# THE UNIVERSAL FAMILY OF SEMI-STABLE $p$-ADIC GALOIS REPRESENTATIONS

URS HARTL AND EUGEN HELLMANN

ABSTRACT. Let $K$ be a finite field extension of $\mathbb{Q}_p$ and let $\mathscr{G}_K$ be its absolute Galois group. We construct the universal family of semi-stable $\mathscr{G}_K$-representations in $\mathbb{Q}_p$-algebras and the moduli space of such representations. It is an Artin stack in adic spaces locally of finite type over $\mathbb{Q}_p$ in the sense of Huber.

CONTENTS



## 1. INTRODUCTION

The study of arithmetic families of $p$-adic Hodge structures and their associated Galois representations was begun by one of us in [Hel13], where a universal family of filtered $\varphi$-modules was constructed and, building on this, a universal family of crystalline representations with Hodge-Tate weights in $\{0,1\}$. The approach is based on Kisin's integral $p$-adic Hodge theory cf. [Kis06]. In the present article we generalize this result to the case of semi-stable representations of the absolute Galois group $\mathscr{G}_K = \mathrm{Gal}(\overline{K}/K)$ of a finite extension $K$ of $\mathbb{Q}_p$ with arbitrary (fixed) Hodge-Tate weights. If the Hodge-Tate weights do not just lie in $\{0,1\}$, Kisin's integral theory does not describe $\mathscr{G}_K$-stable $\mathbb{Z}_p$-lattices in a crystalline resp. semi-stable $\mathscr{G}_K$-representation but all $\mathscr{G}_{K_\infty} = \mathrm{Gal}(\overline{K}/K_\infty)$-stable $\mathbb{Z}_p$-lattices, where $K_\infty$ is a certain Kummer extension of $K$ appearing in [Kis06]. This leads us to consider not only filtered $(\varphi, N)$-modules but rather $(\varphi, N)$-modules together with a so called *Hodge-Pink lattice*. We describe a moduli space (actually a stack) of $(\varphi, N)$-modules with Hodge-Pink lattice. This stack turns out to be a vector bundle over a space of filtered $(\varphi, N)$-modules. The original space of filtered $(\varphi, N)$-modules can be recovered as a section defined by a certain transversality condition in this vector bundle. The inspiration to work with Hodge-Pink lattices instead of filtrations is taken from the analogous theory over function fields; see [Pin97, GL11, Har11]. It was already applied to Kisin's integral $p$-adic Hodge theory by Genestier and Lafforgue [GL12] in the absolute case for $\varphi$-modules over $\mathbb{Q}_p$.

We can define a notion of weak admissibility for $(\varphi, N)$-modules with Hodge-Pink lattice and show that being weakly admissible is an open condition in the set up of adic spaces generalizing the corresponding result for filtered $\varphi$-modules in [Hel13]. Following the method of [Kis06] and [Hel13] we further cut out an open subspace over which an integral structure for the $(\varphi, N)$-modules with Hodge-Pink lattice exists and an open subspace over which a family of $\mathscr{G}_{K_\infty} = \mathrm{Gal}(\overline{K}/K_\infty)$-representation exists. If we restrict ourselves to the subspace of filtered $(\varphi, N)$-modules one can promote this family of $\mathscr{G}_{K_\infty}$-representations to the universal family of semi-stable $\mathscr{G}_K$-representations.

It is worth pointing out that, unlike in the case of filtered $(\varphi, N)$-modules, the Hodge-Tate weights can jump in a family of $(\varphi, N)$-modules with Hodge-Pink lattice, and likewise in a family of $\mathscr{G}_{K_\infty}$-representations (where we define the Hodge-Tate weights to be those of the corresponding Hodge-Pink lattice). As the Hodge-Tate weights of a family of $\mathscr{G}_K$-representations vary $p$-adically continuously, and hence cannot jump, this is certainly an obstruction which forces us to restrict ourselves to the subspace of fixed Hodge-Tate weights in order to construct a family of $\mathscr{G}_K$-representations.

We describe our results in more detail. Let $K$ be a finite extension of $\mathbb{Q}_p$ with absolute Galois group $\mathscr{G}_K$ and maximal unramified sub-extension $K_0$. Let $\mathrm{Frob}_p$ be the $p$-Frobenius on $K_0$. We consider *families of $(\varphi, N)$-modules* over $\mathbb{Q}_p$-schemes $X$, that is finite locally free $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0$-modules $D$ together with a $\varphi := \mathrm{id} \otimes \mathrm{Frob}_p$-linear automorphism $\Phi$ and a linear monodromy operator $N\colon D \to D$ satisfying the usual relation $N\Phi = p\Phi N$. Choosing locally on $X$ a basis of $D$ and considering $\Phi \in \mathrm{GL}_d(\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0)$ and $N \in \mathrm{Mat}_{d\times d}(\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0)$ as matrices, the condition $N\Phi = p\Phi N$ cuts out a closed subscheme $P_{K_0,d} \subset \mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}_d \times_{\mathbb{Q}_p} \mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{Mat}_{d\times d}$. We can describe the geometry of this scheme as follows.

**Theorem 3.2.** *The scheme $P_{K_0,d}$ is equi-dimensional of dimension $[K_0 : \mathbb{Q}_p]\, d^2$. It is reduced, Cohen-Macaulay and generically smooth over $\mathbb{Q}_p$. Its irreducible components are indexed by the possible Jordan types of the (necessarily nilpotent) monodromy operator $N$.*

Further we consider families of $(\varphi, N)$-modules $D$ with a filtration $\mathcal{F}^\bullet$ on $D \otimes_{K_0} K$ and more generally families of $(\varphi, N)$-modules with a Hodge-Pink lattice $\mathfrak{q}$, see Definition 2.5 for the precise definitions. Given a cocharacter $\mu$ of the algebraic group $\mathrm{Res}_{K/\mathbb{Q}_p} \mathrm{GL}_{d,K}$ (or more precisely a cocharacter of the Weil restriction of the diagonal torus which is dominant with respect to the Weil restriction of the upper triangular matrices) we define a notion of a filtration $\mathcal{F}^\bullet$ resp. a Hodge-Pink lattice with *constant Hodge polygon equal to* $\mu$, and a notion of *boundedness by* $\mu$ for a Hodge-Pink lattice $\mathfrak{q}$. With $\mu$ is associated a reflex field $E_\mu$ which is a finite extension of $\mathbb{Q}_p$.

**Theorem 3.6.**

(a) *The stack $\mathscr{H}_{\varphi,N,\preceq\mu}$ parametrizing rank d families of $(\varphi, N)$-modules with Hodge-Pink lattice bounded by $\mu$ on the category of $E_\mu$-schemes is an Artin stack. It is equi-dimensional and generically smooth. Its dimension can be explicitly described in terms of the cocharacter $\mu$ and its irreducible components are indexed by the possible Jordan types of the (nilpotent) monodromy operator.*

(b) *The stack $\mathscr{H}_{\varphi,N,\mu}$ parametrizing rank d families of $(\varphi, N)$-modules with Hodge-Pink lattice with constant Hodge polygon equal to $\mu$, is an open and dense substack of $\mathscr{H}_{\varphi,N,\preceq\mu}$. Further it is reduced and Cohen-Macaulay. It admits a canonical map to the stack $\mathscr{D}_{\varphi,N,\mu}$ of filtered $(\varphi, N)$-modules with filtration of type $\mu$. This map is representable by a vector bundle.*

If we restrict ourselves to the case of vanishing monodromy, i.e. the case $N = 0$, we cut out a single irreducible component $\mathscr{H}_{\varphi,\preceq\mu} \subset \mathscr{H}_{\varphi,N,\preceq\mu}$ and similarly for the other stacks in the theorem. Following [Hel13] we consider the above stacks also as stacks on the category of adic spaces locally of finite type over $\mathbb{Q}_p$, i.e. we consider the adification $\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}$, etc. Passing from $\mathbb{Q}_p$-schemes to adic spaces allows us to generalize Kisin's comparison between filtered $(\varphi, N)$-modules and vector bundles on the open unit disc (together with certain additional structures). To do so we need to fix a uniformizer $\pi$ of $K$ as well as its minimal polynomial $E(u)$ over $K_0$.

**Theorem 4.6.** *For every adic space $X$ locally of finite type over $\mathbb{Q}_p$ there is a natural equivalence of categories between the category of $(\varphi, N)$-modules with Hodge-Pink lattice over $X$ and the category of $(\varphi, N_\nabla)$-modules over $X$, i.e. the category of vector bundles $\mathcal{M}$ on the product of $X$ with the open unit disc $\mathbb{U}$ over $K_0$ together with a semi-linear map $\Phi_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{M}$ that is an isomorphism away from $X \times \{E(u) = 0\} \subset X \times \mathbb{U}$ and a differential operator $N^{\mathcal{M}}_\nabla$ satisfying*

$$N^{\mathcal{M}}_\nabla \circ \Phi_{\mathcal{M}} \circ \varphi = p\,\tfrac{E(u)}{E(0)} \cdot \Phi_{\mathcal{M}} \circ \varphi \circ N^{\mathcal{M}}_\nabla.$$

**Theorem 4.9.** *The differential operator $N^{\mathcal{M}}_\nabla$ defines a canonical meromorphic connection on the vector bundle $\mathcal{M}$. The closed substack $\mathscr{H}^{\nabla}_{\varphi,N,\mu} \subset \mathscr{H}^{\mathrm{ad}}_{\varphi,N,\mu}$ where this connection is holomorphic coincides with the zero section of the vector bundle $\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\mu} \to \mathscr{D}^{\mathrm{ad}}_{\varphi,N,\mu}$.*

Similarly to the case of filtered $\varphi$-modules in [Hel13] there is a notion of weak admissibility for families of $(\varphi, N)$-modules with Hodge-Pink lattice over an adic space. We show that weak admissibility is an open condition.

**Theorem 5.6.** *Let $\mu$ be a cocharacter as above, then the groupoid*

$$X \mapsto \{(D, \Phi, N, \mathfrak{q}) \in \mathscr{H}_{\varphi,N,\preceq\mu}(X) \mid D \otimes \kappa(x) \text{ is weakly admissible for all } x \in X\}$$

*is an open substack $\mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\preceq\mu}$ of $\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}$.*

Following the construction in [Hel13] we construct an open substack $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu} \subset \mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\preceq\mu}$ where an integral model for the $(\varphi, N_\nabla)$-module over the open unit disc exists. Here integral means with respect to the ring of integers $W$ in $K_0$. Dealing with Hodge-Pink lattices instead of filtrations makes is possible to generalize the period morphism of [PR09, § 5] beyond the miniscule case. That is, we consider a stack $\hat{\mathcal{C}}_{\preceq\mu,N,K}$ in the category of formal schemes over $\operatorname{Spf}\mathcal{O}_{E_\mu}$ whose $R$-valued points parameterize tuples $(\mathfrak{M}, \Phi, N)$ where $\mathfrak{M}$ is a finite locally free $(R \otimes_{\mathbb{Z}_p} W)[\![u]\!]$ module, $\Phi$ is a semi-linear morphism $\Phi\colon \mathfrak{M} \to \mathfrak{M}$ which is an isomorphism away from $E(u) = 0$, whose behavior at $E(u)$ is controlled in terms of $\mu$, and $N$ is an endomorphism of $\mathfrak{M}/u\mathfrak{M}$ satisfying $N\Phi = p\Phi N$; see after Remark 6.10 for the precise definition.

Given a $p$-adic formal scheme $\mathcal{X}$ over $\operatorname{Spf}\mathcal{O}_{E_\mu}$ we construct a period morphism

$$\Pi(\mathcal{X})\colon \hat{\mathcal{C}}_{\preceq\mu,N,K}(\mathcal{X}) \longrightarrow \mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}(\mathcal{X}^{\mathrm{ad}})$$

and the substack $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu}$ will serve as the image of this morphism in the following sense.

**Corollary 6.12.** *Let $X$ be an adic space locally of finite type over the reflex field $E_\mu$ of $\mu$ and let $f\colon X \to \mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}$ be a morphism defined by $(D, \Phi, N, \mathfrak{q})$. Then $f$ factors over $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu}$ if and only if there exists an fpqc-covering $(U_i \to X)_{i\in I}$ and formal models $\mathcal{U}_i$ of $U_i$ together with $(\mathfrak{M}_i, \Phi_i) \in \hat{\mathcal{C}}_{\preceq\mu,N,K}(\mathcal{U}_i)$ such that $\Pi(\mathcal{U}_i)(\mathfrak{M}_i, \Phi_i) = (D, \Phi, N, \mathfrak{q})|_{U_i}$.*

Finally we go back to Galois representations. We prove that there is a canonical open subspace $\mathscr{H}^{\mathrm{red,ad,adm}}_{\varphi,N,\preceq\mu}$ of the reduced space underlying $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu}$ which carries a family of $\mathscr{G}_{K_\infty}$-representations. This family is universal in a sense made precise in the body of the article. Roughly this means that a morphism $f\colon X \to \mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu}$ defined by some $(\mathfrak{M}, \Phi, N)$ over a formal model $\mathcal{X}$ of $X$ factors over $\mathscr{H}^{\mathrm{ad,adm}}_{\varphi,N,\preceq\mu}$ if and only there exists a family of $\mathscr{G}_{K_\infty}$-representations $\mathcal{E}$ on $X$ such that the $\varphi$-module of $\mathcal{E}$, in the sense of Fontaine, is (up to inverting $p$) given by the $p$-adic completion of $(\mathfrak{M}, \Phi)[1/u]$. For a finite extension $L$ of $E_\mu$, Kisin's theory implies that we have an equality

$$\mathscr{H}^{\mathrm{red,ad,adm}}_{\varphi,N,\preceq\mu}(L) = \mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu}(L) = \mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\preceq\mu}(L)$$

of $L$-valued points.

If we restrict ourselves to the case of filtrations, i.e. to the above mentioned zero section in the vector bundle $\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\mu} \to \mathscr{D}^{\mathrm{ad}}_{\varphi,N,\mu}$ we may extend this family of $\mathscr{G}_{K_\infty}$-representations to a family of semi-stable $\mathscr{G}_K$-representations which turns out to be the universal family of semi-stable representations (with fixed Hodge-Tate weights).

**Theorem 8.8.** *There is an open substack $\mathscr{D}^{\mathrm{ad,adm}}_{\varphi,N,\mu} \subset \mathscr{D}^{\mathrm{ad}}_{\varphi,N,\mu}$ over which there exists a family $\mathcal{E}$ of semi-stable $\mathscr{G}_K$-representations such that $D_{\mathrm{st}}(\mathcal{E}) = (D, \Phi, N, \mathcal{F}^\bullet)$ is the restriction of the universal family of filtered $(\varphi, N)$-modules on $\mathscr{D}^{\mathrm{ad}}_{\varphi,N,\mu}$ to $\mathscr{D}^{\mathrm{ad,adm}}_{\varphi,N,\mu}$.*
*This family is universal in the following sense: Let $X$ be an adic space locally of finite type over the reflex field $E_\mu$ of $\mu$, and let $\mathcal{E}'$ be a family of semi-stable $\mathscr{G}_K$-representations on $X$ with constant Hodge polygon equal to $\mu$. Then there is a unique morphism $f\colon X \to \mathscr{D}^{\mathrm{ad,adm}}_{\varphi,N,\mu}$ such that $\mathcal{E}' \cong f^*\mathcal{E}$ as families of $\mathscr{G}_K$-representations.*

The corresponding result for crystalline $\mathscr{G}_K$-representations with constant Hodge polygon equal to $\mu$, whose moduli space is $\mathscr{D}^{\mathrm{ad,adm}}_{\varphi,\mu}$, is formulated and proved in Corollary 8.9.

**Notations:** Let $K$ be a finite field extension of the $p$-adic numbers $\mathbb{Q}_p$ and fix an algebraic closure $\overline{K}$ of $K$. We write $\mathbb{C}_p$ for the $p$-adic completion of $\overline{K}$ and let $\mathscr{G}_K = \operatorname{Gal}(\overline{K}/K)$ be the absolute Galois group of

$K$. Let $\widetilde{K}$ be the Galois closure of $K$ inside $\overline{K}$. Let $K_0$ be the maximal unramified subfield of $K$ and $W$ its ring of integers. Set $f := [K_0 : \mathbb{Q}_p]$, and let $\mathrm{Frob}_p$ be the Frobenius automorphism of $K_0$ which induces the $p$-power map on the residue field of $K_0$. We fix once and for all a uniformizer $\pi$ of $K$ and its minimal polynomial $E(u) = \mathrm{mipo}_{\pi/K_0}(u) \in W[u]$ over $K_0$. It is an Eisenstein polynomial, and $K = K_0[u]/(E(u))$. We choose a compatible system $\pi_n$ of $p^n$-th roots of $\pi$ in $\overline{K}$ and write $K_\infty$ for the field obtained from $K$ by adjoining all $\pi_n$.

**Acknowledgements.** The second author acknowledges support of the DFG (German Research Foundation) in form of SFB/TR 45 "Periods, Moduli Spaces and Arithmetic of Algebraic Varieties" and in form of a Forschungsstipendium He 6753/1-1. Both authors were also supported by SFB 878 "Groups, Geometry & Actions" of the DFG. We would further like to thank A. Mézard, M. Rapoport, T. Richarz and P. Scholze for helpful discussions.

## 2. Families of $(\varphi, N)$-modules with Hodge-Pink lattice

Let $R$ be a $\mathbb{Q}_p$-algebra and consider the endomorphism $\varphi := \mathrm{id}_R \otimes \mathrm{Frob}_p$ of $R \otimes_{\mathbb{Q}_p} K_0$. For an $R \otimes_{\mathbb{Q}_p} K_0$-module $M$ we set $\varphi^* M := M \otimes_{R \otimes_{\mathbb{Q}_p} K_0, \varphi} R \otimes_{\mathbb{Q}_p} K_0$. Similar notation is applied to morphisms between $R \otimes_{\mathbb{Q}_p} K_0$-modules. We let $\varphi^* \colon M \to \varphi^* M$ be the $\varphi$-semi-linear map with $\varphi^*(m) = m \otimes 1$.

We introduce the rings

$$\mathbb{B}_R^+ := \varprojlim_i (R \otimes_{\mathbb{Q}_p} K_0[u])/(E(u)^i) \text{ and } \mathbb{B}_R := \mathbb{B}_R^+[\frac{1}{E(u)}].$$

In a certain sense $\mathbb{B}_{\mathbb{Q}_p}^+$ and $\mathbb{B}_{\mathbb{Q}_p}$ are the analogs of Fontaine's rings $\mathbf{B}_{\mathrm{dR}}^+$ and $\mathbf{B}_{\mathrm{dR}}$ in Kisin's theory [Kis06] of $p$-adic Galois representation. By Cohen's structure theorem [Ser79, Theorem II.4.2] the ring $\mathbb{B}_{\mathbb{Q}_p}^+ = \varprojlim K_0[u]/(E(u)^i)$ is isomorphic to $K[\![t]\!]$ under a map sending $t$ to $\frac{E(u)}{E(0)}$. The rings $\mathbb{B}_R^+$ and $\mathbb{B}_R$ are relative versions over $R$, and are isomorphic to $(R \otimes_{\mathbb{Q}_p} K)[\![t]\!]$, respectively $(R \otimes_{\mathbb{Q}_p} K)[\![t]\!][\frac{1}{t}]$. We extend $\varphi$ to $R \otimes_{\mathbb{Q}_p} K_0[u]$ by requiring $\varphi(u) = u^p$ and we define $\varphi^n(\mathbb{B}_R^+) := \varprojlim_i (R \otimes_{\mathbb{Q}_p} K_0[u])/(\varphi^n(E(u))^i)$. Note that we may also identify $\varphi^n(\mathbb{B}_R^+)$ with $\varprojlim (R \otimes_{\mathbb{Q}_p} K(\pi_n)[u])/(1 - \frac{u}{\pi_n})^i = (R \otimes_{\mathbb{Q}_p} K(\pi_n))[\![1 - \frac{u}{\pi_n}]\!]$ under the assignment $\frac{E(u)}{E(0)} \mapsto 1 - \frac{u}{\pi_n}$; compare [Kis06, (1.1.1)]. We extend these rings to sheaves of rings $\varphi^n(\mathbb{B}_X^+) := \varphi^n(\mathbb{B}_{\mathcal{O}_X}^+)$ on $\mathbb{Q}_p$-schemes $X$ or adic spaces $X \in \mathrm{Ad}_{\mathbb{Q}_p}^{\mathrm{lft}}$. Here $\mathrm{Ad}_{\mathbb{Q}_p}^{\mathrm{lft}}$ denotes the category of adic spaces locally of finite type, see [Hub94] for example.

**Remark 2.1.** Note that $\varphi^n(\mathbb{B}_R^+)$ is not a subring of $\mathbb{B}_R^+$. If $X = \mathrm{Spa}(R, R^\circ)$ is an affinoid adic space of finite type over $\mathbb{Q}_p$ one should think of $\varphi^n(\mathbb{B}_R^+)$ as the completion of the structure sheaf on $X \times \mathbb{U}$ along the section defined by $\varphi^n(E(u)) \in \mathbb{U}$. Here $\mathbb{U}$ denotes the open unit disc over $K_0$.

**Definition 2.2.** (a) A *$\varphi$-module $(D, \Phi)$ over $R$* consists of a locally free $R \otimes_{\mathbb{Q}_p} K_0$-module $D$ of finite rank, and an $R \otimes_{\mathbb{Q}_p} K_0$-linear isomorphism $\Phi \colon \varphi^* D \xrightarrow{\sim} D$. A *morphism* $\alpha \colon (D, \Phi) \to (\widetilde{D}, \widetilde{\Phi})$ *of $\varphi$-modules* is an $R \otimes_{\mathbb{Q}_p} K_0$-homomorphism $\alpha \colon D \to \widetilde{D}$ with $\alpha \circ \Phi = \widetilde{\Phi} \circ \varphi^* \alpha$.

(b) A *$(\varphi, N)$-module $(D, \Phi, N)$ over $R$* consists of a $\varphi$-module $(D, \Phi)$ over $R$ and an $R \otimes_{\mathbb{Q}_p} K_0$-linear endomorphism $N \colon D \to D$ satisfying $N \circ \Phi = p \cdot \Phi \circ \varphi^* N$. A *morphism* $\alpha \colon (D, \Phi, N) \to (\widetilde{D}, \widetilde{\Phi}, \widetilde{N})$ *of $(\varphi, N)$-modules* is morphism of $\varphi$-modules with $\alpha \circ N = \widetilde{N} \circ \alpha$.

The rank of $D$ over $R \otimes_{\mathbb{Q}_p} K_0$ is called the *rank* of $(D, \Phi)$, resp. $(D, \Phi, N)$.

Every $\varphi$-module over $R$ can be viewed as a $(\varphi, N)$-module with $N = 0$.

**Lemma 2.3.** (a) *Every $\varphi$-module $(D, \Phi)$ over $R$ is Zariski locally on* $\mathrm{Spec}\, R$ *free over* $R \otimes_{\mathbb{Q}_p} K_0$.

(b) *The endomorphism $N$ of a $(\varphi, N)$-module over $R$ is automatically nilpotent.*

*Proof.* (a) Let $\mathfrak{m} \subset R$ be a maximal ideal. Then $R/\mathfrak{m} \otimes_{\mathbb{Q}_p} K_0$ is a direct product of fields which are transitively permuted by $\mathrm{Gal}(K_0/\mathbb{Q}_p)$. The existence of the isomorphism $\Phi$ implies that $D \otimes_R R/\mathfrak{m}$ is free over $R/\mathfrak{m} \otimes_{\mathbb{Q}_p} K_0$. Now the assertion follows by Nakayama's lemma.

(b) By (a) we may locally on $R$ write $N$ as a matrix with entries in $R \otimes_{\mathbb{Q}_p} K_0$. Set $d := \mathrm{rk}\, D$. If the entries of the $d$-th power $N^d$ lie in $\mathrm{Rad}(0) \otimes_{\mathbb{Q}_p} K_0$, where $\mathrm{Rad}(0) = \bigcap_{\mathfrak{p} \subset R \text{ prime}} \mathfrak{p}$ is the nil-radical, then

$N$ is nilpotent. Thus we may check the assertion in $L = \mathrm{Frac}(R/\mathfrak{p})^{\mathrm{alg}}$ for all primes $\mathfrak{p} \subset R$. We replace $R$ by $L$. Then $D = \prod V_\psi$ splits up into a direct product of $d$-dimensional $L$-vector spaces indexed by the embeddings $\psi\colon K_0 \hookrightarrow L$. For every fixed embedding $\psi$ the $f$-th power $\Phi^f$ restricts to an endomorphism $\Phi_\psi$ of $V_\psi$ satisfying $N\Phi_\psi = p^f \Phi_\psi N$. If $V(\lambda, \Phi_\psi)$ denotes the generalized eigenspace for some $\lambda \in L^\times$, then $N$ maps $V(\lambda, \Phi_\psi)$ to $V(p^f\lambda, \Phi_\psi)$ and hence $N$ is nilpotent, as there are only finitely many non-zero eigenspaces. This implies that $N^d = 0$. □

**Remark 2.4.** If $R$ is even a $K_0$-algebra, we can decompose $R \otimes_{\mathbb{Q}_p} K_0 \cong \prod_{i \in \mathbb{Z}/f\mathbb{Z}} R$ where the $i$-th factor is given by the map $R \otimes_{\mathbb{Q}_p} K_0 \to R$, $a \otimes b \mapsto a\,\mathrm{Frob}_p^{-i}(b)$ for $a \in R, b \in K_0$. For a $(\varphi, N)$-module over $R$ we obtain corresponding decompositions $D = \prod_i D_i$ and $\varphi^*D = \prod_i (\varphi^*D)_i$ with $(\varphi^*D)_i = D_{i-1}$, and hence also $\Phi = (\Phi_i\colon D_{i-1} \xrightarrow{\sim} D_i)_i$ and $N = (N_i\colon D_i \to D_i)_i$ with $p\,\Phi_i \circ N_{i-1} = N_i \circ \Phi_i$, because $(\varphi^*N)_i = N_{i-1}$. If we set $\Psi_i := \Phi_i \circ \ldots \circ \Phi_1 = (\Phi \circ \varphi^*\Phi \circ \ldots \circ \varphi^{(i-1)*}\Phi)_i\colon D_0 = (\varphi^{i*}D)_i \xrightarrow{\sim} D_i$ then $p^i\,\Psi_i \circ N_0 = N_i \circ \Psi_i$ for all $i$, and $\Psi_f = (\Phi^f)_0$. There is an isomorphism of $(\varphi, N)$-modules over $R$

$$(\mathrm{id}_{D_0}, \Psi_1, \ldots, \Psi_{f-1})\colon \Big(\prod_i D_0,\ \big((\Phi^f)_0, \mathrm{id}_{D_0}, \ldots, \mathrm{id}_{D_0}\big),\ (p^i N_0)_i\Big) \xrightarrow{\ \sim\ } \Big(\prod_i D_i,\ (\Phi_i)_i,\ (N_i)_i\Big)\,. \tag{2.1}$$

Thus $(D, \Phi, N)$ is uniquely determined by $(D_0, (\Phi^f)_0, N_0)$ satisfying $p^f(\Phi^f)_0 \circ N_0 = N_0 \circ (\Phi^f)_0$. Further note that under this isomorphism $\Phi^f$ on $(D, \Phi, N)$ corresponds to $\big((\Phi^f)_0, \ldots, (\Phi^f)_0\big)$ on the left hand side.

**Definition 2.5.** (a) A *$K$-filtered $(\varphi, N)$-module* $(D, \Phi, N, \mathcal{F}^\bullet)$ *over* $R$ consists of a $(\varphi, N)$-module $(D, \Phi, N)$ over $R$ together with a decreasing separated and exhaustive $\mathbb{Z}$-filtration $\mathcal{F}^\bullet$ on $D_K := D \otimes_{K_0} K$ by $R \otimes_{\mathbb{Q}_p} K$-submodules such that $\mathrm{gr}^i_{\mathcal{F}} D_K := \mathcal{F}^i D_K / \mathcal{F}^{i+1} D_K$ is locally free as an $R$-module for all $i$. A *morphism* $\alpha\colon (D, \Phi, N, \mathcal{F}^\bullet) \to (\widetilde{D}, \widetilde{\Phi}, \widetilde{N}, \widetilde{\mathcal{F}}^\bullet)$ is a morphism of $(\varphi, N)$-modules with $(\alpha \otimes \mathrm{id})(\mathcal{F}^i D_K) \subset \widetilde{\mathcal{F}}^i \widetilde{D}_K$.

(b) A *$(\varphi, N)$-module with Hodge-Pink lattice* $(D, \Phi, N, \mathfrak{q})$ *over* $R$ consists of a $(\varphi, N)$-module $(D, \Phi, N)$ over $R$ together with a *$\mathbb{B}_R^+$-lattice* $\mathfrak{q} \subset D \otimes_{R \otimes K_0} \mathbb{B}_R$. This means that $\mathfrak{q}$ is a finitely generated $\mathbb{B}_R^+$-submodule, which is a direct summand as $R$-module satisfying $\mathbb{B}_R \cdot \mathfrak{q} = D \otimes_{R \otimes K_0} \mathbb{B}_R$. We call $\mathfrak{q}$ the *Hodge-Pink lattice* of $(D, \Phi, N, \mathfrak{q})$. A *morphism* $\alpha\colon (D, \Phi, N, \mathfrak{q}) \to (\widetilde{D}, \widetilde{\Phi}, \widetilde{N}, \tilde{\mathfrak{q}})$ is a morphism of $(\varphi, N)$-modules with $(\alpha \otimes \mathrm{id})(\mathfrak{q}) \subset \tilde{\mathfrak{q}}$.

