar{s}})$. If $K = K(\mathfrak{n}) \subset K(1)$ and $\eta$ maps $(\widehat{A}^{(v)})^r$ onto $T^{(v)}(\varphi_{\bar{s}})$, then $\overline{\eta}$ is nothing but a level-$\mathfrak{n}$ structure on $(E, \varphi)$; see (2.4). For general $S$, a level-$K^v$ structure on $(E, \varphi)$ is a tuple $\overline{\eta} = (\overline{\eta}_{S_i})_{S_i \in \pi_0(S)}$, where each $\overline{\eta}_{S_i}$ is a level-$K^v$ structure on $(E, \varphi)$ over $S_i$.

Recall that a morphism $\alpha : (E_1, \varphi_1) \to (E_2, \varphi_2)$ of two Drinfeld $A$-modules over $S$ is called an *isogeny* if it is surjective with finite flat kernel. A morphism $\alpha$ is an isogeny if and only if $\alpha \neq 0$ above every connected component of $S$; see [Har19, Proposition 5.4]. For every isogeny $\alpha$, by [Har19, Corollary 5.15] there is an element $a \in A$, $a \neq 0$ and an isogeny $\beta : (E_2, \varphi_2) \to (E_1, \varphi_1)$ with $\alpha\beta = a \cdot \mathrm{id}_{E_2}$ and $\beta\alpha = a \cdot \mathrm{id}_{E_1}$, and $\ker(\alpha) \subset \varphi_1[a]$. We say that an isogeny $\alpha : (E_1, \varphi_1) \to (E_2, \varphi_2)$ is *prime-to-$v$* if there is an element $a \in A$ with $v \nmid a$ such that $\ker \alpha \subset \varphi_1[a]$. Equivalently, let $\mathfrak{a}$ be the kernel of the ring homomorphism $A \to \mathrm{End}(\ker(\alpha))$, sending $a \mapsto \varphi_a$; then $\alpha$ is prime-to-$v$ if and only if $v \nmid \mathfrak{a}$. Since $\ker \alpha \subset \varphi_1[a]$ for some $a \neq 0$, we have $a \in \mathfrak{a}$ and $\mathfrak{a} \neq (0)$.

**Definition 4.10.** Let $\mathbf{M}^r_{K_v K^v}$ denote the moduli functor over $A_{(v)}$ classifying equivalence classes of Drinfeld $A$-modules $(E, \varphi, \bar{\eta})$ with level-$K^v$ structure. Here two objects $(E_1, \varphi_1, \overline{\eta}_1)$ and $(E_2, \varphi_2, \overline{\eta}_2)$ over a base scheme $S$ are said to be *equivalent* if there is a prime-to-$v$ isogeny $\alpha : (E_1, \varphi_1) \to (E_2, \varphi_2)$ over $S$ such that $\alpha_* \overline{\eta}_1 = \overline{\eta}_2$, where $\alpha_* : V^{(v)}(\varphi_{1,\bar{s}}) \to V^{(v)}(\varphi_{2,\bar{s}})$ is induced from $\alpha : \varphi_{1,\bar{s}}[\mathfrak{n}] \to \varphi_{2,\bar{s}}[\mathfrak{n}]$.

We claim that when $K = K_v K^v = G(A_v)K^v \subset G(\mathbb{A}^\infty)$ is fine, the functor $\mathbf{M}^r_{K_v K^v}$ is representable by an affine and smooth $A_{(v)}$-scheme, which we denote again by $\mathbf{M}^r_{K_v K^v}$, with a universal family $(E_K, \varphi_K, \overline{\eta}_K)$. To see this, for any element $g \in G(\mathbb{A}^{\infty v})$, the right translation by $g$ gives an isomorphism of functors:

$$J_g : \mathbf{M}^r_{K_v K^v} \xrightarrow{\sim} \mathbf{M}^r_{(K_v K^v)^g}, \quad (E, \varphi, \overline{\eta}) \mapsto (E, \varphi, \overline{\eta g}).$$

Here we write $K^g := \mathrm{Int}(g^{-1})K = g^{-1}Kg$. In particular, if $g \in K^v$ then $K^g = K$ and $J_g$ is the identity on $\mathbf{M}^r_{K_v K^v}$. One can choose $g \in \{1\}_v \times G(\mathbb{A}^{\infty v})$ such that $(K^v)^g \subset G(\widehat{A}^{(v)})$, and hence that $K^g \subset K(1)$. Then the natural map $i_v : \mathbf{M}^r_{K^g} \otimes_{A[\mathfrak{n}_{K^g}^{-1}]} A_{(v)} \to \mathbf{M}^r_{(K_v K^v)^g}$, sending $(E, \varphi, \lambda K^g) \mapsto (E, \varphi, \bar{\eta})$, where $\eta$ is a lifting of $\lambda$, is an isomorphism, and hence the representability of $\mathbf{M}^r_{(K_v K^v)^g}$, and hence of $\mathbf{M}^r_{K_v K^v}$ via the isomorphism $J_g$, is obtained from Proposition 2.8(2). We then transport the Satake compactification $\overline{\mathbf{M}}^r_{K^g}$ of $\mathbf{M}^r_{K^g}$ and the universal family $(\overline{E}_{K^g}, \overline{\varphi}_{K^g})$ over $\overline{\mathbf{M}}^r_{K^g}$ to $\mathbf{M}^r_{K_v K^v}$. So we obtain the Satake compactification $\overline{\mathbf{M}}^r_{K_v K^v}$ of $\mathbf{M}^r_{K_v K^v}$ over $A_{(v)}$, and the universal family $(\overline{E}_K, \overline{\varphi}_K)$ over $\overline{\mathbf{M}}^r_{K_v K^v}$. By abuse of notation, we also write $\mathbf{M}^r_K$ for $\mathbf{M}^r_{K_v K^v}$ and $\overline{\mathbf{M}}^r_K$ for $\overline{\mathbf{M}}^r_{K_v K^v}$, understanding that they are schemes over $A_{(v)}$, not over $A[\mathfrak{n}_K^{-1}]$.

If $\widetilde{K}^v \subset K^v \subset G(\mathbb{A}^{\infty v})$ are fine open compact subgroups and $\widetilde{K} = G(A_v)\widetilde{K}^v$ and $K = G(A_v)K^v$, then "forgetting the level" induces by Theorem 3.2 a finite surjective and open morphism

$$\pi_{\widetilde{K},K} : \overline{\mathbf{M}}^r_{\widetilde{K}} \longrightarrow \overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} A[\mathfrak{n}_{\widetilde{K}}^{-1}]$$

over $\mathrm{Spec}\, A[\mathfrak{n}_{\widetilde{K}}^{-1}]$ which satisfies $\pi^*_{\widetilde{K},K}(\overline{E}_K, \overline{\varphi}_K) = (\overline{E}_{\widetilde{K}}, \overline{\varphi}_{\widetilde{K}})$ and $\pi^*_{\widetilde{K},K}(\omega_K) = \omega_{\widetilde{K}}$.

The construction of the Satake compactification in Section 3 also shows the following functorial property (see (3.2)): For any $g \in G(\mathbb{A}^{\infty v})$, the Hecke translation $J_g : \mathbf{M}^r_K \to \mathbf{M}^r_{K^g}$, and the the canonical isomorphism $I_g : (E_K, \varphi_K) \to (E_{g^{-1}Kg}, \varphi_{g^{-1}Kg})$ which lifts $J_g$ extend uniquely to isomorphisms $\overline{J}_g$ and $\overline{I}_g$, respectively, that fit into the following commutative diagram:

$$\begin{tikzcd}
(\overline{E}_K, \overline{\varphi}_K) \arrow[r, "\overline{I}_g"] \arrow[d] & (\overline{E}_{K^g}, \overline{\varphi}_{K^g}) \arrow[d] \\
\overline{\mathbf{M}}^r_K \arrow[r, "\overline{J}_g"] & \overline{\mathbf{M}}^r_{K^g}.
\end{tikzcd} \tag{4.13}$$

In particular, if $g \in K^v$ then $K^g = K$ and $\overline{J}_g$ and $\overline{I}_g$ are the identity. Similarly, write $\omega_K := \mathrm{Lie}(\overline{E}_K)^\vee$, which is an ample invertible sheaf on $\overline{\mathbf{M}}^r_K$ over $A_{(v)}$.

**Definition 4.11.** (1) For any integer $k \geq 0$, *fine* open compact subgroup $K = K_v K^v \subset G(\mathbb{A}^\infty)$ with $K_v = G(A_v)$ and $A_{(v)}$-algebra $L$, denote by

$$M_k(r, K, L) := H^0(\overline{\mathbf{M}}^r_K \otimes_{A_{(v)}} L,\ \omega_K^{\otimes k} \otimes L) \tag{4.14}$$

the $L$-module of *algebraic Drinfeld modular forms of rank $r$, weight $k$, level $K$ over $L$*. The definition of $M_k(r, K, L)$ in (4.14) agrees with that in Definition 4.1 (noting $\overline{\mathbf{M}}^r_K \otimes_{A[\mathfrak{n}_K^{-1}]} \otimes L = \overline{\mathbf{M}}^r_{K^v} \otimes_{A_{(v)}} L$). Thus, there should be no danger of confusion.

(2) For $\widetilde{K}^v \subset K^v \subset G(\mathbb{A}^{\infty v})$ fine open compact subgroups, the pull-back under $\pi_{\widetilde{K},K} : \overline{\mathbf{M}}^r_{\widetilde{K}^v} \to \overline{\mathbf{M}}^r_{K^v}$ yields a map $\pi^*_{\widetilde{K},K} : M_k(r, K_v K^v, L) \to M_k(r, K_v \widetilde{K}^v, L)$, which is injective by Lemma 4.6. Define

$$M_k(r, K_v, L) := \varinjlim_{K^v} M_k(r, K_v K^v, L), \quad M(r, K_v, L) := \bigoplus_{k \geq 0} M_k(r, K_v, L). \tag{4.15}$$

(3) Let $k \geq 0$ and $K$ be as in (1). If $L$ is a flat $A_{(v)}$-algebra we set $\widetilde{M}_k(r, K, L) := M_k(r, K, L)$ and $\widetilde{M}_k(r, K_v, L) := M_k(r, K_v, L)$. On the other hand, if $L$ is an $\mathbb{F}_v$-algebra we consider the normalization $\overline{\mathscr{M}}^{r,\mathrm{nor}}_K$ of $\overline{\mathscr{M}}^r_K := \overline{\mathbf{M}}^r_K \otimes_{A_{(v)}} \mathbb{F}_v$ and set

$$\widetilde{M}_k(r, K, L) := H^0(\overline{\mathscr{M}}^{r,\mathrm{nor}}_K \otimes_{\mathbb{F}_v} L,\ \omega_K^{\otimes k} \otimes L)\,. \tag{4.16}$$

By Proposition 4.7 we have $M_k(r, K, L) \subset \widetilde{M}_k(r, K, L)$. Moreover, by Proposition 4.7 the map $\pi^*_{\widetilde{K},K} : \widetilde{M}_k(r, K_v K^v, L) \to \widetilde{M}_k(r, K_v \widetilde{K}^v, L)$ is injective, and we define

$$\widetilde{M}_k(r, K_v, L) := \varinjlim_{K^v} \widetilde{M}_k(r, K_v K^v, L)\,. \tag{4.17}$$

Then $M_k(r, K_v, L) \subset \widetilde{M}_k(r, K_v, L)$.

*Remark* 4.12. Since $\overline{\mathscr{M}}^{r,\mathrm{nor}}_K \to \overline{\mathscr{M}}^r_K$ is finite, the pullback of $\omega_K$ to $\overline{\mathscr{M}}^{r,\mathrm{nor}}_K$, which we again denote $\omega_K$, is ample by [EGA, II, Corollaire 4.6.6 and Proposition 5.1.12]. In particular, the analogously defined $L$-modules $M_k(r, K, L)$ and $\widetilde{M}_k(r, K, L)$ for $k < 0$ are zero by Lemmas 3.9 and 4.2.

*Remark* 4.13. Take the projective limit of the tower of schemes $\overline{\mathbf{M}}^r_{K_v K^v}$

$$\overline{\mathbf{M}}^r_{K_v} := \varprojlim_{K^v} \overline{\mathbf{M}}^r_{K_v K^v}\,. \tag{4.18}$$

Note that $\overline{\mathbf{M}}^r_{K_v}$ is a scheme as the transition maps are affine and it has a right continuous action of $G(\mathbb{A}^{\infty v})$. Then $M_k(r, K_v, L) = H^0(\overline{\mathbf{M}}^r_{K_v} \otimes_{A_{(v)}} L, \omega^{\otimes k} \otimes_{A_{(v)}} L)$. Moreover, for any fine open compact subgroup $K = K_v K^v$, one has $\overline{\mathbf{M}}^r_{K_v}/K^v \simeq \overline{\mathbf{M}}^r_{K_v K^v}$ by Theorem 3.2(2).

We now describe the left action of $G(\mathbb{A}^{\infty v})$ on the $L$-modules $M_k(r, K_v, L)$ and $\widetilde{M}_k(r, K_v, L)$. For any element $g \in G(\mathbb{A}^{\infty v})$, the canonical isomorphisms in (4.13) gives a canonical isomorphism $\omega_K^{\otimes k} \simeq \overline{J}_g^* \omega_{K^g}^{\otimes k}$. Using the adjoint isomorphism $\omega_{K^g}^{\otimes k} \xrightarrow{\sim} \overline{J}_{g*} \overline{J}_g^* \omega_{K^g}^{\otimes k}$, we get isomorphisms

$$\begin{aligned} & H^0(\overline{\mathbf{M}}^r_{K^g} \otimes_{A_{(v)}} L,\ \omega_{K^g}^{\otimes k} \otimes L) \xrightarrow{\sim} H^0(\overline{\mathbf{M}}^r_{K^g} \otimes_{A_{(v)}} L,\ \overline{J}_{g*} \overline{J}_g^* \omega_{K^g}^{\otimes k} \otimes L) \\ & \xrightarrow{\sim} H^0(\overline{\mathbf{M}}^r_K \otimes_{A_{(v)}} L,\ \overline{J}_g^* \omega_{K^g}^{\otimes k} \otimes L) \xrightarrow{\sim} H^0(\overline{\mathbf{M}}^r_K \otimes_{A_{(v)}} L,\ \omega_K^{\otimes k} \otimes L), \end{aligned} \tag{4.19}$$

and an isomorphism $T_g : M_k(r, K^g, L) \xrightarrow{\sim} M_k(r, K, L)$. For $g_1, g_2 \in G(\mathbb{A}^{\infty v})$, one has $T_{g_1} \circ T_{g_2} = T_{g_1 g_2}$. Taking the inductive limit, the group $G(\mathbb{A}^{\infty v})$ naturally acts on $M_k(r, K_v, L)$. The group $G(\mathbb{A}^{\infty v})$ acts as automorphisms of the graded rings on $M(r, K_v, L)$, that is, one has

$$T_g(f_1 \cdot f_2) = T_g(f_1) \cdot T_g(f_2), \quad \forall\, f_1, f_2 \in M(r, K_v, L). \tag{4.20}$$

Likewise, applying base change to $\mathbb{F}_v$ and normalization to the isomorphisms in (4.13) yields a commutative diagram

$$\begin{array}{ccc}(\overline{E}_K, \overline{\varphi}_K) & \xrightarrow{\overline{I}_g} & (\overline{E}_{K^g}, \overline{\varphi}_{K^g}) \\ \downarrow & & \downarrow \\ \overline{\mathscr{M}}_K^{r,\mathrm{nor}} & \xrightarrow{\overline{J}_g} & \overline{\mathscr{M}}_{K^g}^{r,\mathrm{nor}}\,. \end{array}$$

in which the horizontal maps $\overline{I}_g$ and $\overline{J}_g$ are isomorphisms. Here we abuse notation and denote the pullback of $(\overline{E}_K, \overline{\varphi}_K)$ from $\overline{\mathbf{M}}_K^r$ to $\overline{\mathscr{M}}_K^{r,\mathrm{nor}}$ again by $(\overline{E}_K, \overline{\varphi}_K)$. If now $L$ is an $\mathbb{F}_v$-algebra, the same reasoning as above applied to $\overline{\mathscr{M}}_K^{r,\mathrm{nor}}$ produces an isomorphism $T_g : \widetilde{M}_k(r, K^g, L) \xrightarrow{\sim} \widetilde{M}_k(r, K, L)$ and an action of $G(\mathbb{A}^{\infty v})$ on $\widetilde{M}_k(r, K_v, L)$.

**Theorem 4.14.** *(1) Let $L$ be a flat $A_{(v)}$-algebra or a noetherian $\mathbb{F}_v$-algebra. Then the actions of $G(\mathbb{A}^{\infty v})$ on $M_k(r, K_v, L)$ and $\widetilde{M}_k(r, K_v, L)$ are*

- smooth *(in the sense that every element of $M_k(r, K_v, L)$ and $\widetilde{M}_k(r, K_v, L)$ has an open stabilizer) and*
- admissible *(in the sense that for every open compact subgroup $K^v \subset G(\mathbb{A}^{\infty v})$ the fixed points $M_k(r, K_v, L)^{K^v}$ and $\widetilde{M}_k(r, K_v, L)^{K^v}$ form finitely generated $L$-modules).*

*In particular, for every open compact subgroup $K^v \subset G(\mathbb{A}^{\infty v})$ there is an open compact subgroup $\widetilde{K}^v \subset K^v$ such that $M_k(r, K_v, L)^{K^v} \subset M_k(r, K_v\widetilde{K}^v, L)$.*

*(2) If $L$ is a flat $A_{(v)}$-algebra then $M_k(r, K_vK^v, L) = M_k(r, K_v, L)^{K^v}$ for any open compact subgroup $K^v \subset G(\mathbb{A}^{\infty v})$.*

*(3) If $L$ is an $\mathbb{F}_v$-algebra then $\widetilde{M}_k(r, K_vK^v, L) = \widetilde{M}_k(r, K_v, L)^{K^v}$ for any open compact subgroup $K^v \subset G(\mathbb{A}^{\infty v})$.*

Proof. (2) and (3) follow from the isomorphisms (4.8) in Lemma 4.6(3) and (4.9) in Proposition 4.7, respectively.