For every $(\varphi, N)$-module with Hodge-Pink lattice $(D, \Phi, N, \mathfrak{q})$ over $R$ we also consider the tautological $\mathbb{B}_R^+$-lattice $\mathfrak{p} := D \otimes_{R \otimes K_0} \mathbb{B}_R^+$.

**Lemma 2.6.** *Let $\mathfrak{q} \subset D \otimes_{R \otimes K_0} \mathbb{B}_R$ be a $\mathbb{B}_R^+$-submodule. Then $\mathfrak{q}$ is a $\mathbb{B}_R^+$-lattice if and only if $E(u)^n\mathfrak{p} \subset \mathfrak{q} \subset E(u)^{-m}\mathfrak{p}$ for all $n, m \gg 0$ and for any (some) such $n, m$ the quotients $E(u)^{-m}\mathfrak{p}/\mathfrak{q}$ and $\mathfrak{q}/E(u)^n\mathfrak{p}$ are finite locally free $R$-modules.*

*If this is the case then étale locally on* $\operatorname{Spec} R$ *the $\mathbb{B}_R^+$-module $\mathfrak{q}$ is free of the same rank as $\mathfrak{p}$.*

*Proof.* The assertion $E(u)^n\mathfrak{p} \subset \mathfrak{q} \subset E(u)^{-m}\mathfrak{p}$ for all $n, m \gg 0$ is equivalent to $\mathbb{B}_R \cdot \mathfrak{q} = D \otimes_{R \otimes K_0} \mathbb{B}_R$ when $\mathfrak{q}$ is finitely generated. Consider such $n, m$. If $\mathfrak{q}$ is a $\mathbb{B}_R^+$-lattice, hence a direct summand of $D \otimes_{R \otimes K_0} \mathbb{B}_R$ there is an $R$-linear section of the projection $D \otimes_{R \otimes K_0} \mathbb{B}_R \twoheadrightarrow (D \otimes_{R \otimes K_0} \mathbb{B}_R)/\mathfrak{q}$. The composition of this section with the inclusion $E(u)^{-m}\mathfrak{p}/\mathfrak{q} \hookrightarrow (D \otimes_{R \otimes K_0} \mathbb{B}_R)/\mathfrak{q}$ factors through $E(u)^{-m}\mathfrak{p}$ and realizes $E(u)^{-m}\mathfrak{p}/\mathfrak{q}$ as a direct summand of the $R$-module $E(u)^{-m}\mathfrak{p}/E(u)^n\mathfrak{p}$ which is locally free by Lemma 2.3(a). This shows that $E(u)^{-m}\mathfrak{p}/E(u)^n\mathfrak{p} \cong (E(u)^{-m}\mathfrak{p}/\mathfrak{q}) \oplus (\mathfrak{q}/E(u)^n\mathfrak{p})$ and both $E(u)^{-m}\mathfrak{p}/\mathfrak{q}$ and $\mathfrak{q}/E(u)^n\mathfrak{p}$ are finite locally free $R$-modules.

Conversely any isomorphism $E(u)^{-m}\mathfrak{p}/E(u)^n\mathfrak{p} \cong (E(u)^{-m}\mathfrak{p}/\mathfrak{q}) \oplus (\mathfrak{q}/E(u)^n\mathfrak{p})$ together with the decomposition $D \otimes_{R \otimes K_0} \mathbb{B}_R \cong (E(u)^n\mathfrak{p}) \oplus (E(u)^{-m}\mathfrak{p}/E(u)^n\mathfrak{p}) \oplus (D \otimes_{R \otimes K_0} \mathbb{B}_R)/E(u)^{-m}\mathfrak{p}$ realizes $\mathfrak{q}$ as a direct summand of $D \otimes_{R \otimes K_0} \mathbb{B}_R$. Since $E(u)^n\mathfrak{p}$ is finitely generated over $\mathbb{B}_R^+$ and $\mathfrak{q}/E(u)^n\mathfrak{p}$ is finitely generated over $R$, also $\mathfrak{q}$ is finitely generated over $\mathbb{B}_R^+$, hence a $\mathbb{B}_R^+$-lattice.

To prove the local freeness of $\mathfrak{q}$ we may work locally on $R$ and assume by Lemma 2.3(a) that $\mathfrak{p}$ is free over $\mathbb{B}_R^+$, say of rank $d$, and $\mathfrak{q}/E(u)^n\mathfrak{p}$ and $E(u)^{-m}\mathfrak{p}/\mathfrak{q}$ are free over $R$. There is a noetherian subring $\widetilde{R}$ of $R$ and a short exact sequence

$$0 \longrightarrow \widetilde{Q} \longrightarrow \widetilde{P} \longrightarrow \widetilde{N} \longrightarrow 0 \tag{2.2}$$

of $\mathbb{B}^+_{\widetilde{R}}$-modules which are free $\widetilde{R}$-modules, such that the tensor product of (2.2) with $R$ over $\widetilde{R}$ is isomorphic to

$$0 \longrightarrow \mathfrak{q}/E(u)^n\mathfrak{p} \longrightarrow E(u)^{-m}\mathfrak{p}/E(u)^n\mathfrak{p} \longrightarrow E(u)^{-m}\mathfrak{p}/\mathfrak{q} \longrightarrow 0\,. \tag{2.3}$$

Indeed, we can take $\widetilde{R}$ as the finitely generated $\mathbb{Q}_p$-algebra containing all the coefficients appearing in matrix representations of the maps in (2.3) and the action of $K_0[u]/(E(u))^{m+n}$.

Let $\widetilde{\mathfrak{p}}$ be a free $\mathbb{B}^+_{\widetilde{R}}$-module of rank $d$, fix an isomorphism $E(u)^{-m}\widetilde{\mathfrak{p}}\otimes_{\mathbb{B}^+_{\widetilde{R}}}\mathbb{B}^+_{\widetilde{R}}/(E(u))^{m+n}\xrightarrow{\sim}\widetilde{P}$ and let the $\mathbb{B}^+_{\widetilde{R}}$-module $\widetilde{\mathfrak{q}}$ be defined by the exact sequence

$$0 \longrightarrow \widetilde{\mathfrak{q}} \longrightarrow E(u)^{-m}\widetilde{\mathfrak{p}} \longrightarrow \widetilde{N} \longrightarrow 0\,. \tag{2.4}$$

Since $\mathbb{B}^+_{\widetilde{R}}\cong(\widetilde{R}\otimes_{\mathbb{Q}_p}K)[\![t]\!]$ is noetherian, $\widetilde{\mathfrak{q}}$ is finitely generated. Consider a maximal ideal $\mathfrak{m}\subset\mathbb{B}^+_{\widetilde{R}}$. Since $t\in\mathfrak{m}$, it maps to a maximal ideal $\mathfrak{n}$ of $\widetilde{R}$. Since $\mathfrak{n}$ is finitely generated, $\mathbb{B}^+_{\widetilde{R}}\otimes_{\widetilde{R}}\widetilde{R}/\mathfrak{n}\cong(\widetilde{R}/\mathfrak{n}\otimes_{\mathbb{Q}_p}K)[\![t]\!]$ and this is a direct product of discrete valuation rings. Thus $\widetilde{\mathfrak{q}}\otimes_{\widetilde{R}}\widetilde{R}/\mathfrak{n}$ is locally free of rank $d$ by the elementary divisor theorem. Since this holds for all $\mathfrak{m}$, [EGA, IV$_3$, Theorem 11.3.10] implies that $\widetilde{\mathfrak{q}}$ is a projective $\mathbb{B}^+_{\widetilde{R}}$-module and by [EGA, I$_{\text{new}}$, Proposition 10.10.8.6] it is locally on $\operatorname{Spec}\widetilde{R}\otimes_{\mathbb{Q}_p}K$ free over $\mathbb{B}^+_{\widetilde{R}}$. Let $\{\psi\colon K\hookrightarrow\overline{\mathbb{Q}}_p\}$ be the set of all $\mathbb{Q}_p$-homomorphisms and let $\widetilde{K}$ be the compositum of all $\psi(K)$ inside $\overline{\mathbb{Q}}_p$. Then $\widetilde{R}\to\widetilde{R}\otimes_{\mathbb{Q}_p}\widetilde{K}$ is finite étale and the pullback of $\widetilde{\mathfrak{q}}$ under this base change is locally on $\operatorname{Spec}\widetilde{R}\otimes_{\mathbb{Q}_p}\widetilde{K}\otimes_{\mathbb{Q}_p}K$ free over $\mathbb{B}^+_{\widetilde{R}\otimes_{\mathbb{Q}_p}\widetilde{K}}$. Since $\operatorname{Spec}\widetilde{R}\otimes_{\mathbb{Q}_p}\widetilde{K}\otimes_{\mathbb{Q}_p}K=\coprod_\psi\operatorname{Spec}\widetilde{R}\otimes_{\mathbb{Q}_p}\widetilde{K}$ it follows that the pullback of $\widetilde{\mathfrak{q}}$ is already locally on $\operatorname{Spec}\widetilde{R}\otimes_{\mathbb{Q}_p}\widetilde{K}$ free over $\mathbb{B}^+_{\widetilde{R}\otimes_{\mathbb{Q}_p}\widetilde{K}}$.

To finish the proof it remains to show that $\widetilde{\mathfrak{q}}\otimes_{\mathbb{B}^+_{\widetilde{R}}}\mathbb{B}^+_R\cong\mathfrak{q}$. Tensoring (2.4) with $\mathbb{B}^+_R$ over $\mathbb{B}^+_{\widetilde{R}}$ we obtain the top row in the diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Tor}_1^{\mathbb{B}^+_{\widetilde{R}}}(\widetilde{N},\mathbb{B}^+_R) \arrow[r] & \widetilde{\mathfrak{q}}\otimes_{\mathbb{B}^+_{\widetilde{R}}}\mathbb{B}^+_R \arrow[r] \arrow[d, two heads] & E(u)^{-m}\mathfrak{p} \arrow[r] \arrow[d, equal] & \widetilde{N}\otimes_{\mathbb{B}^+_{\widetilde{R}}}\mathbb{B}^+_R \arrow[r] \arrow[d, "\cong"] & 0 \\
 & 0 \arrow[r] & \mathfrak{q} \arrow[r] & E(u)^{-m}\mathfrak{p} \arrow[r] & E(u)^{-m}\mathfrak{p}/\mathfrak{q} \arrow[r] & 0\,.
\end{tikzcd}$$

Abbreviate $\ell:=m+n$. Since the functor $\widetilde{N}\otimes_{\mathbb{B}^+_{\widetilde{R}}}\bullet$ equals the composition of the functors $(\mathbb{B}^+_{\widetilde{R}}/t^\ell)\otimes_{\mathbb{B}^+_{\widetilde{R}}}\bullet$ followed by $\widetilde{N}\otimes_{\mathbb{B}^+_{\widetilde{R}}/t^\ell}\bullet$, the $\operatorname{Tor}_1$-module on the left can be computed from a change of rings spectral sequence and its associated 5-term sequence of low degrees

$$\ldots\ \longrightarrow\ \operatorname{Tor}_1^{\mathbb{B}^+_{\widetilde{R}}}(\mathbb{B}^+_{\widetilde{R}}/t^\ell,\mathbb{B}^+_R)\otimes_{\mathbb{B}^+_{\widetilde{R}}/t^\ell}\widetilde{N}\ \longrightarrow\ \operatorname{Tor}_1^{\mathbb{B}^+_{\widetilde{R}}}(\widetilde{N},\mathbb{B}^+_R)\ \longrightarrow\ \operatorname{Tor}_1^{\mathbb{B}^+_{\widetilde{R}}/t^\ell}(\widetilde{N},\mathbb{B}^+_R/t^\ell)\ \longrightarrow\ 0\,.$$

The right term in this sequence is zero because $\operatorname{Tor}_1^{\mathbb{B}^+_{\widetilde{R}}/t^\ell}(\widetilde{N},\mathbb{B}^+_R/t^\ell)=\operatorname{Tor}_1^{\widetilde{R}}(\widetilde{N},R)$ and $\widetilde{N}$ is flat over $\widetilde{R}$. The left term is zero because $t^\ell$ is a non-zero-divisor both in $\mathbb{B}^+_{\widetilde{R}}$ and $\mathbb{B}^+_R$. This shows that $\operatorname{Tor}_1^{\mathbb{B}^+_{\widetilde{R}}}(\widetilde{N},\mathbb{B}^+_R)=0$ and proves the lemma. □

**Remark 2.7.** (1) Let $R=L$ be a field and let $(D,\Phi,N,\mathfrak{q})$ be a $(\varphi,N)$-module with Hodge-Pink lattice over $L$. The Hodge-Pink lattice $\mathfrak{q}$ gives rise to a $K$-filtration $\mathcal{F}^\bullet_\mathfrak{q}$ as follows. Consider the natural projection

$$\mathfrak{p}\ \twoheadrightarrow\ \mathfrak{p}/E(u)\mathfrak{p}\ =\ D\otimes_{R\otimes K_0}\mathbb{B}^+_R/(E(u))\ =\ D\otimes_{R\otimes K_0}R\otimes_{\mathbb{Q}_p}K\ =\ D_K$$

and let $\mathcal{F}^i_\mathfrak{q}D_K$ be the image of $\mathfrak{p}\cap E(u)^i\mathfrak{q}$ in $D_K$ for all $i\in\mathbb{Z}$, that is

$$\mathcal{F}^i_\mathfrak{q}D_K:=\big(\mathfrak{p}\cap E(u)^i\mathfrak{q}\big)/\big(E(u)\mathfrak{p}\cap E(u)^i\mathfrak{q}\big)\,.$$

Since $L$ is a field $(D,\Phi,N,\mathcal{F}^\bullet_\mathfrak{q})$ is a $K$-filtered $(\varphi,N)$-module over $L$. Note that this functor does not exist for general $R$, because $\operatorname{gr}^i_{\mathcal{F}_\mathfrak{q}}D_K$ will not be locally free over $R$ in general. This is related to the fact that the Hodge polygon of $\mathcal{F}_{\mathfrak{q}}^\bullet$ is locally constant on $R$ whereas the Hodge polygon of $\mathfrak{q}$ is only semi-continuous; see Remark 2.11 below.

(2) However, for general $R$ consider the category of $(\varphi,N)$-modules with Hodge-Pink lattice $(D,\Phi,N,\mathfrak{q})$ over $R$, such that $\mathfrak{p}\subset\mathfrak{q}\subset E(u)^{-1}\mathfrak{p}$. This category is equivalent to the category of $K$-filtered $(\varphi,N)$-modules $(D,\Phi,N,\mathcal{F}^\bullet)$ over $R$ with $\mathcal{F}^0 D_K = D_K$ and $\mathcal{F}^2=0$. Namely, defining $\mathcal{F}_{\mathfrak{q}}^\bullet$ as in (1) we obtain

$$\operatorname{gr}^i_{\mathcal{F}_{\mathfrak{q}}} D_K \;\cong\; \begin{cases} E(u)^{-1}\mathfrak{p}/\mathfrak{q} & \text{for } i=0\,,\\ \mathfrak{q}/\mathfrak{p} & \text{for } i=1\,,\\ 0 & \text{for } i\neq 0,1\,,\end{cases}$$

and so $(D,\Phi,N,\mathcal{F}_{\mathfrak{q}}^\bullet)$ is a $K$-filtered $(\varphi,N)$-module by Lemma 2.6. Conversely, $\mathfrak{q}$ equals the preimage of $\mathcal{F}_{\mathfrak{q}}^1 D_K$ under the morphism $E(u)^{-1}\mathfrak{p}\xrightarrow{\cdot E(u)}\mathfrak{p}\twoheadrightarrow D_K$ and this defines the inverse functor.

(3) Now let $(D,\Phi,N,\mathcal{F}^\bullet)$ be a $K$-filtered $(\varphi,N)$-module over $R$. Using that $\mathbb{B}_R^+=(R\otimes_{\mathbb{Q}_p}K)[\![t]\!]$ is an $R\otimes_{\mathbb{Q}_p}K$-algebra, we can define the Hodge-Pink lattice

$$\mathfrak{q}\;:=\;\mathfrak{q}(\mathcal{F}^\bullet)\;:=\;\sum_{i\in\mathbb{Z}} E(u)^{-i}(\mathcal{F}^i D_K)\otimes_{R\otimes K}\mathbb{B}_R^+.$$

It satisfies $\mathcal{F}_{\mathfrak{q}}^\bullet=\mathcal{F}^\bullet$. Using Lemma 2.6 one easily finds that $\mathfrak{q}(\mathcal{F}^\bullet)$ is indeed a $\mathbb{B}_R^+$-lattice.

**Example 2.8.** The $K$-filtered $(\varphi,N)$-modules over $R=\mathbb{Q}_p$ which correspond to the cyclotomic character $\chi_{\rm cyc}\colon\mathscr{G}_K\to\mathbb{Z}_p^\times$ are $D_{\rm st}(\chi_{\rm cyc})=(K_0,\Phi=p^{-1},N=0,\mathcal{F}^\bullet)$ with $\mathcal{F}^{-1}=K\supsetneq\mathcal{F}^0=(0)$ and its dual $D_{\rm st}^*(\chi_{\rm cyc})=(K_0,\Phi=p,N=0,\mathcal{F}^\bullet)$ with $\mathcal{F}^1=K\supsetneq\mathcal{F}^2=(0)$; see Definition 8.1 and Formula (8.2) below. For both there exists a unique Hodge-Pink lattice which induces the filtration. On $D_{\rm st}(\chi_{\rm cyc})$ it is $\mathfrak{q}=E(u)\mathfrak{p}$ and on $D_{\rm st}^*(\chi_{\rm cyc})$ it is $\mathfrak{q}=E(u)^{-1}\mathfrak{p}$.

We want to introduce Hodge weights and Hodge polygons. Let $d>0$, let $B\subset\mathrm{GL}_d$ be the Borel subgroup of upper triangular matrices and let $T\subset B$ be the maximal torus consisting of the diagonal matrices. Let $\widetilde{G}:=\operatorname{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_{d,K}$, $\widetilde{B}=\operatorname{Res}_{K/\mathbb{Q}_p}B$ and $\widetilde{T}:=\operatorname{Res}_{K/\mathbb{Q}_p}T$ be the Weil restrictions. We consider cocharacters

$$\mu\colon\mathbb{G}_{m,\overline{\mathbb{Q}}_p}\longrightarrow\widetilde{T}_{\overline{\mathbb{Q}}_p}\tag{2.5}$$

which are dominant with respect to the Borel $\widetilde{B}$ of $\widetilde{G}$. In other words on $\overline{\mathbb{Q}}_p$-valued points the cocharacter

$$\mu\colon\overline{\mathbb{Q}}_p^\times\longrightarrow\prod_{\psi\colon K\to\overline{\mathbb{Q}}_p}T(\overline{\mathbb{Q}}_p)\,,$$

where $\psi$ runs over all $\mathbb{Q}_p$-homomorphisms $\psi\colon K\to\overline{\mathbb{Q}}_p$, is given by cocharacters

$$\mu_\psi\colon x\mapsto\operatorname{diag}(x^{\mu_{\psi,1}},\ldots,x^{\mu_{\psi,d}})$$

for some integers $\mu_{\psi,j}\in\mathbb{Z}$ with $\mu_{\psi,j}\geq\mu_{\psi,j+1}$. We define the *reflex field $E_\mu$ of $\mu$* as the fixed field in $\overline{\mathbb{Q}}_p$ of $\{\,\sigma\in\mathscr{G}_{\mathbb{Q}_p}:\mu_{\sigma\psi,j}=\mu_{\psi,j}\ \ \forall\, j,\psi\,\}$. It is a finite extension of $\mathbb{Q}_p$ which is contained in the compositum $\widetilde{K}$ of all $\psi(K)$ inside $\overline{\mathbb{Q}}_p$. For each $j$ the locally constant function $\psi\mapsto\mu_{\psi,j}$ on $\operatorname{Spec}\widetilde{K}\otimes_{\mathbb{Q}_p}K\cong\coprod_{\psi\colon K\to\widetilde{K}}\operatorname{Spec}\widetilde{K}$ descends to a $\mathbb{Z}$-valued function $\mu_j$ on $\operatorname{Spec}E_\mu\otimes_{\mathbb{Q}_p}K$, because $\mu_j$ is constant on the fibers of $\operatorname{Spec}\widetilde{K}\otimes_{\mathbb{Q}_p}K\to\operatorname{Spec}E_\mu\otimes_{\mathbb{Q}_p}K$. In particular, the cocharacter $\mu$ is defined over $E_\mu$. If $R$ is an $E_\mu$-algebra we also view $\mu_j$ as a locally constant $\mathbb{Z}$-valued function on $\operatorname{Spec}R\otimes_{\mathbb{Q}_p}K$.

**Construction 2.9.** Let $\underline{D}=(D,\Phi,N,\mathfrak{q})$ be a $(\varphi,N)$-module with Hodge-Pink lattice of rank $d$ over a field extension $L$ of $\mathbb{Q}_p$. By Lemma 2.3(a) the $L\otimes_{\mathbb{Q}_p}K_0$-module $D$ is free. Since $L\otimes_{\mathbb{Q}_p}K$ is a product of fields, $\mathbb{B}_L^+=(L\otimes_{\mathbb{Q}_p}K)[\![t]\!]$ is a product of discrete valuation rings and $\mathfrak{q}$ is a free $\mathbb{B}_L^+$-module of rank $d$. We choose bases of $D$ and $\mathfrak{q}$. Then the inclusion $\mathfrak{q}\subset D\otimes_{L\otimes K_0}\mathbb{B}_L$ is given by an element $\gamma$ of $\mathrm{GL}_d(\mathbb{B}_L)=\widetilde{G}\big(L(\!(t)\!)\big)$. By the Cartan decomposition for $\widetilde{G}$ there is a uniquely determined dominant cocharacter $\mu_L\colon\mathbb{G}_{m,L}\to\widetilde{T}_L$ over $L$ with $\gamma\in\widetilde{G}\big(L[\![t]\!]\big)\mu_L(t)^{-1}\widetilde{G}\big(L[\![t]\!]\big)$. This cocharacter is independent of the chosen bases. If $L$ contains $\widetilde{K}$, it is defined over $\widetilde{K}$ because $\widetilde{T}$ splits over $\widetilde{K}$. In this case we view it as an element of $X_*(T_{\widetilde K})_{\rm dom}$ and denote it by $\mu_{\underline{D}}(\operatorname{Spec} L)$. It has the following explicit description. Under the decomposition $L\otimes_{\mathbb{Q}_p}K=\prod_{\psi\colon K\to\widetilde K}L$ we have $\widetilde G\big(L((t))\big)=\prod_\psi \mathrm{GL}_d\big(L((t))\big)$, $\mu_L=(\mu_\psi)_\psi$, and $\gamma\in\prod_\psi \mathrm{GL}_d\big(L[\![t]\!]\big)\mu_\psi(t)^{-1}\,\mathrm{GL}_d\big(L[\![t]\!]\big)$. The $t^{-\mu_{\psi,1}},\dots,t^{-\mu_{\psi,d}}$ are the elementary divisors of the $\psi$-component $\mathfrak{q}_\psi$ of $\mathfrak{q}$ with respect to $\mathfrak{p}$. That is, there is an $L[\![t]\!]$-basis $(v_{\psi,1},\dots,v_{\psi,d})$ of the $\psi$-component $\mathfrak{p}_\psi$ of $\mathfrak{p}$ such that $(t^{-\mu_{\psi,1}}v_{\psi,1},\dots,t^{-\mu_{\psi,d}}v_{\psi,d})$ is an $L[\![t]\!]$-basis of $\mathfrak{q}_\psi$.

Let $(D,\Phi,N,\mathcal{F}^\bullet_{\mathfrak{q}})$ be the $K$-filtered $(\varphi,N)$-module associated with $\underline{D}$ by Remark 2.7(1). Then $\mathcal{F}^i_{\mathfrak{q}}D_{K,\psi}=\langle v_{\psi,j}: i-\mu_{\psi,j}\le 0\rangle_L$ and

$$\dim_L \mathrm{gr}^i_{\mathcal{F}_{\mathfrak{q}}}D_{K,\psi}\;=\;\#\{j: i-\mu_{\psi,j}=0\}\,.$$

More generally, for a $K$-filtered $(\varphi,N)$-module $(D,\Phi,N,\mathcal{F}^\bullet)$ over a field extension $L$ of $\widetilde K$ we consider the decomposition $D_K=\prod_\psi D_{K,\psi}$ and define the integers $\mu_{\psi,1}\ge\ldots\ge\mu_{\psi,d}$ by the formula

$$\dim_L \mathrm{gr}^i_{\mathcal{F}}D_{K,\psi}\;=\;\#\{j:\mu_{\psi,j}=i\}\,.$$

We define the cocharacter $\mu_{(D,\Phi,N,\mathcal{F}^\bullet)}(\operatorname{Spec}L):=(\mu_\psi)_\psi$ and view it as an element of $X_*(T_{\widetilde K})_{\rm dom}$.

**Definition 2.10.** (a) Let $R$ be a $\widetilde K$-algebra and consider the decomposition $R\otimes_{\mathbb{Q}_p}K=\prod_{\psi:K\hookrightarrow\widetilde K}R$. Let $\underline{D}$ be a $(\varphi,N)$-module with Hodge-Pink lattice (respectively a $K$-filtered $(\varphi,N)$-module) of rank $d$ over $R$. For every point $s\in\operatorname{Spec}R$ we consider the base change $s^*\underline{D}$ of $\underline{D}$ to $\kappa(s)$. We call the cocharacter $\mu_{\underline{D}}(s):=\mu_{s^*\underline{D}}(\operatorname{Spec}\kappa(s))$ from Construction 2.9 the *Hodge polygon of $\underline{D}$ at $s$* and we consider $\mu_{\underline{D}}$ as a function $\mu_{\underline{D}}:\operatorname{Spec}R\to X_*(T_{\widetilde K})_{\rm dom}$. The integers $-\mu_{\psi,j}(s)$ are called the *Hodge weights of $\underline{D}$ at $s$*.

Now let $\mu:\overline{\mathbb{Q}}_p^\times\to\widetilde T(\overline{\mathbb{Q}}_p)$ be a dominant cocharacter as in (2.5), let $E_\mu$ denote the reflex field of $\mu$, and let $R$ be an $E_\mu$-algebra.

(b) Let $\underline{D}$ be a $(\varphi,N)$-module with Hodge-Pink lattice (respectively a $K$-filtered $(\varphi,N)$-module) of rank $d$ over $R$. We say that *$\underline{D}$ has constant Hodge polygon equal to $\mu$* if $\mu_{\underline{D}}(s)=\mu$ for every point $s\in\operatorname{Spec}(R\otimes_{E_\mu}\widetilde K)$.

(c) Let $\underline{D}=(D,\Phi,N,\mathfrak{q})$ be a $(\varphi,N)$-module with Hodge-Pink lattice over $\operatorname{Spec}R$. We say that $\underline{D}$ has *Hodge polygon bounded by $\mu$* if

$$\bigwedge\nolimits^{j}_{\mathbb{B}_R^+}\mathfrak{q}\;\subset\;E(u)^{-\mu_1-\ldots-\mu_j}\cdot\bigwedge\nolimits^{j}_{\mathbb{B}_R^+}\mathfrak{p}$$

for all $j=1,\dots,d$ with equality for $j=d$, where the $\mu_i$ are the $\mathbb{Z}$-valued functions on $\operatorname{Spec}R\otimes_{\mathbb{Q}_p}K$ determined by $\mu$; see the discussion before Construction 2.9.

Equivalently the condition of being bounded by $\mu$ can be described as follows: Over $\widetilde K$ the cocharacter $\mu$ is described by a decreasing sequence of integers $\mu_{\psi,1}\ge\cdots\ge\mu_{\psi,d}$ for every $\mathbb{Q}_p$-embedding $\psi:K\hookrightarrow\overline{\mathbb{Q}}_p$. Let $R'=R\otimes_{E_\mu}\widetilde K$, then $R'\otimes_{\mathbb{Q}_p}K\cong\prod_{\psi:K\to\widetilde K}R'_\psi$ with each $R'_\psi=R'$ under the isomorphism $a\otimes b\mapsto(a\psi(b))_\psi$, where $\psi:K\to R'$ is given via the embedding into the second factor of $R'=R\otimes_{E_\mu}\widetilde K$. Especially we view $R'_\psi$ as a $K$-algebra via $\psi$. Under this isomorphism $D\otimes_{R\otimes K_0}\mathbb{B}_{R'}=:\mathfrak{p}_{R'}[\frac1t]$ decomposes into a product $\prod_\psi\mathfrak{p}_{R'}[\frac1t]_\psi$, where $\mathfrak{p}_{R'}[\frac1t]_\psi$ is a free $R'_\psi[\![t]\!][\frac1t]$-module and the $\mathbb{B}^+_{R'}$-lattice $\mathfrak{p}_{R'}\subset\mathfrak{p}_{R'}[\frac1t]$ decomposes into a product of $R'_\psi[\![t]\!]$-lattices $\mathfrak{p}_{R',\psi}\subset\mathfrak{p}_{R'}[\frac1t]_\psi$.

Further, under the isomorphism $D\otimes_{R\otimes K_0}\mathbb{B}_{R'}\cong\prod_\psi\mathfrak{p}_{R'}[\frac1t]_\psi$ the Hodge-Pink lattice $\mathfrak{q}_{R'}=\mathfrak{q}\otimes_R R'$ decomposes into a product $\mathfrak{q}_{R'}=\prod_\psi\mathfrak{q}_{R',\psi}$, where $\mathfrak{q}_{R',\psi}$ is an $R'_\psi[\![t]\!]$-lattice in $\mathfrak{p}_{R'}[\frac1t]_\psi$. Then the condition of being bounded by $\mu$ is equivalent to

$$\bigwedge\nolimits^{j}_{\mathbb{B}_{R'}^+}\mathfrak{q}_{R',\psi}\;\subset\;E(u)^{-\mu_{\psi,1}-\ldots-\mu_{\psi,j}}\cdot\bigwedge\nolimits^{j}_{\mathbb{B}_{R'}^+}\mathfrak{p}_{R',\psi}\tag{2.6}$$

for all $\psi$ and all $j=1,\dots,d$ with equality for $j=d$.

Note that by Cramer's rule (e.g. [Bou70, III.8.6, Formulas (21) and (22)]) the condition of Definition 2.10(c), respectively (2.6) is equivalent to

$$\bigwedge_{\mathbb{B}_R^+}^{j} \mathfrak{p} \;\subset\; E(u)^{\mu_{d-j+1}+\ldots+\mu_d} \cdot \bigwedge_{\mathbb{B}_R^+}^{j} \mathfrak{q}\,,$$

respectively

$$\bigwedge_{\mathbb{B}_{R'}^+}^{j} \mathfrak{p}_{R',\psi} \;\subset\; E(u)^{\mu_{\psi,d-j+1}+\ldots+\mu_{\psi,d}} \cdot \bigwedge_{\mathbb{B}_{R'}^+}^{j} \mathfrak{q}_{R',\psi} \tag{2.7}$$

for all $j = 1, \ldots, d$ with equality for $j = d$.

**Remark 2.11.** The Hodge polygon of a $K$-filtered $(\varphi, N)$-module $(D, \Phi, N, \mathcal{F}^\bullet)$ is locally constant on $R$, because $\operatorname{gr}^i_{\mathcal{F}} D_{K,\psi}$ is locally free over $R$ as a direct summand of the locally free $R$-module $\operatorname{gr}^i_{\mathcal{F}} D_K$.

In contrast, the Hodge polygon of a $(\varphi, N)$-module $\underline{D}$ with Hodge-Pink lattice over $R$ is not locally constant in general. Nevertheless, for any cocharacter $\mu$ as in (2.5) the set of points $s \in \operatorname{Spec} R$ such that $\mu_{\underline{D}}(s) \preceq \mu$ in the Bruhat order, is closed in $\operatorname{Spec} R$. This is a consequence of the next

**Proposition 2.12.** *Let $\mu \in X_*(T_{\widetilde{K}})_{\mathrm{dom}}$ be a dominant cocharacter with reflex field $E_\mu$ and let $R$ be an $E_\mu$-algebra. Let $\underline{D} = (D, \Phi, N, \mathfrak{q})$ be a $(\varphi, N)$-module with Hodge-Pink lattice of rank $d$ over $R$.*

(a) *The condition that $\underline{D}$ has Hodge polygon bounded by $\mu$ is representable by a finitely presented closed immersion $(\operatorname{Spec} R)_{\preceq\mu} \hookrightarrow \operatorname{Spec} R$.*

(b) *If $R$ is reduced then $\underline{D}$ has Hodge polygon bounded by $\mu$ if and only if for all points $s \in \operatorname{Spec} R \otimes_{E_\mu} \widetilde{K}$ we have $\mu_{\underline{D}}(s) \preceq \mu$ in the Bruhat order, that is, for all $\psi$ the vector $\mu_\psi - \mu_{\underline{D}}(s)_\psi \in \mathbb{Z}^d$ is a non-negative linear combination of the positive coroots $\check{\alpha}_j = (\ldots, 0, 1, -1, 0, \ldots)$ having the "1" as $j$-th entry.*

(c) *Let $\mu'$ be another dominant cocharacter such that $\mu' \preceq \mu$ in the Bruhat order. Let $E_{\mu'}$ denote its reflex field and let $E = E_\mu E_{\mu'} \subset \widetilde{K}$ be the composite field. Assume that $R$ is an $E$-algebra, then $(\operatorname{Spec} R)_{\preceq\mu'} \hookrightarrow (\operatorname{Spec} R)_{\preceq\mu}$ as closed subschemes of $\operatorname{Spec} R$.*

*Proof.* (a) By Lemma 2.6 we find a large positive integer $n$ such that $E(u)^n \mathfrak{p} \subset \mathfrak{q} \subset E(u)^{-n}\mathfrak{p}$. This implies $\wedge^j \mathfrak{q} \subset E(u)^{-jn} \wedge^j \mathfrak{p}$ for all $j$ and $\wedge^d \mathfrak{p} \subset E(u)^{-dn} \wedge^d \mathfrak{q}$. Viewing $\mu_j \colon \operatorname{Spec} R \otimes_{\mathbb{Q}_p} K \to \mathbb{Z}$ as locally constant function as in the discussion before Construction 2.9, we consider the modules over $\mathbb{B}_R^+ \cong (R \otimes_{\mathbb{Q}_p} K)[\![t]\!]$

$$\begin{aligned} M_0 &:= E(u)^{-dn} \wedge^d \mathfrak{q} \,/\, E(u)^{\mu_1+\ldots+\mu_d} \cdot \wedge^d \mathfrak{q} && \text{and} \\ M_j &:= E(u)^{-jn} \wedge^j \mathfrak{p} \,/\, E(u)^{-\mu_1-\ldots-\mu_j} \cdot \wedge^j \mathfrak{p} && \text{for } 1 \le j \le d\,. \end{aligned} \tag{2.8}$$

As $R$-modules they are finite locally free. Then $\underline{D}$ has Hodge polygon bounded by $\mu$ if and only if for all $j = 1, \ldots, d$ all generators of $\wedge^j \mathfrak{q}$ are mapped to zero in $M_j$ and all generators of $\wedge^d \mathfrak{p}$ are mapped to zero in $M_0$. Since $M := M_0 \oplus \ldots \oplus M_d$ is finite locally free over $R$, this condition is represented by a finitely presented closed immersion into $\operatorname{Spec} R$ by [EGA, $\mathrm{I_{new}}$, Lemma 9.7.9.1].

(b) If $R$ is reduced then also the étale $R$-algebra $R' := R \otimes_{E_\mu} \widetilde{K}$ is reduced and $R \hookrightarrow R' \hookrightarrow \prod_{s \in \operatorname{Spec} R'} \kappa(s)$ is injective. Therefore also $M \hookrightarrow M \otimes_R \big(\prod_{s \in \operatorname{Spec} R'} \kappa(s)\big)$ is injective. So $\underline{D}$ has Hodge polygon bounded by $\mu$ if and only if this holds for the pullbacks $s^*\underline{D}$ to $\operatorname{Spec} \kappa(s)$ at all points $s \in \operatorname{Spec} R'$. By definition of $\mu' := \mu_{\underline{D}}(s)$ there is a $\kappa(s)[\![t]\!]$-basis $(v_{\psi,1}, \ldots, v_{\psi,d})$ of the $\psi$-component $(s^*\mathfrak{p})_\psi$ of $s^*\mathfrak{p}$ such that $(t^{-\mu'_{\psi,1}} v_{\psi,1}, \ldots, t^{-\mu'_{\psi,d}} v_{\psi,d})$ is a $\kappa(s)[\![t]\!]$-basis of $(s^*\mathfrak{q})_\psi$. Therefore condition (2.6) holds if and only if $\mu_{\psi,1} + \ldots + \mu_{\psi,j} \ge \mu'_{\psi,1} + \ldots + \mu'_{\psi,j}$ for all $\psi$ and $j$ with equality for $j = d$. One easily checks that this is equivalent to $\mu' \preceq \mu$.

(c) Again $\mu' \preceq \mu$ implies $\mu_{\psi,1} + \ldots + \mu_{\psi,j} \ge \mu'_{\psi,1} + \ldots + \mu'_{\psi,j}$ for all $\psi$ and $j$ with equality for $j = d$. We view $\mu_j, \mu'_j$ as locally constant $\mathbb{Z}$-valued functions on $\operatorname{Spec} E \otimes_{\mathbb{Q}_p} K$. Then $\mu_1 + \ldots + \mu_j \ge \mu'_1 + \ldots + \mu'_j$ for all $j$ with equality for $j = d$. In terms of (2.8) the $R$-modules $M_j$ for $\mu$ are quotients of the $R$-modules $M'_j$ for $\mu'$ with $M'_0 = M_0$. Therefore $(\operatorname{Spec} R)_{\preceq\mu'} \hookrightarrow \operatorname{Spec} R$ factors through $(\operatorname{Spec} R)_{\preceq\mu}$. $\square$

**Remark 2.13.** The reader should note that $\mu' \preceq \mu$ does not imply a relation between $E_{\mu'}$ and $E_\mu$ as can be seen from the following example. Let $d = 2$ and $[K : \mathbb{Q}_p] = 2$ and $\{\psi : K \hookrightarrow \widetilde{K}\} = \mathrm{Gal}(K/\mathbb{Q}_p) = \{\psi_1, \psi_2\}$. Consider the three cocharacters $\mu, \mu', \mu''$ given by $\mu_{\psi_1} = (2,0)$, $\mu_{\psi_2} = (2,0)$ and $\mu'_{\psi_1} = (2,0)$, $\mu'_{\psi_2} = (1,1)$ and $\mu''_{\psi_1} = (1,1)$, $\mu''_{\psi_2} = (1,1)$. Then $\mu'' \preceq \mu' \preceq \mu$. On the other hand we find $E_\mu = E_{\mu''} = \mathbb{Q}_p$ and $E_{\mu'} = \widetilde{K} = K$.

**Remark 2.14.** In Definition 2.10(a) we assumed that $R$ is a $\widetilde{K}$-algebra to obtain a well defined Hodge polygon $\mu_{\underline{D}}(s) \in X_*(\widetilde{T}_{\widetilde{K}})$. In Definition 2.10(b) we can lower the ground field over which $R$ is defined to $E_\mu$ because $\mathrm{Gal}(\widetilde{K}/E_\mu)$ fixes $\mu$. The ground field cannot be lowered further, as one sees from the following

**Proposition 2.15.** *Let $\underline{D}$ be a $(\varphi, N)$-module with Hodge-Pink lattice (respectively a $K$-filtered $(\varphi, N)$-module) of rank $d$ over a field $L$ such that $\mu_{\underline{D}}(s) = \mu$ for all points $s \in \mathrm{Spec}\, L \otimes_{\mathbb{Q}_p} \widetilde{K}$. Then there is a canonical inclusion of the reflex field $E_\mu \hookrightarrow L$.*

*Proof.* Since every $K$-filtered $(\varphi, N)$-module arises from a $(\varphi, N)$-module with Hodge-Pink lattice as in Remark 2.7(3), it suffices to treat the case where $\underline{D}$ is a $(\varphi, N)$-module with Hodge-Pink lattice. We consider the decomposition $\widetilde{L} := L \otimes_{\mathbb{Q}_p} \widetilde{K} = \prod_{s \in \mathrm{Spec}\, \widetilde{L}} \kappa(s)$ and for each $s$ we denote by $\alpha_s : L \hookrightarrow \kappa(s)$ and $\beta_s : \widetilde{K} \hookrightarrow \kappa(s)$ the induced inclusions. Let $\mu_L : \mathbb{G}_{m,L} \to \widetilde{T}_L$ be the cocharacter over $L$ associated with $\underline{D}$ in Construction 2.9. The assumption of the proposition means that $\alpha_s(\mu_L) = \beta_s(\mu)$ for all $s$. The Galois group $\mathcal{G} := \mathrm{Gal}(\widetilde{K}, \mathbb{Q}_p)$ acts on $\widetilde{L}$. The Galois group $\mathrm{Gal}(\kappa(s)/\alpha_s(L))$ can be identified with the decomposition group $\mathcal{G}_s := \{\sigma \in \mathcal{G} : \sigma(s) = s\}$ under the monomorphism $\mathrm{Gal}(\kappa(s)/\alpha_s(L)) \hookrightarrow \mathcal{G}$, $\tau \mapsto \beta_s^{-1} \circ \tau|_{\beta_s(\widetilde{K})} \circ \beta_s$. Since $\mu_L$ is defined over $L$, each $\tau \in \mathrm{Gal}(\kappa(s)/\alpha_s(L))$ satisfies $\tau(\alpha_s(\mu_L)) = \alpha_s(\mu_L)$, and hence $(\beta_s^{-1} \circ \tau|_{\beta_s(\widetilde{K})} \circ \beta_s)(\mu) = \mu$. By definition of the reflex field $E_\mu$ this implies that $\beta_s^{-1} \circ \tau|_{\beta_s(\widetilde{K})} \circ \beta_s \in \mathrm{Gal}(\widetilde{K}/E_\mu)$ and $\tau|_{\beta_s(E_\mu)} = \mathrm{id}$. So $\beta_s(E_\mu) \subset \alpha_s(L)$ and we get an inclusion $\alpha_s^{-1}\beta_s : E_\mu \hookrightarrow L$. To see that this is independent of $s$ choose a $\sigma \in \mathcal{G}$ with $\sigma(s) = \tilde{s}$. Then $\alpha_{\tilde{s}} = \sigma \circ \alpha_s$ and $\beta_{\tilde{s}} = \sigma \circ \beta_s$. $\square$

## 3. Moduli spaces for $(\varphi, N)$-modules with Hodge-Pink lattice

We will introduce and study moduli spaces for the objects introduced in Chapter 2. Proposition 2.15 suggests to work over the reflex field.

**Definition 3.1.** Let $\mu$ be a cocharacter as in (2.5) and let $E_\mu$ be its reflex field. We define fpqc-stacks $\mathscr{D}_{\varphi,N,\mu}$, resp. $\mathscr{H}_{\varphi,N,\preceq\mu}$, resp. $\mathscr{H}_{\varphi,N,\mu}$ on the category of $E_\mu$-schemes. For an affine $E_\mu$-scheme $\mathrm{Spec}\, R$

(a) the groupoid $\mathscr{D}_{\varphi,N,\mu}(\mathrm{Spec}\, R)$ consists of $K$-filtered $(\varphi, N)$-modules $(D, \Phi, N, \mathcal{F}^\bullet)$ over $R$ of rank $d$ with constant Hodge polygon equal to $\mu$.

(b) the groupoid $\mathscr{H}_{\varphi,N,\preceq\mu}(\mathrm{Spec}\, R)$ consists of $(\varphi, N)$-modules with Hodge-Pink lattice $(D, \Phi, N, \mathfrak{q})$ over $R$ of rank $d$ with Hodge polygon bounded by $\mu$.

(c) the groupoid $\mathscr{H}_{\varphi,N,\mu}(\mathrm{Spec}\, R)$ consists of $(\varphi, N)$-modules with Hodge-Pink lattice $(D, \Phi, N, \mathfrak{q})$ over $R$ of rank $d$ with Hodge polygon bounded by $\mu$ and constant equal to $\mu$.

Let $\mathscr{D}_{\varphi,\mu} \subset \mathscr{D}_{\varphi,N,\mu}$, resp. $\mathscr{H}_{\varphi,\preceq\mu} \subset \mathscr{H}_{\varphi,N,\preceq\mu}$, resp. $\mathscr{H}_{\varphi,\mu} \subset \mathscr{H}_{\varphi,N,\mu}$ be the closed substacks on which $N$ is zero. They classify $\varphi$-modules with $K$-filtration, resp. Hodge-Pink lattice and the corresponding condition on the Hodge polygon.

We are going to show that these stacks are Artin stacks of finite type over $E_\mu$.

Locally on $\mathrm{Spec}\, R$ we may choose an isomorphism $D \cong (R \otimes_{\mathbb{Q}_p} K_0)^d$ by Lemma 2.3(a). Then $\Phi$ and $N$ correspond to matrices $\Phi \in \mathrm{GL}_d(R \otimes_{\mathbb{Q}_p} K_0) = (\mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}_{d,K_0})(R)$ and $N \in \mathrm{Mat}_{d\times d}(R \otimes_{\mathbb{Q}_p} K_0) = (\mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{Mat}_{d\times d})(R)$. The relation $\Phi\, \varphi^* N = p\, N\, \Phi$ is represented by a closed subscheme

$$P_{K_0,d} \subset (\mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}_{d,K_0}) \times_{\mathrm{Spec}\, \mathbb{Q}_p} (\mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{Mat}_{d\times d}).$$

**Theorem 3.2.** (a) *The $\mathbb{Q}_p$-scheme $P_{K_0,d}$ is reduced, Cohen-Macaulay, generically smooth and equidimensional of dimension $fd^2$. In the notation of Remark 2.4 the matrix $(\Phi^f)_0$ has no multiple eigenvalues at the generic points of the irreducible components of $P_{K_0,d}$.*

(b) *The generic points of $P_{K_0,d}$ are in bijection with the partitions $d = k_1 + \ldots + k_m$ for integers $m$ and $1 \leq k_1 \leq \ldots \leq k_m$. To such a partition corresponds the generic point at which the* *suitably ordered eigenvalues* $\lambda_1,\ldots,\lambda_d$ *of* $(\Phi^f)_0$ *satisfy* $p^f\lambda_i=\lambda_j$ *if and only if* $j=i+1$ *and* $i\notin\{k_1,\,k_1+k_2,\,\ldots\,,\,k_1+\ldots+k_m\}$. *Equivalently to such a partition corresponds the generic point at which the nilpotent endomorphism* $N_0$, *in the notation of 2.4, has Jordan canonical form with* $m$ *Jordan blocks of size* $k_1,\ldots,k_m$.

For the proof we will need the following lemma.

**Lemma 3.3.** *Let* $r_1,\ldots,r_n$ *be integers with* $r_1+\ldots+r_n\ge n$. *Then* $\sum_{i=1}^n r_i^2-\sum_{i=1}^{n-1}r_i\,r_{i+1}>1$, *except for the case when* $r_1=\ldots=r_n=1$.

*Proof.* We multiply the claimed inequality with 2 and write it as $r_1^2+\sum_{i=1}^{n-1}(r_i-r_{i+1})^2+r_n^2>2$. There are the following three critical cases

(a) $\sum_i(r_i-r_{i+1})^2=0$,
(b) $\sum_i(r_i-r_{i+1})^2=1$,
(c) $\sum_i(r_i-r_{i+1})^2=2$.

In case (a) we have $r_1=\ldots=r_n$. Since $r_1=\ldots=r_n=1$ was excluded and $r_1\le 0$ contradicts $r_1+\ldots+r_n\ge n$, we have $r_1^2+r_n^2>2$.

In case (b) there is exactly one index $1\le i<n$ with $r_1=\ldots=r_i\ne r_{i+1}=\ldots=r_n$ and $|r_i-r_{i+1}|=1$. If $r_1\ne 0\ne r_n$ then $r_1^2+\sum_{i=1}^{n-1}(r_i-r_{i+1})^2+r_n^2>2$. On the other hand, if $r_1=\pm1$ and $r_n=0$, then $\sum_i r_i=\pm i<n$. And if $r_1=0$ and $r_n=\pm1$, then $\sum_i r_i=\pm(n-i)<n$. Both are contradictions.

In case (c) there are exactly two indices $1\le i<j<n$ with $r_1=\ldots=r_i$ and $r_{i+1}=\ldots=r_j$ and $r_{j+1}=\ldots=r_n$, as well as $|r_i-r_{i+1}|=1=|r_j-r_{j+1}|$. If in addition $r_1=r_n=0$ then $\sum_i r_i=\pm(j-i)<n$, which is a contradiction. Therefore $r_1^2+r_n^2>0$ and $r_1^2+\sum_{i=1}^{n+1}(r_i-r_{i+1})^2+r_n^2>2$ as desired. □

*Proof of Theorem 3.2.* 1. By [EGA, $\mathrm{IV}_2$, Proposition 6.5.3, Corollaires 6.3.5(ii), 6.1.2, and $\mathrm{IV}_4$, Proposition 17.7.1] the statement may be checked after the finite étale base change $\operatorname{Spec}K_0\to\operatorname{Spec}\mathbb{Q}_p$. We will use throughout that after this base change, Remark 2.4 allows to decompose $\Phi=(\Phi_i)_i$ and $N=(N_i)_i$ such that $p\,\Phi_i\circ N_i=N_{i+1}\circ\Phi_i$.

2. We first prove that all irreducible components of $P_{K_0,d}$ have dimension greater or equal to $fd^2$. Sending $(\Phi,N)$ to the entries of the matrices $\Phi_i,N_i$ embeds $P_{K_0,d}\times_{\mathbb{Q}_p}K_0$ into affine space $\mathbb{A}_{K_0}^{2fd^2}$ as a locally closed subscheme cut out by the $fd^2$ equations $p\,\Phi_i\circ N_i=N_{i+1}\circ\Phi_i$ for $i=0,\ldots,f-1$. Therefore the codimension of $P_{K_0,d}\times_{\mathbb{Q}_p}K_0$ in $\mathbb{A}_{K_0}^{2fd^2}$ is less or equal to $fd^2$ by Krull's principal ideal theorem [Eis95, Theorem 10.2], and all irreducible components of $P_{K_0,d}$ have dimension greater or equal to $fd^2$ by [Eis95, Corollary 13.4].

3. We next prove the assertion on the generic points. Let $y=(\Phi,N)$ be the generic point of an irreducible component $Y$ of $P_{K_0,d}$. After passing to an algebraic closure $L$ of $\kappa(y)$ we may use Remark 2.4 to find a base change matrix $S\in\mathrm{GL}_d(L\otimes_{\mathbb{Q}_p}K_0)$ such that $S^{-1}\Phi\,\varphi(S)=\big((\Phi^f)_0,\mathrm{Id}_d,\ldots,\mathrm{Id}_d\big)$ and $(\Phi^f)_0$ is a block diagonal matrix in Jordan canonical form

$$(\Phi^f)_0=\begin{pmatrix}J_1&&\\&\ddots&\\&&J_r\end{pmatrix}\quad\text{with}\quad J_i=\begin{pmatrix}\rho_i&1&\\&\ddots&1\\&&\rho_i\end{pmatrix}\quad\text{and}\quad N_0=\begin{pmatrix}N_{11}&\cdots&N_{1r}\\\vdots&&\vdots\\N_{r1}&\cdots&N_{rr}\end{pmatrix}.$$

Note that a priory some of the $\rho_i$ can be equal. Let $s_i$ be the size of the Jordan block $J_i$. Then $N_{ij}$ is an $s_i\times s_j$-matrix. The condition $p^f(\Phi^f)_0\circ N_0=N_0\circ(\Phi^f)_0$ is equivalent to $p^fJ_i\,N_{ij}=N_{ij}\,J_j$ for all $i,j$. It yields $N_{ij}=(0)$ for $p^f\rho_i\ne\rho_j$. By renumbering the $J_i$ we may assume that $N_{ij}\ne(0)$ implies $i<j$. We set $N_{ij}=\big(n_{\mu,\nu}^{(ij)}\big)_{\mu=1\ldots s_i,\,\nu=1\ldots s_j}$. When $p^f\rho_i=\rho_j$ it follows from

$$\begin{pmatrix}p^fn_{2,1}&\cdots&p^fn_{2,s_j}\\\vdots&&\vdots\\p^fn_{s_i,1}&\cdots&p^fn_{s_i,s_j}\\0&\cdots&0\end{pmatrix}=p^f(J_i-\rho_i)N_{ij}=N_{ij}(J_j-\rho_j)=\begin{pmatrix}0&n_{1,1}&\cdots&n_{1,s_j-1}\\\vdots&\vdots&&\vdots\\0&n_{s_i,1}&\cdots&n_{s_i,s_j-1}\end{pmatrix}$$

that $p^f n^{(ij)}_{\mu,\nu} = n^{(ij)}_{\mu-1,\nu-1}$ for all $\mu,\nu \geq 2$ and $n^{(ij)}_{\mu,\nu} = 0$ whenever $\mu - \nu > \min\{0, s_i - s_j\}$. We set $s := \max\{s_i\}$. The assertion of the theorem says that $s = 1$ and that all $\rho_i$ are pairwise different.

First assume that $s > 1$. We exhibit a morphism $\operatorname{Spec} L[z, z^{-1}] \to P_{K_0,d}$ which sends the point $\{z = 1\}$ to $y$ and the generic point $\operatorname{Spec} L(z)$ to a point at which the maximal size of the Jordan blocks is strictly less than $s$. Since $y$ was a generic point of $P_{K_0,d}$ this is impossible. The morphism $\operatorname{Spec} L[z, z^{-1}] \to P_{K_0,d}$ is given by matrices $\widetilde{S}$, $(\widetilde{\Phi}^f)_0$ and $\widetilde{N}_0$ as follows. We set $\widetilde{S} := S$. For all $i$ with $s_i = s$ we set

$$\widetilde{J}_i := \begin{pmatrix} \rho_i & 1 & & \\ & \ddots & \ddots & \\ & & \rho_i & 1 \\ & & & z\rho_i \end{pmatrix}$$

and for all $i$ with $s_i < s$ we set $\widetilde{J}_i := J_i$. When $p^f\rho_i \neq \rho_j$ we set $\widetilde{N}_{ij} := (0)$. To define $\widetilde{N}_{ij}$ when $p^f\rho_i = \rho_j$, and hence $i < j$, we distinguish the following cases

(a) If $s_i, s_j < s$ we set $\widetilde{N}_{ij} = N_{ij}$.
(b) If $s_i = s > s_j$ we set $\widetilde{N}_{ij} = N_{ij}$.
(c) If $s_i < s = s_j$ we set $\widetilde{N}_{ij} = \left(\tilde{n}^{(ij)}_{\mu,\nu}\right)_{\mu,\nu}$ with $\tilde{n}^{(ij)}_{\mu,s_j} := n^{(ij)}_{\mu,s_j}$ for all $\mu$, with $\tilde{n}^{(ij)}_{\mu,\nu} := 0$ whenever $\mu > \nu + s_i - s_j + 1$, and with $\tilde{n}^{(ij)}_{\mu,\nu} := n^{(ij)}_{\mu,\nu} + (1-z)p^{(s_j-1-\nu)f} \cdot \rho_j \cdot n^{(ij)}_{\mu-\nu+s_j-1,\,s_j}$ for $\nu < s_j$ and $\mu \leq \nu + s_i - s_j + 1$.
(d) If $s_i = s_j = s$ we set $\widetilde{N}_{ij} = \left(\tilde{n}^{(ij)}_{\mu,\nu}\right)_{\mu,\nu}$ with $\tilde{n}^{(ij)}_{\mu,s} := n^{(ij)}_{\mu,s}$ for all $\mu$, with $\tilde{n}^{(ij)}_{\mu,\nu} := 0$ whenever $\mu > \nu$, and with $\tilde{n}^{(ij)}_{\mu,\nu} := n^{(ij)}_{\mu,\nu} + (1-z)p^{(s-1-\nu)f} \cdot \rho_j \cdot n^{(ij)}_{\mu-\nu+s-1,\,s}$ for all $\mu \leq \nu < s$.

We have to check that $p^f\widetilde{J}_i\,\widetilde{N}_{ij} = \widetilde{N}_{ij}\widetilde{J}_j$ for all $i,j$ with $p^f\rho_i = \rho_j$. In case (a) this is obvious and in case (b) it follows from the fact that the bottom row of $N_{ij}$ is zero. For case (c) we compute

$$p^f(\widetilde{J}_i - \rho_i)\widetilde{N}_{ij} = \begin{pmatrix} p^f\tilde{n}_{2,1} & \cdots & p^f\tilde{n}_{2,s_j} \\ \vdots & & \vdots \\ p^f\tilde{n}_{s_i,1} & \cdots & p^f\tilde{n}_{s_i,s_j} \\ 0 & \cdots & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \tilde{n}_{1,1} & \cdots & \tilde{n}_{1,s_j-2} & \tilde{n}_{1,s_j-1} + (z-1)\rho_j\,\tilde{n}_{1,s_j} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \tilde{n}_{s_i,1} & \cdots & \tilde{n}_{s_i,s_j-2} & \tilde{n}_{s_i,s_j-1} + (z-1)\rho_j\,\tilde{n}_{s_i,s_j} \end{pmatrix} = \widetilde{N}_{ij}(\widetilde{J}_j - \rho_j)\,.$$

Finally for case (d) we compute

$$p^f(\widetilde{J}_i - \rho_i)\widetilde{N}_{ij} = \begin{pmatrix} p^f\tilde{n}_{2,1} & \cdots & p^f\tilde{n}_{2,s-1} & p^f\tilde{n}_{2,s} \\ \vdots & & \vdots & \vdots \\ p^f\tilde{n}_{s,1} & \cdots & p^f\tilde{n}_{s,s-1} & p^f\tilde{n}_{s,s} \\ 0 & \cdots & 0 & (z-1)p^f\rho_i\,\tilde{n}_{s,s} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \tilde{n}_{1,1} & \cdots & \tilde{n}_{1,s-2} & \tilde{n}_{1,s-1} + (z-1)\rho_j\,\tilde{n}_{1,s} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \tilde{n}_{s,1} & \cdots & \tilde{n}_{s,s-2} & \tilde{n}_{s,s-1} + (z-1)\rho_j\,\tilde{n}_{s,s} \end{pmatrix} = \widetilde{N}_{ij}(\widetilde{J}_j - \rho_j)\,.$$

Altogether this defines the desired morphism $\operatorname{Spec} L[z, z^{-1}] \to P_{K_0,d}$.

So we have shown that $s = 1$ at the generic point $y$ and that $(\Phi^f)_0$ is a diagonal matrix. We still have to show that all diagonal entries are pairwise different. For this purpose we rewrite $(\Phi^f)_0$ and $N_0$ as

$$(\Phi^f)_0 \;=\; \begin{pmatrix} \lambda_1 \operatorname{Id}_{r_1} & & \\ & \ddots & \\ & & \lambda_n \operatorname{Id}_{r_n} \end{pmatrix} \qquad \text{and} \qquad N_0 \;=\; \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \vdots \\ M_{n1} & \cdots & M_{nn} \end{pmatrix}.$$

We denote the multiplicity of the eigenvalue $\lambda_i$ by $r_i \geq 1$. Then $M_{ij}$ is an $r_i \times r_j$-matrix. By renumbering the $\lambda_i$ we may assume that there are indices $0 = l_0 < l_1 < \ldots < l_m = d$ such that $p^f\lambda_i = \lambda_j$ if and only if $j = i+1$ and $i \notin \{l_1, \ldots, l_m\}$.