(1) To prove smoothness let $f \in M_k(r, K_v, L)$ or $f \in \widetilde{M}_k(r, K_v, L)$. Then there exists an open compact subgroup $K^v \subset G(\mathbb{A}^{\infty v})$ such that $f \in M(r, K_vK^v, L)$ or $f \in \widetilde{M}(r, K_vK^v, L)$, respectively, and then $K^v$ stabilizes $f$.

To prove admissibility, let $K^v \subset G(\mathbb{A}^{\infty v})$ be an open compact subgroup. If $L$ is a flat $A_{(v)}$-algebra then (2) implies that $\widetilde{M}_k(r, K_v, L)^{K^v} := M_k(r, K_v, L)^{K^v} = M_k(r, K_vK^v, L)$ is a finitely generated $L$-module. On the other hand, if $L$ is an $\mathbb{F}_v$-algebra then (3) implies that $\widetilde{M}_k(r, K_v, L)^{K^v} = \widetilde{M}_k(r, K_vK^v, L)$ is a finitely generated $L$-module. If $L$ is moreover noetherian, then $M_k(r, K_v, L)^{K^v} \subset \widetilde{M}_k(r, K_v, L)^{K^v} = \widetilde{M}_k(r, K_vK^v, L)$ is also finitely generated.

The last assertion of (1) follows from the finiteness of $M_k(r, K_v, L)^{K^v}$ as $L$-submodule of the inductive limit $M_k(r, K_v, L) = \varinjlim M_k(r, K_v\widetilde{K}^v, L)$. ∎

## 4.3. Systems of Hecke eigenvalues in $\overline{\mathbb{F}}_v$.

From [Her11, HV15] we recall the following

**Definition 4.15.** Set $\mathbb{G} := G(\mathbb{A}^{\infty v})$, let $K^v \subset \mathbb{G}$ be an open compact subgroup, and let $R$ be any commutative ring. The *Hecke algebra* $\mathcal{H}_R(G(\mathbb{A}^{\infty v}), K^v)$ is the convolution $R$-algebra

$$\mathcal{H}_R(G(\mathbb{A}^{\infty v}), K^v) := \big\{\, h\colon \mathbb{G} \longrightarrow R \text{ functions with compact support } \big| \; h(k_1gk_2) = h(g) \text{ for all } g \in G(\mathbb{A}^{\infty v}) \text{ and } k_1, k_2 \in K^v \,\big\}\,.$$

Here compact support means that $h$ is zero outside a finite union of cosets $K^vgK^v$. The multiplication is defined by *convolution*

$$(\tilde{h} * h)(g) \;:=\; \sum_{\tilde{g} \in \mathbb{G}/K^v} \tilde{h}(\tilde{g}) \cdot h(\tilde{g}^{-1}g) \;=\; \int_{\mathbb{G}} \tilde{h}(\tilde{g}) \cdot h(\tilde{g}^{-1}g)\, d\tilde{g}\,,$$

where $d\tilde{g}$ is the left-invariant Haar measure on $\mathbb{G}$ with $\mathrm{vol}(K^v)=1$. The characteristic function $\mathbf{1}_{K^v}$ of $K^v$ is the unit element.

The Hecke algebra is isomorphic to the endomorphism algebra $\mathrm{End}_{R[\mathbb{G}]}\bigl(\mathrm{ind}_{K^v}^{\mathbb{G}}(\mathbb{1})\bigr)$ of the compact induction

$$\mathrm{ind}_{K^v}^{\mathbb{G}}(\mathbb{1}) \;:=\; \bigl\{\, f\colon \mathbb{G}\to R \text{ functions with compact support} \bigm| f(kg)=f(g) \text{ for all } k\in K^v, g\in\mathbb{G}\bigr\}$$

of the trivial $K^v$-representation $\mathbb{1}=R$. Here $\mathrm{ind}_{K^v}^{\mathbb{G}}(\mathbb{1})$ is an $R[\mathbb{G}]$-module by right translation, that is, $\tilde{g}\in\mathbb{G}$ maps $f\in\mathrm{ind}_{K^v}^{\mathbb{G}}(\mathbb{1})$ to $\rho_{\tilde{g}}(f)$ which is defined by $\rho_{\tilde{g}}(f)(g)=f(g\tilde{g})$. The isomorphism $\mathcal{H}_R(G(\mathbb{A}^{\infty v}),K^v)\xrightarrow{\sim}\mathrm{End}_{R[\mathbb{G}]}\bigl(\mathrm{ind}_{K^v}^{\mathbb{G}}(\mathbb{1})\bigr)$ sends $h\in\mathcal{H}_R(G(\mathbb{A}^{\infty v}),K^v)$ to the endomorphism $h$ of $\mathrm{ind}_{K^v}^{\mathbb{G}}(\mathbb{1})$

$$h\colon f\;\longmapsto\; h*f\,,\ \text{defined by}\quad (h*f)(g)\;:=\;\sum_{\tilde{g}\in\mathbb{G}/K^v} h(\tilde{g})\cdot f(\tilde{g}^{-1}g)\;=\;\int_{\mathbb{G}} h(\tilde{g})\cdot f(\tilde{g}^{-1}g)\,d\tilde{g}\,.$$

From now on we assume that $K^v=\prod_{v'\nmid\infty v}K_{v'}$ with open compact subgroups $K_{v'}\subset G(F_{v'})$ with $K_{v'}=G(A_{v'})$ for almost all $v'$. Then the Hecke algebra $\mathcal{H}_R(G(\mathbb{A}^{\infty v}),K^v)$ decomposes into local Hecke algebras

$$\mathcal{H}_R(G(\mathbb{A}^{\infty v}),K^v)\;=\;\sideset{}{'}\bigotimes_{v'\nmid v\infty}\mathcal{H}_R(G(F_{v'}),K_{v'})$$

which are defined analogously. Here $\otimes'$ denotes the restricted tensor product with respect to the unit elements $\mathbf{1}_{K_{v'}}\in\mathcal{H}_R(G(F_{v'}),K_{v'})$ for almost all places $v'$.

Let $\mathfrak{n}\subset A$ be a non-zero ideal, prime to $v$ such that $K(\mathfrak{n})$ is contained in a conjugate of $K=K_vK^v$. When $v'\nmid\infty v\mathfrak{n}$, then the open compact subgroup $K_{v'}$ is maximal and hyperspecial, and $\mathcal{H}_R(G(F_{v'}),K_{v'})$ is commutative by [HV15, § 1.5 Theorem and Remark], or [Her11, Theorem 2.6]. This can also be seen in an elementary way called "Gelfand's trick" as follows. Let $g'\in G(F_{v'})$ be such that ${g'}^{-1}K_{v'}g'=G(A_{v'})$. Then there is an isomorphism of $R$-algebras

$$\begin{aligned}\mathcal{H}_R(G(F_{v'}),K_{v'})\;&\xrightarrow{\;\sim\;}\;\mathcal{H}_R\bigl(G(F_{v'}),G(A_{v'})\bigr)\\ h\;&\longmapsto\; h\circ\mathrm{Int}_{g'}\qquad\text{where}\qquad h\circ\mathrm{Int}_{g'}: g\mapsto h(g'g{g'}^{-1})\,.\end{aligned}$$

The $R$-algebra $\mathcal{H}_R\bigl(G(F_{v'}),G(A_{v'})\bigr)$ has an involution

$$\iota\colon h\longmapsto {}^{\iota}h\qquad\text{where}\qquad {}^{\iota}h\colon g\mapsto h({}^tg)$$

and ${}^tg\in G(F_{v'})$ denotes the transpose of $g\in G(F_{v'})$. This means that ${}^{\iota}(\tilde{h}*h)={}^{\iota}h*{}^{\iota}\tilde{h}$. By the elementary divisor theorem every double coset $G(A_{v'})gG(A_{v'})$ has a representative $g$ which is a diagonal matrix, and hence satisfies ${}^tg=g$. This shows that $\iota$ is the identity on $\mathcal{H}_R\bigl(G(F_{v'}),G(A_{v'})\bigr)$ and proves the commutativity $\tilde{h}*h=h*\tilde{h}$ of $\mathcal{H}_R\bigl(G(F_{v'}),G(A_{v'})\bigr)$.

For brevity we put $\mathcal{H}_L^{\infty v}:=\mathcal{H}_L(G(\mathbb{A}^{\infty v}),K^v)$ for any $\mathbb{F}_v$-algebra $L$. By Theorem 4.14, the spaces $M_k(r,K_v,L)^{K^v}$ and $\widetilde{M}_k(r,K_v,L)^{K^v}=\widetilde{M}_k(r,K_vK^v,L)$ are finite $L$-modules equipped with an $\mathcal{H}_L^{\infty v}$-module structure given for $f\in M_k(r,K_v,L)^{K^v}$ or $f\in\widetilde{M}_k(r,K_v,L)^{K^v}$ and $h=\sum_i n_i\cdot\mathbf{1}_{g_iK^v}\in\mathcal{H}_L^{\infty v}$ with $n_i\in L$ by the rule

$$T(h)(f)\;=\;\sum_{\tilde{g}\in\mathbb{G}/K^v}h(\tilde{g})\cdot T_{\tilde{g}}(f)\;=\;\sum_i n_i\cdot T_{g_i}(f)\;=\;\int_{\mathbb{G}}h(\tilde{g})\cdot T_{\tilde{g}}(f)\,d\tilde{g}\tag{4.21}$$

where again $d\tilde{g}$ is the left-invariant Haar measure on $\mathbb{G}$ with $\mathrm{vol}(K^v) = 1$. In particular, with $g_1 := \tilde{g}g$ and $g = \tilde{g}^{-1}g_1$ one computes as usual

$$
\begin{aligned}
(4.22) \qquad T(\tilde{h})\big(T(h)(f)\big) &= \int_{\mathbb{G}} \tilde{h}(\tilde{g}) \cdot T_{\tilde{g}}\big(\int_{\mathbb{G}} h(g) \cdot T_g(f)\, dg\big)\, d\tilde{g} \\
(4.23) \qquad &= \int_{\mathbb{G}}\int_{\mathbb{G}} \tilde{h}(\tilde{g}) \cdot h(g) \cdot T_{\tilde{g}} \circ T_g(f)\, dg\, d\tilde{g} \\
&= \int_{\mathbb{G}}\int_{\mathbb{G}} \tilde{h}(\tilde{g}) \cdot h(\tilde{g}^{-1}g_1) \cdot T_{g_1}(f)\, d\tilde{g}\, dg_1 \\
&= T(\tilde{h} * h)(f)\,.
\end{aligned}
$$

We now specialize to the case where $L = \overline{\mathbb{F}}_v$. The commutativity of the Hecke algebra $\mathcal{H}^{\infty v\mathfrak{n}}_{\overline{\mathbb{F}}_v} := \bigotimes'_{v'\nmid\infty v\mathfrak{n}} \mathcal{H}_{\overline{\mathbb{F}}_v}(G(F_{v'}), K_{v'})$ allows us to make the following definition.

**Definition 4.16.** Let $\mathfrak{n}$ and $K = K_vK^v$ be as above. A Drinfeld modular form $0 \neq f \in M_k(r, K_v, \overline{\mathbb{F}}_v)^{K^v}$ is said to be a *Hecke eigenform for prime-to-$v\mathfrak{n}$ Hecke operators* if for any place $v' \nmid \infty v\mathfrak{n}$ and any element $h \in \mathcal{H}_{v',\overline{\mathbb{F}}_v} := \mathcal{H}_{\overline{\mathbb{F}}_v}(G(F_{v'}), K_{v'})$, one has

$$(4.24) \qquad T(h)(f) \;=\; a_{v'}(h)f, \quad \text{for some } a_{v'}(h) \in \overline{\mathbb{F}}_v.$$

Then by formula (4.22) the map $a_{v'} \colon \mathcal{H}_{v',\overline{\mathbb{F}}_v} \to \overline{\mathbb{F}}_v$ is a homomorphism of $\overline{\mathbb{F}}_v$-algebras, and is called the *character* of $f$ at $v'$. The collection $(a_{v'})_{v'\nmid\infty v\mathfrak{n}}$ of characters $a_{v'}$, or equivalently, the character

$$(a_{v'})_{v'} \,:\, \mathcal{H}^{\infty v\mathfrak{n}}_{\overline{\mathbb{F}}_v} := \sideset{}{'}\bigotimes_{v\nmid\infty v\mathfrak{n}} \mathcal{H}_{v',\overline{\mathbb{F}}_v} \longrightarrow \overline{\mathbb{F}}_v$$

is called the *system of Hecke eigenvalues (or the Hecke eigensystem) of $f$.*

We make the analogous definition for $f \in \widetilde{M}_k(r, K_v, \overline{\mathbb{F}}_v)^{K^v} = \widetilde{M}_k(r, K_vK^v, \overline{\mathbb{F}}_v)$.

We are interested in Hecke eigensystems arising from algebraic Drinfeld Hecke eigenforms over $\overline{\mathbb{F}}_v$ for *all* weights and will determine them in Theorem 6.10.

*Remark* 4.17. Our discussions also cover the case where $L = \mathbb{C}_\infty := \widehat{\overline{F_\infty}}$ the completion of an algebraic closure of the completion $F_\infty$ of $F$ at $\infty$. Namely, we have the space $M_k(r, K, \mathbb{C}_\infty)$ of algebraic Drinfeld modular forms of rank $r$ and level $K$ over $\mathbb{C}_\infty$, and can consider Hecke eigenforms and Hecke eigensystems over $\mathbb{C}_\infty$. The comparison theorem for algebraic and analytic Drinfeld modular forms proved by Basson, Breuer and Pink [BBP18b, Theorem 10.9] (cf. our Theorem 4.8) indicates an alternative way of studying the action of Hecke operators on these modular forms by analysis. This has been accomplished for $r = 2$ by Gekeler and others. We explain how to compare Hecke eigensystems arising from $M_k(r, K, \mathbb{C}_\infty)$ and $M_k(r, K, \overline{\mathbb{F}}_v)$. Suppose $f \in M_k(r, K, \mathbb{C}_\infty)$ is a prime-to-$v\mathfrak{n}$ eigenform and let $(a_{v'}) : \mathcal{H}^{\infty v\mathfrak{n}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}_\infty \to \mathbb{C}_\infty$ be the associated Hecke character, where $\mathcal{H}^{\infty v\mathfrak{n}}_{\mathbb{Z}} := \mathcal{H}_{\mathbb{Z}}(G(\mathbb{A}^{\infty v\mathfrak{n}}), K^{v\mathfrak{n}})$ and $K^{v\mathfrak{n}}$ is the prime-to-$\mathfrak{n}$ part of $K^v$. Now $M_k(r, K, \mathbb{C}_\infty) = M_k(r, K, A_{(v)}) \otimes_{A_{(v)}} \mathbb{C}_\infty$ by [EGA, $\mathrm{I}_{\text{new}}$, Proposition 9.3.2] and any prime-to-$v\mathfrak{n}$ Hecke operator $T(h)$ for $h \in \mathcal{H}^{\infty v\mathfrak{n}}_{\mathbb{Z}}$ leaves the $A_{(v)}$-module $M_k(r, K, A_{(v)})$ invariant, which is finite by Theorem 4.14. Consider the $A_{(v)}$-subalgebra $\mathcal{T}_{A_{(v)}}$ of $\mathrm{End}_{A_{(v)}}\big(M_k(r, K, A_{(v)})\big)$ generated by all $T(h)$ for $h \in \mathcal{H}^{\infty v\mathfrak{n}}_{\mathbb{Z}}$, or equivalently, $\mathcal{T}_{A_{(v)}}$ is the image of $\mathcal{H}^{\infty v\mathfrak{n}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} A_{(v)} \to \mathrm{End}_{A_{(v)}}\big(M_k(r, K, A_{(v)})\big)$. Then $\mathcal{T}_{A_{(v)}}$ is an $A_{(v)}$-algebra which is finite as an $A_{(v)}$-module, because the same holds for $\mathrm{End}_{A_{(v)}}\big(M_k(r, K, A_{(v)})\big)$ and $A_{(v)}$ is noetherian. It follows that $\mathcal{T}_{A_{(v)}} \otimes_{A_{(v)}} \overline{\mathbb{F}}_v$ surjects onto the image of $\mathcal{H}^{\infty v\mathfrak{n}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_v \to \mathrm{End}_{\overline{\mathbb{F}}_v}\big(M_k(r, K, A_{(v)}) \otimes_{A_{(v)}} \overline{\mathbb{F}}_v\big)$. Note that for $k \gg 0$ we have $M_k(r, K, A_{(v)}) \otimes_{A_{(v)}} \overline{\mathbb{F}}_v = M_k(r, K, \overline{\mathbb{F}}_v)$ and $M_k(r, K, A_{(v)})$ is finite projective by Corollary 4.5. In this case the kernel of $\mathcal{T}_{A_{(v)}} \otimes_{A_{(v)}} \overline{\mathbb{F}}_v \to \mathrm{End}_{\overline{\mathbb{F}}_v}\big(M_k(r, K, \overline{\mathbb{F}}_v)\big)$ is a nilpotent ideal by [Bel10, Proposition I.4.1] and we denote the image by $\mathcal{T}_{\overline{\mathbb{F}}_v}$.