We compute $\dim Y = \operatorname{trdeg}_{\mathbb{Q}_p} \kappa(y) = \operatorname{trdeg}_{\mathbb{Q}_p} L$ as follows. The eigenvalues $\lambda_{l_1}, \ldots, \lambda_{l_m}$ contribute at most the summand $m$ to $\operatorname{trdeg}_{\mathbb{Q}_p} L$.

The matrix $S \in \operatorname{GL}_d(L \otimes_{\mathbb{Q}_p} K_0)$ is determined only up to multiplication on the right with an element of the $\varphi$-centralizer $\mathcal{C}(L) := \{\, S \in \operatorname{GL}_d(L \otimes_{\mathbb{Q}_p} K_0) \colon S\big((\Phi^f)_0, \operatorname{Id}_d, \ldots, \operatorname{Id}_d\big) = \big((\Phi^f)_0, \operatorname{Id}_d, \ldots, \operatorname{Id}_d\big)\varphi(S)\,\}$ of $\big((\Phi^f)_0, \operatorname{Id}_d, \ldots, \operatorname{Id}_d\big)$. Writing $S = (S_0, \ldots, S_{f-1})$ this condition implies that $S_i \overset{!}{=} (\varphi(S))_i := S_{i-1}$ for $i = 1, \ldots, f-1$ and $S_0(\Phi^f)_0 \overset{!}{=} (\Phi^f)_0(\varphi(S))_0 := (\Phi^f)_0 S_{f-1} = (\Phi^f)_0 S_0$. Therefore $\mathcal{C}$ has dimension $\sum_i r_i^2$ and the entries of $S \in (\operatorname{Res}_{K_0/\mathbb{Q}_p} \operatorname{GL}_{d,K_0})/\mathcal{C}$ contribute at most the summand $fd^2 - \sum_i r_i^2$ to $\operatorname{trdeg}_{\mathbb{Q}_p} L$.

The condition $p^f(\Phi^f)_0 \circ N_0 = N_0 \circ (\Phi^f)_0$ is equivalent to $p^f\lambda_i \, M_{ij} = \lambda_j M_{ij}$ for all $i, j$. This implies that there is no condition on $M_{ij}$ when $j = i+1$ and $i \notin \{l_1, \ldots, l_m\}$, and that all other $M_{ij}$ are zero. So the entries of the $M_{ij}$ contribute at most the summand $\sum_{i \notin \{l_1, \ldots, l_m\}} r_i \, r_{i+1}$ to $\operatorname{trdeg}_{\mathbb{Q}_p} L$.

Adding all summands and comparing with our estimate in part 2 above, we obtain

$$\begin{aligned} fd^2 \;\leq\; \dim Y \;=\; \operatorname{trdeg}_{\mathbb{Q}_p} \kappa(y) \;&\leq\; m + fd^2 - \sum_{i=1}^{n} r_i^2 + \sum_{i \notin \{l_1, \ldots, l_m\}} r_i \, r_{i+1} \\ &=\; fd^2 + \sum_{\nu=0}^{m-1} \Big( 1 - \sum_{i=1+l_\nu}^{l_{\nu+1}} r_i^2 + \sum_{i=1+l_\nu}^{l_{\nu+1}-1} r_i \, r_{i+1} \Big). \end{aligned}$$

By Lemma 3.3 the parentheses are zero when all $r_i = 1$, and negative otherwise. So we have proved that $r_1 = \ldots = r_n = 1$. In other words, all diagonal entries of $(\Phi^f)_0$ are pairwise different. Let $k_\nu := l_\nu - l_{\nu-1}$ for $\nu = 1, \ldots, m$. Then the generic point $y$ corresponds to the partition $d = k_1 + \ldots + k_m$ under the description of the generic points in the theorem. As we have noticed above the $1 \times 1$ matrices $M_{ij}$ vanish at $y$ unless $j = i+1$ and $i \notin \{l_1, \ldots, l_m\}$ and in the latter case we must have $M_{ij}(y) \neq 0$. This implies the claim on the Jordan type of $N_0$ at the generic points of the irreducible components.

Moreover, it follows that $\dim Y = fd^2$ for all irreducible components $Y$ of $P_{K_0,d}$. By [Eis95, Proposition 18.13] this also implies that $P_{K_0,d} \times_{\mathbb{Q}_p} K_0$ is Cohen-Macaulay.

4. It remains to show that $P_{K_0,d}$ is generically smooth over $\mathbb{Q}_p$. From this it follows that it is reduced, because it is Cohen-Macaulay. Let again $y$ be the generic point of an irreducible component of $P_{K_0,d} \times_{\mathbb{Q}_p} K_0$ and let $L$ be an algebraic closure of $\kappa(y)$. As above, Remark 2.4 allows us to change the basis over $L$ and assume that $\Phi = \big((\Phi_0^f), \mathrm{id}, \ldots, \mathrm{id}\big)$ and $N = (p^i N_0)_i$ with $(\Phi^f)_0 = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$ and $\lambda_i \neq \lambda_j$ for all $i \neq j$. We write $F^{(0)} := (\Phi^f)_0$ and $N_0^{(0)} := N_0 = (n_{ij})_{ij}$. The condition $N_0^{(0)} F^{(0)} = p^f F^{(0)} N_0^{(0)}$ implies that $n_{ij} = 0$ if $p^f \lambda_i \neq \lambda_j$. And conversely $n_{ij} \neq 0$ if $p^f \lambda_i = \lambda_j$ by our explicit description of $N_0$ at $y$ above.

We claim that for every $n \geq 1$, any deformation $(F^{(n-1)}, N_0^{(n-1)}) \in P_{K_0,d}(L[\varepsilon]/\varepsilon^n)$ of $(F^{(0)}, N_0^{(0)})$ can be lifted further to $(F^{(n)}, N_0^{(n)}) \in P_{K_0,d}(L[\varepsilon]/\varepsilon^{n+1})$. This implies that $P_{K_0,d}$ is smooth at $y$, as it follows that any tangent vector $\mathcal{O}_{P_{K_0,d},y} \to L[\varepsilon]/\varepsilon^2$ comes from a map $\mathcal{O}_{P_{K_0,d},y} \to L[\![\varepsilon]\!]$ and hence the image of

$$\operatorname{Spec}\big(\textstyle\bigoplus_{i\geq 0} (\mathfrak{m}^i_{P_{K_0,d},y}/\mathfrak{m}^{i+1}_{P_{K_0,d},y})\big) \to \operatorname{Spec}\big(\operatorname{Sym}^\bullet(\mathfrak{m}_{P_{K_0,d},y}/\mathfrak{m}^2_{P_{K_0,d},y})\big)$$

contains any tangent vector. This means that the tangent cone at $y$ equals the tangent space, and hence by [Mum99, III, §4 Definition 2 and Corollary 1] $P_{K_0,d}$ is smooth at $y$. Let us take any deformation $(\widetilde{F}, \widetilde{N}_0) \in \operatorname{GL}_d(L[\varepsilon]/\varepsilon^{n+1}) \times \operatorname{Mat}_{d\times d}(L[\varepsilon]/\varepsilon^{n+1})$ of $(F^{(n-1)}, N_0^{(n-1)})$. Then we have $\widetilde{N}_0 \widetilde{F} - p^f \widetilde{F} \widetilde{N}_0 \in$

$\varepsilon^n \mathrm{Mat}_{d\times d}(L)$. Changing $(\widetilde{F},\widetilde{N}_0)$ to $(F^{(n)},N_0^{(n)})=(\widetilde{F}+\varepsilon^n F',\widetilde{N}_0+\varepsilon^n N_0')$ with $F',N_0'\in\mathrm{Mat}_{d\times d}(L)$ we find

$$N_0^{(n)}F^{(n)}-p^fF^{(n)}N_0^{(n)}=(\widetilde{N}_0\widetilde{F}-p^f\widetilde{F}\widetilde{N}_0)+\varepsilon^n(N_0^{(0)}F'+N_0'F^{(0)}-p^f(F'N_0^{(0)}+F^{(0)}N_0'))$$

and hence it suffices to show that the map $h:\mathrm{Mat}_{d\times d}(L)\times\mathrm{Mat}_{d\times d}(L)\to\mathrm{Mat}_{d\times d}(L)$ given by

$$h:(F',N_0')\longmapsto(N_0'F^{(0)}-p^fF^{(0)}N_0')+(N_0^{(0)}F'-p^fF'N_0^{(0)})$$

is surjective. For this purpose let $N_0'=(b_{ij})_{ij}$ and $F'=\mathrm{diag}(a_1,\dots,a_d)$. Then we find that

$$h_{ij}(F',N_0')\;:=\;h(F',N_0')_{ij}\;=\;(\lambda_j-p^f\lambda_i)b_{ij}+a_j\,n_{ij}-p^fa_i\,n_{ij}\,.$$

Whenever $\lambda_j-p^f\lambda_i\in L^\times$ and hence $n_{ij}=0$ we obtain the surjectivity of $h_{ij}$. By permuting the indices we may assume that $n_{ij}\neq0$ implies $j=i+1$. Treating every Jordan block of $N_0^{(0)}$ separately we may further assume that $n_{i,i+1}\neq0$ for all $i$. It then follows that we have

$$h_{i,i+1}(F',N_0')=(a_{i+1}-p^fa_i)n_{i,i+1}$$

which suffices to see that the map $h$ is surjective. □

**Remark 3.4.** The scheme $P_{K_0,d}$ is in general not normal. For example if $K=\mathbb{Q}_p$ and $d=2$ then $P_{\mathbb{Q}_p,2}$ has two generic points. In one of them $\Phi$ has eigenvalues $\lambda,p\lambda$ and $N\neq0$. In the other $\Phi$ has eigenvalues $\lambda_1,\lambda_2$ with $\lambda_j\neq p\lambda_i$ for all $i,j$ and $N=0$. Both irreducible components meet in the codimension one point where $\lambda_2=p\lambda_1$ and $N=0$.

Let $\Delta$ denote the set of simple roots (defined over $\overline{\mathbb{Q}}_p$) of $\widetilde{G}:=\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_{d,K}$ with respect to the Borel subgroup $\widetilde{B}$ and denote by $\Delta_\mu\subset\Delta$ the set of all simple roots $\alpha$ such that $\langle\alpha,\mu\rangle=0$. Here $\langle-,-\rangle$ is the canonical pairing between characters and cocharacters. We write $P_\mu$ for the parabolic subgroup of $\widetilde{G}$ containing $\widetilde{B}$ and corresponding to $\Delta_\mu\subset\Delta$. This parabolic subgroup is defined over $E_\mu$, and the quotient by this parabolic is a projective $E_\mu$-variety

$$\mathrm{Flag}_{K,d,\mu}=\widetilde{G}_{E_\mu}/P_\mu \tag{3.1}$$

representing the functor

$$R\mapsto\{\text{filtrations }\mathcal{F}^\bullet\text{ of }R\otimes_{\mathbb{Q}_p}K^{\oplus d}\text{ with constant Hodge polygon equal to }\mu\}$$

Thus $\mathscr{D}_{\varphi,N,\mu}$ and $\mathscr{D}_{\varphi,\mu}$ are isomorphic to the stack quotients

$$\begin{aligned}\mathscr{D}_{\varphi,N,\mu}&\;\cong\;(P_{K_0,d}\times_{\mathrm{Spec}\,\mathbb{Q}_p}\mathrm{Flag}_{K,d,\mu})\;/\;(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}\qquad\text{and}\\ \mathscr{D}_{\varphi,\mu}&\;\cong\;(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0}\times_{\mathrm{Spec}\,\mathbb{Q}_p}\mathrm{Flag}_{K,d,\mu})\;/\;(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}\end{aligned}$$

where $g\in(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}$ acts on $(\Phi,N,\mathcal{F}^\bullet)\in P_{K_0,d}\times_{\mathrm{Spec}\,\mathbb{Q}_p}\mathrm{Flag}_{K,d,\mu}$ by

$$(\Phi,N,\mathcal{F}^\bullet)\;\longmapsto\;\big(g^{-1}\Phi\,\varphi(g)\,,\,g^{-1}Ng\,,\,g^{-1}\mathcal{F}^\bullet\big)\,.$$

We next describe the moduli space for the Hodge-Pink lattice $\mathfrak{q}$. Fix the integers $m=\max\{\,\mu_{\psi,1}\,|\,\psi:K\to\widetilde{K}\,\}$ and $n=\max\{\,-\mu_{\psi,d}\,|\,\psi:K\to\widetilde{K}\,\}$. Then by Cramer's rule $E(u)^n\mathfrak{p}\subset\mathfrak{q}\subset E(u)^{-m}\mathfrak{p}$. So $\mathfrak{q}$ is determined by the epimorphism

$$pr\colon R\otimes_{\mathbb{Q}_p}(K[t]/t^{m+n})^{\oplus d}\;\xrightarrow{\;\sim\;}\;E(u)^{-m}\mathfrak{p}/E(u)^n\mathfrak{p}\;\twoheadrightarrow\;E(u)^{-m}\mathfrak{p}/\mathfrak{q} \tag{3.2}$$

which is induced by choosing an isomorphism $D\cong(R\otimes_{\mathbb{Q}_p}K_0)^d$ locally on $R$. The quotient $E(u)^{-m}\mathfrak{p}/\mathfrak{q}$ is a finite locally free $R$-module and of finite presentation over $R\otimes_{\mathbb{Q}_p}K[t]/t^{m+n}$ by Lemma 2.6. Therefore it is an $R$-valued point of Grothendieck's Quot-scheme $\mathrm{Quot}_{\mathcal{O}^d\,|\,K[t]/t^{m+n}\,|\,\mathbb{Q}_p}$; see [FGA, n°221, Theorem 3.1] or [AK80, Theorem 2.6]. This Quot-scheme is projective over $\mathbb{Q}_p$. The boundedness by $\mu$ is represented by a closed subscheme $Q_{K,d,\preceq\mu}$ of $\mathrm{Quot}_{\mathcal{O}^d\,|\,K[t]/t^{m+n}\,|\,\mathbb{Q}_p}\times_{\mathrm{Spec}\,\mathbb{Q}_p}\mathrm{Spec}\,E_\mu$ according to Proposition 2.12(a). Thus $\mathscr{H}_{\varphi,N,\preceq\mu}$ and $\mathscr{H}_{\varphi,\preceq\mu}$ are isomorphic to the stack quotients

$$\begin{aligned}\mathscr{H}_{\varphi,N,\preceq\mu}&\;\cong\;(P_{K_0,d}\times_{\mathrm{Spec}\,\mathbb{Q}_p}Q_{K,d,\preceq\mu})\;/\;(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}\qquad\text{and}\\ \mathscr{H}_{\varphi,\preceq\mu}&\;\cong\;(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0}\times_{\mathrm{Spec}\,\mathbb{Q}_p}Q_{K,d,\preceq\mu})\;/\;(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}\end{aligned}$$

where $g \in (\mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}_{d,K_0})_{E_\mu}$ acts on $(\Phi, N, pr) \in P_{K_0,d} \times_{\mathrm{Spec}\,\mathbb{Q}_p} Q_{K,d,\preceq\mu}$ with $pr$ from (3.2) by

$$(\Phi, N, pr) \longmapsto \big(g^{-1}\Phi\,\varphi(g)\,,\, g^{-1}Ng\,,\, pr \circ (g \otimes_{\mathbb{Q}_p} \mathrm{id}_{\mathbb{Q}_p[t]/t^{m+n}})\big)\,.$$

Let $Q_{K,d,\mu}$ be the complement in $Q_{K,d,\preceq\mu}$ of the image of $\bigcup_{\mu'\prec\mu} Q_{K,d,\preceq\mu'} \times_{\mathrm{Spec}\,E_{\mu'}} \mathrm{Spec}\,\widetilde{K}$ under the finite étale projection $Q_{K,d,\preceq\mu} \times_{\mathrm{Spec}\,E_\mu} \mathrm{Spec}\,\widetilde{K} \to Q_{K,d,\preceq\mu}$. Here the union is taken over all dominant cocharacters $\mu' : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to \widetilde{T}_{\overline{\mathbb{Q}}_p}$ which are strictly less than $\mu$ in the Bruhat order; see Proposition 2.12(b). Since there are only finitely many such $\mu'$ the scheme $Q_{K,d,\mu}$ is an open subscheme of $Q_{K,d,\preceq\mu}$ and quasi-projective over $E_\mu$. By Proposition 2.12(a) the stacks $\mathscr{H}_{\varphi,N,\mu} \subset \mathscr{H}_{\varphi,N,\preceq\mu}$ and $\mathscr{H}_{\varphi,\mu} \subset \mathscr{H}_{\varphi,\preceq\mu}$ are therefore open substacks and isomorphic to the stack quotients

$$\mathscr{H}_{\varphi,N,\mu} \;\cong\; (P_{K_0,d} \times_{\mathrm{Spec}\,\mathbb{Q}_p} Q_{K,d,\mu}) \,/\, (\mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}_{d,K_0})_{E_\mu} \qquad \text{and}$$

$$\mathscr{H}_{\varphi,\mu} \;\cong\; (\mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}_{d,K_0} \times_{\mathrm{Spec}\,\mathbb{Q}_p} Q_{K,d,\mu}) \,/\, (\mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}_{d,K_0})_{E_\mu}\,.$$

There is another description of $Q_{K,d,\preceq\mu}$ in terms of the affine Grassmannian. Consider the infinite dimensional affine group schemes $L^+\mathrm{GL}_d$ and $L^+\widetilde{G}$ over $\mathbb{Q}_p$, and the sheaves $L\mathrm{GL}_d$ and $L\widetilde{G}$ for the fpqc-topology on $\mathbb{Q}_p$ whose sections over a $\mathbb{Q}_p$-algebra $R$ are given by

$$\begin{aligned}
L^+\mathrm{GL}_d(R) &= \mathrm{GL}_d(R[\![t]\!])\,,\\
L^+\widetilde{G}(R) &= \widetilde{G}(R[\![t]\!]) \;=\; \mathrm{GL}_d(R \otimes_{\mathbb{Q}_p} K[\![t]\!]) \;=\; \mathrm{GL}_d(\mathbb{B}_R^+)\,,\\
L\mathrm{GL}_d(R) &= \mathrm{GL}_d(R[\![t]\!][\tfrac{1}{t}])\,,\\
L\widetilde{G}(R) &= \widetilde{G}(R[\![t]\!][\tfrac{1}{t}]) \;=\; \mathrm{GL}_d(R \otimes_{\mathbb{Q}_p} K[\![t]\!][\tfrac{1}{t}]) \;=\; \mathrm{GL}_d(\mathbb{B}_R)\,.
\end{aligned}$$

$L^+\mathrm{GL}_d$ and $L^+\widetilde{G}$ are called the *group of positive loops*, and $L\mathrm{GL}_d$ and $L\widetilde{G}$ are called the *loop group* of $\mathrm{GL}_d$, resp. $\widetilde{G}$. The *affine Grassmannian* of $\mathrm{GL}_d$, resp. $\widetilde{G}$ is the quotient sheaf for the fppf-topology on $\mathbb{Q}_p$

$$\mathrm{Gr}_{\mathrm{GL}_d} \;:=\; L\mathrm{GL}_d\,/L^+\mathrm{GL}_d\,, \qquad \text{resp.} \qquad \mathrm{Gr}_{\widetilde{G}} \;:=\; L\widetilde{G}/L^+\widetilde{G}\,.$$

They are ind-schemes over $\mathbb{Q}_p$ which are ind-projective; see [BD, §4.5], [BL94], [LS97], [HV11].

We set $\mathrm{Gr}_{\mathrm{GL}_d,\widetilde{K}} := \mathrm{Gr}_{\mathrm{GL}_d} \times_{\mathrm{Spec}\,\mathbb{Q}_p} \mathrm{Spec}\,\widetilde{K}$. Then there are morphisms

$$Q_{K,d,\preceq\mu} \longrightarrow \mathrm{Gr}_{\widetilde{G}} \times_{\mathrm{Spec}\,\mathbb{Q}_p} \mathrm{Spec}\,E_\mu \;=:\; \mathrm{Gr}_{\widetilde{G},E_\mu} \qquad \text{and} \tag{3.3}$$

$$Q_{K,d,\preceq\mu} \times_{\mathrm{Spec}\,E_\mu} \mathrm{Spec}\,\widetilde{K} \longrightarrow \prod_{\psi:K\to\widetilde{K}} \mathrm{Gr}_{\mathrm{GL}_d,\widetilde{K}}\,,$$

which are defined as follows. Let $\mathfrak{q} \subset (\mathbb{B}_{Q_{K,d,\preceq\mu}})^{\oplus d}$ be the universal Hodge-Pink lattice over $Q_{K,d,\preceq\mu}$. Then by Lemma 2.6 there is an étale covering $f : \mathrm{Spec}\,R \to Q_{K,d,\preceq\mu}$ such that $f^*\mathfrak{q}$ is free over $\mathbb{B}_R^+$. With respect to a basis of $f^*\mathfrak{q}$ the equality $\mathbb{B}_R \cdot f^*\mathfrak{q} = D \otimes_{R\otimes K_0} \mathbb{B}_R$ corresponds to a matrix $A \in \mathrm{GL}_d(\mathbb{B}_R) = L\widetilde{G}(R)$. The image of $A$ in $\mathrm{Gr}_{\widetilde{G}}(R)$ is independent of the basis and by étale descend defines the first factor of the map $Q_{K,d,\preceq\mu} \to \mathrm{Gr}_{\widetilde{G}} \times_{\mathrm{Spec}\,\mathbb{Q}_p} \mathrm{Spec}\,E_\mu$. The base change of this map along the finite étale morphism $\mathrm{Spec}\,\widetilde{K} \to \mathrm{Spec}\,E_\mu$ defines the second map in (3.3), using the splitting $\widetilde{G} \times_{\mathbb{Q}_p} \widetilde{K} = \prod_\psi \mathrm{GL}_{d,\widetilde{K}}$ which induces similar splittings for $L^+\widetilde{G}$, $L\widetilde{G}$, and $\mathrm{Gr}_{\widetilde{G}}$.

The boundedness by $\mu$ is represented by closed ind-subschemes

$$\mathrm{Gr}^{\preceq\mu}_{\widetilde{G},E_\mu} \qquad \text{and} \qquad \mathrm{Gr}^{\preceq\mu}_{\widetilde{G},E_\mu} \times_{\mathrm{Spec}\,E_\mu} \mathrm{Spec}\,\widetilde{K} \;=\; \prod_\psi \mathrm{Gr}^{\preceq\mu_\psi}_{\mathrm{GL}_d,\widetilde{K}}$$

of $\mathrm{Gr}_{\widetilde{G},E_\mu}$, resp. $\prod_\psi \mathrm{Gr}_{\mathrm{GL}_d,\widetilde{K}}$ through which the maps (3.3) factor. Conversely the universal matrix $A$ over $L\widetilde{G}$ defines a $\mathbb{B}^+_{L\widetilde{G}}$-lattice $\mathfrak{q} = A \cdot (\mathbb{B}^+_{L\widetilde{G}})^d$. Its restriction to $\mathrm{Gr}^{\preceq\mu}_{\widetilde{G},E_\mu}$ has Hodge polygon bounded by $\mu$ and corresponds to the inverses of the maps (3.3). This yields canonical isomorphisms $Q_{K,d,\preceq\mu} \;\cong\; \mathrm{Gr}^{\preceq\mu}_{\widetilde{G},E_\mu}$

and $Q_{K,d,\preceq\mu} \times_{\operatorname{Spec} E_\mu} \operatorname{Spec} \widetilde{K} \;\cong\; \prod_\psi \mathrm{Gr}^{\preceq\mu_\psi}_{\mathrm{GL}_d,\widetilde{K}}$. These isomorphisms restrict to isomorphisms of open subschemes $Q_{K,d,\mu} \;\cong\; \mathrm{Gr}^{\mu}_{\widetilde{G},E_\mu}$ and $Q_{K,d,\mu} \times_{\operatorname{Spec} E_\mu} \operatorname{Spec} \widetilde{K} \;\cong\; \prod_\psi \mathrm{Gr}^{\mu_\psi}_{\mathrm{GL}_d,\widetilde{K}}$.

In view of [HV11, §4], especially Lemma 4.3, the boundedness by $\mu$ on $\prod_\psi \mathrm{Gr}^{\preceq\mu_\psi}_{\mathrm{GL}_d,\widetilde{K}}$ can be phrased in terms of Weyl module representations of $\mathrm{GL}_{d,\widetilde{K}}$. In this formulation it was proved by Varshavsky [Var04, Proposition A.9] that $\mathrm{Gr}^{\mu_\psi}_{\mathrm{GL}_d,\widetilde{K}}$ is reduced. Therefore this locally closed subscheme is determined by its underlying set of points. Reasoning with the elementary divisor theorem as in Construction 2.9 shows that $\mathrm{Gr}^{\mu_\psi}_{\mathrm{GL}_d,\widetilde{K}}$ is equal to the locally closed Schubert cell $L^+\mathrm{GL}_{d,\widetilde{K}} \cdot \mu_\psi(t)^{-1} \cdot L^+\mathrm{GL}_{d,\widetilde{K}} \,/\, L^+\mathrm{GL}_{d,\widetilde{K}}$ and is a homogeneous space under $L^+\mathrm{GL}_{d,\widetilde{K}}$. This description descends to $Q_{K,d,\mu}$ and shows that the latter is reduced and isomorphic to the locally closed Schubert cell $L^+\widetilde{G}_{E_\mu} \cdot \mu(t)^{-1} \cdot L^+\widetilde{G}_{E_\mu} \,/\, L^+\widetilde{G}_{E_\mu}$ which is a homogeneous space under $L^+\widetilde{G}_{E_\mu} := L^+\widetilde{G} \times_{\operatorname{Spec}\mathbb{Q}_p} \operatorname{Spec} E_\mu$.

These homogeneous spaces can be described more explicitly. Set

$$S_{\mathrm{GL}_d,\mu_\psi} \;:=\; L^+\mathrm{GL}_{d,\widetilde{K}} \;\cap\; \mu_\psi(t)^{-1} \cdot L^+\mathrm{GL}_{d,\widetilde{K}} \cdot \mu_\psi(t) \;\subset\; L^+\mathrm{GL}_{d,\widetilde{K}} \qquad \text{and}$$

$$S_{\widetilde{G},\mu} \;:=\; L^+\widetilde{G}_{E_\mu} \;\cap\; \mu(t)^{-1} \cdot L^+\widetilde{G}_{E_\mu} \cdot \mu(t) \;\subset\; L^+\widetilde{G}_{E_\mu}\,.$$

These are closed subgroup schemes and the homogeneous spaces are isomorphic to the quotients

$$L^+\mathrm{GL}_{d,\widetilde{K}} / S_{\mathrm{GL}_d,\mu_\psi} \;\xrightarrow{\;\sim\;}\; L^+\mathrm{GL}_{d,\widetilde{K}} \cdot \mu_\psi(t)^{-1} \cdot L^+\mathrm{GL}_{d,\widetilde{K}} \,/\, L^+\mathrm{GL}_{d,\widetilde{K}} \qquad \text{and}$$

$$L^+\widetilde{G}_{E_\mu} / S_{\widetilde{G},\mu} \;\xrightarrow{\;\sim\;}\; L^+\widetilde{G}_{E_\mu} \cdot \mu(t)^{-1} \cdot L^+\widetilde{G}_{E_\mu} \,/\, L^+\widetilde{G}_{E_\mu} \;\cong\; Q_{K,d,\mu}\,.$$

Consider the closed normal subgroup $L^{++}\widetilde{G}_{E_\mu}(R) \;:=\; \{\, A \in L^+\widetilde{G}_{E_\mu}(R) : A \equiv 1 \bmod t \,\}$. Then the parabolic subgroup $P_\mu$ from (3.1) equals

$$P_\mu \;=\; S_{\widetilde{G},\mu} \cdot L^{++}\widetilde{G}_{E_\mu} \,/\, L^{++}\widetilde{G}_{E_\mu} \;\subset\; L^+\widetilde{G}_{E_\mu} \,/\, L^{++}\widetilde{G}_{E_\mu} \;=\; \widetilde{G}_{E_\mu}$$

and this yields a morphism

$$Q_{K,d,\mu} \;=\; L^+\widetilde{G}_{E_\mu}/S_{\widetilde{G},\mu} \;\twoheadrightarrow\; L^+\widetilde{G}_{E_\mu}/S_{\widetilde{G},\mu} \cdot L^{++}\widetilde{G}_{E_\mu} \;=\; \widetilde{G}_{E_\mu}/P_\mu \;=\; \mathrm{Flag}_{K,d,\mu}\,, \tag{3.4}$$

with fibers isomorphic to $S_{\widetilde{G},\mu} \cdot L^{++}\widetilde{G}_{E_\mu}/S_{\widetilde{G},\mu}$. The latter is an affine space because we may consider the base change from $E_\mu$ to $\widetilde{K}$ and the decomposition

$$\big(S_{\widetilde{G},\mu} \cdot L^{++}\widetilde{G}_{E_\mu}/S_{\widetilde{G},\mu}\big) \times_{\operatorname{Spec} E_\mu} \operatorname{Spec}\widetilde{K} \;=\; \prod_\psi \big(S_{\mathrm{GL}_d,\mu_\psi} \cdot L^{++}\,\mathrm{GL}_{d,\widetilde{K}} / S_{\mathrm{GL}_d,\mu_\psi}\big).$$

Each component is an affine space whose $R$-valued points are in bijection with the matrices

$$\begin{pmatrix} 1 & & & \\ a_{21} & 1 & & \\ \vdots & \ddots & \ddots & \\ a_{d1} & \dots & a_{d,d-1} & 1 \end{pmatrix}$$

where $a_{ij} \in \bigoplus_{k=1}^{\mu_{\psi,j}-\mu_{\psi,i}-1} t^k R$. The Galois group $\mathrm{Gal}(\widetilde{K}/E_\mu)$ canonically identifies the components with the same values for $\mu_\psi$. Therefore $S_{\widetilde{G},\mu} \cdot L^{++}\widetilde{G}_{E_\mu}/S_{\widetilde{G},\mu}$ is an affine space.

We show that $Q_{K,d,\mu}$ is a geometric vector bundle over $\mathrm{Flag}_{K,d,\mu}$ by exhibiting its zero section. The projection $L^+\widetilde{G}_{E_\mu} \twoheadrightarrow \widetilde{G}_{E_\mu}$ has a section given on $R$-valued points by the map $\widetilde{G}_{E_\mu}(R) \to L^+\widetilde{G}_{E_\mu}(R) = \widetilde{G}_{E_\mu}(R[\![t]\!])$ induced from the natural inclusion $R \hookrightarrow R[\![t]\!]$. Since $L^+P_\mu \subset S_{\widetilde{G},\mu}$ by definition of $P_\mu$, this section induces a section

$$\mathrm{Flag}_{K,d,\mu} \;\longrightarrow\; L^+\widetilde{G}_{E_\mu}/L^+P_\mu \;\longrightarrow\; L^+\widetilde{G}_{E_\mu}/S_{\widetilde{G},\mu} \;=\; Q_{K,d,\mu}\,.$$

This is the zero section of the geometric vector bundle $Q_{K,d,\mu}$ over $\mathrm{Flag}_{K,d,\mu}$. Using lattices the section coincides with the map $\mathcal{F}^\bullet \mapsto \mathfrak{q}(\mathcal{F}^\bullet)$ defined in Remark 2.7(3) and the projection $Q_{K,d,\mu} \to \mathrm{Flag}_{K,d,\mu}$ coincides with the map $\mathfrak{q} \mapsto \mathcal{F}^\bullet_{\mathfrak{q}}$ from Remark 2.7(1). Let us summarize.

**Proposition 3.5.** (a) *$Q_{K,d,\preceq\mu}$ is projective over $E_\mu$ of dimension $\sum_{\psi,j}(d+1-2j)\mu_{\psi,j}$ and contains $Q_{K,d,\mu}$ as dense open subscheme. Both schemes are irreducible.*

(b) *$Q_{K,d,\mu}$ is smooth over $E_\mu$ and isomorphic to the homogeneous space $L^+\widetilde{G}_{E_\mu}/S_{\widetilde{G},\mu}$ which is a geometric vector bundle over $\mathrm{Flag}_{K,d,\mu}$.*

*Proof.* Everything was proved above, except the formula for the dimension and the density of $Q_{K,d,\mu}$ which follow from [BD, 4.5.8 and 4.5.12]. The irreducibility of $Q_{K,d,\preceq\mu}$ is a consequence of the density statement. □

**Theorem 3.6.** (a) *The moduli stacks $\mathscr{D}_{\varphi,N,\mu}$, $\mathscr{D}_{\varphi,\mu}$, $\mathscr{H}_{\varphi,N,\preceq\mu}$, $\mathscr{H}_{\varphi,\preceq\mu}$, $\mathscr{H}_{\varphi,N,\mu}$ and $\mathscr{H}_{\varphi,\mu}$ are noetherian Artin stacks of finite type over $E_\mu$.*

(b) *The stack $\mathscr{H}_{\varphi,N,\mu}$ is a dense open substack of $\mathscr{H}_{\varphi,N,\preceq\mu}$ and projects onto $\mathscr{D}_{\varphi,N,\mu}$. The morphism $\mathscr{H}_{\varphi,N,\mu}\to\mathscr{D}_{\varphi,N,\mu}$ has a section and is relatively representable by a vector bundle.*

(c) *The stack $\mathscr{H}_{\varphi,\mu}$ is a dense open substack of $\mathscr{H}_{\varphi,\preceq\mu}$ and projects onto $\mathscr{D}_{\varphi,\mu}$. The morphism $\mathscr{H}_{\varphi,\mu}\to\mathscr{D}_{\varphi,\mu}$ has a section and is relatively representable by a vector bundle.*

(d) *The stacks $\mathscr{H}_{\varphi,\preceq\mu}$, $\mathscr{H}_{\varphi,\mu}$ are irreducible of dimension $\sum_{\psi,j}(d+1-2j)\mu_{\psi,j}$, and $\mathscr{D}_{\varphi,\mu}$ is irreducible of dimension $\sum_\psi \#\{(i,j):\mu_{\psi,i}>\mu_{\psi,j}\}$. The stacks $\mathscr{H}_{\varphi,\mu}$ and $\mathscr{D}_{\varphi,\mu}$ are smooth over $E_\mu$.*

(e) *The stacks $\mathscr{H}_{\varphi,N,\preceq\mu}$, $\mathscr{H}_{\varphi,N,\mu}$ are equi-dimensional of dimension $\sum_{\psi,j}(d+1-2j)\mu_{\psi,j}$, and $\mathscr{D}_{\varphi,N,\mu}$ is equi-dimensional of dimension $\sum_\psi \#\{(i,j):\mu_{\psi,i}>\mu_{\psi,j}\}$. The stacks $\mathscr{H}_{\varphi,N,\mu}$ and $\mathscr{D}_{\varphi,N,\mu}$ are reduced, Cohen-Macaulay and generically smooth over $E_\mu$. The irreducible components of $\mathscr{H}_{\varphi,N,\preceq\mu}$, $\mathscr{H}_{\varphi,N,\mu}$ and $\mathscr{D}_{\varphi,N,\mu}$ are indexed by the possible Jordan types of the nilpotent endomorphism $N$.*

*Proof.* (a) The stacks are quotients of noetherian schemes of finite type over $E_\mu$ by the action of the smooth group scheme $(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}$ and hence are noetherian Artin stacks of finite type by [LM00, 4.6.1, 4.7.1, 4.14].

(b) and (c) follow from the corresponding statements for $Q_{K,d,\mu}$ in Proposition 3.5.

(d) The covering spaces

$$\begin{aligned}&\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0}\times_{\mathrm{Spec}\,\mathbb{Q}_p} Q_{K,d,\preceq\mu},\ \text{resp.}\\&\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0}\times_{\mathrm{Spec}\,\mathbb{Q}_p} Q_{K,d,\mu},\ \text{resp.}\\&\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0}\times_{\mathrm{Spec}\,\mathbb{Q}_p} \mathrm{Flag}_{K,d,\mu}\end{aligned}$$

of these stacks are irreducible because $\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0}$ is geometrically irreducible. This implies the irreducibility of the stacks. The formulas for the dimension follow from [LM00, pp. 98f] and Proposition 3.5, respectively the well known dimension formula for partial flag varieties. The smoothness follows from the smoothness of $\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0}\times_{\mathrm{Spec}\,\mathbb{Q}_p} Q_{K,d,\mu}$, resp. $\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0}\times_{\mathrm{Spec}\,\mathbb{Q}_p}\mathrm{Flag}_{K,d,\mu}$ by [LM00, 4.14].

(e) As in (d) these results are direct consequences of the corresponding results on the covering spaces, which follow from Theorem 3.2. We only need to convince ourselves that the action of $(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}$ does not identify irreducible components of $P_{K_0,d}$. However this follows from the fact that the Jordan canonical forms of the nilpotent endomorphism $N$ at two distinct generic points $y_1$ and $y_2$ of $P_{K_0,d}$ are distinct by the description in Theorem 3.2. □

**Remark 3.7.** These stacks are not separated. Namely, let $\underline{D},\underline{D}'$ be two $(\varphi,N)$-modules with Hodge-Pink lattice (respectively two $K$-filtered $(\varphi,N)$-modules) over $R$. Then $\mathrm{Isom}(\underline{D},\underline{D}')$ is representable by an algebraic space, separated and of finite type over $R$; see [LM00, Lemme 4.2]. The above stacks are separated over $E_\mu$ if and only if all these algebraic spaces $\mathrm{Isom}(\underline{D},\underline{D}')$ are proper. This is not the case in general. For example let $R$ be a discrete valuation ring with fraction field $L$, let $D=D'=R\otimes_{\mathbb{Q}_p}K_0^d$ with $\Phi=\mathrm{id}$ and $N=0$. Then every element $f\in L$ is an automorphism of $D\otimes_R L$, compatible with $\Phi$ and $N$. However, it extends to an automorphism of $D$ only if $f\in R^\times$.

## 4. Vector bundles on the open unit disc

In [Kis06] Kisin related $K$-filtered $(\varphi,N)$-modules over $\mathbb{Q}_p$ to vector bundles on the open unit disc. This was generalized in [Hel13, §5] to families of $K$-filtered $\varphi$-modules with Hodge-Tate weights 0 and $-1$. In this section we generalize it to arbitrary families of $(\varphi,N)$-modules with Hodge-Pink lattice. For this purpose we work in the category $\mathrm{Ad}^{\mathrm{lft}}_{\mathbb{Q}_p}$ of adic spaces locally of finite type over $\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$; see [Hub93, Hub94, Hub96] and [Hel13, §2.2]. Since the stacks $\mathscr{D}_{\varphi,\mu}$, $\mathscr{D}_{\varphi,N,\mu}$, $\mathscr{H}_{\varphi,\preceq\mu}$, $\mathscr{H}_{\varphi,N,\preceq\mu}$, $\mathscr{H}_{\varphi,\mu}$ and $\mathscr{H}_{\varphi,N,\mu}$ are quotients of quasi-projective schemes over $E_\mu$ they give rise to stacks on $\mathrm{Ad}^{\mathrm{lft}}_{E_\mu}$ which we denote by $\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\mu}$, etc.

For $0\le r<1$ we write $\mathbb{B}_{[0,r]}$ for the closed disc of radius $r$ over $K_0$ in the category of adic spaces and denote by

$$\mathbb{U}=\varinjlim_{r\to 1}\mathbb{B}_{[0,r]}$$

the open unit disc. This is an open subspace of the closed unit disc (which is *not* identified with the set of all points $x$ in the closed unit disc with $|x|<1$ in the adic setting). In the following we will always write $u$ for the coordinate function on $\mathbb{B}_{[0,r]}$ and $\mathbb{U}$, i.e. we view

$$\mathbf{B}^{[0,r]}:=\Gamma(\mathbb{B}_{[0,r]},\mathcal{O}_{\mathbb{B}_{[0,r]}})$$

and $\mathbf{B}^{[0,1)}:=\Gamma(\mathbb{U},\mathcal{O}_{\mathbb{U}})$ as sub-rings of $K_0[\![u]\!]$.

Let $X\in\mathrm{Ad}^{\mathrm{lft}}_{\mathbb{Q}_p}$ be an adic space over $\mathbb{Q}_p$ and write

$$\mathscr{B}^{[0,r]}_X=\mathcal{O}_X\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}^{[0,r]}=\mathrm{pr}_{X,*}\,\mathcal{O}_{X\times\mathbb{B}_{[0,r]}}$$
$$\mathscr{B}^{[0,1)}_X=\mathcal{O}_X\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}^{[0,1)}=\mathrm{pr}_{X,*}\,\mathcal{O}_{X\times\mathbb{U}}$$

for the sheafified versions of the rings $\mathbf{B}^{[0,r]}$ and $\mathbf{B}^{[0,1)}$ where $\mathrm{pr}_X$ is the projection onto $X$. These are sheaves of topological $\mathcal{O}_X$-algebras on $X$.

We introduce the function

$$\lambda\;:=\;\prod_{n\ge 0}\varphi^n(E(u)/E(0))\;\in\;\mathbf{B}^{[0,1)}\,. \tag{4.1}$$

and the differential operator $N_\nabla:=-u\lambda\frac{d}{du}:\mathbf{B}_{[0,1)}\to\mathbf{B}_{[0,1)}$. For any adic space $X\in\mathrm{Ad}^{\mathrm{lft}}_{\mathbb{Q}_p}$ we view $\lambda$ as a section of $\mathscr{B}^{[0,1)}_X$ and $N_\nabla$ as a differential operator on $\mathscr{B}^{[0,1)}_X$. The Frobenius $\varphi$ on $\mathcal{O}_X\otimes_{\mathbb{Q}_p}K_0[u]$ extends to a Frobenius endomorphism of $\mathscr{B}^{[0,1)}_X$ again denoted by $\varphi$ by means of $\varphi(u)=u^p$. These operators satisfy the relation

$$N_\nabla\,\varphi=p\,\tfrac{E(u)}{E(0)}\cdot\varphi\,N_\nabla\,. \tag{4.2}$$

**Definition 4.1.** A $(\varphi,N_\nabla)$-*module* $(\mathcal{M},\Phi_{\mathcal{M}},N^{\mathcal{M}}_\nabla)$ over an adic space $X\in\mathrm{Ad}^{\mathrm{lft}}_{\mathbb{Q}_p}$ consists of a locally free sheaf $\mathcal{M}$ of finite rank on $X\times_{\mathbb{Q}_p}\mathbb{U}$, a differential operator $N^{\mathcal{M}}_\nabla:\mathcal{M}\to\mathcal{M}[\frac{1}{\lambda}]$ over $N_\nabla$, that is $N^{\mathcal{M}}_\nabla(fm)=-u\lambda\frac{df}{du}\cdot m+f\cdot N^{\mathcal{M}}_\nabla(m)$ for all sections $f$ of $\mathcal{O}_{X\times_{\mathbb{Q}_p}\mathbb{U}}$ and $m$ of $\mathcal{M}$, and an $\mathcal{O}_{X\times_{\mathbb{Q}_p}\mathbb{U}}$-linear isomorphism $\Phi_{\mathcal{M}}:(\varphi^*\mathcal{M})[\frac{1}{E(u)}]\xrightarrow{\sim}\mathcal{M}[\frac{1}{E(u)}]$, satisfying $N^{\mathcal{M}}_\nabla\circ\Phi_{\mathcal{M}}\circ\varphi=p\,\frac{E(u)}{E(0)}\cdot\Phi_{\mathcal{M}}\circ\varphi\circ N^{\mathcal{M}}_\nabla$.

A *morphism* $\alpha:(\mathcal{M},\Phi_{\mathcal{M}},N^{\mathcal{M}}_\nabla)\to(\mathcal{N},\Phi_{\mathcal{N}},N^{\mathcal{N}}_\nabla)$ between $(\varphi,N_\nabla)$-modules over $X$ is a morphism $\alpha:\mathcal{M}\to\mathcal{N}$ of sheaves satisfying $\alpha\circ\Phi_{\mathcal{M}}=\Phi_{\mathcal{N}}\circ\varphi^*(\alpha)$ and $N^{\mathcal{N}}_\nabla\circ\alpha=\alpha\circ N^{\mathcal{M}}_\nabla$.

**Remark 4.2.** (1) Note that it is not clear whether a $(\varphi,N_\nabla)$-module $\mathcal{M}$ is locally on $X$ free over $X\times\mathbb{U}$ and hence it is not clear whether $\mathrm{pr}_{X,*}\mathcal{M}$ is locally on $X$ a free $\mathscr{B}^{[0,1)}_X$-module. However it follows from [KPX12, Proposition 2.1.15] that $\mathrm{pr}_{X,*}\mathcal{M}$ is a finitely presented $\mathscr{B}^{[0,1)}_X$-module.
(2) The differential operator $N^{\mathcal{M}}_\nabla$ can be equivalently described as a connection $\nabla_{\mathcal{M}}:\mathcal{M}\to\mathcal{M}\otimes u^{-1}\Omega^1_{X\times\mathbb{U}/X}[\frac{1}{\lambda}]$ when we set $\nabla_{\mathcal{M}}(m):=-\frac{1}{\lambda}N^{\mathcal{M}}_\nabla(m)\otimes\frac{du}{u}$. Then $N^{\mathcal{M}}_\nabla$ is recovered as the composition of $\nabla_{\mathcal{M}}$ followed by the map $u^{-1}\Omega^1_{X\times\mathbb{U}/X}[\frac{1}{\lambda}]\to\mathcal{M}$, $du\mapsto-u\lambda$.

Let $X\in\mathrm{Ad}^{\mathrm{lft}}_{\mathbb{Q}_p}$ be an adic space. We will show that the category of $(\varphi,N_\nabla)$-modules over $X$ is equivalent to the category of $(\varphi,N)$-modules with Hodge-Pink lattice over $X$ by defining two mutually quasi-inverse functors $\underline{\mathcal{M}}$ and $\underline{D}$.

To define $\underline{\mathcal{M}}$ let $\underline{D}=(D,\Phi,N,\mathfrak{q})$ be a $(\varphi,N)$-module with Hodge-Pink lattice over $X$. We denote by $\mathrm{pr}:X\times_{\mathbb{Q}_p}\mathbb{U}\to X\times_{\mathbb{Q}_p}K_0$ the projection and set $(\mathcal{D},\Phi_{\mathcal{D}}):=\mathrm{pr}^*(D,\Phi)$. Then

$$\mathrm{pr}_{X,*}(\mathcal{D},\Phi_{\mathcal{D}})=(D,\Phi)\otimes_{(\mathcal{O}_X\otimes K_0)}\mathscr{B}^{[0,1)}_X.$$

We choose a $\mathbb{B}^+_{\mathcal{O}_X}$-automorphism $\eta_{\underline{D}}$ of $\mathfrak{p} := D \otimes_{\mathcal{O}_X \otimes K_0} \mathbb{B}^+_{\mathcal{O}_X}$ and we let $\iota_0 : \mathcal{D} \hookrightarrow D \otimes_{\mathcal{O}_X \otimes K_0} \mathbb{B}^+_{\mathcal{O}_X}$ be the embedding obtained as the composition of the natural inclusion $D \otimes_{\mathcal{O}_X \otimes K_0} \mathscr{B}_X^{[0,1)} \hookrightarrow D \otimes_{\mathcal{O}_X \otimes K_0} \mathbb{B}^+_{\mathcal{O}_X}$ composed with the automorphism $\eta_{\underline{D}}$. Here we follow Kisin [Kis06, § 1.2] and choose

$$
\begin{aligned}
\eta_{\underline{D}} :\ D \otimes_{\mathcal{O}_X \otimes K_0} \mathbb{B}^+_{\mathcal{O}_X} &\xrightarrow{\ \sim\ } D \otimes_{\mathcal{O}_X \otimes K_0} \mathbb{B}^+_{\mathcal{O}_X}\,, \\
d_0 \otimes f &\longmapsto \textstyle\sum_i N^i(d_0) \otimes \frac{(-1)^i}{i!} \log\big(1 - \frac{E(u)}{E(0)}\big)^i \cdot f\,.
\end{aligned}
\tag{4.3}
$$

**Remark 4.3.** (1) Actually, Kisin introduces a formal variable $\ell_u$ over $\mathscr{B}_X^{[0,1)}$ which formally acts like $\log u$. He extends $\varphi$ to $\mathscr{B}_X^{[0,1)}[\ell_u]$ via $\varphi(\ell_u) = p\,\ell_u$, extends $N_\nabla$ to a derivation on $\mathscr{B}_X^{[0,1)}[\ell_u]$ via $N_\nabla(\ell_u) = -\lambda$, and defines $N$ as the $\mathscr{B}_X^{[0,1)}$-linear derivation on $\mathscr{B}_X^{[0,1)}[\ell_u]$ that acts as the differentiation of the formal variable $\ell_u$. Under the $\Phi$-equivariant identification

$$
\begin{array}{rcl}
D[\ell_u]^{N=0} &:=& \big\{ \sum_{i=0}^{<\infty} d_i \ell_u^i :\ d_i \in D \text{ with } N(\sum_i d_i \ell_u^i) = 0 \big\} \xleftarrow{\ \cong\ } D \\
&& \sum_{i=0}^{<\infty} \frac{(-1)^i}{i!} N^i(d_0) \ell_u^i \longleftarrow\!\!\!\!\!\!\!\!\!\!\!\!\!\longleftarrow\!\!\!\dashv\ d_0
\end{array}
$$

Kisin's map $\iota_0 : D[\ell_u]^{N=0} \otimes_{\mathcal{O}_X \otimes K_0} \mathscr{B}_X^{[0,1)} \hookrightarrow \mathfrak{p}$, $\sum_i d_i \ell_u^i \otimes f \mapsto \sum_i d_i \otimes f \cdot (\log \frac{u}{\pi})^i$ corresponds to our $\iota_0$, because we identify $\frac{E(u)}{E(0)}$ with $1 - \frac{u}{\pi}$.

(2) Instead of the above $\eta_{\underline{D}}$ one could also choose $\eta_{\underline{D}} = \mathrm{id}_{\mathfrak{p}}$. This would lead to a few changes which we will comment on in Remark 4.11. Note that our $\eta_{\underline{D}}$ from (4.3) is different from $\mathrm{id}_{\mathfrak{p}}$ if $N \neq 0$.

For all $n \geq 0$ we now consider the map

$$
\mathrm{pr}_{X,*}\,\mathcal{D}[\tfrac{1}{\lambda}] \xrightarrow{\Phi_{\mathcal{D}}^{-j}} \mathrm{pr}_{X,*}\,\varphi^{j*}(\mathcal{D}[\tfrac{1}{\lambda}]) =\!= \mathrm{pr}_{X,*}\,\mathcal{D}[\tfrac{1}{\lambda}] \otimes_{\mathscr{B}_X^{[0,1)},\varphi^j} \mathscr{B}_X^{[0,1)} \xrightarrow{\varphi^{j*}\iota_0} \mathfrak{p}[\tfrac{1}{E(u)}] \otimes_{\mathbb{B}^+_{\mathcal{O}_X}} \varphi^j(\mathbb{B}^+_{\mathcal{O}_X})
$$

where we write $\varphi^{j*}\iota_0$ for $\iota_0 \otimes \mathrm{id}$. We set

$$
\mathrm{pr}_{X,*}\,\mathcal{M} := \big\{ m \in \mathrm{pr}_{X,*}\,\mathcal{D}[\tfrac{1}{\lambda}] :\ \varphi^{j*}\iota_0 \circ \Phi_{\mathcal{D}}^{-j}(m) \in \mathfrak{q} \otimes_{\mathbb{B}^+_{\mathcal{O}_X}} \varphi^j(\mathbb{B}^+_{\mathcal{O}_X}) \text{ for all } j \geq 0 \big\}\,.
\tag{4.4}
$$

and we let $\mathcal{M}$ be the induced sheaf on $X \times_{\mathbb{Q}_p} \mathbb{U}$. Since $\lambda = \frac{E(u)}{E(0)}\varphi(\lambda)$ the isomorphism $\Phi_{\mathcal{D}}$ induces an isomorphism $\Phi_{\mathcal{M}} : (\varphi^*\mathcal{M})[\frac{1}{E(u)}] \xrightarrow{\sim} \mathcal{M}[\frac{1}{E(u)}]$.

We want to show that $\mathcal{D}$ and $\mathcal{M}$ are locally free sheaves of finite rank on $X \times_{\mathbb{Q}_p} \mathbb{U}$. For $\mathcal{D}$ this follows from $\mathcal{D}|_{X \times \mathbb{B}_{[0,r]}} = D \otimes_{(\mathcal{O}_X \otimes K_0)} \mathcal{O}_{X \times \mathbb{B}_{[0,r]}}$. We work on a covering of $X$ by affinoids $Y = \mathrm{Spa}(A, A^+)$. Let $h \in \mathbb{Z}$ be such that $\mathfrak{q} \subset E(u)^{-h}\mathfrak{p}$ on $Y$ and let $n$ be maximal such that $\varphi^n(E(u))$ is not a unit in $\mathbf{B}_{[0,r]}$, that is, such that $r^{p^n} \geq |\pi|$. Then $\mathcal{M}|_{Y \times \mathbb{B}_{[0,r]}}$ is defined by the exact sequence

$$
0 \longrightarrow \mathcal{M}|_{Y \times \mathbb{B}_{[0,r]}} \longrightarrow \lambda^{-h}\mathcal{D}|_{Y \times \mathbb{B}_{[0,r]}} \xrightarrow{\bigoplus_{j=0}^n \varphi^{j*}\iota_0 \circ \Phi_{\mathcal{D}}^{-j}} \bigoplus_{j=0}^{n} (E(u)^{-h}\mathfrak{p}/\mathfrak{q}) \otimes_{\mathbb{B}^+_A,\varphi^j} \varphi^j(\mathbb{B}^+_A) \longrightarrow 0\,.
$$

The $A \otimes_{\mathbb{Q}_p} K_0[u]$-module $E(u)^{-h}\mathfrak{p}/\mathfrak{q}$ is locally free over $A$, say of rank $k$. The endomorphism $\varphi : K_0[u] \to K_0[u]$ makes the target $K_0[u]$ into a free module of rank $p$ over the source $K_0[u]$. Therefore $(E(u)^{-h}\mathfrak{p}/\mathfrak{q}) \otimes_{\mathbb{B}^+_A,\varphi^j} \varphi^j(\mathbb{B}^+_A)$ is locally free over $A$ of rank $p^j k$. Since the affinoid algebra $A$ is noetherian and $\mathbf{B}_{[0,r]}$ is a principal ideal domain by [Laz62, Corollary of Proposition 4] also $\Gamma(Y \times \mathbb{B}_{[0,r]}, \mathcal{O}_{Y \times \mathbb{B}_{[0,r]}}) = A \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{[0,r]}$ is noetherian. So $\Gamma(Y \times \mathbb{B}_{[0,r]}, \mathcal{M})$ is finitely generated over $A \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{[0,r]}$ and flat over $A$. The residue field of each maximal ideal $\mathfrak{m} \subset A \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{[0,r]}$ is finite over $\mathbb{Q}_p$ by [BGR84, Corollary 6.1.2/3]. Therefore $\mathfrak{n} = \mathfrak{m} \cap A$ is a maximal ideal of $A$. By the elementary divisor theorem $A/\mathfrak{n} \otimes_A \Gamma(Y \times \mathbb{B}_{[0,r]}, \mathcal{M})$ is free over the product of principal ideal domains $A/\mathfrak{n} \otimes_{\mathbb{Q}_p} \mathbf{B}_{[0,r]}$. Therefore $\Gamma(Y \times \mathbb{B}_{[0,r]}, \mathcal{M})$ is locally free of rank $d$ over $A \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{[0,r]}$ by [EGA, $\mathrm{IV}_3$, Theorem 11.3.10]. This shows that $\mathcal{M}$ is a locally free sheaf of rank $d$ on $X \times_{\mathbb{Q}_p} \mathbb{U}$.

We equip $\mathcal{M}$ with a differential operator $N_\nabla^{\mathcal{M}}$ over $N_\nabla$. On $\lambda^{-h}\mathcal{D} = D \otimes_{(\mathcal{O}_X \otimes K_0)} \lambda^{-h}\mathscr{B}_X^{[0,1)}$ we have the differential operator $N_\nabla^{\mathcal{D}} := N \otimes \lambda + \mathrm{id}_D \otimes N_\nabla$

$$\begin{aligned} \lambda^{-h}\mathcal{D} &\xrightarrow{\;N\otimes\lambda+\mathrm{id}_D\otimes N_\nabla\;} \lambda^{-h}\mathcal{D} \\ d\otimes\lambda^{-h}f &\longmapsto N(d)\otimes\lambda^{1-h}f + d\otimes\big(hu\lambda^{-h}f\tfrac{d\lambda}{du} - u\lambda^{1-h}\tfrac{df}{du}\big) \end{aligned} \tag{4.5}$$

with $d \in D$ and $f \in \mathscr{B}_X^{[0,1)}$. Its image lies in $\lambda^{-h}\mathcal{D}$. If $E(u)^n\mathfrak{p} \subset \mathfrak{q} \subset E(u)^{-h}\mathfrak{p}$ then $\lambda^n\mathcal{D} \subset \mathcal{M} \subset \lambda^{-h}\mathcal{D}$. Thus $N_\nabla^{\mathcal{D}}(\mathcal{M}) \subset \lambda^{-h}\mathcal{D} \subset \lambda^{-h-n}\mathcal{M}$ and we let $N_\nabla^{\mathcal{M}}$ be the restriction of $N_\nabla^{\mathcal{D}}$ to $\mathcal{M}$. The equation $N_\nabla^{\mathcal{M}} \circ \Phi_{\mathcal{M}} \circ \varphi = p\,\frac{E(u)}{E(0)} \cdot \Phi_{\mathcal{M}} \circ \varphi \circ N_\nabla^{\mathcal{M}}$ is satisfied because it is satisfied on $\mathcal{D}$ by (4.5). Therefore we have constructed a $(\varphi, N_\nabla)$-module $\underline{\mathcal{M}}(D,\Phi,N,\mathfrak{q}) := (\mathcal{M}, \Phi_{\mathcal{M}}, N_\nabla^{\mathcal{M}})$ over $X$. Note that in terms of Kisin's description of $\mathcal{D} \cong D[\ell_u]^{N=0} \otimes_{(\mathcal{O}_X\otimes K_0)} \mathscr{B}_X^{[0,1)}$ the differential operator $N_\nabla^{\mathcal{M}}$ is given as $\mathrm{id}_D \otimes N_\nabla$.