For every $h \in \mathcal{H}^{\infty v\mathfrak{n}}_{\mathbb{Z}}$ the image of $T(h)$ in $\mathcal{T}_{A_{(v)}}$ satisfies a monic polynomial with coefficients in $A_{(v)}$ by [Eis95, Theorem 4.3], and hence the eigenvalues of $T(h)$ on $M_k(r, K, \mathbb{C}_\infty)$ are all $v$-adically integral. Let $F(f) \subset \mathbb{C}_\infty$ be the field generated by the eigenvalues of all $T(h)$ on $f$. Since $\mathcal{T}_{A_{(v)}}$ is finitely generated, $F(f)$ is a finite field extension of $F$ and the character $(a_{v'})_{v'} : \mathcal{H}_{\mathbb{Z}}^{\infty v \mathrm{n}} \to \mathbb{C}_\infty$ factors through the integral closure $R_f$ of $A_v$ in $F(f)$. In other words, $f$ defines a character $\chi_f \colon \mathcal{T}_{A_{(v)}} \to R_f$. Modulo any maximal ideal $\mathfrak{m}$ of $R_f$ we obtain a Hecke eigensystem $(\bar{a}_{v'}^{\mathfrak{m}})_{v'}$ with values in $\overline{\mathbb{F}}_v$. We say the collection of characters $(\bar{a}_{v'}^{\mathfrak{m}})_{v'}$ for all maximal ideals $\mathfrak{m}$ of $R_f$ is the *reduction modulo* $v$ of $(a_{v'})_{v'}$. This collection is a subset of $\mathrm{Hom}_{A_{(v)}}(\mathcal{T}_{A_{(v)}}, \overline{\mathbb{F}}_v) = \mathrm{Hom}_{\overline{\mathbb{F}}_v}(\mathcal{T}_{A_{(v)}} \otimes_{A_{(v)}} \overline{\mathbb{F}}_v, \overline{\mathbb{F}}_v)$. Assume now that $M_k(r, K, A_{(v)}) \otimes_{A_{(v)}} \overline{\mathbb{F}}_v = M_k(r, K, \overline{\mathbb{F}}_v)$ and $M_k(r, K, A_{(v)})$ is finite projective. Then every character $(\bar{a}_{v'}^{\mathfrak{m}})_{v'}$ for fixed $\mathfrak{m}$ induces an element of $\mathrm{Hom}_{\overline{\mathbb{F}}_v}(\mathcal{T}_{\overline{\mathbb{F}}_v}, \overline{\mathbb{F}}_v)$, as the kernel of $\mathcal{T}_{A_{(v)}} \otimes_{A_{(v)}} \overline{\mathbb{F}}_v \to \mathcal{T}_{\overline{\mathbb{F}}_v}$ is nilpotent. Then $(\bar{a}_{v'}^{\mathfrak{m}})_{v'}$ is the Hecke eigensystem of a Hecke eigenform $\bar{g}$ in $M_k(r, K, \overline{\mathbb{F}}_v)$ by [Bel10, Theorem I.5.9]. But note that there is no reduction map of Hecke eigenforms in general producing $\bar{g}$ from $f$. Many more such considerations can be found in [Bel10], for instance on the question of lifting Hecke eigensystems from $\overline{\mathbb{F}}_v$ to finite extensions of the completion of $A_{(v)}$ in [Bel10, Lemma I.7.9].

## 5. The Supersingular locus and Hecke modules

Let $F$, $\infty$, $A$ be as in previous sections. Let $v$ be a finite place and $\mathfrak{p} \subset A$ the corresponding prime ideal. Denote by $\mathbb{F}_v := A/\mathfrak{p}$ the residue field at $v$ and $\overline{\mathbb{F}}_v$ its algebraic closure, regarded as $A$-fields. The cardinality of $\mathbb{F}_v$ is denoted by $q_v$.

### 5.1. The $v$-rank stratification.

**Definition 5.1.** Let $\varphi$ be a Drinfeld $A$-module over an $\mathbb{F}_v$-field $L$.

(1) The *$v$-rank* of $\varphi$, denoted by $v$-rank$(\varphi)$, is the non-negative integer $j$ with $\varphi[\mathfrak{p}](\bar{L}) \simeq (\mathfrak{p}^{-1}/A)^j$. The integer $h = r - j$ is called the *height* of $\varphi$, where $r$ is the rank of $\varphi$.

(2) We call $\varphi$ *supersingular* if $v$-rank$(\varphi) = 0$.

The $v$-rank $j$ of $\varphi$ satisfies $0 \le j \le r-1$ and drops under specialization as $j$ is the étale rank of $\varphi[\mathfrak{p}]$.

Let $K = K_v K^v$ where $K_v = G(A_v)$. Let $\mathbf{M}_K^r = \mathbf{M}_{K_v K^v}^r$ be the Drinfeld moduli scheme of rank $r$ over $A_{(v)}$ with level-$K^v$ structure as in Section 4.2. It is a coarse moduli scheme if $K$ is not fine. Denote by $\mathscr{M}_K^r := \mathbf{M}_K^r \otimes_{A_{(v)}} \mathbb{F}_v$ the special fiber of $\mathbf{M}_K^r$. Let

$$(\mathscr{M}_K^r)_{\le j}(\overline{\mathbb{F}}_v) := \{(\varphi, \bar{\eta}) \in (\mathscr{M}_K^r)(\overline{\mathbb{F}}_v) \mid v\text{-rank}(\varphi) \le j\} \tag{5.1}$$

be the closed subset consisting of all Drinfeld $A$-modules of $v$-rank $\le j$. We regard $(\mathscr{M}_K^r)_{\le j}$ as a closed subscheme of $\mathscr{M}_K^r$ with the induced reduced structure. One can show that each stratum $(\mathscr{M}_K^r)_{\le j}$ is stable under the $\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v)$-action. Thus, each $(\mathscr{M}_K^r)_{\le j}$ is defined over $\mathbb{F}_v$. Put $(\mathscr{M}_K^r)_{(j)} := (\mathscr{M}_K^r)_{\le j} - (\mathscr{M}_K^r)_{\le j-1}$, which is a reduced locally closed subscheme consisting of all Drinfeld $A$-modules of $v$-rank $j$. One has the $v$-rank stratification

$$\mathscr{M}_K^r = \coprod_{0 \le j \le r-1} (\mathscr{M}_K^r)_{(j)}. \tag{5.2}$$

For each Drinfeld $A$-module $\varphi$ over $\overline{\mathbb{F}}_v$, the associated $v$-divisible $A_v$-module $\varphi[\mathfrak{p}^\infty] = \varphi[\mathfrak{p}^\infty]^{\mathrm{loc}} \oplus \varphi[\mathfrak{p}^\infty]^{\mathrm{et}}$ canonically decomposes into the local and étale parts. The étale part is isomorphic to $(F_v/A_v)^j$, where $j$ is the $v$-rank of $\varphi$, while the local part is a formal $A_v$-module of height $h$. By [Dri76, Proposition 1.17], any two formal $A_v$-modules of the same height $h$ over $\overline{\mathbb{F}}_v$ are isomorphic. Thus, the associated $v$-divisible $A_v$-modules $\varphi[\mathfrak{p}^\infty]$ are geometrically constant on each $v$-rank stratum. So each $v$-rank stratum $(\mathscr{M}_K^r)_{(j)}$ is a *central leaf* in the sense of Oort.

**Theorem 5.2.** *Assume that $K = K_v K^v$ is fine, where $K_v = G(A_v)$. For any integer $j$ with $0 \le j \le r-1$, the subscheme $(\mathscr{M}_K^r)_{(j)}$ is non-empty, smooth over $\mathbb{F}_v$, of pure dimension $j$. The closed scheme $(\mathscr{M}_K^r)_{\le j}$ is smooth over $\mathbb{F}_v$ of pure dimension $j$.*

Proof. See [Boy99, Theorem 10.33]. ∎

For each integer $h$ with $1 \leq h \leq r$, we write $(\mathscr{M}_K^r)^{\geq h}$ and $(\mathscr{M}_K^r)^{(h)}$ for $(\mathscr{M}_K^r)_{\leq r-h}$ and $(\mathscr{M}_K^r)_{(r-h)}$, respectively. Then the $v$-rank stratification is the same as the stratification by height: $\mathscr{M}_K^r = \coprod_{1\leq h\leq r}(\mathscr{M}_K^r)^{(h)}$. However, we will see in Section 6.2 that the height stratification behaves better when we work with the Satake compactification $\overline{\mathscr{M}}_K^r$ of $\mathscr{M}_K^r$. For example, we have $(\overline{\mathscr{M}}_K^r)_{(r-1)} = (\mathscr{M}_K^r)_{(r-1)}$, because $\mathrm{rank}(\varphi) \geq 1 + v\text{-rank}(\varphi)$, and hence the boundary $\partial\overline{\mathscr{M}}_K^r$ is contained in $(\overline{\mathscr{M}}_K^r)_{\leq r-2}$. The $v$-rank stratum $(\overline{\mathscr{M}}_K^r)_{(r-2)}$ then consists of two parts: $(\mathscr{M}_K^r)_{(r-2)}$ and $(\partial\overline{\mathscr{M}}_K^r)_{(r-2)}$. We will see that $(\mathscr{M}_K^r)_{(r-2)}$ is of dimension $r-2$. By (the proof of) Proposition 2.10 also $(\partial\overline{\mathscr{M}}_K^r)_{(r-2)}$ is of dimension $r-2$. Therefore, $(\mathscr{M}_K^r)_{(r-2)}$ will not be dense in $(\overline{\mathscr{M}}_K^r)_{(r-2)}$.

5.2. **Supersingular Drinfeld modules.** Let $\mathbb{F}_{v^m} \subset \overline{\mathbb{F}}_v$ be the finite extension of $\mathbb{F}_v$ of degree $m$.

**Proposition 5.3.** *Let $\varphi$ be a Drinfeld $A$-module of rank $r$ over one of the finite fields $\mathbb{F}_{v^m}$. The following statements are equivalent.*

- *(a) $\varphi$ is supersingular.*
- *(b) The endomorphism algebra $D := \mathrm{End}^0(\varphi \otimes \overline{\mathbb{F}}_v) := \mathrm{End}(\varphi \otimes \overline{\mathbb{F}}_v) \otimes_A F$ of $\varphi$ over $\overline{\mathbb{F}}_v$ is a central division $F$-algebra of dimension $r^2$.*
- *(c) Some power of the Frobenius endomorphism of $\varphi$ lies in $A$.*

*In this case the Hasse invariants of $D$ are $\mathrm{inv}_v D = 1/r$ and $\mathrm{inv}_\infty D = -1/r$.*

Proof. See [Gek91, Prop. 4.1].

We recall the function field analogue of the Honda-Tate theorem proved by Drinfeld [Dri77]. We also refer to [Yu95] for a clear exposition with detailed proofs.

**Definition 5.4.** A *Weil number over $\mathbb{F}_{v^m}$ of rank $r$* is an element $\pi$ of $\overline{F}$ satisfying the following property.

1. (1) $\pi$ is integral over $A$.
2. (2) There is only one place $w$ of $F(\pi)$ which is a zero of $\pi$, i.e. with $w(\pi) > 0$. This place lies over $v$.
3. (3) There is only one place of $F(\pi)$ lying over $\infty$.
4. (4) $|\pi|_\infty = \#\mathbb{F}_v^{m/r}$, where $|\ |_\infty$ is the unique extension to $F(\pi)$ of the normalized absolute value $|\ |_\infty$ on $F$.
5. (5) $[F(\pi) : F]$ divides $r$.

Let $W^r_{v^m}$ denote the set of Galois conjugacy classes of Weil numbers of rank $r$ over $\mathbb{F}_{v^m}$. The analogous Honda-Tate theorem [Dri77, Yu95] states that the map sending $\varphi$ to its Frobenius endomorphism $\pi_\varphi$ induces a bijection

$$\{\text{isogeny classes of Drinfeld } A\text{-modules of rank } r \text{ over } \mathbb{F}_{v^m}\} \xrightarrow{\sim} W^r_{v^m}$$

Note that if $\pi$ is a Weil number of rank $r$ over $\mathbb{F}_{v^m}$ then it is also a Weil number of rank $nr$ over $\mathbb{F}_{v^{nm}}$ for any integer $n \geq 1$; namely, one has $W^{nr}_{v^{nm}} = W^r_{v^m}$.

A Weil number $\pi$ is said to be *supersingular* if the corresponding isogeny class of Drinfeld modules is supersingular. By Proposition 5.3, a Weil number $\pi$ is supersingular if and only if $\pi^n \in A$ for some $n \geq 1$.

Observe that if $\pi \in W^r_{v^m}$, then the element $N_{F(\pi)/F}(\pi) \in A$ generates the ideal $\mathfrak{p}^{m[F(\pi):F]/r}$; this follows from Defn. 5.4 (4) and the product formula. In particular, the ideal $\mathfrak{p}^m$ must be principal, because $[F(\pi) : F]|r$ by (5) and $\mathfrak{p}^m$ is a power of the principal ideal $\mathfrak{p}^{m[F(\pi):F]/r}$. Therefore, if $\pi$ is a Weil number over $\mathbb{F}_v$ then $\mathfrak{p}$ is necessarily principal. Let $m_v$ be the order of $\mathfrak{p}$ in the ideal class group $\mathrm{Cl}(A)$. Then any generator $P$ of $\mathfrak{p}^{m_v}$ is a supersingular Weil number of rank 1 over $\mathbb{F}_{v^{m_v}}$, or a supersingular Weil number of rank $r$ over $\mathbb{F}_{v^{m_v r}}$ for any positive integer $r$. In particular, there always exists a supersingular Drinfeld $A$-module over $\overline{\mathbb{F}}_v$ of rank $r$ for any $r \geq 1$.

**Lemma 5.5.** *Let $P$ be a generator of the principal ideal $\mathfrak{p}^{m_v}$. Every supersingular Drinfeld $A$-module $\varphi$ of rank $r$ over $\overline{\mathbb{F}}_v$ admits a unique model $\varphi'$ over $\mathbb{F}_{v^{m_v r}}$, up to $\mathbb{F}_{v^{m_v r}}$-isomorphism, with Frobenius endomorphism $\pi_{\varphi'} = P$.*

Lemma 5.5 is an analogous result of [Ghi04, Proposition 6] that every supersingular elliptic curve over $\overline{\mathbb{F}}_p$ admits a canonical $\mathbb{F}_{p^2}$-model. The proof is similar and omitted. We call $\varphi'$ the canonical model of $\varphi$ over $\mathbb{F}_{v^{m_v r}}$. It defines a natural $\mathbb{F}_{v^{m_v r}}$-structure on the space $\omega(\varphi) = \mathrm{Lie}(\varphi)^\vee$ of invariant differential forms of $\varphi$. Lemma 5.5 shows that the canonical model exists over $\mathbb{F}_{v^{m_v r}}$ for all $\varphi$; $\mathbb{F}_{v^{m_v r}}$ is the smallest field of definition by the Honda-Tate theory. We will show by another method in the next subsection that $\omega(\varphi)$ can actually have a natural $\mathbb{F}_{v^r}$-structure for all $\varphi$.

5.3. **Supersingular and algebraic Drinfeld modular forms mod $v$.** Let $\Sigma(r,v)$ be the set of isomorphism classes of supersingular Drinfeld $A$-modules of rank $r$ over $\overline{\mathbb{F}}_v$. We fix a member $\varphi_0$ in $\Sigma(r,v)$. Let $\mathcal{O}_D := \mathrm{End}(\varphi_0)$ and $D := \mathrm{End}(\varphi_0) \otimes F$. Then $D$ is the central division $F$-algebra (unique up to isomorphism) ramified precisely at $\infty$ and $v$, with invariants $\mathrm{inv}_\infty(D) = -1/r$ and $\mathrm{inv}_v(D) = 1/r$, see Proposition 5.3, and $\mathcal{O}_D$ is a maximal $A$-order in $D$; see [Dri76, Prop. 1.7] and [Yu95, Theorem 1]. Denote by $G'$ the group scheme over $A$ associated to the multiplicative group of $\mathcal{O}_D$: For any $A$-algebra $R$, the group of $R$-valued points of $G'$ is $G'(R) = (\mathcal{O}_D \otimes_A R)^\times$. By [Gek91, Theorem 4.3] (cf. [YuYu04, Corollary 3.3]), there is a natural bijection

$$\Sigma(r,v) \simeq G'(F)\backslash G'(\mathbb{A}^\infty)/G'(\widehat{A}). \tag{5.3}$$

Let $\mathscr{S}_K \subset \mathscr{M}^r_K$ be the supersingular locus with $K = K_v K^v$. It is an affine scheme, finite over $\mathbb{F}_v$. Let $\mathscr{S}_{K_v} := \lim_{\substack{\longleftarrow \\ K^v}} \mathscr{S}_{K_v K^v}$, where $K^v$ runs through open compact subgroups of $G(\mathbb{A}^{\infty v})$. The scheme $\mathscr{S}_{K_v}$ has a right continuous action of $G(\mathbb{A}^{\infty v})$, and for any $K^v \subset G(\mathbb{A}^{\infty v})$, one has $\mathscr{S}_{K_v}/K^v = \mathscr{S}_{K_v K^v}$. Fix an isomorphism $\eta_0 : (\mathbb{A}^{\infty v})^r \xrightarrow{\sim} V^{(v)}(\varphi_0)$ such that $\eta_0(\widehat{A}^{(v)})^r = T^{(v)}(\varphi_0)$. This isomorphism induces an identification

$$G(\mathbb{A}^{\infty v}) \xrightarrow{\sim} G'(\mathbb{A}^{\infty v}), \quad g \mapsto g' = \eta_0 g \eta_0^{-1}, \quad g \in G(\mathbb{A}^{\infty v}), \tag{5.4}$$

and gives a base point $(\varphi_0, \eta_0)$ in $\mathscr{S}_{K_v}$. We shall use it to identify the open compact subgroups of $G(\mathbb{A}^{\infty v})$ and $G'(\mathbb{A}^{\infty v})$. Let

$$\mathscr{S}^{\mathrm{rig}}_{K_v} := \big\{ (\varphi, \eta, \alpha) \,\big|\, (\varphi,\eta) \in \mathscr{S}_{K_v}, \alpha : \varphi \to \varphi_0 \text{ is a quasi-isogeny} \big\}. \tag{5.5}$$

Here "rig" indicates that the pairs $(\varphi, \eta)$ are rigidified by a quasi-isogeny to the base object $\varphi_0$. This space admits a natural left action of $G'(F)$. We will describe $\mathscr{S}^{\mathrm{rig}}_{K_v}$ in terms of $G'(\mathbb{A}^\infty)$. Since any two supersingular Drinfeld modules are isogenous, the natural map $\mathscr{S}^{\mathrm{rig}}_{K_v} \to \mathscr{S}_{K_v}$ is surjective, and it induces an isomorphism $G'(F)\backslash \mathscr{S}^{\mathrm{rig}}_{K_v} \simeq \mathscr{S}_{K_v}$, because for a given Drinfeld module $\varphi$ over $\overline{\mathbb{F}}_v$ the set $\{\alpha\colon \varphi \to \varphi_0 \text{ a quasi-isogeny}\}$ is a principal homogeneous space under $\mathrm{End}^0(\varphi_0 \otimes \overline{\mathbb{F}}_v)^\times = G'(F)$.