**Example 4.4.** The $(\varphi, N)$-modules with Hodge-Pink lattice from Example 2.8, corresponding to the cyclotomic character, give rise to the following $(\varphi, N_\nabla)$-modules of rank 1 over $X = \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. For $\underline{D} = (K_0, \Phi = p^{-1}, N = 0, \mathfrak{q} = E(u)\mathfrak{p})$ we obtain $\mathcal{D} = (\mathscr{B}_X^{[0,1)}, \Phi_{\mathcal{D}} = p^{-1}, N_\nabla)$ and $\mathcal{M} = \lambda\mathscr{B}_X^{[0,1)}$. On the basis vector $\lambda$ of $\mathcal{M}$ the actions of $\Phi_{\mathcal{D}}$ and $N_\nabla$ are given by $\Phi_{\mathcal{D}}(\varphi(\lambda)) = p^{-1}\varphi(\lambda) = \frac{E(0)}{pE(u)}\,\lambda$ and $N_\nabla(\lambda) = -u\frac{d\lambda}{du}\,\lambda$. So we find $\underline{\mathcal{M}}(\underline{D}) \cong (\mathscr{B}_X^{[0,1)}, \Phi_{\mathcal{M}} = \frac{E(0)}{pE(u)}, N_\nabla^{\mathcal{M}})$ with $N_\nabla^{\mathcal{M}}(f) = N_\nabla(f) - u\frac{d\lambda}{du}\,f$. Similarly for $\underline{D} = (K_0, \Phi = p, N = 0, \mathfrak{q} = E(u)^{-1}\mathfrak{p})$ we obtain $\mathcal{D} = (\mathscr{B}_X^{[0,1)}, \Phi_{\mathcal{D}} = p, N_\nabla)$ and $\mathcal{M} = \lambda^{-1}\mathscr{B}_X^{[0,1)}$ which leads to $\underline{\mathcal{M}}(\underline{D}) \cong (\mathscr{B}_X^{[0,1)}, \Phi_{\mathcal{M}} = \frac{pE(u)}{E(0)}, N_\nabla^{\mathcal{M}})$ with $N_\nabla^{\mathcal{M}}(f) = N_\nabla(f) + u\frac{d\lambda}{du}\,f$.

To define the quasi-inverse functor $\underline{D}$ let $(\mathcal{M}, \Phi_{\mathcal{M}}, N_\nabla^{\mathcal{M}})$ be a $(\varphi, N_\nabla)$-module over $X$. We denote by $e : X \times_{\mathbb{Q}_p} K_0 \to X \times_{\mathbb{Q}_p} \mathbb{U}$ the isomorphism $x \mapsto (x, 0)$ onto the closed subspace defined by $u = 0$. Let $(D, \Phi, N) := e^*(\mathcal{M}, \Phi_{\mathcal{M}}, N_\nabla^{\mathcal{M}})$. It is a $(\varphi, N)$-module over $X$ because $N$ is clearly $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0$-linear and $e^*\big(\frac{E(u)}{E(0)}\big) = 1$ implies $N \circ \Phi = p \cdot \Phi \circ \varphi^* N$. By [PR09, Proposition 5.2] there is a unique $\mathcal{O}_{X\times_{\mathbb{Q}_p}\mathbb{U}}$-linear isomorphism

$$\xi : \mathrm{pr}^* D[\tfrac{1}{\lambda}] \xrightarrow{\;\sim\;} \mathcal{M}[\tfrac{1}{\lambda}] \tag{4.6}$$

satisfying $\xi \circ \mathrm{pr}^*\Phi = \Phi_{\mathcal{M}} \circ \varphi^*\xi$ and $e^*\xi = \mathrm{id}_D$. In particular the composition $\mathrm{pr}^*\Phi \circ (\varphi^*\xi)^{-1} = \xi^{-1} \circ \Phi_{\mathcal{M}}$ induces an isomorphism $\varphi^*\mathcal{M} \otimes \mathbb{B}^+_{\mathcal{O}_X} \xrightarrow{\;\sim\;} D \otimes_{(\mathcal{O}_X\otimes K_0)} \mathbb{B}^+_{\mathcal{O}_X} = \mathfrak{p}$ of $\mathbb{B}^+_{\mathcal{O}_X}$-modules. We set $\mathfrak{q} := \eta_{\underline{D}} \circ (\xi \otimes \mathrm{id}_{\mathbb{B}_{\mathcal{O}_X}})^{-1}(\mathcal{M} \otimes \mathbb{B}^+_{\mathcal{O}_X})$. Then $\underline{D}(\mathcal{M}, \Phi_{\mathcal{M}}, N_\nabla^{\mathcal{M}}) := (D, \Phi, N, \mathfrak{q})$ is a $(\varphi, N)$-module with Hodge-Pink lattice over $X$ by Lemma 2.6 and the following lemma.

**Lemma 4.5.** *Locally on a covering of $X$ by affinoids $Y = \mathrm{Spa}(A, A^+)$ there exist integers $h, n$ with $E(u)^n\Phi_{\mathcal{M}}(\varphi^*\mathcal{M}) \subset \mathcal{M} \subset E(u)^{-h}\Phi_{\mathcal{M}}(\varphi^*\mathcal{M})$ such that the quotients*

$$E(u)^{-h}\Phi_{\mathcal{M}}(\varphi^*\mathcal{M})/\mathcal{M} \text{ and } \mathcal{M}/E(u)^n\Phi_{\mathcal{M}}(\varphi^*\mathcal{M})$$

*are finite locally free over $A$.*

*Proof.* The existence of $h$ and $n$ follows from the finiteness of $\mathcal{M}$ and $\varphi^*\mathcal{M}$. Let $\mathfrak{m} \subset A$ be a maximal ideal and set $L = A/\mathfrak{m}$. Let $|\pi| < r < 1$ and set $\widetilde{\mathcal{M}} := \Gamma(Y \times_{\mathbb{Q}_p} \mathbb{B}_{[0,r]}, \mathcal{M})$ and $\widetilde{\varphi^*\mathcal{M}} := \Gamma(Y \times_{\mathbb{Q}_p} \mathbb{B}_{[0,r]}, \varphi^*\mathcal{M})$. Then $\mathcal{M}/E(u)^n\Phi_{\mathcal{M}}(\varphi^*\mathcal{M}) \cong \widetilde{\mathcal{M}}/E(u)^n\Phi_{\mathcal{M}}(\widetilde{\varphi^*\mathcal{M}})$. Consider the exact sequence

$$0 \longrightarrow E(u)^n\widetilde{\varphi^*\mathcal{M}} \xrightarrow{\;\Phi_{\mathcal{M}}\;} \widetilde{\mathcal{M}} \longrightarrow \mathcal{M}/E(u)^n\Phi_{\mathcal{M}}(\varphi^*\mathcal{M}) \longrightarrow 0$$

in which the first map is injective because $E(u)$ is a non-zero-divisor in $A\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}_{[0,r]}$. We tensor the sequence with $L$ over $A$ to obtain the exact sequence of $L \otimes_{\mathbb{Q}_p} \mathbf{B}_{[0,r]}$-modules

$$0 \longrightarrow T \longrightarrow L \otimes_A E(u)^n\widetilde{\varphi^*\mathcal{M}} \xrightarrow{\;\mathrm{id}_L\otimes\Phi_{\mathcal{M}}\;} L \otimes_A \widetilde{\mathcal{M}} \longrightarrow L \otimes_A \big(\mathcal{M}/E(u)^n\Phi_{\mathcal{M}}(\varphi^*\mathcal{M})\big) \longrightarrow 0$$

with $T = \mathrm{Tor}_1^A\big(L, \mathcal{M}/E(u)^n\Phi_{\mathcal{M}}(\varphi^*\mathcal{M})\big)$. Since $L \otimes_{\mathbb{Q}_p} K_0$ is a product of fields, $L \otimes_{\mathbb{Q}_p} \mathbf{B}_{[0,r]}$ is a product of principal ideal domains by [Laz62, Corollary of Proposition 4]. Since $E(u)^{n+h}$ annihilates

$L\otimes_A \mathcal{M}/E(u)^n\Phi_{\mathcal{M}}(\varphi^*\mathcal{M})$ the latter is a torsion module over $L\otimes_{\mathbb{Q}_p}\mathbf{B}_{[0,r]}$. By the elementary divisor theorem $\mathrm{id}_L\otimes\Phi_{\mathcal{M}}$ is injective and $T=0$. Since $\mathcal{M}/E(u)^n\Phi_{\mathcal{M}}(\varphi^*\mathcal{M})$ is finite over the noetherian ring $A$ it is locally free by the local criterion of flatness [Eis95, Theorem 6.8]. From the exact sequence

$$0\longrightarrow E(u)^{-h}\Phi_{\mathcal{M}}(\varphi^*\mathcal{M})/\mathcal{M}\xrightarrow{\cdot E(u)^{n+h}}\mathcal{M}/E(u)^{n+h}\mathcal{M}\longrightarrow \mathcal{M}/E(u)^n\Phi_{\mathcal{M}}(\varphi^*\mathcal{M})\longrightarrow 0$$

it follows that also $E(u)^{-h}\Phi_{\mathcal{M}}(\varphi^*\mathcal{M})/\mathcal{M}$ is finite locally free over $A$. □

**Theorem 4.6.** *For every adic space $X\in\mathrm{Ad}^{\mathrm{lft}}_{\mathbb{Q}_p}$ the functors $\underline{\mathcal{M}}$ and $\underline{D}$ constructed above are mutually quasi-inverse equivalences between the category of $(\varphi,N)$-modules with Hodge-Pink lattice over $X$ and the category of $(\varphi,N_\nabla)$-modules over $X$.*

*Proof.* We must show that the functors are mutually quasi-inverse. To prove one direction let $(D,\Phi,N,\mathfrak{q})$ be a $(\varphi,N)$-module with Hodge-Pink lattice over $X$ and let $(\mathcal{M},\Phi_{\mathcal{M}},N_\nabla^{\mathcal{M}})=\underline{\mathcal{M}}(D,\Phi,N,\mathfrak{q})$. By construction $e^*\mathcal{M}=D$, and under this equality $e^*\Phi_{\mathcal{M}}$ corresponds to $\Phi$. Since $e^*\lambda=1$, formula (4.5) shows that $e^*N_\nabla^{\mathcal{M}}$ corresponds to $N$ on $D$. By the uniqueness of the map $\xi$ from (4.6), its inverse $\xi^{-1}$ equals the inclusion $\mathcal{M}\hookrightarrow\mathcal{D}[\tfrac{1}{\lambda}]$, by which we defined $\mathcal{M}$. This shows that $\eta_{\underline{D}}\circ(\xi\otimes\mathrm{id}_{\mathbb{B}_{\mathcal{O}_X}})^{-1}(\mathcal{M}\otimes\mathbb{B}^+_{\mathcal{O}_X})$ equals $\mathfrak{q}$ and that $\underline{D}\circ\underline{\mathcal{M}}=\mathrm{id}$.

Conversely let $(\mathcal{M},\Phi_{\mathcal{M}},N_\nabla^{\mathcal{M}})$ be a $(\varphi,N_\nabla)$-module over $X$ and let $(D,\Phi,N,\mathfrak{q})=\underline{D}(\mathcal{M},\Phi_{\mathcal{M}},N_\nabla^{\mathcal{M}})$. Via the isomorphism $\xi$ from (4.6), $\mathcal{M}$ is a $\varphi$-submodule of $\mathrm{pr}^*D[\tfrac{1}{\lambda}]$. By construction of $\mathfrak{q}$ and

$$\underline{\mathcal{M}}(\underline{D}(\mathcal{M},\Phi_{\mathcal{M}},N_\nabla^{\mathcal{M}}))\subset\mathrm{pr}^*D[\tfrac{1}{\lambda}],$$

the latter submodule coincides with $\mathcal{M}$ modulo all powers of $E(u)$. Since both submodules have a Frobenius which is an isomorphism outside $\mathrm{V}(E(u))$ they are equal on all of $X\times_{\mathbb{Q}_p}\mathbb{U}$. It remains to show that $N_\nabla^{\mathcal{M}}$ is compatible with $N_\nabla^{\mathrm{pr}^*D}$ under the isomorphism $\xi:\mathrm{pr}^*D[\tfrac{1}{\lambda}]\xrightarrow{\sim}\mathcal{M}[\tfrac{1}{\lambda}]$. We follow [Kis06, Lemma 1.2.12(3)] and let $\sigma:=\xi\circ N_\nabla^{\mathrm{pr}^*D}-N_\nabla^{\mathcal{M}}\circ\xi$. Then $\sigma:\mathrm{pr}^*D[\tfrac{1}{\lambda}]\to\mathcal{M}[\tfrac{1}{\lambda}]$ is $\mathcal{O}_{X\times_{\mathbb{Q}_p}\mathbb{U}}$-linear and it suffices to show that $\sigma(D)=0$. By (4.5) both $N_\nabla^{\mathrm{pr}^*D}$ and $N_\nabla^{\mathcal{M}}$ reduce to $N$ modulo $u$. Therefore $\sigma(D)\subset u\mathcal{M}[\tfrac{1}{\lambda}]$. One checks that $\sigma\circ\Phi_{\mathrm{pr}^*D}\circ\varphi=p\frac{E(u)}{E(0)}\cdot\Phi_{\mathcal{M}}\circ\varphi\circ\sigma$ and this implies

$$\sigma(D)\;=\;\sigma\circ\Phi_{\mathrm{pr}^*D}(\varphi^*D)\;=\;p\tfrac{E(u)}{E(0)}\cdot\Phi_{\mathcal{M}}\circ\varphi^*(u\mathcal{M}[\tfrac{1}{\lambda}])\;\subset\;u^p\mathcal{M}[\tfrac{1}{\lambda}].$$

By induction $\sigma(D)\subset u^{p^i}\mathcal{M}[\tfrac{1}{\lambda}]$ for all $i$ and hence $\sigma(D)=0$. This shows that also $\underline{\mathcal{M}}\circ\underline{D}$ is isomorphic to the identity and proves the theorem. □

**Corollary 4.7.** *The stack $\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}$ is isomorphic to the stack whose groupoid of $X$-valued points for $X\in\mathrm{Ad}^{\mathrm{lft}}_{E_\mu}$ consists of $(\varphi,N_\nabla)$-modules $(\mathcal{M},\Phi_{\mathcal{M}},N_\nabla^{\mathcal{M}})$ over $X$ satisfying*

$$\bigwedge^{j}_{\mathcal{O}_{X\times\mathbb{U}}}\mathcal{M}\;\subset\;E(u)^{-\mu_1-\ldots-\mu_j}\cdot\bigwedge^{j}_{\mathcal{O}_{X\times\mathbb{U}}}\Phi_{\mathcal{M}}(\varphi^*\mathcal{M})$$

*with equality for $j=\mathrm{rk}\,\mathcal{M}$. Here $\mu_i$ is viewed as a $\mathbb{Z}$-valued function on $X\times_{\mathbb{Q}_p}K$.*

*Proof.* This follows from the definition of the functor $\underline{D}$, in particular the definition of the Hodge-Pink lattice. □

**Definition 4.8.** We define substacks $\mathscr{H}^{\nabla}_{\varphi,N,\mu}\subset\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\mu}$, resp. $\mathscr{H}^{\nabla}_{\varphi,N,\preceq\mu}\subset\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}$, resp. $\mathscr{H}^{\nabla}_{\varphi,\mu}\subset\mathscr{H}^{\mathrm{ad}}_{\varphi,\mu}$, resp. $\mathscr{H}^{\nabla}_{\varphi,\preceq\mu}\subset\mathscr{H}^{\mathrm{ad}}_{\varphi,\preceq\mu}$. For an adic space $X\in\mathrm{Ad}^{\mathrm{lft}}_{E_\mu}$ the groupoid $\mathscr{H}^{\nabla}_{\varphi,N,\mu}(X)$ consists of those $(D,\varphi,N,\mathfrak{q})\in\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\mu}(X)$ for which the associated $(\varphi,N_\nabla)$-module $(\mathcal{M},\Phi_{\mathcal{M}},N_\nabla^{\mathcal{M}})$ satisfies $N_\nabla^{\mathcal{M}}(\mathcal{M})\subset\mathcal{M}$. The groupoids $\mathscr{H}^{\nabla}_{\varphi,N,\preceq\mu}(X)$, resp. $\mathscr{H}^{\nabla}_{\varphi,\mu}(X)$, resp. $\mathscr{H}^{\nabla}_{\varphi,\preceq\mu}(X)$ are defined by the same condition. (Note that on the latter two $N=0$, but $N_\nabla^{\mathcal{M}}\neq0$.)

**Theorem 4.9.** *The substacks $\mathscr{H}^{\nabla}_{\varphi,N,\mu}\subset\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\mu}$, resp. $\mathscr{H}^{\nabla}_{\varphi,N,\preceq\mu}\subset\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}$, resp. $\mathscr{H}^{\nabla}_{\varphi,\mu}\subset\mathscr{H}^{\mathrm{ad}}_{\varphi,\mu}$, resp. $\mathscr{H}^{\nabla}_{\varphi,\preceq\mu}\subset\mathscr{H}^{\mathrm{ad}}_{\varphi,\preceq\mu}$ are Zariski closed substacks. The substack $\mathscr{H}^{\nabla}_{\varphi,N,\mu}$ coincides with the image of the zero section of the vector bundle $\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\mu}\to\mathscr{D}^{\mathrm{ad}}_{\varphi,N,\mu}$.*

**Remark 4.10.** We can consider a family of $(\varphi, N_\nabla)$-modules over $\mathscr{D}^{\mathrm{ad}}_{\varphi,N,\mu}$ namely we pull back the canonical family of $(\varphi, N_\nabla)$-modules on $\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\mu}$ along the zero section. Then for $x \in \mathscr{D}^{\mathrm{ad}}_{\varphi,N,\mu}(\mathbb{Q}_p)$ the fiber of this family at $x$ coincides with the $(\varphi, N_\nabla)$-module that Kisin [Kis06] associates with the filtered $(\varphi, N)$-module defined by $x$.

**Remark 4.11.** If instead of the isomorphism $\eta_{\underline{D}}$ from (4.3) we choose $\eta_{\underline{D}} = \mathrm{id}_{\mathfrak{p}}$ as in Remark 4.3(2), the above results remain valid, except that $\mathscr{H}^{\nabla}_{\varphi,N,\mu}$ coincides with the image of a different section. This section is obtained by composing the zero section with the inverse ${\eta_{\underline{D}}}^{-1} : d\otimes f \mapsto \sum_i N^i(d)\otimes \frac{1}{i!}\log\bigl(1-\frac{E(u)}{E(0)}\bigr)^i\cdot f$ of the automorphism $\eta_{\underline{D}}$. It sends a filtration $\mathcal{F}^\bullet$ to ${\eta_{\underline{D}}}^{-1}\bigl(\sum_{i\in\mathbb{Z}} E(u)^{-i}(\mathcal{F}^i D_K)\otimes_{R\otimes K}\mathbb{B}^+_R\bigr)$. Note that both sections coincide on the closed substack $\mathscr{D}^{\mathrm{ad}}_{\varphi,\mu}$ where $N = 0$.

*Proof of Theorem 4.9.* To prove that the substacks are closed let $\underline{D} \in \mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}(X)$ for an adic space $X \in \mathrm{Ad}^{\mathrm{lft}}_{E_\mu}$ and let $(M, \Phi_{\mathcal{M}}, N^{\mathcal{M}}_\nabla) = \underline{\mathcal{M}}(\underline{D})$ be the associated $(\varphi, N_\nabla)$-module over $X$. Locally on $X$ there is an integer $h$ with $N^{\mathcal{M}}_\nabla(\mathcal{M}) \subset \lambda^{-h}\mathcal{M}$ by Lemma 2.6 and the construction of $N^{\mathcal{M}}_\nabla$. The quotients $\bigl(\lambda^{-h}\mathcal{M}/\mathcal{M}\bigr)\otimes\bigl(\mathscr{B}^{[0,1)}_X/(\varphi^n(E(u))^h)\bigr)$ are finite locally free as $\mathcal{O}_X$-modules for all $n \geq 0$. Now the condition $N^{\mathcal{M}}_\nabla(\mathcal{M}) \subset \mathcal{M}$ is equivalent to the vanishing of the images under $N^{\mathcal{M}}_\nabla$ of a set of generators of $\mathcal{M}$ in $\bigl(\lambda^{-h}\mathcal{M}/\mathcal{M}\bigr)\otimes\bigl(\mathscr{B}^{[0,1)}_X/(\varphi^n(E(u))^h)\bigr)$ for each $n \geq 0$. Due to [EGA, $\mathrm{I}_{\mathrm{new}}$, Lemma 9.7.9.1] the latter is represented by a Zariski closed subspace of $X$.

We show that the closed substack $\mathscr{H}^{\nabla}_{\varphi,N,\mu}$ of $\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\mu}$ coincides with the image of the zero section. Since $N^{\mathcal{D}}_\nabla$ on $\mathcal{D} := D\otimes_{\mathcal{O}_X\otimes K_0}\mathscr{B}^{[0,1)}_{\mathcal{O}_X}$ induces the differential operator $\mathrm{id}_D\otimes N_\nabla$ on $\mathfrak{p} := D\otimes_{\mathcal{O}_X\otimes K_0}\mathbb{B}^+_{\mathcal{O}_X}$ under the map $i_0 = \eta_{\underline{D}}\circ\mathrm{inclusion} : \mathcal{D}\hookrightarrow\mathfrak{p}$ from (4.3), it follows directly that the image of the zero section is contained in $\mathscr{H}^{\nabla}_{\varphi,N,\mu}$. To prove the converse we may work on the coverings $X := (P_{K_0,d}\times_{\mathbb{Q}_p} Q_{K,d,\mu})^{\mathrm{ad}}$ of $\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\mu}$ and $(P_{K_0,d}\times_{\mathbb{Q}_p}\mathrm{Flag}_{K,d,\mu})^{\mathrm{ad}}$ of $\mathscr{D}^{\mathrm{ad}}_{\varphi,N,\mu}$ because the zero section and $\mathscr{H}^{\nabla}_{\varphi,N,\mu}$ are both invariant under the action of $(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}$. We first claim that both have the same underlying topological space. By [BGR84, Corollary 6.1.2/3] this can be checked on $L$-valued points of $X$ for finite extensions $L$ of $E_\mu$. For those it was proved by Kisin [Kis06, Lemma 1.2.12(4)] that the universal Hodge-Pink lattice $\mathfrak{q}$ at $L$ lies in the image of the zero section if the pullback $\underline{\mathcal{M}}$ to $L$ of the universal $(\varphi, N_\nabla)$-module on $X$ has holomorphic $N^{\mathcal{M}}_\nabla$. From this our claim follows.

To prove equality as closed subspaces of $X$ we look at a closed point $x \in X$ and its complete local ring $\widehat{\mathcal{O}}_{X,x}$. Let $\mathfrak{m}_x \subset \widehat{\mathcal{O}}_{X,x}$ be the maximal ideal, let $I \subset \widehat{\mathcal{O}}_{X,x}$ be the ideal defining $\mathscr{H}^{\nabla}_{\varphi,N,\mu}$, and set $R_n := \widehat{\mathcal{O}}_{X,x}/(\mathfrak{m}_x^n + I)$. Then $R_n$ is a finite dimensional $\mathbb{Q}_p$-vector space by [BGR84, Corollary 6.1.2/3]. We consider the universal $\underline{D}_{R_n} = (D, \Phi, N, \mathfrak{q})$ over $R_n$ by restriction of scalars from $R_n$ to $\mathbb{Q}_p$ as a $(\varphi, N)$-module $\underline{\widetilde{D}}$ with Hodge-Pink lattice over $\mathbb{Q}_p$ of rank $(\dim_{\mathbb{Q}_p} R_n)(\mathrm{rk}_{R_n\otimes K_0} D) = \dim_{K_0} D$. It is equipped with a ring homomorphism $R_n \to \mathrm{End}(\underline{\widetilde{D}})$. Since $N^{\mathcal{M}}_\nabla$ is holomorphic on $\underline{\mathcal{M}}(\underline{D}_{R_n})$, Kisin [Kis06, Lemma 1.2.12(4)] tells us again that $\mathfrak{q} = \mathfrak{q}(\mathcal{F}^\bullet)$ for the filtration $\mathcal{F}^\bullet = \mathcal{F}^\bullet_{\mathfrak{q}}$ from Remark 2.7. This shows that the ideal $J$ defining the zero section in $X$ vanishes in $R_n$. Since this holds for all $n$, the ideals $I$ and $J$ are equal in $\widehat{\mathcal{O}}_{X,x}$. As $x$ was arbitrary, they coincide on all of $X$ and this proves the theorem. ☐

## 5. Weak admissibility

Similar to the case of filtrations, one can define a notion of weak admissibility for $(\varphi, N)$-modules with Hodge-Pink lattice and develop a Harder-Narasimhan formalism. Compare also [Hel11, § 2] for the following. Recall that $f = [K_0 : \mathbb{Q}_p]$ and $e = [K : K_0]$.

**Definition 5.1.** Let $L$ be a field with a valuation $v_L : L \to \Gamma_L \cup \{0\}$ in the sense of Huber, see [Hub93, § 2, Definition] and set $\Gamma^{\mathbb{Q}}_L := \Gamma_L \otimes_{\mathbb{Z}} \mathbb{Q}$.
(i) Let $\underline{D} = (D, \Phi, N)$ be a $(\varphi, N)$-module over $L$. Then define

$$t_N(\underline{D}) \;:=\; v_L(\det\nolimits_L \Phi^f)^{1/f^2} \;\in\; \Gamma^{\mathbb{Q}}_L.$$

If $L \supset K_0$ we are in the situation of Remark 2.4 and have $t_N(\underline{D}) = v_L(\det_L(\Phi^f)_0)^{1/f}$.

(ii) Let $\underline{D} = (D, \Phi, N, \mathcal{F}^\bullet)$ be a $K$-filtered $(\varphi, N)$-module over $L$. Then

$$t_H(\underline{D}) \;:=\; \tfrac{1}{ef} \sum_{i \in \mathbb{Z}} i \dim_L(\mathcal{F}^i D_K / \mathcal{F}^{i+1} D_K) \;\in\; \mathbb{Q}.$$

(iii) Let $\underline{D} = (D, \Phi, N, \mathfrak{q})$ be a $(\varphi, N)$-module with Hodge-Pink lattice of rank $d$ over $L$. Then we set

$$t_H(\underline{D}) \;:=\; \tfrac{1}{ef}\big(\dim_L(\mathfrak{q}/t^n\mathfrak{p}) - \dim_L(\mathfrak{p}/t^n\mathfrak{p})\big) \;=\; \tfrac{1}{ef}\dim_L(\mathfrak{q}/t^n\mathfrak{p}) \;-\; n\,\mathrm{rk}\,\underline{D} \;\in\; \mathbb{Q}$$

for $n \gg 0$, which is independent of $n$ whenever $t^n\mathfrak{p} \subset \mathfrak{q}$. If $L$ is an extension of $\widetilde{K}$ and $(\mu_\psi)_\psi = \mu_{\underline{D}}(\operatorname{Spec} L)$ is the Hodge polygon of $\underline{D}$, see Definition 2.10, then $t_H(\underline{D}) := \frac{1}{ef}\sum_\psi \mu_{\psi,1} + \ldots + \mu_{\psi,d}$. If the $\psi$-component $\mathfrak{q}_\psi$ satisfies $\wedge^d \mathfrak{q}_\psi = t^{-h_\psi} \wedge^d \mathfrak{p}_\psi$ then $t_H(\underline{D}) = \frac{1}{ef}\sum_\psi h_\psi$. Moreover $t_H(\underline{D}) = t_H(D, \Phi, N, \mathcal{F}^\bullet_{\mathfrak{q}})$.

(iv) Let $\underline{D}$ be a $(\varphi, N)$-module with Hodge-Pink lattice (resp. a $K$-filtered $(\varphi, N)$-module) over $L$. Then its *slope* is defined to be

$$\lambda(\underline{D}) \;:=\; \big(v_L(p)^{t_H(\underline{D})} \cdot t_N(\underline{D})^{-1}\big)^{1/d} \;\in\; \Gamma_L^{\mathbb{Q}}.$$

**Definition 5.2.** (i) A $(\varphi, N)$-module with Hodge-Pink lattice $\underline{D} = (D, \Phi, N, \mathfrak{q})$ over a field $L$ endowed with a valuation is called *semi-stable* if $\lambda(\underline{D}') \geq \lambda(\underline{D})$ for all $\underline{D}' = \big(D', \Phi|_{\varphi^* D'}, N|_{D'}, \mathfrak{q} \cap D' \otimes_{L \otimes_{\mathbb{Q}_p} K_0} \mathbb{B}_L\big)$ where $D' \subset D$ is a free $L \otimes_{\mathbb{Q}_p} K_0$-submodule stable under $\Phi$ and $N$.

(ii) A $K$-filtered $(\varphi, N)$-module $\underline{D} = (D, \Phi, N, \mathcal{F}^\bullet)$ over $L$ is called *semi-stable* if $\lambda(\underline{D}') \geq \lambda(\underline{D})$ for all $\underline{D}' = \big(D', \Phi|_{\varphi^* D'}, N|_{D'}, \mathcal{F}^\bullet \cap D'_K\big)$ where $D' \subset D$ is a free $L \otimes_{\mathbb{Q}_p} K_0$-submodule stable under $\Phi$ and $N$.

(iii) A $(\varphi, N)$-module with Hodge-Pink lattice (resp. a $K$-filtered $(\varphi, N)$-module) is called *weakly admissible* if it is semi-stable of slope 1.