Write $A_v = \mathbb{F}_v[[z_v]]$, where $z_v$ is a fixed uniformizer of $A_v$. (One can of course choose $z_v \in A_{(v)}$, if necessary.) The completions of the maximal unramified extensions of $A_v$ and $F_v$ are $\overline{\mathbb{F}}_v[[z_v]]$ and $\overline{\mathbb{F}}_v((z_v))$, respectively. Let $\sigma_v$ be the Frobenius map on $\overline{\mathbb{F}}_v[[z_v]]$ and $\overline{\mathbb{F}}_v((z_v))$ induced by the map $x \mapsto x^{\#\mathbb{F}_v}$ on $\overline{\mathbb{F}}_v$ and satisfying $\sigma_v(z_v) = z_v$.

**Definition 5.6.** A *covariant Dieudonné module of rank* $r$ over $\overline{\mathbb{F}}_v$ is a free $\overline{\mathbb{F}}_v[[z_v]]$-module $M$ of rank $r$ together with a $\sigma_v^{-1}$-semilinear map $V\colon M \to M$ such that $z_v \cdot M \subset V(M)$. Here $\sigma_v^{-1}$-semilinear means that $V(fm) = \sigma_v^{-1}(f) \cdot V(m)$ for $f \in \overline{\mathbb{F}}_v[[z_v]]$ and $m \in M$.

The covariant equi-characteristic Dieudonné modules are the twisted linear duals of contravariant Dieudonné modules defined in Laumon [Lau96]. More precisely, for a covariant Dieudonné module $(M,V)$ as in Definition 5.6 the pair consisting of $M^\vee := \mathrm{Hom}_{\overline{\mathbb{F}}_v[[z_v]]}(M, \overline{\mathbb{F}}_v[[z_v]])$ and $V^\vee\colon M^\vee \to M^\vee$ is a contravariant Dieudonné module as in [Lau96]. Here $V^\vee$ is $\sigma_v$-semilinear in the sense that $V^\vee(fm^\vee) = \sigma_v(f)\cdot V^\vee(m^\vee)$. In terms of [HaKi19, Har09, Har11] the pair $(M^\vee, V^\vee)$ is a *local shtuka*. The following argument can also be reformulated in terms of local shtukas.

Let $(M_0, V_0)$ be the covariant Dieudonné module of $\varphi_0$ and extend $V_0$ to $N_0 := M_0[1/z_v]$. One has $M_0/V_0M_0 = \mathrm{Lie}(\varphi_0)$ and $\mathrm{End}(M_0) = \mathcal{O}_{D_v} = G'(A_v)$. Let

$$X_v := \{\,\overline{\mathbb{F}}_v[[z_v]]\text{-lattices } M \text{ in } N_0, \text{ such that } (M, V_0) \text{ is a Dieudonné module}\,\}.$$

and let $X^v$ be the set of pairs $(L^{(v)},\eta)$, where $L^{(v)}\subset V^{(v)}(\varphi_0)$ is an $\widehat{A}^{(v)}$-lattice and $\eta : (\widehat{A}^{(v)})^r \xrightarrow{\sim} L^{(v)}$ is an isomorphism.

**Lemma 5.7.** *Let $F$ be a global function field with finite constant field $\mathbb{F}_q$ and let $D$ be a finite dimensional division algebra over $F$ with center $F$. Let $G'$ be the algebraic group over $F$ defined on $F$-algebras $R$ by $G'(R) = (D\otimes_F R)^\times$. Let $S$ be a non-empty set of places of $F$ and let $\mathbb{A}^S$ be the prime to $S$ adele ring of $F$. Then the topological space $G'(F)\backslash G'(\mathbb{A}^S)$ is compact.*

Proof. Let $N_{D/F}\colon D\to F$ be the reduced norm of $D$ and let $G'(\mathbb{A})_1$ be the kernel of the group homomorphism

$$|N_{D/F}(\,.\,)|\colon G'(\mathbb{A}) \longrightarrow q^{\mathbb{Z}},\quad g=(g_x)_x \mapsto \prod_x |N_{D/F}(g_x)|_x$$

where the product runs over all places $x$ of $F$, $g_x$ is the component of the adele $g$ at the place $x$ and $|\,.\,|_x\colon F_x \to q^{\deg(x)\mathbb{Z}}$ is the normalized absolute value on $F_x$. By the product formula [Cas67, Chapter II, § 12, Theorem] the group $G'(F)$ is contained in $G'(\mathbb{A})_1$. Since the center of $G'$ is the maximal $F$-split torus in $G'$, the quotient $G'(F)\backslash G'(\mathbb{A})_1$ is a compact topological space by [Har69, Korollar 2.2.7], see also [Con12, Theorem A.5.5(i)]. Therefore also its quotient $G'(F)\backslash G'(\mathbb{A})_1/\big(G'(\mathbb{A})_1\cap \prod'_{x\in S} G'(F_x)\big) = G'(F)\backslash\big(G'(\mathbb{A})_1\cdot \prod'_{x\in S} G'(F_x)\big)/\prod'_{x\in S} G'(F_x)$ is compact.

By [Rei03, (33.4) Theorem] the map $N_{D/F}\colon G'(F_x)\to F_x^\times$ is surjective for every place $x$. Thus the quotient $G'(\mathbb{A})/\big(G'(\mathbb{A})_1\cdot\prod'_{x\in S}G'(F_x)\big)\hookrightarrow q^{\mathbb{Z}/d\mathbb{Z}}$ is finite, where $d\neq 0$ is the greatest common divisor of $\deg(x)$ for all $x\in S$. It follows that $G'(F)\backslash G'(\mathbb{A}^S) = G'(F)\backslash G'(\mathbb{A})/\prod'_{x\in S}G'(F_x)$ is a finite disjoint union of cosets of the compact topological space

$$G'(F)\backslash\big(G'(\mathbb{A})_1\cdot\textstyle\prod'_{x\in S}G'(F_x)\big)/\prod'_{x\in S}G'(F_x).$$

This proves the lemma.

**Lemma 5.8.** *There are natural $G(\mathbb{A}^{\infty v})$-equivariant isomorphisms*

$$\xi^{\mathrm{rig}} : \mathscr{S}^{\mathrm{rig}}_{K_v} \xrightarrow{\sim} X_v\times X^v \xrightarrow{\sim} G'(\mathbb{A}^\infty)/G'(A_v)\qquad \textit{and}$$

$$\xi : \mathscr{S}_{K_v} \xrightarrow{\sim} G'(F)\backslash G'(\mathbb{A}^\infty)/G'(A_v) \tag{5.6}$$

*which send the base point $(\varphi_0,\eta_0)$ to the class of $1\in G'(\mathbb{A}^\infty)$. In particular, for any open compact subgroup $K^v\subset G(\mathbb{A}^{\infty v})\cong G'(\mathbb{A}^{\infty v})$, there is an isomorphism*

$$\xi_{K^v} : \mathscr{S}_K \xrightarrow{\sim} G'(F)\backslash G'(\mathbb{A}^\infty)/G'(A_v)K^v \tag{5.7}$$

*which is compatible with the prime-to-$v$ Hecke action.*

Proof. The proof is similar to that of [Ghi04] or [Yu03, Yu10]. For a member $(\varphi,\eta,\alpha)$ in $\mathscr{S}^{\mathrm{rig}}_{K_v}$, we can replace $(\varphi,\alpha)$ by a prime-to-$v$ quasi-isogeny so that $\eta$ induces an isomorphism $\eta : (\widehat{A}^{(v)})^r \xrightarrow{\sim} T^{(v)}(\varphi)$. Then $\mathscr{S}^{\mathrm{rig}}_{K_v}$ can be also interpreted as the set of isomorphism classes of such triples $(\varphi,\eta,\alpha)$ such that $\eta$ satisfies the integrality above. Taking the Dieudonné and prime-to-$v$ Tate modules, we obtain an isomorphism $\mathscr{S}^{\mathrm{rig}}_{K_v}\simeq X_v\times X^v$. Since any two supersingular Dieudonné modules are isomorphic [Dri76, Proposition 1.17], the action of $G'(F_v)$ on $X_v$ is transitive and one has an isomorphism $G'(F_v)/G'(A_v)\xrightarrow{\sim} X_v$, $g\mapsto gM_0$. For each element $g\in G'(\mathbb{A}^{\infty v})$, one associates a pair $(L^{(v)},\eta)$ in $X^v$ by taking $L^{(v)} := g\cdot T^{(v)}(\varphi_0)$ and $\eta =$

$g\eta_0 : (\widehat{A}^{(v)})^r \xrightarrow{\sim} L^{(v)}$. This gives an isomorphism $G'(\mathbb{A}^{\infty v}) \simeq X^v$ and we have proven $\mathscr{S}^{\mathrm{rig}}_{K_v} \simeq G'(F_v)/G'(A_v) \times G'(\mathbb{A}^{\infty v})$. Everything else follows immediately. ∎

For $M \in X_v$, we define the skeleton $M^\diamond$ of $M$ by $M^\diamond := \{m \in M | V_0^r m = z_v m\}$. This is an equi-characteristic Dieudonné module over $\mathbb{F}_{v^r}$ (as in Definition 5.6 but with $\overline{\mathbb{F}}_v$ replaced by $\mathbb{F}_{v^r}$, which is the field extension of $\mathbb{F}_v$ of degree $r$) and one has $M^\diamond \otimes_{A_{v^r}} \overline{\mathbb{F}}_v[[z_v]] = M$. The construction $M \mapsto M^\diamond$ is functorial and it defines an $\mathbb{F}_{v^r}$-subspace $\omega(M)^\diamond \subset \omega(M) := (M/V_0 M)^\vee$. The endomorphism ring $\mathrm{End}(M_0) = \mathcal{O}_{D_v}$ acts on $M/V_0M$ and this induces an isomorphism $\mathbb{F}_{D_v} := \mathcal{O}_{D_v}/\mathrm{rad}(\mathcal{O}_{D_v}) \simeq \mathrm{End}((M/V_0M)^\diamond) = \mathbb{F}_{v^r}$. Set

$$G(v) := \mathbb{F}_{D_v}^\times \cong \mathbb{F}_{v^r}^\times, \quad U(v) := \ker\big(G'(A_v) = \mathcal{O}_{D_v}^\times \to G(v)\big). \tag{5.8}$$

The above isomorphism identifies $G(v)$ with $\widetilde{G}'(\mathbb{F}_v)$, where $\widetilde{G}'$ is the maximal reductive quotient of $G' \otimes_A (A_v/z_v)$.

Consider the space $X_v^\omega$ which consists of pairs $(M, e)$ where $M \in X_v$ and $e \in \omega(M)^\diamond$ is an $\mathbb{F}_{v^r}$-generator. Fix a base point $(M_0, e_0) \in X_v^\omega$. The group $G'(F_v)$ acts transitively on $X_v^\omega$ and one has a bijection $G'(F_v)/U(v) \simeq X_v^\omega$.

For any finite-dimensional vector space $W$ over $\overline{\mathbb{F}}_v$, denote by $C^\infty(G'(F)\backslash G'(\mathbb{A}^\infty), W)$ the space of locally constant functions $f' : G'(F)\backslash G'(\mathbb{A}^\infty) \to W$. We equip it with the right regular translation of $G'(\mathbb{A}^\infty)$, that is $(g \cdot f')(x) := f'(xg)$ for $g \in G'(\mathbb{A}^\infty)$ and $x \in G'(F)\backslash G'(\mathbb{A}^\infty)$. Then $C^\infty(G'(F)\backslash G'(\mathbb{A}^\infty), W)$ is an admissible smooth representation of $G'(\mathbb{A}^\infty)$. Indeed, the quotient $G'(F)\backslash G'(\mathbb{A}^\infty)$ is a compact topological space by Lemma 5.7. Thus, every vector $f' \in C^\infty(G'(F)\backslash G'(\mathbb{A}^\infty), W)$ takes on only finitely many values in $W$, and hence is fixed by an open subgroup of $G'(\mathbb{A}^\infty)$. Moreover, for each open compact subgroup $K' \subset G'(\mathbb{A}^\infty)$, the subspace $C^\infty(G'(F)\backslash G'(\mathbb{A}^\infty), W)^{K'}$ of $K'$-fixed vectors is equal to $C^\infty(G'(F)\backslash G'(\mathbb{A}^\infty)/K', W)$, which is finite dimensional, because the set $G'(F)\backslash G'(\mathbb{A}^\infty)/K'$ is finite.

Now assume that $W$ is equipped with a finite dimensional irreducible representation $\rho\colon G'(A_v) \to \mathrm{Aut}(W)$ of $G'(A_v)$. Following [Gro99], we define *the space $M_\rho^{\mathrm{alg}}(G')$ of algebraic modular forms (mod $v$) of weight $\rho$ on $G'$* by

$$\begin{aligned} M_\rho^{\mathrm{alg}}(G'; W) := \{ & f' \in C^\infty(G'(F)\backslash G'(\mathbb{A}^\infty), W) \mid \\ & f'(xk_v) = \rho(k_v^{-1}) f'(x), \forall x \in G'(F)\backslash G'(\mathbb{A}^\infty),\ k_v \in G'(A_v)\}. \end{aligned} \tag{5.9}$$

If $K^v \subset G'(\mathbb{A}^{\infty v}) = G(\mathbb{A}^{\infty v})$ is an open compact subgroup, we write

$$\begin{aligned} & M_\rho^{\mathrm{alg}}(G', K^v; W) := M_\rho^{\mathrm{alg}}(G'; W)^{K^v} = \\ & \qquad \{f' \in M_\rho^{\mathrm{alg}}(G'; W) \mid f'(xk^v) = f'(x), \forall x \in G'(F)\backslash G'(\mathbb{A}^\infty),\ k^v \in K^v\} \end{aligned}$$

for the subspace of *algebraic modular forms with level $K^v$*.

Let $S_k(r, K_vK^v, \overline{\mathbb{F}}_v) := H^0(\mathscr{S}_K \otimes_{\mathbb{F}_v} \overline{\mathbb{F}}_v, i^*\omega_K^{\otimes k} \otimes \overline{\mathbb{F}}_v)$ be the space of supersingular Drinfeld modular forms of rank $r$, weight $k$ with level $K^v$ over $\overline{\mathbb{F}}_v$, where $i : \mathscr{S}_K \to \mathscr{M}_K^r$ is the inclusion map. Note that $\mathscr{S}_{K_v}/K^v = \mathscr{S}_{K_vK^v}$ implies $S_k(r, K_vK^v, \overline{\mathbb{F}}_v) \simeq H^0(\mathscr{S}_{K_v} \otimes_{\mathbb{F}_v} \overline{\mathbb{F}}_v, i^*\omega^{\otimes k} \otimes \overline{\mathbb{F}}_v)^{K^v}$, and $H^0(\mathscr{S}_{K_v} \otimes_{\mathbb{F}_v} \overline{\mathbb{F}}_v, i^*\omega^{\otimes k} \otimes \overline{\mathbb{F}}_v) = \varinjlim_{\widetilde{K}^v} S_k(r, K_v\widetilde{K}^v, \overline{\mathbb{F}}_v)$.