**Lemma 5.3.** *Let $(D, \Phi, N, \mathcal{F}^\bullet)$ be a $K$-filtered $(\varphi, N)$-module over a valued field $L$ and let $(D, \Phi, N, \mathfrak{q})$ denote the $(\varphi, N)$-module with Hodge-Pink lattice associated to $(D, \Phi, N, \mathcal{F}^\bullet)$ by the zero section $\mathcal{F}^\bullet \mapsto \mathfrak{q} = \mathfrak{q}(\mathcal{F}^\bullet)$ of Remark 2.7(3). Then $(D, \Phi, N, \mathcal{F}^\bullet)$ is weakly admissible if and only if $(D, \Phi, N, \mathfrak{q})$ is.*

*Proof.* It is obvious from the definitions that $t_H(D, \Phi, N, \mathcal{F}^\bullet) = t_H(D, \Phi, N, \mathfrak{q}(\mathcal{F}^\bullet))$. Further we have to test on the same sub-objects $D' \subset D$. Hence the claim follows from the fact

$$\mathfrak{q}(\mathcal{F}^\bullet \cap D'_K) = \mathfrak{q}(\mathcal{F}^\bullet) \cap D' \otimes_{L \otimes_{\mathbb{Q}_p} K_0} \mathbb{B}_L\,,$$

which is obvious from the description of $\mathfrak{q}(-)$ in Remark 2.7(3) by choosing an $L \otimes_{\mathbb{Q}_p} K$-basis of $D_K$ adapted to the submodules $\mathcal{F}^i D'_K$ and $\mathcal{F}^i D_K$. □

**Proposition 5.4.** *Let $(D, \Phi, N, \mathfrak{q})$ be a $(\varphi, N)$-module with Hodge-Pink lattice defined over some valued field $L$. Then there is a unique Harder-Narasimhan filtration*

$$0 = D_0 \subset D_1 \subset \cdots \subset D_r = D$$

*of $(D, \Phi, N, \mathfrak{q})$, by free $L \otimes_{\mathbb{Q}_p} K_0$-submodules stable under $\Phi$ and $N$ such that the subquotients $D_i/D_{i-1}$ with their induced Hodge-Pink lattice are semi-stable of slope $\lambda_i \in \Gamma_L \otimes \mathbb{Q}$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_r$.*

*Proof.* This is the usual Harder-Narasimhan formalism, see [FF, 3] for a fairly general exposition. See also [Hel11, Proposition 2.19]. □

**Corollary 5.5.** *Let $(D, \Phi, N, \mathfrak{q})$ be a $(\varphi, N)$-module with Hodge-Pink lattice over $L$ and let $L'$ be an extension of $L$ with valuation $v_{L'}$ extending the valuation $v_L$. Then $(D, \Phi, N, \mathfrak{q})$ is weakly admissible if and only if $(D', \Phi', N', \mathfrak{q}') = (D \otimes_L L', \Phi \otimes \mathrm{id}, N \otimes \mathrm{id}, \mathfrak{q} \otimes_L L')$ is weakly admissible.*

*Proof.* This is similar to [Hel11, Corollary 2.22].

If $(D', \Phi', N', \mathfrak{q}')$ is weakly admissible, then every $(\Phi, N)$-stable subobject $D_1 \subset D$ defines a $(\Phi', N')$-stable subobject $D'_1 = D_1 \otimes_L L'$ of $D'$ such that

$$\lambda(D_1, \Phi|_{\varphi^* D_1}, N|_{D_1}, \mathfrak{q} \cap D_1 \otimes_{L \otimes K_0} \mathbb{B}_L) = \lambda(D'_1, \Phi'|_{\varphi^* D'_1}, N'|_{D'_1}, \mathfrak{q}' \cap D'_1 \otimes_{L' \otimes K_0} \mathbb{B}_{L'}).$$

It follows that $(D, \Phi, N, \mathfrak{q})$ is weakly admissible, as $(D', \Phi', N', \mathfrak{q}')$ is.

Now assume that $(D, \Phi, N, \mathfrak{q})$ is weakly admissible. We may reduce to the case where $L'$ is finitely generated over $L$ and $\mathrm{Aut}(L'/L)$ is large enough. As in the proof of [Hel11, Corollary 2.22] one shows that the action of $\mathrm{Aut}(L'/L)$ preserves the slope of $\Phi'$-stable sub-objects of $D'$. Hence the Harder-Narasimhan filtration of $D'$ descends to $D$. As $D$ is weakly admissible, this filtration can only have one step. □

**Theorem 5.6.** *Let $\mu$ be a cocharacter as in (2.5) with reflex field $E_\mu$. Then the groupoid*

$$X \mapsto \{(D, \Phi, N, \mathfrak{q}) \in \mathscr{H}_{\varphi,N,\preceq\mu}(X) \mid D \otimes \kappa(x) \text{ is weakly admissible for all } x \in X\}$$

*is an open substack $\mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\preceq\mu}$ of $\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}$ on the category of adic spaces locally of finite type over $E_\mu$.*

*Proof.* This is similar to the proof of [Hel13, Theorem 4.1].

It follows from Corollary 5.5 that $\mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\preceq\mu}$ is indeed a stack, i.e. weak admissibility may be checked over an fpqc-covering. Hence it suffices to show that the weakly admissible locus is open in

$$X_\mu := P_{K_0,d} \times_{\mathbb{Q}_p} Q_{K,d,\preceq\mu}.$$

Let us denote by $Z_i$ the projective $P_{K_0,d}$-scheme whose $S$-valued points are given by pairs $(x, U)$ with $x = (g, N) \in P_{K_0,d}(S) \subset (\mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}_{d,K_0}) \times (\mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{Mat}_{d\times d})$ and an $\mathcal{O}_S \otimes K_0$-subspace $U \subset \mathcal{O}_S \otimes K_0^{\oplus d}$ which is locally on $S$ free of rank $i$, a direct summand as $\mathcal{O}_S$-module, and stable under the action of $\Phi_g = g\cdot\varphi$ and $N$. This is a closed subscheme of the product $P_{K_0,d} \times_{\mathbb{Q}_p} \mathrm{Quot}_{\mathcal{O}^d \mid K_0 \mid \mathbb{Q}_p}$ (where $\mathrm{Quot}_{\mathcal{O}^d \mid K_0 \mid \mathbb{Q}_p}$ is Grothendieck's Quot-scheme which is projective over $\mathbb{Q}_p$; see [FGA, n°221, Theorem 3.1] or [AK80, Theorem 2.6]), cut out by the invariance conditions under $\Phi_g$ and $N$. Further write $f_i \in \Gamma(Z_i, \mathcal{O}_{Z_i})$ for the global section defined by

$$f_i(g, U) = \det(g \cdot \varphi)^f|_U = \det\big(g \cdot \varphi(g) \cdot \ldots \cdot \varphi^{f-1}(g)\big)^f|_U,$$

where $f = [K_0 : \mathbb{Q}_p]$, and where the determinant is the determinant as $\mathcal{O}_{Z_i}$-modules. Write $U$ for the pullback of the universal $(\Phi, N)$-invariant subspace on $Z_i$ to the product $Z_i \times Q_{K,d,\preceq\mu}$, write $\mathfrak{q}$ for the pullback of the universal $\mathbb{B}^+$-lattice on $Q_{K,d,\preceq\mu}$ to $Z_i \times Q_{K,d,\preceq\mu}$, and write $\mathfrak{p} = (\mathbb{B}^+)^{\oplus d}$ for the pullback of the tautological $\mathbb{B}^+$-lattice $D \otimes \mathbb{B}^+$ on $P_{K_0,d}$ to $Z_i \times Q_{K,d,\preceq\mu}$. Fix integers $n, h$ with $t^n\mathfrak{p} \subset \mathfrak{q} \subset t^{-h}\mathfrak{p}$ and consider the complex of finite locally free sheaves on $Z_i \times Q_{K,d,\preceq\mu}$

$$P_\bullet : \quad P_1 := t^{-h}\mathfrak{p}/t^n\mathfrak{p} \xrightarrow{\ \delta\ } t^{-h}\mathfrak{p}/\mathfrak{q} \oplus \big(D/U \otimes t^{-h}\mathbb{B}^+/t^n\mathbb{B}^+\big) =: P_0$$

given by the canonical projection $D \twoheadrightarrow D/U$ in the second summand. Let $T_1$ be the functor from the category of quasi-coherent sheaves on $Z_i \times Q_{K,d,\preceq\mu}$ to itself defined by

$$T_1 : M \longmapsto \ker\big(\delta \otimes \mathrm{id}_M : P_1 \otimes M \to P_0 \otimes M\big).$$

If $M = \kappa(y)$ for a point $y = (g_y, N_y, U_y, \mathfrak{q}_y) \in Z_i \times Q_{K,d,\preceq\mu}$ then $T_1(\kappa(y)) = (\mathfrak{q}_y \cap \mathfrak{p}_{i,y}[\tfrac{1}{t}])/t^n\mathfrak{p}_{i,y}$ where we write $\mathfrak{p}_{i,y} := U_y \otimes_{\kappa(y)\otimes K_0} \mathbb{B}^+_{\kappa(y)}$. We consider the function

$$\begin{aligned} h_i : Z_i \times Q_{K,d,\preceq\mu} &\longrightarrow \mathbb{Q}, \\ y &\longmapsto \tfrac{1}{ef} \dim_{\kappa(y)} T_1(\kappa(y)) - ni = t_H\big(U_y, g_y(\mathrm{id}\otimes\varphi)|_{U_y}, N|_{U_y}, \mathfrak{q}_y \cap \mathfrak{p}_{i,y}[\tfrac{1}{t}]\big). \end{aligned} \tag{5.1}$$

We write $Z_i^{\mathrm{ad}}$ resp. $Q^{\mathrm{ad}}_{K,d,\preceq\mu}$ for the adic spaces associated to the varieties $Z_i$ and $Q_{K,d,\preceq\mu}$. Similarly we write $h_i^{\mathrm{ad}}$ for the function on the adic spaces $Z_i^{\mathrm{ad}} \times Q^{\mathrm{ad}}_{K,d,\preceq\mu}$ defined by the same formula as in (5.1). By semi-continuity [EGA, III$_2$, Théorème 7.6.9], the sets

$$Y_{i,m} = \{y \in Z_i^{\mathrm{ad}} \times Q^{\mathrm{ad}}_{K,d,\preceq\mu} \mid h_i(y) \geq m\}$$

are closed and hence proper over $X_\mu^{\mathrm{ad}} = P^{\mathrm{ad}}_{K_0,d} \times_{\mathbb{Q}_p} Q^{\mathrm{ad}}_{K,d,\leq\mu}$. We write

$$\mathrm{pr}_{i,m} : Y_{i,m} \longrightarrow X_\mu^{\mathrm{ad}}$$

for the canonical, proper projection.

If we write $X_0 \subset X_\mu^{\mathrm{ad}}$ for the open subset of all $(D, \Phi, N, \mathfrak{q})$ such that $\lambda(D, \Phi, N, \mathfrak{q}) = 1$, then

$$X_0 \smallsetminus X_\mu^{\mathrm{wa}} = X_0 \cap \bigcup_{i,m} \mathrm{pr}_{i,m}\left(\{y \in Y_{i,m} \mid v_y(f_i) > v_y(p)^{f^2 m}\}\right), \tag{5.2}$$

where the union runs over $1 \leq i \leq d-1$ and $m \in \mathbb{Z}$. Indeed: Let $x = (D, \Phi, N, \mathfrak{q})$ be an $L$-valued point of $X_0$, then any proper $(\Phi, N)$-stable subspace of $D' \subset D$ defines (for some $1 \leq i \leq d-1$) a point $y = (D', \mathfrak{q})$ of $Z_i \times Q_{K,d,\leq\mu}$ mapping to $x$. This subspace violates the weak admissibility condition if and only if

$$v_y(f_i) = t_N(D', \Phi|_{\varphi^* D'})^{f^2} > v_y(p)^{f^2\, t_H(D', \Phi, \mathfrak{q}\cap(D'\otimes\mathbb{B}_{\kappa(x)}))} = v_y(p)^{f^2 h_i(y)},$$

and hence (5.2) follows. On the other hand the union

$$\bigcup_{i,m} \mathrm{pr}_{i,m}\left(\{y \in Y_{i,m} \mid v_y(f_i) > v_y(p)^{f^2 m}\}\right)$$

is a finite union, because $Y_{i,m} = \emptyset$ for $m > hi$ and $Y_{i,m} = Z_i^{\mathrm{ad}} \times Q_{K,d,\preceq\mu}^{\mathrm{ad}}$ for $m \leq -ni$. Therefore the union is closed by properness of the map $\mathrm{pr}_{i,m}$ and the definition of the topology on an adic space. The theorem follows from this. □

We define sub-groupoids $\mathscr{H}_{\varphi,\preceq\mu}^{\mathrm{ad,wa}} \subset \mathscr{H}_{\varphi,\preceq\mu}^{\mathrm{ad}}$, resp. $\mathscr{H}_{\varphi,N,\mu}^{\mathrm{ad,wa}} \subset \mathscr{H}_{\varphi,N,\mu}^{\mathrm{ad}}$, resp. $\mathscr{H}_{\varphi,\mu}^{\mathrm{ad,wa}} \subset \mathscr{H}_{\varphi,\mu}^{\mathrm{ad}}$, resp. $\mathscr{D}_{\varphi,N,\mu}^{\mathrm{ad,wa}} \subset \mathscr{D}_{\varphi,N,\mu}^{\mathrm{ad}}$, resp. $\mathscr{D}_{\varphi,\mu}^{\mathrm{ad,wa}} \subset \mathscr{D}_{\varphi,\mu}^{\mathrm{ad}}$ as follows: Given an adic space $X$ and $(D,\Phi,N,\mathfrak{q}) \in \mathscr{H}_{\varphi,\preceq\mu}^{\mathrm{ad}}$, we say that $(D,\Phi,N,\mathfrak{q}) \in \mathscr{H}_{\varphi,\preceq\mu}^{\mathrm{ad,wa}}$ if and only if $(D,\Phi,N,\mathfrak{q}) \otimes \kappa(x)$ is weakly admissible for all points $x \in X$. The other sub-groupoids are defined in the same manner.

**Corollary 5.7.** *The sub-groupoids* $\mathscr{H}_{\varphi,\preceq\mu}^{\mathrm{ad,wa}} \subset \mathscr{H}_{\varphi,\preceq\mu}^{\mathrm{ad}}$, *resp.* $\mathscr{H}_{\varphi,N,\mu}^{\mathrm{ad,wa}} \subset \mathscr{H}_{\varphi,N,\mu}^{\mathrm{ad}}$, *resp.* $\mathscr{H}_{\varphi,\mu}^{\mathrm{ad,wa}} \subset \mathscr{H}_{\varphi,\mu}^{\mathrm{ad}}$, *resp.* $\mathscr{D}_{\varphi,N,\mu}^{\mathrm{ad,wa}} \subset \mathscr{D}_{\varphi,N,\mu}^{\mathrm{ad}}$, *resp.* $\mathscr{D}_{\varphi,\mu}^{\mathrm{ad,wa}} \subset \mathscr{D}_{\varphi,\mu}^{\mathrm{ad}}$ *are open substacks.*

*Proof.* This follows by pulling back $\mathscr{H}_{\varphi,N,\preceq\mu}^{\mathrm{ad,wa}} \subset \mathscr{H}_{\varphi,N,\preceq\mu}^{\mathrm{ad}}$ along the morphisms $\mathscr{H}_{\varphi,\preceq\mu}^{\mathrm{ad}} \to \mathscr{H}_{\varphi,N,\preceq\mu}^{\mathrm{ad}}$, resp. $\mathscr{H}_{\varphi,N,\mu}^{\mathrm{ad}} \to \mathscr{H}_{\varphi,N,\preceq\mu}^{\mathrm{ad}}$, resp. $\mathscr{H}_{\varphi,\mu}^{\mathrm{ad}} \to \mathscr{H}_{\varphi,N,\preceq\mu}^{\mathrm{ad}}$, resp. $\mathscr{D}_{\varphi,N,\mu}^{\mathrm{ad}} \to \mathscr{H}_{\varphi,N,\preceq\mu}^{\mathrm{ad}}$, resp. $\mathscr{D}_{\varphi,\mu}^{\mathrm{ad}} \to \mathscr{H}_{\varphi,N,\preceq\mu}^{\mathrm{ad}}$. Here we use the fact that the zero sections $\mathscr{D}_{\varphi,N,\mu}^{\mathrm{ad}} \to \mathscr{H}_{\varphi,N,\preceq\mu}^{\mathrm{ad}}$ and $\mathscr{D}_{\varphi,\mu}^{\mathrm{ad}} \to \mathscr{H}_{\varphi,N,\preceq\mu}^{\mathrm{ad}}$ preserve weak admissibility by Lemma 5.3. □

**Remark 5.8.** Note that the projection

$$\mathrm{pr} : \mathscr{H}_{\varphi,N,\mu}^{\mathrm{ad}} \longrightarrow \mathscr{D}_{\varphi,N,\mu}^{\mathrm{ad}}$$

does not preserve weak admissibility. We always have $\mathrm{pr}^{-1}(\mathscr{D}_{\varphi,N,\mu}^{\mathrm{ad,wa}}) \subset \mathscr{H}_{\varphi,N,\mu}^{\mathrm{ad,wa}}$ and hence especially any section of the vector bundle $\mathscr{H}_{\varphi,N,\mu}^{\mathrm{ad}} \to \mathscr{D}_{\varphi,N,\mu}^{\mathrm{ad}}$ maps the weakly admissible locus to the weakly admissible locus.

Indeed, let $\underline{D} = (D,\Phi,N,\mathfrak{q})$ be a point of $\mathscr{H}_{\varphi,N,\mu}^{\mathrm{ad}}$ over a field $L$ whose image $(D,\Phi,N,\mathcal{F}_{\mathfrak{q}}^\bullet)$ in $\mathscr{D}_{\varphi,N,\mu}^{\mathrm{ad}}$ is weakly admissible. Let $D' \subset D$ be an $L \otimes_{\mathbb{Q}_p} K_0$-submodule which is stable under $\Phi$ and $N$, and set $\mathfrak{p}' := D' \otimes_{L\otimes K_0} \mathbb{B}_L^+$. Then $\mathfrak{q}' := \mathfrak{q} \cap \mathfrak{p}'[\frac{1}{t}]$ satisfies $t^i\mathfrak{q}' \cap \mathfrak{p}' \subset t^i\mathfrak{q} \cap \mathfrak{p}$ and $\mathcal{F}_{\mathfrak{q}'}^i D_K' \subset \mathcal{F}_{\mathfrak{q}}^i D_K \cap D_K'$. This implies

$$t_H(D',\Phi|_{\varphi^*D'},N|_{D'},\mathfrak{q}') \;=\; t_H(D',\Phi|_{\varphi^*D'},N|_{D'},\mathcal{F}_{\mathfrak{q}'}^\bullet) \;\leq\; t_H(D',\Phi|_{\varphi^*D'},N|_{D'},\mathcal{F}_{\mathfrak{q}}^\bullet \cap D_K')$$

with equality for $D' = D$, and

$$v_L(p)^{t_H(D',\Phi|_{\varphi^*D'},N|_{D'},\mathfrak{q}')} \;\geq\; v_L(p)^{t_H(D',\Phi|_{\varphi^*D'},N|_{D'},\mathcal{F}_{\mathfrak{q}}^\bullet\cap D_K')} \;\geq\; t_N(D',\Phi|_{\varphi^*D'})$$

with equality for $D' = D$, because $(D,\Phi,N,\mathcal{F}_{\mathfrak{q}}^\bullet)$ is weakly admissible. Therefore also $\underline{D}$ is weakly admissible. This proves that $\mathrm{pr}^{-1}(\mathscr{D}_{\varphi,N,\mu}^{\mathrm{ad,wa}}) \subset \mathscr{H}_{\varphi,N,\mu}^{\mathrm{ad,wa}}$. However, in general this inclusion is strict as can be seen from the following example.

**Example 5.9.** Let $K = K_0 = \mathbb{Q}_p$, $d = 2$ and $\mu = (2,0)$. We consider points $\underline{D} = (D,\Phi,N,\mathcal{F}^\bullet)$ in $\mathscr{D}_{\varphi,N,\mu}^{\mathrm{ad}}$ over a field $L$ with $\Phi = p\,\mathrm{Id}_2$ and $N = 0$. The filtration is of the form $D = \mathcal{F}^0D \supset \mathcal{F}^1D = \mathcal{F}^2D = \binom{u}{v}\cdot L \supset \mathcal{F}^3D = (0)$. None of these points is weakly admissible, because the subspace $D' = \binom{u}{v}\cdot L \subset D$ has $t_N(\underline{D}') = v_L(p)$ and $\mathcal{F}^2D' = D'$, whence $t_H(\underline{D}') = 2$ and $\lambda(\underline{D}') = v_L(p) < 1$.

The preimage of such a point in $\mathscr{H}_{\varphi,N,\mu}^{\mathrm{ad}}$ is given by a Hodge-Pink lattice $\mathfrak{q}$ with $\mathfrak{p} \subset \mathfrak{q} \subset t^{-2}\mathfrak{p}$ with Hodge weights 0 and $-2$. This means that $\mathfrak{q} = \mathfrak{p} + \binom{u+tu'}{v+tv'}\cdot t^{-2}\mathbb{B}_L^+$. If the vectors $\binom{u}{v}$ and $\binom{u'}{v'}$ are linearly dependent then $\underline{D} = (D,\Phi,N,\mathfrak{q})$ is not weakly admissible, because the subspace $D' = \binom{u}{v}\cdot L \subset D$ has $t_N(\underline{D}') = v_L(p)$ and $\mathfrak{q}' := \mathfrak{q} \cap D' \otimes_L \mathbb{B}_L = t^{-2}D' \otimes_L \mathbb{B}_L^+$, whence $t_H(\underline{D}') = 2$ and $\lambda(\underline{D}') = v_L(p) < 1$.

On the other hand, if the vectors $\binom{u}{v}$ and $\binom{u'}{v'}$ are linearly *independent* then $\underline{D} = (D,\Phi,N,\mathfrak{q})$ is weakly admissible, because then $\mathfrak{q}' \subset t^{-1}D' \otimes_L \mathbb{B}_L^+$ for any subspace $D' = \binom{a}{b}\cdot L \subset D$, whence $t_N(\underline{D}') = v_L(p)$, $t_H(\underline{D}') \leq 1$ and $\lambda(\underline{D}') \geq 1$. Indeed, $\binom{a}{b}\cdot t^{-2} \in \mathfrak{q}'$ would imply that $\binom{a}{b}\cdot t^{-2} \equiv \binom{u+tu'}{v+tv'}\cdot t^{-2}\cdot(c+tc') \equiv c\binom{u}{v}\cdot t^{-2} + (c'\binom{u}{v} + c\binom{u'}{v'})t^{-1} \mod \mathfrak{p}$ for $c, c' \in L$. This implies $\binom{a}{b} = c\binom{u}{v}$ and $c'\binom{u}{v} + c\binom{u'}{v'} = 0$ contradicting the linear independence.

Thus the weakly admissible locus $\mathscr{H}_{\varphi,\mu}^{\mathrm{ad,wa}}$ in the fiber of $\mathscr{H}_{\varphi,N,\mu}$ over the point $(\Phi, N) = (p\,\mathrm{Id}_2, 0)$ in $P_{\mathbb{Q}_p,2}$ equals the complement of the zero section, while this fiber in $\mathscr{D}_{\varphi,N,\mu}^{\mathrm{ad,wa}}$ is empty; see also Lemma 5.3.

We end this section by remarking that the weakly admissible locus is determined by the rigid analytic points, i.e. those points of an adic space whose residue field is a finite extension of $\mathbb{Q}_p$.

**Lemma 5.10.** *Let $X$ be an adic space locally of finite type over $E_\mu$ and let $f : X \to \mathscr{H}_{\varphi,N,\preceq\mu}^{\mathrm{ad}}$ be a morphism defined by a $(\varphi, N)$-module with Hodge-Pink lattice $\underline{D}$. Then $f$ factors over $\mathscr{H}_{\varphi,N,\preceq\mu}^{\mathrm{ad,wa}}$ if and only if $\underline{D} \otimes \kappa(x)$ is weakly admissible for all rigid analytic points $x \in X$.*

*Proof.* One implication is obvious and the other one is an easy application of the maximum modulus principle. It is proven along the same lines as [Hel13, Proposition 4.3]. ☐

**Remark 5.11.** The analogous statements for the stacks $\mathscr{H}_{\varphi,\preceq\mu}^{\mathrm{ad,wa}} \subset \mathscr{H}_{\varphi,\preceq\mu}^{\mathrm{ad}}$, resp. $\mathscr{H}_{\varphi,N,\mu}^{\mathrm{ad,wa}} \subset \mathscr{H}_{\varphi,N,\mu}^{\mathrm{ad}}$, resp. $\mathscr{H}_{\varphi,\mu}^{\mathrm{ad,wa}} \subset \mathscr{H}_{\varphi,\mu}^{\mathrm{ad}}$, resp. $\mathscr{D}_{\varphi,N,\mu}^{\mathrm{ad,wa}} \subset \mathscr{D}_{\varphi,N,\mu}^{\mathrm{ad}}$, resp. $\mathscr{D}_{\varphi,\mu}^{\mathrm{ad,wa}} \subset \mathscr{D}_{\varphi,\mu}^{\mathrm{ad}}$ are also true and are a direct consequence of their construction.

## 6. The étale locus

Let us denote by $\mathbb{B}_{[r,s]}$ the closed annulus over $K_0$ of inner radius $r$ and outer radius $s$ for some $r, s \in [0,1) \cap p^{\mathbb{Q}}$. For an adic space $X \in \mathrm{Ad}_{\mathbb{Q}_p}^{\mathrm{lft}}$ we write

$$\mathscr{A}_X^{[0,1)} = \mathrm{pr}_{X,*}\,\mathcal{O}_{X\times\mathbb{U}}^+ \subset \mathscr{B}_X^{[0,1)} = \mathrm{pr}_{X,*}\,\mathcal{O}_{X\times\mathbb{U}}$$
$$\mathscr{A}_X^{[r,s]} = \mathrm{pr}_{X,*}\,\mathcal{O}_{X\times\mathbb{B}_{[r,s]}}^+ \subset \mathscr{B}_X^{[r,s]} = \mathrm{pr}_{X,*}\,\mathcal{O}_{X\times\mathbb{B}_{[r,s]}}$$

The Frobenius $\varphi$ on $\mathscr{B}_X^{[0,1)}$ restricts to a ring homomorphism $\varphi$ on $\mathscr{A}_X^{[0,1)}$. For this section we adapt the notation from [Hel13] and write $r_i = r^{1/p^i}$. Then $\varphi$ restricts to a homomorphism

$$\varphi : \mathscr{B}_X^{[r,s]} \longrightarrow \mathscr{B}_X^{[r_1,s_1]}.$$

**Lemma 6.1.** *If $A$ is a reduced $p$-adically complete $\mathbb{Z}_p$-algebra topologically of finite type, then*

$$\Gamma(\mathrm{Spa}(A[1/p], A), \mathscr{A}_{\mathrm{Spa}(A[1/p],A)}^{[0,1)}) = (A \otimes_{\mathbb{Z}_p} W)[\![u]\!].$$

*Proof.* Let $X = \mathrm{Spa}(A[1/p], A)$. We need to compute the global sections of $\Gamma(X \times \mathbb{U}, \mathcal{O}_{X\times\mathbb{U}}^+)$. By definition we have

$$\Gamma(X \times \mathbb{U}, \mathcal{O}_{X\times\mathbb{U}}^+) = \varprojlim_r \Gamma(X \times \mathbb{B}_{[0,r]}, \mathcal{O}_{X\times\mathbb{U}}^+) \subset \varprojlim_r \Gamma(X \times \mathbb{B}_{[0,r]}, \mathcal{O}_{X\times\mathbb{U}}) \subset (A \otimes_{\mathbb{Z}_p} K_0)[\![u]\!].$$

The inclusion $(A \otimes_{\mathbb{Z}_p} W)[\![u]\!] \subset \Gamma(X \times \mathbb{U}, \mathcal{O}_{X\times\mathbb{U}})$ is obvious. Let

$$f = \sum\nolimits_i f_i u^i \in \Gamma(X \times \mathbb{U}, \mathcal{O}_{X\times\mathbb{U}}^+) \subset (A \otimes_{\mathbb{Z}_p} K_0)[\![u]\!].$$

Then $f_i(x) \in \Gamma(X \otimes_{\mathbb{Z}_p} W, \mathcal{O}_{X\otimes W}^+) = A \otimes_{\mathbb{Z}_p} W \subset A \otimes_{\mathbb{Z}_p} K_0$ for all $i$. The claim follows from this. ☐

**Definition 6.2.** A *$\varphi$-module of finite height* over $\mathscr{A}_X^{[0,1)}$ is an $\mathscr{A}_X^{[0,1)}$-module $\mathfrak{M}$ which is locally on $X$ free of finite rank over $\mathscr{A}_X^{[0,1)}$ together with an injective morphism $\Phi : \varphi^*\mathfrak{M} \to \mathfrak{M}$ of $\mathscr{A}_X^{[0,1)}$-modules such that $\mathrm{coker}\,\Phi$ is killed by some power of $E(u) \in W[\![u]\!] \subset \mathscr{A}_X^{[0,1)}$.

Inspired by Example 4.4 we define the $(\varphi, N_\nabla)$-module $\mathscr{B}_X^{[0,1)}(1)$ over $X$ to be $(\mathscr{B}_X^{[0,1)}, \Phi_{\mathcal{M}} = \frac{pE(u)}{E(0)}, N_\nabla^{\mathcal{M}})$ with $N_\nabla^{\mathcal{M}}(f) = N_\nabla(f) + u\frac{d\lambda}{du} f$. For an integer $n \in \mathbb{Z}$ we set $\mathscr{B}_X^{[0,1)}(n) := \mathscr{B}_X^{[0,1)}(1)^{\otimes n} = (\mathscr{B}_X^{[0,1)}, (\frac{pE(u)}{E(0)})^n, N_\nabla^{\mathcal{M}})$ with $N_\nabla^{\mathcal{M}}(f) = N_\nabla(f) + nu\frac{d\lambda}{du} f$. Given a $(\varphi, N_\nabla)$-module $(\mathcal{M}, \Phi_{\mathcal{M}})$ on $X$ we write $(\mathcal{M}, \Phi_{\mathcal{M}})(n)$ for the twist $\mathcal{M} \otimes_{\mathscr{B}_X^{[0,1)}} \mathscr{B}_X^{[0,1)}(n)$. Note that $\frac{p}{E(0)} \in W^\times$ since $E(u)$ is an Eisenstein polynomial. Thus for $n \geq 0$ we have an obvious integral model $\mathscr{A}_X^{[0,1)}(n)$ for $\mathscr{B}^{[0,1)}(n)$ which is a $\varphi$-module of finite height over $\mathscr{A}_X^{[0,1)}$ (by forgetting the $N_\nabla$-action). Further we write $\mathbf{A}^{[0,1)}(n) = \mathscr{A}_{\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)}(n)$ for the $W[\![u]\!]$-module of rank 1 with basis $e$ on which $\Phi$ acts via $\Phi(e) = (\frac{E(u)}{pE(0)})^n e$.