**Proposition 5.9.** *Let $\chi : G'(A_v) \to \overline{\mathbb{F}}_v^\times$ be the character of the 1-dimensional representation $\omega(\varphi_0)$. For any integer $k \geq 1$ and open compact subgroup $K^v \subset G(\mathbb{A}^{\infty v})$, there is an isomorphism $S_k(r, K_vK^v, \overline{\mathbb{F}}_v) \simeq M_{\chi^k}^{\mathrm{alg}}(G', K^v; \overline{\mathbb{F}}_v)$ which is compatible with the prime-to-$v$ Hecke action.*

Proof. By what was said before the proposition, it is equivalent to prove that there is a $G(\mathbb{A}^{\infty v})$-equivalent isomorphism $H^0(\mathscr{S}_{K_v} \otimes_{\mathbb{F}_v} \overline{\mathbb{F}}_v, i^*\omega^{\otimes k} \otimes \overline{\mathbb{F}}_v) \simeq M_{\chi^k}^{\mathrm{alg}}(G'; \overline{\mathbb{F}}_v)$. By Lemma 5.8, the first space consists of all $G'(F)$-invariant locally constant sections $f$ on $\mathscr{S}_{K_v}^{\mathrm{rig}} = X_v \times G(\mathbb{A}^{\infty v})$ with $f(M, g^v) \in \omega(M)^{\otimes k}$. We lift each section $f$ to a function $f' : X_v^\omega \times G(\mathbb{A}^{\infty v}) \to \overline{\mathbb{F}}_v$ by $f'((M, e), g^v) := (e^k)^{-1} f(M, g^v)$, where $e^k := e \otimes \cdots \otimes e$ ($k$ times) is an element in $\omega(M^\diamond)^{\otimes k}$

and it induces an isomorphism $e^k : \overline{\mathbb{F}}_v \xrightarrow{\sim} \omega(M)^{\otimes k}$. By $X_v^\omega = G'(F_v)/U(v)$, this defines an $G(\mathbb{A}^{\infty v})$-equivariant map

$$H^0(\mathscr{S}_{K_v} \otimes_{\mathbb{F}_v} \overline{\mathbb{F}}_v, i^*\omega^{\otimes k} \otimes \overline{\mathbb{F}}_v) \to C^\infty(G'(F)\backslash[G'(F_v)/U(v) \times G(\mathbb{A}^{\infty v})], \overline{\mathbb{F}}_v),$$

which is injective, because $f$ can be recovered as $f(M, g^v) = e^k \cdot f'((M,e), g^v)$. For $g_v \in G'(F_v)$ and $k_v \in G'(A_v)$, if $g_v(M_0, e_0) = (M, e)$ then $g_v k_v(M_0, e_0) = (M, g\chi(k_v)e_0) = (M, \chi(k_v)ge_0) = (M, \chi(k_v)e)$. It is easy to see that

$$f'(g_v k_v, g^v) = (\chi(k_v)^k e^k)^{-1} \cdot f(M, g^v) = \chi^k(k_v)^{-1} f'(g_v, g^v). \tag{5.10}$$

Therefore, we obtain an injection $H^0(\mathscr{S}_{K_v} \otimes_{\mathbb{F}_v} \overline{\mathbb{F}}_v, i^*\omega^{\otimes k} \otimes \overline{\mathbb{F}}_v) \xrightarrow{\sim} M^{\mathrm{alg}}_{\chi^k}(G'; \overline{\mathbb{F}}_v)$ which is $G(\mathbb{A}^{\infty v})$-equivariant. To see that it is surjective, let $f' \in M^{\mathrm{alg}}_{\chi^k}(G'; \overline{\mathbb{F}}_v)$, then

$$f' \in C^\infty(G'(F)\backslash[G'(F_v)/U(v) \times G(\mathbb{A}^{\infty v})], \overline{\mathbb{F}}_v) = C^\infty(X_v^\omega \times G(\mathbb{A}^{\infty v}), \overline{\mathbb{F}}_v)^{G'(F)}$$

because $\chi^k(U(v)) = \{1\}$. We define $f$ on $((M,e), g^v)$ as $f((M,e), g^v) := e^k(f'((M,e), g^v)) \in \omega(M)^{\otimes k}$. As in (5.10) we see that this does not depend on the choice of $e$ because every other $\tilde{e}$ is of the form $c \cdot e$ with $c \in \mathbb{F}_{v^r}^\times = \chi(G'(A_v))$, that is, $c = \chi(k_v)$ for $k_v \in G'(A_v)$. So $f$ descends to a section $(f : (M, g^v) \mapsto f((M,e), g^v)) \in H^0(\mathscr{S}_{K_v} \otimes_{\mathbb{F}_v} \overline{\mathbb{F}}_v, i^*\omega^{\otimes k} \otimes \overline{\mathbb{F}}_v)$. ∎

Since $\chi$ is of order $q_v^r - 1 = \#F_{v^r}^\times$, where $q_v := \#\mathbb{F}_v$, the characters $\chi^k$ for $k = 1, \dots, q_v^r - 1$ are all distinct irreducible representations of $G(v)$. Therefore,

$$C^\infty(G'(F)\backslash G'(\mathbb{A}^\infty)/U(v)K^v, \overline{\mathbb{F}}_v) = \bigoplus_{k=1}^{q_v^r - 1} M^{\mathrm{alg}}_{\chi^k}(G', K^v). \tag{5.11}$$

As a corollary of Proposition 5.9, we get a prime-to-$v$ Hecke equivariant isomorphism

$$\bigoplus_{k=1}^{q_v^r - 1} S_k(r, K_v K^v, \overline{\mathbb{F}}_v) \simeq C^\infty(G'(F)\backslash G'(\mathbb{A}^\infty)/U(v)K^v, \overline{\mathbb{F}}_v). \tag{5.12}$$

Let $\mathbf{1}$ be the constant function in $C^\infty(G'(F)\backslash G'(\mathbb{A}^\infty)/U(v)K^v, \overline{\mathbb{F}}_v)$ with value 1 on each double coset. This element maps under projection to an element still denoted by $\mathbf{1} \in H^0(\mathscr{S}_K \otimes_{\mathbb{F}_v} \overline{\mathbb{F}}_v, i^*\omega_K^{\otimes q_v^r - 1} \otimes \overline{\mathbb{F}}_v)$. Multiplication by $\mathbf{1}$ gives a prime-to-$v$ Hecke equivariant isomorphism

$$\mathbf{1} : H^0(\mathscr{S}_K \otimes_{\mathbb{F}_v} \overline{\mathbb{F}}_v, i^*\omega_K^{\otimes k} \otimes \overline{\mathbb{F}}_v) \xrightarrow{\sim} H^0(\mathscr{S}_K \otimes_{\mathbb{F}_v} \overline{\mathbb{F}}_v, i^*\omega_K^{\otimes k + q_v^r - 1} \otimes \overline{\mathbb{F}}_v). \tag{5.13}$$

Recall that the *mass* for $\Sigma(r, v)$ is defined by

$$\mathrm{Mass}(\Sigma(r,v)) := \sum_{\psi \in \Sigma(r,v)} \frac{1}{\#\,\mathrm{Aut}(\psi)}. \tag{5.14}$$

For any open compact subgroup $K' \subset G'(A^\infty)$, the arithmetic mass is defined by

$$\mathrm{Mass}(G', K') := \sum_{i=1}^{h} \frac{1}{\#\Gamma_i}, \quad \Gamma_i := c_i K' {c_i}^{-1} \cap G'(F), \tag{5.15}$$

where $c_1 \dots, c_h$ are coset representatives for the finite double coset space $G'(F)\backslash G'(\mathbb{A}^\infty)/K'$. The mass formula (see [Gek92, 2.5 and 5,11] and [YuYu04, Theorem 2.1], also see [WeYu12] and [Yu15] for generalizations) states that

$$\mathrm{Mass}(\Sigma(r,v)) = \mathrm{Mass}(G', G'(\widehat{A})) = \frac{h(A)}{q-1} \prod_{i=1}^{r-1} \zeta_F^{\infty v}(-i), \tag{5.16}$$

where $h(A)$ is the class number of $A$ and $\zeta_F^{\infty v}(s) := \prod_{w \neq \infty, v} = (1 - \#\mathbb{F}_w^{-s})^{-1}$ is the zeta function of $F$ with factors at $\infty$ and $v$ removed.

*Remark* 5.10. In [Gek92] Gekeler proved a recursive formula (referred as "the transfer principle") which computes explicitly the class number $h(\mathcal{O}_D) = \#\Sigma(v,r)$ for the case $F = \mathbb{F}_q(t)$ and $A = \mathbb{F}_q[t]$. Gekeler's transfer principle was generalized by F.-T Wei and the second author for an arbitrary hereditary $A$-order $R$ in any central division $F$-algebra $D$ definite at $\infty$ (namely, $D_\infty$ is still a central division $F_\infty$-algebra but $D$ can be ramified at several finite places of $F$); see [WeYu15, Theorem 1.1]. Using the recursive formulas in loc. cit., one can compute the class number $h(R)$ of $R$ explicitly.

**Lemma 5.11.** *Suppose $K = K_v K^v \subset G(\widehat{A})$ is fine with $K_v = G(A_v)$, then*

$$\begin{aligned} &\dim_{\overline{\mathbb{F}}_v} C^\infty(G'(F)\backslash G'(\mathbb{A}^\infty)/U(v)K^v, \overline{\mathbb{F}}_v) \\ &= [G(\widehat{A}^{(v)}) : K^v] h(A) \frac{q_v^r - 1}{q-1} \prod_{i=1}^{r-1} \zeta_F^{\infty v}(-i). \end{aligned} \tag{5.17}$$

Proof. Since $K$ is fine, the automorphism group of any object in $\mathscr{S}_K$ is trivial by Lemma 2.7. Then $\#\mathscr{S}_K = \mathrm{Mass}(\mathscr{S}_K) = [G(\widehat{A}^{(v)}) : K^v] \cdot \mathrm{Mass}(\Sigma(r,v))$. The picture for the arithmetic side is true, namely arithmetic subgroups $\Gamma_i$ for the level group $K' = G'(A_v)K^v$ defined in (5.15) are all trivial, and $\#G'(F)\backslash G'(\mathbb{A}^\infty)/U(v)K^v = \mathrm{Mass}(G', U(v)K^v)$. Then $[G'(A_v) : U(v)] = q_v^r - 1$ and $\#G'(F)\backslash G'(\mathbb{A}^\infty)/U(v)K^v = (q_v^r - 1) \cdot \#G'(F)\backslash G'(\mathbb{A}^\infty)/G'(A_v)K^v$. By Lemma 5.8 and the mass formula (5.16), $\#G'(F)\backslash G'(\mathbb{A}^\infty)/U(v)K^v$ is given by the formula (5.17). ■

## 6. Generalized Hasse invariants and $v$-rank strata

6.1. **Coefficient modular forms.** We keep the notation $F$, $\infty$, $A$, $v$ and $\mathfrak{p} \subset A$ from Section 5. As in Section 4.2 let $(\overline{E}_K, \overline{\varphi}_K)$ be the universal family on $\overline{\mathbf{M}}_K^r$ over $A_{(v)}$, where $K = K_v K^v \subset G(\mathbb{A}^\infty)$ is an open compact subgroup with $K_v = G(A_v)$. For any element $a \in A$, write

$$\overline{\varphi}_{K,a} = \sum_{i=0}^{r \deg a} \overline{\varphi}_{K,a,i} \cdot \tau^i.$$

Then each $\overline{\varphi}_{K,a,i} \in H^0(\overline{\mathbf{M}}_K^r, \overline{E}_K^{\,1-q^i}) = H^0(\overline{\mathbf{M}}_K^r, \omega_K^{\otimes q^i - 1})$ is a Drinfeld modular form of rank $r$, and weight $q^i - 1$ over $A_{(v)}$. These are called *coefficient modular forms*. Coefficient modular forms of rank 2 were studied by Gekeler [Gek88] and of higher rank by Basson, Breuer and Pink [BBP18c].

Suppose $K(\mathfrak{n}) \subset K$ for a non-zero ideal $\mathfrak{n}$ of $A$. Then the moduli scheme $\overline{\mathbf{M}}_K^r$ and Drinfeld modular forms $\overline{\varphi}_{K,a,i}$ are even defined over $A[\mathfrak{n}^{-1}]$ and not just over $A_{(v)}$.

Note that $\overline{\mathbf{M}}_K^r$ is normal and therefore the notions of Cartier divisors and Weil divisors of $\overline{\mathbf{M}}_K^r$ are the same. One can consider the (Cartier) divisor $V(\overline{\varphi}_{K,a,i})$ which is defined as the zero section of $\overline{\varphi}_{K,a,i}$ on $\overline{\mathbf{M}}_K^r$, or the intersection of several such divisors. For example, if $a \notin A[\mathfrak{n}^{-1}]^\times$. then $V(\overline{\varphi}_{K,a,0}) = \overline{\mathbf{M}}_K^r \otimes_{A[\mathfrak{n}^{-1}]} A[\mathfrak{n}^{-1}]/(a)$, because $\varphi_{K,a,0} = \gamma(a)$. If the prime ideal $\mathfrak{p} = (a,b)$ is generated by elements $a$ and $b$, then the intersection $V(\overline{\varphi}_{K,a,0}) \cap V(\overline{\varphi}_{K,b,0})$ is the fiber of $\overline{\mathbf{M}}_K^r$ over $\operatorname{Spec} \mathbb{F}_v$.

By Theorem 3.2, we have $\pi^*_{\widetilde{K},K}(\overline{\varphi}_{K,a,i}) = \overline{\varphi}_{\widetilde{K},a,i}$ for fine open compact subgroups $\widetilde{K} \subset K$. Thus, the image of $\overline{\varphi}_{K,a,i}$ in $M_{q^i-1}(r, K_v, A_{(v)})$ is well-defined, see (4.15) in Definition 4.11. We denote it by $\overline{\varphi}_{a,i}$.

Let $L$ be an $A_{(v)}$-algebra. Recall from Theorem 4.14 that $M_k(r, K_v, L)^{K^v}$ is an $\mathcal{H}_L^{\infty v}$-module, where $\mathcal{H}_L^{\infty v} := \mathcal{H}_L(G(\mathbb{A}^{\infty v}), K^v)$, and if $L$ is an $\mathbb{F}_v$-algebra, also $\widetilde{M}_k(r, K_v K^v, L) = \widetilde{M}_k(r, K_v, L)^{K^v}$ is an $\mathcal{H}_L^{\infty v}$-module.

**Lemma 6.1.**

(1) *For any* $a \in A$, $0 \le i \le r \deg a$ *and* $g \in G(\mathbb{A}^{\infty v})$, *one has* $T_g \cdot \overline{\varphi}_{K^g,a,i} = \overline{\varphi}_{K,a,i}$. *The Drinfeld modular form* $\overline{\varphi}_{a,i}$ *is fixed by* $G(\mathbb{A}^{\infty v})$. *We have*

$$\mathbf{1}_{K^v g K^v} * \overline{\varphi}_{K,a,i} = \#(K^v g K^v / K^v) \cdot \overline{\varphi}_{K,a,i},$$

*where* $\mathbf{1}_{K^v g K^v} \in \mathcal{H}^{\infty v}_{\mathbb{Z}}$ *is the characteristic function of* $K^v g K^v$.

(2) *For every* $A_{(v)}$*-algebra* $L$ *the multiplication by* $\overline{\varphi}_{K,a,i}$ *gives rise to a morphism of Hecke modules*

$$\overline{\varphi}_{K,a,i} : M_k(r, K_v, L)^{K^v} \to M_{k+q^i-1}(r, K_v, L)^{K^v}.$$

(3) *For every* $\mathbb{F}_v$*-algebra* $L$ *the multiplication by* $\overline{\varphi}_{K,a,i}$ *gives rise to a morphism of Hecke modules*

$$\overline{\varphi}_{K,a,i} : \widetilde{M}_k(r, K_v K^v, L) \to M_{k+q^i-1}(r, K_v K^v, L).$$

Proof. (1) The first statement follows from the functorial property of the Satake compactification; see (4.13) and (4.19). It follows from $T_g \cdot \overline{\varphi}_{K^g,a,i} = \overline{\varphi}_{K,a,i}$ that $T_g \cdot \overline{\varphi}_{a,i} = \overline{\varphi}_{a,i}$, which proves the second statement. Write $K^v g K^v = \coprod_{j=1}^m g_j K^v$, where $m = \#(K^v g K^v / K^v)$, then

$$\mathbf{1}_{K^v g K^v} * \overline{\varphi}_{K,a,i} = \sum_{j=1}^m T_{g_j}(\overline{\varphi}_{a,i}) = m \cdot \overline{\varphi}_{K,a,i}.$$

(2) Let $f \in M_k(r, K_v, L)^{K^v}$ and $h \in \mathcal{H}^{\infty v}_L$. It suffices to check the case where $h$ is of the form $\mathbf{1}_{K^v g K^v}$, because these form an $L$-basis of $\mathcal{H}^{\infty v}_L$. Let $h = \mathbf{1}_{K^v g K^v}$ and write $K^v g K^v = \coprod_{j=1}^m g_j K^v$, then we compute in $M_k(r, K_v, L)$ which contains $M_k(r, K_v, L)^{K^v}$:

$$\begin{aligned} h * (\overline{\varphi}_{a,i} \cdot f) &= \sum_{j=1}^m T_{g_j}(\overline{\varphi}_{a,i}) \cdot T_{g_j}(f) = \sum_{j=1}^m \overline{\varphi}_{a,i} \cdot T_{g_j}(f) \\ &= \overline{\varphi}_{a,i} \cdot \sum_{j=1}^m T_{g_j}(f) = \overline{\varphi}_{a,i} \cdot (h * f). \end{aligned} \tag{6.1}$$

(3) is proved in the same way as (2). ∎

One may ask how many Drinfeld modular forms are produced from coefficient modular forms. Let us consider the modular forms over $F$ and write $\overline{\varphi}_{K,a,i}$ for their restriction to the generic fiber $\overline{M}^r_K := \overline{\mathbf{M}}^r_K \otimes_{A_{(v)}} F$. Let $M^c(r, K, F) \subset M(r, K, F)$ be the graded subring generated by all coefficient modular forms $\overline{\varphi}_{K,a,i}$. As in Lemma 6.1, $T_g(\overline{\varphi}_{K,a,i}) = \overline{\varphi}_{K,a,i}$ for all $g \in K(1)$. Then

$$M^c(r, K, F) \subset M(r, K, F)^{K(1)} = M(r, K(1), F) \tag{6.2}$$

by Lemma 4.6(3).

Now suppose $A = \mathbb{F}_q[t]$ and $K = K(t)$, By [Pin13, Theorem 7.4], one has

$$M(r, K, F) = F \otimes_{\mathbb{F}_q} R_r = F[1/v; v \in V^0]. \tag{6.3}$$

On the other hand, since $A$ is generated by $t$ over $\mathbb{F}_q$, one has $M^c(r, K, F) = F[\overline{\varphi}_{K,t,1}, \dots, \overline{\varphi}_{K,t,r}]$. From this, we see that $M^c(r, K, F) \neq M(r, K, F)$ at least when $(q, r) \neq (2, 1)$, because $M^c(r, K, F)$ does not contain all elements of degree one in $M(r, K, F)$. We show that $M^c(r, K(t), F)$ contains sufficiently many modular forms. By [Pin13, Theorem 8.1 (c) and Remark 8.3], one has $M^c(r, K(t), F) = M(r, K(1), F) = M(r, K(t), F)^{\mathrm{GL}_r(A/t)}$. It is proved [PiSc14, Theorem 1.7] that $M(r, K(t), F)$ is a normal domain. It follows that $M(r, K(t), F)$ is the normalization of $M^c(r, K(t), F)$ in the quotient field $\mathrm{Frac}(M(r, K(t), F))$. Thus, $M(r, K(t), F)$ is a finite $M^c(r, K(t), F)$-module and hence $M^c(r, K(t), F)$ contains many Drinfeld modular forms.

6.2. **Generalized Hasse invariants.** Recall that $\mathfrak{p} \subset A$ is the prime ideal corresponding to the finite place $v$ and $\deg(v) = [\mathbb{F}_v : \mathbb{F}_q]$.

**Definition 6.2.** For any element $a \in \mathfrak{p}$ and integer $0 \le i \le r-1$, define the *i-th a-Hasse invariant* on $\overline{\mathbf{M}}_K^r$ over $A_{(v)}$ by

$$H_i^a := \overline{\varphi}_{K,a,i\,v(a)\deg(v)} \in H^0(\overline{\mathbf{M}}_K^r, \omega_K^{\otimes q^{i\,v(a)\deg(v)}-1}). \tag{6.4}$$

As in the previous sections, let $\overline{\mathscr{M}}_K^r := \overline{\mathbf{M}}_K^r \otimes_{A_{(v)}} \mathbb{F}_v$. For any $1 \le h \le r$ and $a \in \mathfrak{p}$, let $V(H_1^a, \dots, H_{h-1}^a)$ be the closed subscheme of $\overline{\mathscr{M}}_K^r$ defined as the vanishing locus of $H_1^a, \cdots, H_{h-1}^a$, that is, of the sheaf of ideals $\sum_{i=1}^{h-1} H_i^a \cdot \omega_K^{\otimes 1-q^{i\,v(a)\deg(v)}} \subset \mathcal{O}_{\overline{\mathscr{M}}_K^r}$. Note that $H_0^a = \gamma(a)$ is already zero in $\mathbb{F}_v$.

**Lemma 6.3.** *Let $(\overline{E}, \overline{\varphi})$ be a generalized Drinfeld $A$-module of rank $\le r$ over an $A$-scheme $S$ whose structure morphism $S \to \operatorname{Spec} A$ factors through $\mathbb{F}_v$. Let $a \in A$ with $v(a) = 1$ and write $\overline{\varphi}_a = \sum_{i=0}^{r\cdot\deg(a)} \overline{\varphi}_{a,i}\tau^i$ with $\overline{\varphi}_{a,i} \in H^0(S, \overline{E}^{\otimes 1-q^i})$. Then for every $i,j$ with $(i-1)\deg(v) \le j < i\deg(v)$ the coefficient $\overline{\varphi}_{a,j}$ lies in the subsheaf of $\overline{E}^{\otimes 1-q^j}$ generated by $(H_0^a \cdot \overline{E}^{\otimes 1-q^j}, \dots, H_{i-1}^a \cdot \overline{E}^{\otimes q^{(i-1)\deg v}-q^j})$, where the $H_i^a := \overline{\varphi}_{a,i\deg(v)}$ are defined as in (6.2).*

Proof. The usual proof also works for generalized Drinfeld modules. First, the statement is local on $S$, so we may assume that $S = \operatorname{Spec} R$ is affine and that there is an isomorphism $\overline{E} \simeq \mathcal{O}_{\operatorname{Spec} R}$. We use it to view all $\overline{\varphi}_{a,i}$ as elements of $R$. The statement is equivalent to showing that for any $j < i\deg(v)$ we have $\overline{\varphi}_{a,j} = 0$ in $\overline{R}_i := R/(H_0^a, \dots H_{i-1}^a)$. Let $b \in A$ such that the image $\bar{b}$ of $b$ in $\mathbb{F}_v$ generates the multiplicative group $\mathbb{F}_v^\times$. Then $\bar{b}^n = 1$ in $\mathbb{F}_v$ if and only if $(q^{\deg(v)} - 1)|n$. Let $n = \min\{j : \overline{\varphi}_{a,j} \ne 0 \text{ in } \overline{R}_i\}$. From

$$\begin{aligned} \overline{\varphi}_{ab} &= \overline{\varphi}_a\overline{\varphi}_b = (\overline{\varphi}_{a,n}\tau^n + \dots)(\gamma(\bar{b})\tau^0 + \dots) = \overline{\varphi}_{a,n} \cdot \gamma(\bar{b})^{q^n}\tau^n + \dots \\ \overline{\varphi}_{ba} &= \overline{\varphi}_b\overline{\varphi}_a = (\gamma(\bar{b})\tau^0 + \dots)(\overline{\varphi}_{a,n}\tau^n + \dots) = \gamma(\bar{b}) \cdot \overline{\varphi}_{a,n}\tau^n + \dots \end{aligned}$$

we deduce $(\gamma(\bar{b})^{q^n} - \gamma(\bar{b})) \cdot \overline{\varphi}_{a,n} = 0$ in $\overline{R}_i$. Now write $n = k\deg(v) + m$ with $k \in \mathbb{Z}$ and $0 \le m < \deg(v)$ and use $\bar{b}^{q^{\deg(v)}} = \bar{b}$. If $m \ne 0$ then $\bar{b}^{q^n} - \bar{b} = \bar{b}^{(q^{\deg(v)})^k q^m} - \bar{b} = \bar{b}^{q^m} - \bar{b} \in \mathbb{F}_v^\times$, whence $\gamma(\bar{b})^{q^n} - \gamma(\bar{b}) \in R^\times$ and $\overline{\varphi}_{a,n} = 0$. Since $\overline{\varphi}_{a,k\deg(v)} = H_k^a = 0$ in $\overline{R}_i$ for $0 \le k < i$, it follows that $n \ge i\deg(v)$ and the lemma is proved. ∎

**Lemma 6.4.** *Suppose $v(a) = 1$.*

(1) *For any integer $1 \le h \le r$, the closed subscheme $V(H_1^a, \dots, H_{h-1}^a)$ of $\overline{\mathscr{M}}_K^r$ is independent from the choice of $a$ (satisfying $v(a) = 1$).*

(2) *For any point $x$ in $\overline{\mathscr{M}}_K^r$, the Drinfeld $A$-module $\overline{\varphi}_{K,x}$ over the point $x$ has height $\ge h$ if and only if $x \in V(H_1^a, \dots, H_{h-1}^a)$.*

Proof. Statement (2) follows immediately from the definition of the height of a Drinfeld module $\varphi$, as the exponent of the leading term of $\overline{\varphi}_a = \overline{\varphi}_{a,h}\tau^h + \cdots$ in the associated formal $A$-module $\overline{\varphi}$ when $a$ is a uniformizer; see [Str10, (1.1), p. 529].

(1) Suppose $\widetilde{a} \in A$ is another element with $v(\widetilde{a}) = 1$. On $\overline{\mathscr{M}}_K^r$, where $H_0^a = 0$, we have

$$\overline{\varphi}_a := \overline{\varphi}_{K,a} = H_1^a\tau^{\deg v} + \cdots + H_2^a\tau^{2\deg v} + \cdots,$$

such that for every $j \in \{(i-1)\deg(v), \dots, i\deg(v)-1\}$ the coefficient $\overline{\varphi}_{K,a,j}$, which is a section of $\omega_K^{\otimes q^j-1}$ lies in the submodule generated by $(H_0^a \cdot \omega_K^{\otimes q^j-1}, \dots, H_{i-1}^a \cdot \omega_K^{\otimes q^j - q^{(i-1)\deg v}})$ by Lemma 6.3. Then $a/\widetilde{a} \in A_{(v)}^\times$ and $a/\widetilde{a} = c/\widetilde{c}$ with $c, \widetilde{c} \in A$ and $v(c) = v(\widetilde{c}) = 0$. Write $\overline{\varphi}_{\widetilde{c}} = \gamma(\widetilde{c})\tau^0 + \dots$ and likewise for $\overline{\varphi}_c$. We have

$$\begin{aligned}\overline{\varphi}_{\widetilde{c}a} &= \overline{\varphi}_{\widetilde{c}}\overline{\varphi}_a = \gamma(\widetilde{c})H_1^a \cdot \tau^{\deg v} + \cdots \\ \overline{\varphi}_{c\widetilde{a}} &= \overline{\varphi}_c\overline{\varphi}_{\widetilde{a}} = \gamma(c)H_1^{\widetilde{a}} \cdot \tau^{\deg v} + \cdots .\end{aligned}$$

From $\widetilde{c}a = c\widetilde{a}$ and $\gamma(c), \gamma(\widetilde{c}) \in \mathbb{F}_v^\times$, we get $\gamma(\widetilde{c})H_1^a = \gamma(c)H_1^{\widetilde{a}}$ and $V(H_1^a) = V(H_1^{\widetilde{a}})$. We now proceed by induction. For $1 \le i \le h-1$, we have

$$\begin{aligned}\overline{\varphi}_{\widetilde{c}a} \mod (H_0^a, \dots, H_{i-1}^a) &= \gamma(\widetilde{c})H_i^a \cdot \tau^{i\deg v} + \cdots \\ \overline{\varphi}_{c\widetilde{a}} \mod (H_0^{\widetilde{a}}, \dots, H_{i-1}^{\widetilde{a}}) &= \gamma(c)H_i^{\widetilde{a}} \cdot \tau^{i\deg v} + \cdots .\end{aligned} \tag{6.5}$$

By $\overline{\varphi}_{\widetilde{c}a} = \overline{\varphi}_{c\widetilde{a}}$ and the induction hypothesis $(H_1^a, \dots, H_{i-1}^a) = (H_1^{\widetilde{a}}, \dots, H_{i-1}^{\widetilde{a}})$, we obtain the equality $(H_1^a, \dots, H_i^a) = (H_1^{\widetilde{a}}, \dots, H_i^{\widetilde{a}})$. Therefore, $(\overline{\mathscr{M}}{}_K^r)^{\ge h} := V(H_1^a, \dots, H_{h-1}^a)$ is independent of $a$. ∎

**Definition 6.5.** For $1 \le h \le r$, we define $(\overline{\mathscr{M}}{}_K^r)^{\ge h}$ (resp. $(\mathscr{M}_K^r)^{\ge h}$) as the closed subscheme of $\overline{\mathscr{M}}{}_K^r$ (resp. $\mathscr{M}_K^r$) defined by the Hasse invariants $H_1^a, \dots, H_{h-1}^a$ for any $a \in \mathfrak{p}$ with $v(a) = 1$. Let $(\overline{\mathscr{M}}{}_K^r)^{(h)} := (\overline{\mathscr{M}}{}_K^r)^{\ge h} - (\overline{\mathscr{M}}{}_K^r)^{\ge h+1}$ and $(\mathscr{M}_K^r)^{(h)} := (\mathscr{M}_K^r)^{\ge h} - (\mathscr{M}_K^r)^{\ge h+1}$ be the locally closed subschemes.

**Theorem 6.6.** *Let $h$ be an integer with $1 \le h \le r$ and let $a \in \mathfrak{p}$ with $v(a) = 1$.*

(1) *The subschemes $(\overline{\mathscr{M}}{}_K^r)^{\ge h}$ and $(\overline{\mathscr{M}}{}_K^r)^{(h)}$ are of pure dimension $r-h$ and $(\mathscr{M}_K^r)^{(h)}$ is Zariski dense in $(\overline{\mathscr{M}}{}_K^r)^{\ge h}$, in $(\mathscr{M}_K^r)^{\ge h}$ and in $(\overline{\mathscr{M}}{}_K^r)^{(h)}$.*

(2) *The subschemes $(\overline{\mathscr{M}}{}_K^r)^{(h)}$ and $(\mathscr{M}_K^r)^{(h)}$ are affine.*

(3) *$(H_0^a, \dots, H_{r-1}^a)$ is a regular sequence on $\overline{\mathbf{M}}{}_K^r$.*

(4) *For every $1 \le h \le r$ the closed subscheme $X_h := V(H_1^a, \dots, H_h^a)$ of $\overline{\mathbf{M}}{}_K^r$ is flat over $A_{(v)}$.*

(5) *For every $h < r$, every irreducible component of $(\overline{\mathscr{M}}{}_K^r)^{\ge h}$ meets $(\overline{\mathscr{M}}{}_K^r)^{\ge h+1}$.*

(6) *If $\overline{\mathbf{M}}{}_K^r$ is Cohen-Macaulay, then so is each subscheme $(\overline{\mathscr{M}}{}_K^r)^{\ge h}$.*

(7) *If $\overline{\mathbf{M}}{}_K^r$ is Cohen-Macaulay and $h \ne r$, then the natural map $\pi_0((\overline{\mathscr{M}}{}_K^r)^{\ge h}) \to \pi_0((\overline{\mathscr{M}}{}_K^r)^{\ge h-1})$ of connected components is bijective.*

Proof. (1) We first show that $(\overline{\mathscr{M}}{}_K^r)^{\ge h}$ is of pure dimension $r-h$. When $h = 1$, consider for a moment the scheme $\overline{\mathbf{M}}{}_K^r$ over $A[\mathfrak{n}_K^{-1}]$ from Theorem 3.2 and let $\operatorname{Spec} \widetilde{A} := \operatorname{Spec} A[\mathfrak{n}_K^{-1}a^{-1}] \cup \{\mathfrak{p}\}$. Then the projective variety $(\overline{\mathscr{M}}{}_K^r)^{\ge 1} = \overline{\mathscr{M}}{}_K^r = \overline{\mathbf{M}}{}_K^r \otimes_{A[\mathfrak{n}_K^{-1}]} \mathbb{F}_v$ is of pure codimension 1 in $\overline{\mathbf{M}}{}_K^r \otimes_{A[\mathfrak{n}_K^{-1}]} \widetilde{A}$ by [EGA, IV$_4$, Corollaire 21.12.7], because the inclusion of its complement $\overline{\mathbf{M}}{}_K^r \otimes_{A[\mathfrak{n}_K^{-1}]} \widetilde{A}[1/a] \hookrightarrow \overline{\mathbf{M}}{}_K^r \otimes_{A[\mathfrak{n}_K^{-1}]} \widetilde{A}$ is an affine morphism. Since $\overline{\mathbf{M}}{}_K^r \otimes_{A[\mathfrak{n}_K^{-1}]} \widetilde{A}$ is irreducible of dimension $r$ we conclude that $\overline{\mathscr{M}}{}_K^r$ is pure of dimension $r-1$; use [Eis95, Corollary 13.4]. Since the subvariety $(\overline{\mathscr{M}}{}_K^r)^{\ge h}$ is cut out by $h-1$ equations from $\overline{\mathscr{M}}{}_K^r$, every irreducible component has dimension $\ge r-h$ by [Har77, Theorem I.7.2]. We stratify the scheme $\overline{\mathscr{M}}{}_K^r = \coprod_{1 \le r' \le r} S_{r'}$ by ranks $r'$, that is, $S_{r'}$ is the locally closed reduced subscheme consisting of all points where the universal Drinfeld $A$-module has rank $r'$. By adding a level-$\mathfrak{n}$ structure on $S_{r'}$ for each $r'$, there exist a finite étale cover $\widetilde{S}_{r'}$ of $S_{r'}$ and a morphism $\widetilde{S}_{r'} \to \mathscr{M}_{K'(\mathfrak{n})}^{r'}$ for $K'(\mathfrak{n}) = \ker\big(\mathrm{GL}_{r'}(\widehat{A}) \to \mathrm{GL}_{r'}(A/\mathfrak{n})\big)$ induced by the universal property of the moduli scheme $\mathscr{M}_{K'(\mathfrak{n})}^{r'}$. The latter morphism is quasi-finite, because $\overline{\varphi}$ is weakly separating on $\overline{\mathscr{M}}{}_K^r$. Since the stratum $(\mathscr{M}_{K'(\mathfrak{n})}^{r'})^{\ge h}$ is of pure dimension $r'-h$ by Theorem 5.2, the stratum $S_{r'}^{\ge h} = (\overline{\mathscr{M}}{}_K^r)^{\ge h} \cap S_{r'}$ has dimension $\le r'-h \le r-h$. Therefore, $\dim(\overline{\mathscr{M}}{}_K^r)^{\ge h} = r-h$ and every irreducible component has the same dimension. Moreover, the complement of $(\mathscr{M}_K^r)^{(h)}$ in $(\overline{\mathscr{M}}{}_K^r)^{\ge h}$ equals $(\mathscr{M}_K^r)^{\ge h+1} \cup \coprod_{1 \le r' < r} S_{r'}^{\ge h}$ and is of dimension $< r-h$ by the above. So every irreducible component of $(\overline{\mathscr{M}}{}_K^r)^{\ge h}$ meets $(\mathscr{M}_K^r)^{(h)}$ and the Zariski-density is proved.

(2) Since the stratum $(\overline{\mathscr{M}}^r_K)^{(h)}$ is the complement of an effective ample divisor defined by $H^a_h = 0$ in the projective scheme $V(H^a_1, \dots, H^a_{h-1})$, it is affine. Here we use that on the latter scheme $\omega_K$ is ample by [EGA, II, Proposition 4.6.13 (i bis)]. Since $\mathscr{M}^r_K$ is affine and $\overline{\mathscr{M}}^r_K$ is separated, the intersection $(\mathscr{M}^r_K)^{(h)} = \mathscr{M}^r_K \cap (\overline{\mathscr{M}}^r_K)^{(h)}$ is also affine.

(3) For any point $x \in (\overline{\mathscr{M}}^r_K)^{\geq h}$, one has $\mathcal{O}_{(\overline{\mathscr{M}}^r_K)^{\geq h},x} = \mathcal{O}_{\overline{\mathbf{M}}^r_K,x}/(H^a_0, \dots, H^a_h)$. It follows from (1) that for any $0 \leq i \leq r-2$, one has

$$\dim \mathcal{O}_{\overline{\mathbf{M}}^r_K,x}/(H^a_0, \dots, H^a_i) = \dim \mathcal{O}_{\overline{\mathbf{M}}^r_K,x}/(H^a_0, \dots, H^a_{i+1}) + 1.$$

Thus by [Har77, Theorem II.8.21A], $H^a_0, \dots, H^a_{r-1}$ form a regular sequence in the local ring $\mathcal{O}_{\overline{\mathbf{M}}^r_K,x}$.

(4) By [Eis95, Theorem 18.17(a)] we must show that $\operatorname{depth}(H^a_0 \cdot \mathcal{O}_{X_h,x}, \mathcal{O}_{X,x}) = \dim A_{(v)} = 1$ for every point $x \in V(H^a_0, \dots, H^a_h)$. By (3) and [Eis95, Corollary 17.2] also the sequence $H^a_1, \dots, H^a_h, H^a_0$ is a regular sequence in $\mathcal{O}_{\overline{\mathbf{M}}^r_K,x}$. It follows that $H^a_0$ is a non-zero-divisor in $\mathcal{O}_{X_h,x}$ and $\operatorname{depth}(H^a_0 \cdot \mathcal{O}_{X_h,x}, \mathcal{O}_{X,x}) = 1$ as desired.

(5) Let $X \subset (\overline{\mathscr{M}}^r_K)^{\geq h}$ be an irreducible component with reduced subscheme structure. Then $\omega_K|_X$ is ample on $X$ by [EGA, II, Proposition 4.6.13 (i bis)]. If $V_X(H^a_h) = (\overline{\mathscr{M}}^r_K)^{\geq h+1} \cap X = \emptyset$, then $H^a_h$ induces an isomorphism $\mathcal{O}_X \xrightarrow{\sim} (\omega_K|_X)^{\otimes q^{h \deg(v)} - 1}$. Then $\mathcal{O}_X$ is ample by [EGA, II, Proposition 4.5.6(i)] and $X$ is quasi-affine by [EGA, II, Proposition 5.1.2]. Since $X$ is a projective $\overline{\mathbb{F}}_v$-scheme it is finite over $\overline{\mathbb{F}}_v$ by [GöWe10, Corollary 13.82]. This contradicts that its dimension is $r - h \geq 1$.

(6) Let $x \in (\overline{\mathscr{M}}^r_K)^{\geq h}$. Since $H^a_0, \dots, H^a_{h-1}$ is a regular sequence in $\mathcal{O}_{\overline{\mathbf{M}}^r_K,x}$ by (3) the assertion follows from [Eis95, Proposition 18.13]

(7) The map is surjective by (5). The injectivity follows from Lemma 6.7 below and (6), because $(\overline{\mathscr{M}}^r_K)^{\geq h}$ is the subscheme of $(\overline{\mathscr{M}}^r_K)^{\geq h-1}$ cut out by the generalized Hasse invariant $H^a_h$ and $H^a_h$ is a global section of an ample invertible sheaf. ∎

**Lemma 6.7.** *If $X$ is a connected projective Cohen-Macaulay scheme over a field $k$ of pure dimension $\geq 2$, and $Y$ is a closed subset which is the support of an effective ample divisor, then $Y$ is connected.*

PROOF. Since the support does not change when we replace an effective divisor by a power of it, we may assume that $Y$ is the support of a very ample divisor $D$. Let $\mathcal{O}(1)$ be the corresponding very ample invertible sheaf. For each $q > 0$, let $Y_q$ be the closed subscheme supported on $Y$ corresponding to the divisor $qD$. Then we have an exact sequence

$$0 \to \mathcal{O}_X(-q) \to \mathcal{O}_X \to \mathcal{O}_{Y_q} \to 0.$$

Taking cohomology we have an exact sequence

$$H^0(X, \mathcal{O}_X) \xrightarrow{\alpha} H^0(Y, \mathcal{O}_{Y_d}) \to H^1(X, \mathcal{O}_X(-q)).$$

As $X$ is Cohen-Macaulay and equi-dimensional $H^i(X, \mathcal{O}_X(-q)) = 0$ for $i < \dim X$ and $q \gg 0$, by [Har77, Chap. III, Theorem 7.6(b)]. Note that the assumption of loc. cit. that $k$ is algebraically closed is not needed, because cohomology commutes with the flat base change from $k$ to an algebraic closure by [EGA, $\mathrm{I}_{\mathrm{new}}$, Proposition 9.3.2]. Thus, for $q \gg 0$, we have $H^1(X, \mathcal{O}_X(-q)) = 0$ and the map $\alpha$ is surjective. But $H^0(X, \mathcal{O}_X)$ is a finite local $k$-algebra as $X$ is connected, and $H^0(Y, \mathcal{O}_{Y_q})$ contains $k$, so we conclude that $H^0(Y, \mathcal{O}_{Y_d})$ is also a finite local $k$-algebra. Therefore, $Y$ is connected. ∎

*Remark* 6.8.

(1) When $h = r$, the closed stratum $(\overline{\mathscr{M}}^r_K)^{\geq r}$ is the zero dimensional supersingular locus and it has many connected components (points) as given by the mass formula (5.16); see Section 5.3.

(2) When $A = \mathbb{F}_q[t]$ and $K = K(t)$, Pink and Schieder showed that $\overline{\mathbf{M}}^r_K$ is Cohen-Macaulay. We expect this is also true for arbitrary $\overline{\mathbf{M}}^r_K$.

(3) Very recently Fukaya, Kato and Sharifi [FKS20] have constructed toroidal compactifications of the Drinfeld moduli scheme $\mathbf{M}_K^r$ over $A$ for $A = \mathbb{F}_q[t]$ and $K = K(\mathfrak{n})$ with a nonzero ideal $\mathfrak{n} \subset A$. This leads to the following results.

**Proposition 6.9.** *Let $A = \mathbb{F}_q[t]$, $K = K(\mathfrak{n})$ with a nonzero ideal $\mathfrak{n} \subset A$, $\mathfrak{p} \nmid \mathfrak{n}$ a prime ideal of $A$ with corresponding place $v$.*

*(1) The moduli space $\mathscr{M}_K^r = \mathbf{M}_K^r \otimes_{A_{(v)}} \otimes \overline{\mathbb{F}}_v$ has $|(A/\mathfrak{n})^\times/\mathbb{F}_q^\times|$ connected components.*

*(2) Let $\mathscr{M}_K^{r,\mathrm{ord}} := (\mathscr{M}_K^r)^{(1)}$ be the ordinary locus of $\mathscr{M}_K^r$ with universal family $(E, \varphi, \bar{\eta})$. For every $m \geq 0$, let $Ig_m := \mathrm{Isom}_{\mathscr{M}_K^{r,\mathrm{ord}}}((\mathfrak{p}^{-m}/A)^{r-1}, \varphi[\mathfrak{p}^m]^{\mathrm{et}})$ be the Igusa cover of level $m$, where $\varphi[\mathfrak{p}^m]^{\mathrm{et}}$ is the etale part of $\varphi[\mathfrak{p}^m]$. Then the natural map $\pi : Ig_m \to \mathscr{M}_K^{r,\mathrm{ord}}$ induces a bijection $\pi_0(Ig_m) \simeq \pi_0(\mathscr{M}_K^{r,\mathrm{ord}})$. Therefore, the Igusa cover $Ig_m$ has $|(A/\mathfrak{n})^\times/\mathbb{F}_q^\times|$ connected components.*

Proof. (1) By [FKS20], there is a proper smooth compactification $\overline{\mathbf{M}}_{K,\Sigma}^r$ of $\mathbf{M}_K^r$ over $A_{(v)}$. Therefore, there is a bijection $\pi_0(\overline{\mathbf{M}}_{K,\Sigma}^r \otimes_{A_{(v)}} \overline{\mathbb{F}}_v) \simeq \pi_0(\overline{\mathbf{M}}_{K,\Sigma}^r \otimes_{A_{(v)}} \overline{F})$. Since $\mathbf{M}_K^r \subset \overline{\mathbf{M}}_{K,\Sigma}^r$ is fiber-wise open and dense and $\overline{\mathbf{M}}_{K,\Sigma}^r$ is smooth over $A_{(v)}$, we get bijections:

$$\pi_0(\mathscr{M}_K^r) = \pi_0(\overline{\mathbf{M}}_{K,\Sigma}^r \otimes_{A_{(v)}} \overline{\mathbb{F}}_v) \simeq \pi_0(\overline{\mathbf{M}}_{K,\Sigma}^r \otimes_{A_{(v)}} \overline{F}) = \pi_0(\mathbf{M}_K^r \otimes_{A_{(v)}} \overline{F}).$$

By Proposition 2.4, the latter set is isomorphic to $(\mathbb{A}^\infty)^\times/(F^\times(1+\mathfrak{n}\widehat{A}))$, which is isomorphic to $(A/\mathfrak{n})^\times/\mathbb{F}_q^\times$ because $A = \mathbb{F}_q[t]$ is a principal ideal domain. Thus, $\mathscr{M}_K^r$ has $|(A/\mathfrak{n})^\times/\mathbb{F}_q^\times|$ connected components.

(2) It suffices to show that for every component $S \in \pi_0(\mathscr{M}_K^{r,\mathrm{ord}})$, the cover $\pi^{-1}(S)$ over $S$ is connected. Let $\bar{s}$ be a geometric point of $S$, the action of the fundamental group $\pi_1(S, \bar{s})$ on the fiber $\pi^{-1}(S)_{\bar{s}}$ gives a global monodromy $\rho_S : \pi_1(S, \bar{s}) \to \mathrm{GL}_{r-1}(A/\mathfrak{p}^m)$. By Theorem 6.5(5), every component $S$ contains in its closure $\overline{S} \subset \mathscr{M}_K^r$ a supersingular point $x \in \overline{S}(\overline{\mathbb{F}}_v)$. By the analog of the Serre-Tate theorem for Drinfeld $A$-modules, the completed local ring $\widehat{\mathcal{O}}_{\overline{S},x}$ is the universal deformation ring of the one-dimensional formal $A_v$-module attached to the supersingular Drinfeld module $(E_x, \varphi_x)$ over the point $x$. Let $s'$ and $\bar{s}'$ be the generic point and geometric generic point of $\operatorname{Spec} \widehat{\mathcal{O}}_{\overline{S},x}$, and denote their residue fields by $k(s')$ and $k(\bar{s}')$, respectively. Then $s'$ and $\bar{s}'$ map into $S$. We may change the initial geometric base point $\bar{s}$ of $S$ and assume that $\bar{s} = \bar{s}'$. Then the action of the Galois group $\mathrm{Gal}(k(\bar{s}')/k(s'))$ on the fiber $\pi^{-1}(S)_{\bar{s}'}$ gives a local monodromy $\rho_x : \mathrm{Gal}(k(\bar{s}')/k(s')) \to \mathrm{GL}_{r-1}(A/\mathfrak{p}^m)$ and it factors through the global monodromy $\rho_S$:

$$\rho_x : \mathrm{Gal}(k(\bar{s}')/k(s')) \longrightarrow \pi_1(S, \bar{s}') \longrightarrow \mathrm{GL}_{r-1}(A/\mathfrak{p}^m).$$

By [Str10, Theorem 2.1], the local monodromy $\rho_x$ is surjective. It follows that the global monodromy $\rho_S$ is surjective and that $\pi^{-1}(S)$ is connected. ∎

6.3. **Hecke eigensystems of Drinfeld modular forms modulo $v$.** In this final section we determine the Hecke eigensystems arising from $M_k(r, K_v, \overline{\mathbb{F}}_v)^{K^v}$, see Definition 4.16. Recall the group $G'$ over $F$ from Section 5.3 and the isomorphism $G(\mathbb{A}^{\infty v}) \simeq G'(\mathbb{A}^{\infty v})$ from (5.4). Let

$$\mathcal{A}(G', \overline{\mathbb{F}}_v) := \left\{ f : G'(F)\backslash G'(\mathbb{A})/G'(F_\infty) \longrightarrow \overline{\mathbb{F}}_v \quad \text{locally constant functions} \right\}$$

and recall the group $U(v)$ from (5.8). For brevity we write $\mathcal{H}_{\overline{\mathbb{F}}_v}^{\infty v\mathfrak{n}} = \mathcal{H}_{\overline{\mathbb{F}}_v}(G(\mathbb{A}^{\infty v\mathfrak{n}}), K^{v\mathfrak{n}}) \simeq \mathcal{H}_{\overline{\mathbb{F}}_v}(G'(\mathbb{A}^{\infty v\mathfrak{n}}), K^{v\mathfrak{n}})$ for the prime-to-$v\mathfrak{n}$ spherical Hecke algebra over $\overline{\mathbb{F}}_v$. We consider the smooth admissible $G(\mathbb{A}^{\infty v})$-modules $M_k(r, K_v, \overline{\mathbb{F}}_v) \subset \widetilde{M}_k(r, K_v, \overline{\mathbb{F}}_v)$ from Theorem 4.14.

**Theorem 6.10.** *Let $K_v = K_v(1) = G(A_v)$ and let $K^v \subset G(\mathbb{A}^{\infty v})$ be an open compact subgroup. Let $\mathfrak{n} \subset A$ be a non-zero ideal, prime to $v$ such that $K(\mathfrak{n})$ is contained in a conjugate of $K_v K^v$. Consider three sets of prime-to-$v\mathfrak{n}$ Hecke eigensystems $\mathcal{H}_{\overline{\mathbb{F}}_v}^{\infty v\mathfrak{n}} \to \overline{\mathbb{F}}_v$ arising from*

*(1) algebraic Drinfeld modular forms in $M_k(r, K_v, \overline{\mathbb{F}}_v)^{K^v}$ for all $k \geq 0$,*

*(2) elements of* $\widetilde{M}_k(r,K_v,\overline{\mathbb{F}}_v)^{K^v} = \widetilde{M}_k(r,K_vK^v,\overline{\mathbb{F}}_v) = H^0(\overline{\mathscr{M}}^{r,\mathrm{nor}}_{K_vK^v}\otimes_{\mathbb{F}_v}\overline{\mathbb{F}}_v, \omega^{\otimes k}_{K_vK^v}\otimes\overline{\mathbb{F}}_v)$ *for all* $k\geq 0$*, and*

*(3) elements of* $\mathcal{A}(G',\overline{\mathbb{F}}_v)^{U(v)K^v}$*, respectively.*

*Then all three sets of Hecke eigensystems are equal. In particular, there are only finitely many Hecke eigensystems of algebraic Drinfeld modular forms over* $\overline{\mathbb{F}}_v$ *of a fixed level and all weights.*

PROOF. For any $K$ consider the inclusion maps $i_h : (\overline{\mathscr{M}}^r_{K,\overline{\mathbb{F}}_v})^{\geq h} := V(H^a_1,\ldots,H^a_{h-1}) \to \overline{\mathscr{M}}^r_{K,\overline{\mathbb{F}}_v} := \mathscr{M}^r_K\otimes_{\mathbb{F}_v}\overline{\mathbb{F}}_v$ and $i_h : (\overline{\mathscr{M}}^{r,\mathrm{nor}}_{K,\overline{\mathbb{F}}_v})^{\geq h} := V(H^a_1,\ldots,H^a_{h-1}) \to \overline{\mathscr{M}}^{r,\mathrm{nor}}_{K,\overline{\mathbb{F}}_v} := \mathscr{M}^{r,\mathrm{nor}}_K\otimes_{\mathbb{F}_v}\overline{\mathbb{F}}_v$. Since the Hasse invariant $H^a_h$ is a non-zero divisor on $(\overline{\mathscr{M}}^r_{K,\overline{\mathbb{F}}_v})^{\geq h}$ by Theorem 6.6(3), the multiplication by $H^a_h$ gives a short exact sequence of coherent sheaves on $(\overline{\mathscr{M}}^r_{K,\overline{\mathbb{F}}_v})^{\geq h}$ and on $(\overline{\mathscr{M}}^{r,\mathrm{nor}}_{K,\overline{\mathbb{F}}_v})^{\geq h}$, respectively:

$$0 \longrightarrow i_h^*\omega_K^{\otimes(k-q^{h\deg(v)}+1)} \xrightarrow{H^a_h} i_h^*\omega_K^{\otimes k} \longrightarrow i_{h+1}^*\omega_K^{\otimes k}\big|_{(\overline{\mathscr{M}}^r_{K,\overline{\mathbb{F}}_v})^{\geq h+1}} \longrightarrow 0.$$

This gives an exact sequence of global sections

$$\begin{aligned}(6.6)\quad 0 &\longrightarrow H^0\big((\overline{\mathscr{M}}^r_{K,\overline{\mathbb{F}}_v})^{\geq h}, i_h^*\omega_K^{\otimes(k-q^{h\deg(v)}+1)}\otimes\overline{\mathbb{F}}_v\big) \xrightarrow{H^a_h} H^0\big((\overline{\mathscr{M}}^r_{K,\overline{\mathbb{F}}_v})^{\geq h}, i_h^*\omega_K^{\otimes k}\otimes\overline{\mathbb{F}}_v\big)\\ &\xrightarrow{\ \mathbf{r}\ } H^0\big((\overline{\mathscr{M}}^r_{K,\overline{\mathbb{F}}_v})^{\geq h+1}, i_{h+1}^*\omega_K^{\otimes k}\otimes\overline{\mathbb{F}}_v\big),\end{aligned}$$

where $\mathbf{r}$ is the restriction map onto $(\overline{\mathscr{M}}^r_{K,\overline{\mathbb{F}}_v})^{\geq h+1}$ and similarly for $(\overline{\mathscr{M}}^{r,\mathrm{nor}}_{K,\overline{\mathbb{F}}_v})^{\geq h+1}$. We now consider this sequence for $K = K_v\widetilde{K}^v$ for varying open compact subgroups $\widetilde{K}^v \subset G(\mathbb{A}^{\infty v})$ and also for $(\overline{\mathscr{M}}^{r,\mathrm{nor}}_{K,\overline{\mathbb{F}}_v})^{\geq h}$ instead of $(\overline{\mathscr{M}}^r_{K,\overline{\mathbb{F}}_v})^{\geq h}$. For $1\leq h\leq r$ and all $k\in\mathbb{Z}$ we define

$$\begin{aligned} V(h,k) &:= \Big(\varinjlim_{\widetilde{K}^v} H^0\big((\overline{\mathscr{M}}^r_{K_v\widetilde{K}^v,\overline{\mathbb{F}}_v})^{\geq h}, i_h^*\omega^{\otimes k}_{K_v\widetilde{K}^v}\otimes\overline{\mathbb{F}}_v\big)\Big)^{K^v} \quad\text{and}\\ \widetilde{V}(h,k) &:= \Big(\varinjlim_{\widetilde{K}^v} H^0\big((\overline{\mathscr{M}}^{r,\mathrm{nor}}_{K_v\widetilde{K}^v,\overline{\mathbb{F}}_v})^{\geq h}, i_h^*\omega^{\otimes k}_{K_v\widetilde{K}^v}\otimes\overline{\mathbb{F}}_v\big)\Big)^{K^v},\end{aligned}$$

where $K^v$ is the subgroup which was fixed in the theorem. Note that we do not know whether $V(h,k) = \widetilde{V}(h,k) = (0)$ for $k<0$ and $1<h<r$, because $(\overline{\mathscr{M}}^r_{K_v\widetilde{K}^v,\overline{\mathbb{F}}_v})^{\geq h}$ might not be reduced and then Lemma 4.2 cannot be applied. For $h=r$ we have $V(h,k) = \widetilde{V}(h,k) \simeq V(r,k+q^{r\deg v}-1)$ by the periodic property (5.13) for all $k\in\mathbb{Z}$.

Since taking the inductive limit is an exact functor and taking $K^v$-invariants is left exact with $H^a_h$ fixed under $K^v$ by Lemma 6.1(1), sequence (6.6) and its analog for $(\overline{\mathscr{M}}^{r,\mathrm{nor}}_{K,\overline{\mathbb{F}}_v})^{\geq h}$ yield exact sequences of $\mathcal{H}^{\infty v\mathfrak{n}}_{\overline{\mathbb{F}}_v}$-modules

$$(6.7)\qquad 0 \longrightarrow V(h,k-q^{h\deg(v)}+1) \xrightarrow{H^a_h} V(h,k) \xrightarrow{\ \mathbf{r}\ } V(h+1,k),$$

$$(6.8)\qquad 0 \longrightarrow \widetilde{V}(h,k-q^{h\deg(v)}+1) \xrightarrow{H^a_h} \widetilde{V}(h,k) \xrightarrow{\ \mathbf{r}\ } \widetilde{V}(h+1,k).$$

Note that for $h>1$ we do not know whether the canonical map $V(h,k)\to\widetilde{V}(h,k)$ is injective, because if $(\overline{\mathscr{M}}^r_{K_v\widetilde{K}^v,\overline{\mathbb{F}}_v})^{\geq h}$ is not reduced the proof in Proposition 4.7 fails for $(\overline{\mathscr{M}}^r_{K_v\widetilde{K}^v,\overline{\mathbb{F}}_v})^{\geq h}$. For all $1\leq h\leq r$ and $k\in\mathbb{Z}$ let $H(h,k), \widetilde{H}(h,k)\subset \mathrm{Hom}_{\overline{\mathbb{F}}_v}(\mathcal{H}^{\infty v\mathfrak{n}}_{\overline{\mathbb{F}}_v},\overline{\mathbb{F}}_v)$ be the subsets of all prime-to-$v\mathfrak{n}$ Hecke eigensystems arising from the Hecke modules $V(h,k)$ and $\widetilde{V}(h,k)$ respectively.

When $h=1$, $(\overline{\mathscr{M}}^r_{K,\overline{\mathbb{F}}_v})^{\geq h} = \overline{\mathscr{M}}^r_{K,\overline{\mathbb{F}}_v}$ and the union $\bigcup_{k\geq 0} H(1,k)$ is the set of all prime-to-$v\mathfrak{n}$ Hecke eigensystems arising from the Hecke modules $V(1,k) = M_k(r,K_v,\overline{\mathbb{F}}_v)^{K^v}$ for all $k\geq 0$. Likewise, $(\overline{\mathscr{M}}^{r,\mathrm{nor}}_{K,\overline{\mathbb{F}}_v})^{\geq 1} = \overline{\mathscr{M}}^{r,\mathrm{nor}}_{K,\overline{\mathbb{F}}_v}$ and the union $\bigcup_{k\geq 0}\widetilde{H}(1,k)$ is the set of all prime-to-$v\mathfrak{n}$ Hecke eigensystems arising from the Hecke modules $\widetilde{V}(1,k) = \widetilde{M}_k(r,K_v,\overline{\mathbb{F}}_v)^{K^v}$ for all $k\geq 0$. In this case $V(1,k)\subset\widetilde{V}(1,k)$ by Proposition 4.7, and hence $H(1,k)\subset\widetilde{H}(1,k)$. Moreover, when $k<0$ then $V(1,k) = \widetilde{V}(1,k) = (0)$ by Lemma 4.2, and hence $H(1,k) = \widetilde{H}(1,k) = \emptyset$.

On the other hand, when $h = r$, $(\overline{\mathscr{M}}^{r,\mathrm{nor}}_{K,\overline{\mathbb{F}}_v})^{\geq r} = (\overline{\mathscr{M}}^{r}_{K,\overline{\mathbb{F}}_v})^{\geq r}$ equals the supersingular set $\mathscr{S}_{K,\overline{\mathbb{F}}_v} := \mathscr{S}_K \otimes_{\mathbb{F}_v} \overline{\mathbb{F}}_v$ from Section 5.3, because $\mathscr{S}_K$ is contained in $\mathscr{M}^r_K$ and the normalization is an isomorphism on $\mathscr{M}^r_K$. Therefore, $V(r,k) = \widetilde{V}(r,k) = S_k(r, K_vK^v, \overline{\mathbb{F}}_v)$, and $H(r,k) = \widetilde{H}(r,k)$ equals the set of Hecke eigensystems of the supersingular Hecke modules $S_k(r, K_vK^v, \overline{\mathbb{F}}_v)$ for all $k \geq 0$, which we studied in Proposition 5.9. To prove the theorem we next show

(a) for any integer $j$, one has $\bigcup_{k\leq j} \widetilde{H}(h,k) \subset \bigcup_{k\leq j} \widetilde{H}(h+1,k)$ for all $1 \leq h \leq r-1$,
(b) there is a positive integer $k_0$ such that $H(r,k) = \widetilde{H}(r,k) \subset H(1,k)$ for all $k \geq k_0$,
(c) $H(1,k) \subset \widetilde{H}(1,k)$ for all $k \geq 0$ and $H(1,k) = \widetilde{H}(1,k) = \emptyset$ for all $k < 0$.

(c) was already observed above.

(a) Let $0 \neq f \in \widetilde{V}(h,k)$ be a prime-to-$v\mathfrak{n}$ Hecke eigenform defined on $(\overline{\mathscr{M}}^{r,\mathrm{nor}}_{K_v\widetilde{K}^v,\overline{\mathbb{F}}_v})^{\geq h}$ for some $\widetilde{K}^v$. If $\mathbf{r}(f) \neq 0$ in sequence (6.8), then $\mathbf{r}(f)$ gives rise to the same Hecke eigensystem as $f$. Otherwise, $f$ is divisible by $H^a_h$. We show that $f$ cannot be arbitrarily often divisible by $H^a_h$. Namely, let $x$ be a point in $V(H^a_h) \subset (\overline{\mathscr{M}}^{r,\mathrm{nor}}_{K_v\widetilde{K}^v,\overline{\mathbb{F}}_v})^{\geq h}$. The Krull intersection theorem [Eis95, Corollary 5.4] for an affine open neighborhood of $x$ on which $\omega_K$ is trivial shows that $f$ can only be arbitrarily often divisible by $H^a_h$, if $f$ is zero in an open neighborhood $U$ of $V(H^a_h)$. Let $X$ be a connected component of $(\overline{\mathscr{M}}^{r,\mathrm{nor}}_{K_v\widetilde{K}^v,\overline{\mathbb{F}}_v})^{\geq h}$. Then $U \cap X \neq \emptyset$ because $V(H^a_h)$ meets $X$ by Theorem 6.6(5). Since $f$ is a section of an invertible sheaf, the set on which $f \neq 0$ is open in $X$ by [EGA, $\mathrm{I_{new}}$, Lemma 9.7.9.1]. So $U \cap X$ and $\{f \neq 0\} \cap X$ form a disjoint open covering of the connected $X$, and hence $\{f \neq 0\} \cap X = \emptyset$ and $f = 0$ on $X$. It follows that $f = 0$ on all of $(\overline{\mathscr{M}}^{r,\mathrm{nor}}_{K_v\widetilde{K}^v,\overline{\mathbb{F}}_v})^{\geq h}$, which is a contradiction. Therefore, $f$ cannot be arbitrarily often divisible by $H^a_h$ and there is an integer $s$ and an element $0 \neq f' \in \widetilde{V}(h, k - s(q^{h\deg(v)} - 1))$ such that $(H^a_h)^s \cdot f' = f$ and $\mathbf{r}(f') \neq 0$. Since the multiplication by $H^a_h$ is Hecke equivariant (by the analog of Lemma 6.1(2) for $(\overline{\mathscr{M}}^{r,\mathrm{nor}}_{K})^{\geq h}$ ), the form $f'$ and hence $\mathbf{r}(f')$ give rise to the same Hecke eigensystem as $f$. This proves (i), as the eigenform $\mathbf{r}(f')$ lies in $\widetilde{V}(h+1,j)$ for $j := k - s(q^{h\deg(v)} - 1) \leq k$.

(b) Let $K = K_vK^v$. Since $\omega_K$ is ample, [Har77, Proposition III.5.3] yields a positive integer $k_0$ such that for any integer $k \geq k_0$ and any $1 \leq h < r$ we have $H^1\big((\overline{\mathscr{M}}^{r}_{K,\overline{\mathbb{F}}_v})^{\geq h}, i_h^*\omega_K^{\otimes(k-q^{h\deg(v)}+1)} \otimes \overline{\mathbb{F}}_v\big) = 0$. Therefore, the restriction maps $\mathbf{r}$ in sequence (6.6) are surjective for every $1 \leq h < r$, and hence their composition

$$\mathbf{r} : H^0(\overline{\mathscr{M}}^{r}_{K,\overline{\mathbb{F}}_v}, \omega_K^{\otimes k} \otimes \overline{\mathbb{F}}_v) \to H^0(\mathscr{S}_{K,\overline{\mathbb{F}}_v}, i_r^*\omega_K^{\otimes k} \otimes \overline{\mathbb{F}}_v) \tag{6.9}$$

is likewise surjective. Since $H^0(\overline{\mathscr{M}}^{r}_{K,\overline{\mathbb{F}}_v}, \omega_K^{\otimes k} \otimes \overline{\mathbb{F}}_v) \subset V(1,k)$ by Lemma 4.6(2), it follows that the map of $\mathcal{H}^{\infty v\mathfrak{n}}_{\overline{\mathbb{F}}_v}$-modules $\mathbf{r}\colon V(1,k) \to V(r,k)$ is surjective for every $k \geq k_0$. Since $\mathcal{H}^{\infty v\mathfrak{n}}_{\overline{\mathbb{F}}_v}$ is commutative, both $\mathcal{H}^{\infty v\mathfrak{n}}_{\overline{\mathbb{F}}_v}$-modules decompose as the direct sums of their common generalized $\mathcal{H}^{\infty v\mathfrak{n}}_{\overline{\mathbb{F}}_v}$-eigenspaces $V(1,k) = \bigoplus_\chi V(1,k)_\chi$ and $V(r,k) = \bigoplus_\chi V(r,k)_\chi$, respectively. Moreover, $\mathbf{r}(V(1,k)_\chi) = V(r,k)_\chi$. In particular, if $\chi = (a_{v'})_{v'}$ is the Hecke eigensystem of an eigenform $f \in V(r,k)$, then $V(1,k)_\chi \neq 0$ and there is an eigenform $f \in V(1,k)_\chi$ with Hecke eigensystem $\chi$. This proves (ii).

It follows from the periodic property (5.13): $H(r,k) = H(r, k + q^{r\deg v} - 1)$ that

$$\bigcup_{k\geq k_0} H(r,k) = \bigcup_{k\geq 0} H(r,k) = \bigcup_{1\leq k\leq q^{r\deg v}-1} H(r,k). \tag{6.10}$$

Combining (i), (ii) and (iii), we prove that that $\bigcup_{k\geq 0} H(1,k) = \bigcup_{k\geq 0} \widetilde{H}(1,k)$ is the same set of prime-to-$v\mathfrak{n}$ Hecke eigensystems as that arising from $S_k(r, K_vK^v, \overline{\mathbb{F}}_v)$ for $k = 1, \ldots, q^{r\deg v} - 1$. The theorem then follows from (5.12). Note that the vector space $\mathcal{A}(G', \overline{\mathbb{F}}_v)^{U(v)K^v}$ has finite dimension given by (5.17) in Lemma 5.11, so we have the finiteness of the Hecke eigensystems. ∎

**Corollary 6.11.** *Let $\mathfrak{n}$ be a non-zero ideal of $A$ with $v \nmid \mathfrak{n}$, and $N(r,\mathfrak{n},v)$ the number of prime-to-$\mathfrak{n}v$ Hecke eigensystems arising from $M_k(r, K_v, \overline{\mathbb{F}}_v)^{K(\mathfrak{n})^v}$ for all $k \geq 1$. Then with $q_v = q^{\deg v}$ we have*

$$N(r,\mathfrak{n},v) \leq \dim \mathcal{A}(G', \overline{\mathbb{F}}_v)^{U(v)K(\mathfrak{n})^v}$$

$$= \# \operatorname{GL}_r(A/\mathfrak{n}) \cdot h(A) \frac{q_v^r - 1}{q-1} \prod_{i=1}^{r-1} \zeta_F^{\infty v}(-i). \tag{6.11}$$

Proof. This follows from Theorem 6.10 and the dimension formula (5.17). ∎

Put $\zeta_A(s) := \zeta_F^{\infty}(s) = \zeta_F(s)(1 - q_\infty^{-s})$ and

$$c(r, A, \mathfrak{n}) := \# \operatorname{GL}_r(A/\mathfrak{n}) \cdot \frac{h(A)}{q-1} \prod_{i=1}^{r-1} |\zeta_A(-i)|.$$

Then $|\zeta_F^{\infty v}(-i)| = |\zeta_A(-i)|(q_v^i - 1)$ and $N(r,\mathfrak{n},v) \leq c(r,A,\mathfrak{n}) \prod_{i=1}^r (q_v^i - 1)$. Thus, we obtain the asymptotic behavior for $N(r,\mathfrak{n},v)$ when $v$ varies:

$$N(r,\mathfrak{n},v) = O(q_v^{r(r+1)/2}) \quad \text{as } q_v \to +\infty. \tag{6.12}$$

*Remark* 6.12. Theorem 6.10 is the function field analogue of a theorem of Serre [Ser96] which describes elliptic modular forms modulo $p$ by quaternion algebras. In [Ghi04] Ghitza generalized Serre's theorem to Siegel modular forms (mod $p$). Ghitza followed Serre's idea by restricting modular forms (mod $p$) to the superspecial locus, but he also gives an argument which applies the Kodaira-Spencer map, due to the lack of generalized Hasse invariants. Instead of using the Kodaira-Spencer map argument, we use generalized Hasse invariants; this is more direct and also close to Serre's original proof. Our proof also shows that the prime-to-$v\mathfrak{n}$ Hecke eigensystems arising from every intermediate stratum are the same. The argument used by Ghitza (also used by Reduzzi [Red13]) should lead to another proof of Theorem 6.10, which does not rely on the existence of generalized Hasse invariants. We leave the details to the reader. We remark that the construction of generalized Hasse invariants for Shimura varieties of PEL-type is known recently due to the work of Boxer [Box15] and of Goldring and Koskivirta [GoKo19].

## Acknowledgements

The article was prepared during the second named author's visits at Münster University. He wishes to thank the institution for warm hospitality, generous support and excellent working conditions. Our intelligent debt to Richard Pink is obvious. It is a great pleasure to thank Richard Pink and Simon Schieder for their inspiring articles [Pin13, PiSc14] on which the present article is partially based. We also thank Richard Pink, David Rydh and Zhiwei Yun for helpful discussions.

Both authors acknowledge the support of the DAAD-MoST-PPP "G-Shtukas: Geometry and Arithmetic". Furthermore, both authors are grateful for the support of the DFG (German Research Foundation) in form of Project-ID 427320536 – SFB 1442 and Germany's Excellence Strategy EXC 2044–390685587 "Mathematics Münster: Dynamics–Geometry–Structure". The second named author is partially support by the MoST grants 107-2115-M-001-001-MY2 and 109-2115-M-001-002-MY3

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