**Definition 6.3.** Let $(\mathcal{M}, \Phi_{\mathcal{M}}, N_{\nabla}^{\mathcal{M}})$ be a $(\varphi, N_{\nabla})$-module over an adic space $X \in \mathrm{Ad}_{\mathbb{Q}_p}^{\mathrm{lft}}$.
(i) The module $\mathcal{M}$ is called *étale* if there exists an fpqc-covering $(U_i \to X)$, an integer $n \geq 0$ and $\varphi$-modules $(\mathfrak{M}_i, \Phi_{\mathfrak{M}_i})$ of finite height over $\mathscr{A}_{U_i}^{[0,1)}$ such that

$$(\mathcal{M}, \Phi_{\mathcal{M}})(n)|_{U_i} = (\mathfrak{M}_i, \Phi_{\mathfrak{M}_i}) \otimes_{\mathscr{A}_{U_i}^{[0,1)}} \mathscr{B}_{U_i}^{[0,1)}.$$

(ii) Let $x \in X$, then $\mathcal{M}$ is called *étale at $x$* if there exists an integer $n \geq 0$ and a $(\kappa(x)^+ \otimes_{\mathbb{Z}_p} W)[\![u]\!]$-lattice $\mathfrak{M} \subset \mathcal{M}(n) \otimes \kappa(x)$ such that $E(u)^h \mathfrak{M} \subset \Phi_{\mathcal{M}(n)}(\varphi^* \mathfrak{M}) \subset \mathfrak{M}$ for some integer $h \geq 0$.

**Theorem 6.4.** *Let $X$ be an adic space locally of finite type over $\mathbb{Q}_p$ and let $(\mathcal{M}, \Phi)$ be a $(\varphi, N_{\nabla})$-module. Then the subset*

$$X^{\mathrm{int}} = \{x \in X \mid \mathcal{M} \text{ is étale at } x\}$$

*is open and the restriction $\mathcal{M}|_{X^{\mathrm{int}}}$ is étale.*

This is similar to the proof of [Hel13, Theorem 7.6]. However, we need to make a few generalizations as we cannot rely on a reduced universal case. Given an affinoid algebra $A$ and $r, s \in [0,1) \cap p^{\mathbb{Q}}$ we write

$$\mathbf{B}_A^{[r,s]} = \Gamma(\mathrm{Spa}(A, A^\circ), \mathscr{B}_{\mathrm{Spa}(A,A^\circ)}^{[r,s]}) = A \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}^{[r,s]} = A_W \langle T/s, r/T \rangle,$$
$$\mathbf{A}_A^{[r,s]} = \Gamma(\mathrm{Spa}(A, A^\circ), \mathscr{A}_{\mathrm{Spa}(A,A^\circ)}^{[r,s]}) = A^\circ \widehat{\otimes}_{\mathbb{Z}_p} \mathbf{A}^{[r,s]} = A_W^\circ \langle T/s, r/T \rangle.$$

The following is the analogue of [Hel13, Theorem 6.9] in the non-reduced case.

**Theorem 6.5.** *Let $X$ be an adic space locally of finite type over $\mathbb{Q}_p$ and let $\mathcal{N}$ be a family of free $\varphi$-modules of rank $d$ over $\mathscr{B}_X^{[r,r_2]}$. Assume that there exists $x \in X$ and an $\mathscr{A}_X^{[r,r_2]} \otimes \kappa(x)^\circ$-lattice $N_x \subset \mathcal{N} \otimes \kappa(x)$ such that $\Phi$ induces an isomorphism*

$$\Phi : \varphi^*(N_x \otimes_{\mathscr{A}_X^{[r,r_2]}} \mathscr{A}_X^{[r,r_1]}) \xrightarrow{\sim} N_x \otimes_{\mathscr{A}_X^{[r,r_2]}} \mathscr{A}_X^{[r_1,r_2]}. \tag{6.1}$$

*Then there exists an open neighborhood $U \subset X$ of $x$ and a locally free $\mathscr{A}_U^{[r,r_2]}$-submodule $N \subset \mathcal{N}$ of rank $d$ such that*

$$N \otimes \kappa(x)^\circ = N_x$$
$$\Phi(\varphi^* N|_{U \times \mathbb{B}_{[r,r_1]}}) = N|_{U \times \mathbb{B}_{[r_1,r_2]}}$$
$$N \otimes_{\mathscr{A}_U^{[r,r_2]}} \mathscr{B}_U^{[r,r_2]} = \mathcal{N}|_U.$$

*Proof.* We may assume that $X = \mathrm{Spa}(A, A^\circ)$ is affinoid and we may choose a Banach norm $||\cdot||$ and a $\mathbb{Z}_p$-subalgebra $A^+ = \{x \in A \mid ||x|| \leq 1\} \subset A^\circ$ such that $A = A^+[1/p]$ and $X = \mathrm{Spa}(A, A^+) = \mathrm{Spa}(A, A^\circ)$.

Choose a basis $\underline{e}_x$ of $N_x$ and denote by $D_0 \in \mathrm{GL}_d(\mathscr{A}_X^{[r,r_2]} \otimes \kappa(x)^\circ)$ the matrix of $\Phi$ in this basis. After shrinking $X$ if necessary we may lift the matrix $D_0$ to a matrix $D$ with coefficients in $\Gamma(X, \mathscr{A}_X^{[r,r_2]})$. Localizing further we may assume that $D$ is invertible over $\Gamma(X, \mathscr{A}_X^{[r,r_2]})$, as we only need to ensure that the inverse of its determinant has coefficients $a_i \in A^+$, i.e. $||a_i|| \leq 1$ for some Banach norm $||\cdot||$ corresponding to $A^+$. However, this is clear for all but finitely many coefficients and hence we may localize further to ensure that the finitely many remaining coefficients are small, as this is true at $x$.

Fixing a basis $\underline{b}$ of $\mathcal{N}$ we denote by $S \in \mathrm{GL}_d(\mathbf{B}_A^{[r,r]})$ the matrix of $\Phi$ in this basis. Further we denote by $V$ a lift of the change of basis matrix from the basis $\underline{e}_x$ to the basis $\underline{b}$ mod $x$. From now on the proof is the same as the proof of [Hel13, Theorem 6.9]. □

**Proposition 6.6.** *Let $X = \mathrm{Spa}\,(A, A^+)$ be an affinoid adic space of finite type over $\mathbb{Q}_p$. Let $r > |\pi|$ with $r \in p^{\mathbb{Q}}$ and set $r_i = r^{1/p^i}$. Let $\mathcal{M}_r$ be a free vector bundle on $X \times \mathbb{B}_{[0,r_2]}$ together with an injection*

$$\Phi : \varphi^*(\mathcal{M}_r|_{X \times \mathbb{B}_{[0,r_1]}}) \longrightarrow \mathcal{M}_r$$

*with cokernel supported at the point defined by $E(u)$ Assume that there exists a free $\mathbf{A}_A^{[r,r_2]} = A^+\langle T/r_2, r/T \rangle$ submodule*

$$N_r \subset \mathcal{N}_r := \mathcal{M}_r \otimes_{\mathbf{B}_A^{[0,r_2]}} \mathbf{B}_A^{[r,r_2]}$$

*of rank d, containing a basis of $\mathcal{N}_r$ such that*

$$\Phi(\varphi^*(N_r \otimes_{\mathbf{A}_A^{[r,r_2]}} \mathbf{A}_A^{[r,r_1]})) = N_r \otimes_{\mathbf{A}_A^{[r,r_2]}} \mathbf{A}_A^{[r_1,r_2]}.$$

*Then fpqc-locally on $X$ there exists a free $\mathbf{A}_A^{[0,r_2]}$-submodule $M_r \subset \mathcal{M}_r$ of rank d, containing a basis of $\mathcal{M}_r$ such that*

$$\Phi : \varphi^*(M_r \otimes_{\mathbf{A}_A^{[0,r_2]}} \mathbf{A}_A^{[0,r_1]}) \longrightarrow M_r \tag{6.2}$$

*is injective with cokernel killed by $E(u)$.*

*Proof.* This is the generalization of [Hel13, Proposition 7.7] to our context. We also write $\mathcal{M}_r$ for the global sections of the vector bundle. Write $M_r' = \mathcal{M}_r \cap N_r \subset \mathcal{N}_r$. This is an $A^+\langle T/r_2\rangle$-module. Further we set

$$M_r = (M_r' \otimes_{\mathbf{A}_A^{[0,r_2]}} \mathbf{A}_A^{[r,r_2]}) \cap M_r'[\tfrac{1}{p}] \subset \mathcal{N}_r.$$

Then $M_r$ is a finitely generated $A^+\langle T/r_2\rangle$-module as the ring is noetherian. First we need to make some modification in order to assure that $M_r$ is flat. Let $\mathcal{Y} = \mathrm{Spf}\, W\langle T/r_2\rangle$ denote the formal model of $\mathbb{B}_{[0,r_2]}$ and let $\mathcal{Y}' = \mathrm{Spf}\, W\langle T/r_2, r/T\rangle$ denote the formal model of $\mathbb{B}_{[r,r_2]}$. By [BL93, Theorem 4.1] there exists a blow-up $\widetilde{\mathcal{X}}$ of $\mathrm{Spf}(A)$ such that the strict transform $\widetilde{M}_r$ of $M_r$ in $\widetilde{\mathcal{X}} \times \mathcal{Y}$ is flat over $\widetilde{\mathcal{X}}$. We write $\mathcal{M}_{r,\widetilde{\mathcal{X}}}$ (resp. $N_{r,\widetilde{\mathcal{X}}}$) for the pullback of $\mathcal{M}_r$ (resp. $N_r$) to the generic fiber of $\widetilde{\mathcal{X}} \times \mathcal{Y}$ (resp. to $\widetilde{\mathcal{X}} \times \mathcal{Y}'$). If we set $\widetilde{M}_r' = \mathcal{M}_{r,\widetilde{\mathcal{X}}} \cap N_{r,\widetilde{\mathcal{X}}}$ then one easily finds

$$\widetilde{M}_r = (\widetilde{M}_r' \otimes_{\mathscr{A}_{\widetilde{\mathcal{X}}}^{[0,r_2]}} \mathscr{A}_{\widetilde{\mathcal{X}}}^{[r,r_2]}) \cap \widetilde{M}_r'[\tfrac{1}{p}].$$

If follows that $\widetilde{M}_r$ is stable under $\Phi$. Further, as $\widetilde{M}_r$ is flat, it has no $p$-power torsion and hence we find that the formation $(\mathcal{M}_{r,\widetilde{\mathcal{X}}}, N_{r,\widetilde{\mathcal{X}}}) \mapsto \widetilde{M}_r$ commutes with base change $\mathrm{Spf}\,\mathcal{O} \hookrightarrow \widetilde{\mathcal{X}}$ for any finite flat $\mathbb{Z}_p$-algebra $\mathcal{O}$, compare the proof of [Hel13, Proposition 7.7]. Especially this pullback is free over $\mathcal{O} \otimes_{\mathbb{Z}_p} W\langle T/r_2\rangle$ and the cokernel of $\Phi$ is annihilated by $E(u)^{k_\mu}$ for some $k_\mu \gg 0$ depending only on the Hodge polygon $\mu$ (for an arbitrary finite flat $\mathbb{Z}_p$-algebra $\mathcal{O}$ this follows by forgetting the $\mathcal{O}$-structure and only considering the $\mathbb{Z}_p$-structure).

It follows that the restriction of $\widetilde{M}_r$ to the reduced special fiber $\widetilde{\mathcal{X}}_0$ of $\widetilde{\mathcal{X}}$ is locally free over $\widetilde{\mathcal{X}}_0 \times \mathbb{A}^1$ and hence, as in the proof of [Hel13, Proposition 7.7] we may locally lift a basis and find that $\widetilde{M}_r$ is locally on $\widetilde{\mathcal{X}}$ free over $\widetilde{\mathcal{X}} \times \mathcal{Y}$.

It is only left to show that $E(u)^{k_\mu}\,\mathrm{coker}\,\Phi = 0$ over $\widetilde{\mathcal{X}}$. To do so we may localize and assume that $\widetilde{\mathcal{X}}$ is affine. By abuse of language we denote it again by $\mathrm{Spf}\, A^+$ and write $N = E(u)^{k_\mu}\,\mathrm{coker}\,\Phi$. If $I$ denotes the ideal of nilpotent elements in $A^+$, we need to show that the multiplication $I \otimes_{A^+} N \to N$ is the zero map. Then the claim follows from the above. However for some $k \gg 0$ we know that $I^k \otimes_{A^+} N \to N$ is the zero map, as $I$ is nilpotent. Then $N = N/I^k$ and the multiplication map $I^{k-1} \otimes_{A^+} N \to N$ factors over $I^{k-1}/I^k \otimes_{A^+} N \to N$ and this is a map of finitely generated $A^+/I$-modules which vanishes after pulling back to a quotient $A^+/I \to \mathcal{O}_L$ onto the ring of integers in some finite extension $L$ of $\mathbb{Q}_p$. This can be seen as follows. The map on this pull back is induced by the pullback of the multiplication to a quotient of $A^+$ which is finite flat over $\mathbb{Z}_p$, and where $N$ is known to vanish by the above. It follows that $I^{k-1} \otimes_{A^+} N \to N$ is the zero map and by descending induction we find that $I$ acts trivial on $N$. $\square$

*Proof of Theorem 6.4.* Fix some $r > |\pi|$ and re-define

$$X^{\mathrm{int}} = \{x \in X \mid \mathcal{M}|_{X \times \mathbb{B}_{[r,r_2]}} \otimes \kappa(x) \text{ is étale in the sense of (6.1)}\}.$$

By Theorem 6.5 this subset is open and we need to show that the restriction $\mathcal{M}|_{X^{\mathrm{int}}}$ is étale. Then is follows directly that $X^{\mathrm{int}}$ coincides with the characterization in the theorem, as the notion of being étale at points may be checked fpqc-locally by [Hel13, Proposition 6.14].

However Proposition 6.6 provides (locally on $X^{\mathrm{int}}$) an integral model $\mathfrak{M}_{[0,r_2]}$ over $X \times \mathbb{B}_{[0,r_2]}$. Now we can glue $\mathfrak{M}_{[0,r_2]}$ and $\varphi^*\mathfrak{M}_{[0,r_2]}$ over $X \times \mathbb{B}_{[r_2,r_3]}$ and hence extend $\mathfrak{M}_{[0,r_2]}$ to a model $\mathfrak{M}_{[0,r_3]}$ over $X \times \mathbb{B}_{[0,r_3]}$. Proceeding by induction we get a model $\mathfrak{M}$ on $X \times \mathbb{U}$ and [Hel13, Proposition 6.5] guarantees that $\mathfrak{M}$ is locally in $X$ free over $\mathscr{A}_X^{[0,1)}$. Hence it is the desired étale model. $\square$

**Corollary 6.7.** *Let $\mu$ be a cocharacter as in (2.5) with reflex field $E_\mu$. Then there is an open substack $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu} \subset \mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}$ such that $f: X \to \mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}$ factors over $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu}$ if and only if the family $(\mathcal{M}, \Phi_{\mathcal{M}}, N_\nabla^{\mathcal{M}})$ defined by $f$ and $\underline{\mathcal{M}}$ is étale.*

*Proof.* Let $\underline{\mathcal{M}}(\underline{D})$ be the universal $(\varphi, N_\nabla)$-module over $\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}$. By Theorem 6.4 the set $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu} := \{x \in \mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu} : \underline{\mathcal{M}}(\underline{D}) \text{ is étale at } x\}$ is open and above it $\underline{\mathcal{M}}(\underline{D})$ is étale. If $f$ factors over $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu}$ then $(\mathcal{M}, \Phi_{\mathcal{M}}, N_\nabla^{\mathcal{M}})$ is the pullback of the universal $\underline{\mathcal{M}}(\underline{D})$ and hence is étale. Conversely if $(\mathcal{M}, \Phi_{\mathcal{M}}, N_\nabla^{\mathcal{M}})$ is étale, then it is étale at all points and $f$ factors over $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu}$, because the notion of being étale at points may be checked fpqc-locally by [Hel13, Proposition 6.14]. □

**Proposition 6.8.** *Let $L$ be a finite extension of $E_\mu$, then $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu}(L) = \mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\preceq\mu}(L)$ and hence $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu} \subset \mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\preceq\mu}$.*

*Proof.* We show that being weakly admissible translates into being pure of slope zero over the Robba ring (in the sense of [Ked08]) under the equivalence of categories from Theorem 4.6. However, the proof is the same as in [Kis06, Theorem 1.3.8]. One easily verifies that the functor $\underline{\mathcal{M}}$ preserves the slope and that the slope filtration on the base change of $\underline{\mathcal{M}}(D, \Phi, N, \mathfrak{q})$ to the Robba ring extends to all of $\underline{\mathcal{M}}(D, \Phi, N, \mathfrak{q})$. Compare [Kis06, Proposition 1.3.7].

As in [Hel13, Theorem 7.6 (ii)] the second part is now a consequence of the fact that $\mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\preceq\mu} \subset \mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}$ is the maximal open subspace whose rigid analytic points are exactly the weakly admissible ones, see Lemma 5.10. □

In the miniscule case Pappas and Rapoport [PR09, 5.b] define a *period morphism* from a stack of integral data to a stack of filtered $\varphi$-modules as follows. Let $d > 0$ and let $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to \widetilde{T}_{\overline{\mathbb{Q}}_p}$ be a cocharacter as in (2.5). Pappas and Rapoport [PR09, 3.d] define an fpqc-stack $\hat{\mathcal{C}}_{\mu,K}$ on the category $\mathrm{Nil}_{\mathcal{O}_{E_\mu}}$ of schemes over the ring of integers $\mathcal{O}_{E_\mu}$ of $E_\mu$ on which $p$ is locally nilpotent. If $R$ is an $\mathcal{O}_{E_\mu}$-algebra, we set $R_W = R \otimes_{\mathbb{Z}_p} W$ and denote by $\varphi : R_W(\!(u)\!) \to R_W(\!(u)\!)$ the ring homomorphism that is the identity on $R$, the $p$-Frobenius on $W$ and that maps $u$ to $u^p$. Now the $R$-valued points of the stack $\hat{\mathcal{C}}_{\preceq\mu,K}$ are given by a subset

$$\hat{\mathcal{C}}_{\preceq\mu,K}(R) \subset \{\mathfrak{M},\ \Phi\colon \varphi^*\mathfrak{M}[1/u] \xrightarrow{\sim} \mathfrak{M}[1/u]\}$$

where $\mathfrak{M}$ is an $R_W[\![u]\!] = (R \otimes_{\mathbb{Z}_p} W)[\![u]\!]$-module that is fpqc-locally on $\operatorname{Spec} R$ free as an $R_W[\![u]\!]$-module of rank $d$. This subset is cut out by a condition prescribing the relative position of $\Phi(\varphi^*\mathfrak{M})$ with respect to $\mathfrak{M}$ at the locus $E(u) = 0$ in terms of the cocharacter $\mu$, see [PR09, 3.c,d] for the precise definition.

**Remark 6.9.** In [PR09] the stack $\hat{\mathcal{C}}_{\preceq\mu,K}$ is only defined in the case of a minuscule cocharacter and is named $\hat{\mathcal{C}}_{\mu,K}$ in loc. cit. However, we can use the same definition, using the local model $M^{\mathrm{loc}}_{\mu,K}$, for a general cocharacter. This yields an integral version of the condition "being bounded by $\mu$". In fact this boundedness condition is the defining condition of the generic fiber of the local model $M^{\mathrm{loc}}_{\mu,K}$ and then $M^{\mathrm{loc}}_{\mu,K}$ is defined by taking the flat closure (inside some affine Grassmannian).

If $\mu$ is miniscule, this condition implies that $E(u)\mathfrak{M} \subset \Phi(\varphi^*\mathfrak{M}) \subset \mathfrak{M}$.

Now let $\mu$ be arbitrary and let $R$ be a $p$-adically complete $\mathcal{O}_{E_\mu}$-algebra topologically of finite type over $\mathcal{O}_{E_\mu}$ and let $(\mathfrak{M}, \Phi) \in \hat{\mathcal{C}}_{\mu,K}(\operatorname{Spf} R)$. The construction of section 4 associates to

$$(\mathcal{M}, \Phi_{\mathcal{M}}) = (\mathfrak{M}, \Phi) \otimes_{R_W[\![u]\!]} \mathscr{B}^{[0,1)}_{\mathrm{Spa}(R[1/p],R)} \tag{6.3}$$

a $K$-filtered $\varphi$-module over $\mathrm{Spa}(R[1/p], R)$. Given a formal scheme $\mathcal{X}$ locally topologically of finite type over $\mathcal{O}_{E_\mu}$, this yields a period functor

$$\Pi(\mathcal{X}) : \hat{\mathcal{C}}_{\mu,K}(\mathcal{X}) \longrightarrow \mathscr{D}^{\mathrm{ad}}_{\varphi,\mu}(\mathcal{X}^{\mathrm{rig}}),$$

where $\mathcal{X}^{\mathrm{rig}}$ denotes the generic fiber of the formal scheme $\mathcal{X}$ in the sense of rigid geometry (or in the sense of adic spaces). If $\mu$ is not miniscule, this construction breaks down, as the family of vector bundles on the open unit disc defined by (6.3) is not necessarily associated to a filtered $\varphi$-module: the monodromy operator $N_\nabla^{\mathcal{M}}$ is not necessarily holomorphic.

If we use $\varphi$-modules with Hodge-Pink lattices instead of filtered $\varphi$-modules we can overcome this problem. Using section 4, the same construction as in [PR09] gives a period functor

$$\Pi(\mathcal{X}) : \hat{\mathcal{C}}_{\preceq\mu,K}(\mathcal{X}) \longrightarrow \mathscr{H}^{\mathrm{ad}}_{\varphi,\preceq\mu}(\mathcal{X}^{\mathrm{rig}}). \tag{6.4}$$

When $\mathcal{X} = \operatorname{Spf}\mathcal{O}_L$ for a finite field extension $L$ of $E_\mu$, it was shown by Genestier and Lafforgue [GL12, Théorème 0.6] that $\Pi(\operatorname{Spf}\mathcal{O}_L)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is fully faithful, and surjective onto $\mathscr{H}^{\mathrm{ad,wa}}_{\varphi,\preceq\mu}(L) = \mathscr{H}^{\mathrm{ad,int}}_{\varphi,\preceq\mu}(L)$.

**Remark 6.10.** From the point of view of Galois representations it is not surprising that we can not define a general period morphism using filtered $\varphi$-modules. If $R$ is finite over $\mathcal{O}_{E_\mu}$, then the points of $\hat{\mathcal{C}}_{\mu,K}(R)$ correspond to $\mathscr{G}_{K_\infty}$-representations rather than to $\mathscr{G}_K$-representations. This also explains why the target of the period map is $\mathscr{H}^{\mathrm{ad}}_{\varphi,\mu}$ instead of $\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\mu}$: the $\mathscr{G}_{K_\infty}$-representation does not see the monodromy.

If we want to take the monodromy into account we have to consider a stack $\hat{\mathcal{C}}_{\preceq\mu,N,K}$ whose $\mathcal{X}$-valued points are given by $(\mathfrak{M},\Phi,N)$ with $(\mathfrak{M},\Phi)\in\hat{\mathcal{C}}_{\preceq\mu,K}(\mathcal{X})$ and $N:\mathfrak{M}/u\mathfrak{M}\to\mathfrak{M}/u\mathfrak{M}$ satisfying

$$N\circ\overline{\Phi}(n) = p\cdot\overline{\Phi}(n)\circ N. \tag{6.5}$$

Here $(\mathfrak{M}(n),\Phi(n)) = (\mathfrak{M},\Phi)\otimes_{W[\![u]\!]}\mathbf{A}^{[0,1)}(n)$ is the twist of $(\mathfrak{M},\Phi)$ with the object $\mathbf{A}^{[0,1)}(n)$ defined before Definition 6.3 and $n\gg 0$ is some integer such that $\Phi(n)(\varphi^*\mathfrak{M})\subset\mathfrak{M}$ and $\overline{\Phi}$ denotes the reduction of $\Phi$ modulo $u$. Note that given $\mu$ we may choose an $n$ like that for all $(\mathfrak{M},\Phi)\in\hat{\mathcal{C}}_{\preceq\mu,K}(\mathcal{X})$ and the map $\overline{\Phi}$ (and hence the equation (6.5)) makes sense after this twist. Further the condition defined by (6.5) is independent of the chosen $n$.

**Remark 6.11.** (i) Using (2.7) we observe that if $\mu_{\psi,d}\geq 0$ for all $\psi$, and if $L$ is a finite extension of $E_\mu$, a $\operatorname{Spf}\mathcal{O}_L$-valued point of the stack $\hat{\mathcal{C}}_{\preceq\mu,N,K}$ gives rise to an object of the category $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$ in the sense of Kisin [Kis06, (1.3.12)]. We only use the twist in order to define the stack in the general case (i.e. if $\Phi(\varphi^*\mathfrak{M})$ is not contained in $\mathfrak{M}$). Kisin's definition takes place in the generic fiber. However, we can not use this as a good definition as our stack is defined for $p$-power torsion objects.

(ii) Note that we do not know much about the stack $\hat{\mathcal{C}}_{\preceq\mu,N,K}$ and its definition is rather ad hoc. Especially we doubt that it is flat over $\operatorname{Spf}\mathbb{Z}_p$. This means that there is no reason to expect that we can reconstruct Kisin's semi-stable deformation rings [Kis08] by using a similar construction as in [PR09, § 4].

In this general case described above we obtain a similar period morphism

$$\hat{\mathcal{C}}_{\preceq\mu,N,K}(\mathcal{X}) \longrightarrow \mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}(\mathcal{X}^{\mathrm{rig}}). \tag{6.6}$$

As in [Hel13, Theorem 7.8] the above allows us to determine the image of the period morphism. Recall that a valued field $(L,v_L)$ over $\mathbb{Q}_p$ is called *of $p$-adic type* if it is complete, topologically finitely generated over $\mathbb{Q}_p$ and if for all $f_1,\dots,f_m\in L$ the closure of $\mathbb{Q}_p[f_1\dots,f_m]$ inside $L$ is a Tate algebra, i.e. the quotient of some $\mathbb{Q}_p\langle T_1,\dots,T_{m'}\rangle$.

**Corollary 6.12.** *The substack $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu}$ is the image of the period morphism* (6.6) *in the following sense:*
(i) *If $\mathcal{X}$ is a $p$-adic formal scheme and $(\mathfrak{M},\Phi,N)\in\hat{\mathcal{C}}_{\preceq\mu,N,K}(X)$, then $\Pi(\mathcal{X})(\mathfrak{M},\Phi,N)\in\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu}(\mathcal{X}^{\mathrm{rig}})$.*
(ii) *Let $L$ be a field of $p$-adic type over $E_\mu$ and $(D,\Phi,N,\mathfrak{q})\in\mathscr{H}_{\varphi,N,\preceq\mu}(L)$. Then there exists $(\mathfrak{M},\Phi,N)\in\hat{\mathcal{C}}_{\preceq\mu,N,K}(\operatorname{Spf} L^+)$ such that $\Pi(\operatorname{Spf} L^+)(\mathfrak{M},\Phi,N) = (D,\Phi,N,\mathfrak{q})$ if and only if*

$$\underline{\mathcal{M}}(D) = \mathfrak{M}\otimes_{L^+_W[\![u]\!]}\mathscr{B}^{[0,1)}_L.$$

*is étale, if and only if* $\operatorname{Spa}(L,L^+)\to\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}$ *factors over* $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu}$.
(iii) *Let $X\in\mathrm{Ad}^{\mathrm{lft}}_{E_\mu}$ and let $f:X\to\mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}$ be a morphism defined by $(D,\Phi,N,\mathfrak{q})$. Then $f$ factors over $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu}$ if and only if there exists a fpqc-covering $(U_i\to X)_{i\in I}$ and formal models $\mathcal{U}_i$ of $U_i$ together with $(\mathfrak{M}_i,\Phi_i,N)\in\hat{\mathcal{C}}_{\preceq\mu,N,K}(\mathcal{U}_i)$ such that $\Pi(\mathcal{U}_i)(\mathfrak{M}_i,\Phi_i,N) = (D,\Phi,N,\mathfrak{q})|_{U_i}$.*

**Remark 6.13.** If we consider the period morphism without monodromy, then we obtain a similar characterization of the stack $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,\preceq\mu}\subset\mathscr{H}^{\mathrm{ad}}_{\varphi,\preceq\mu}$ as the image of the period morphism (6.4).

## 7. Sheaves of period rings and the admissible locus

We recall the definition of some sheafified period rings from [Hel13]. Let $R = \varprojlim \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ be the inverse limit with transition maps given by the $p$-th power. Given a reduced $p$-adically complete $\mathbb{Z}_p$-algebra $A^+$ topologically of finite type, we define

$$A^+\widehat{\otimes}W(R) = \varprojlim_i A^+\widehat{\otimes}_{\mathbb{Z}_p}W_i(R),$$

where the completed tensor product on the right hand side means completion with respect to the canonical topology on the truncated Witt vectors $W_i(R)$ and the discrete topology on $A^+/p^iA^+$.

If $X$ is a reduced adic space locally of finite type over $\mathbb{Q}_p$, then there are sheaves $\mathcal{O}_X^+\widehat{\otimes}W(R)$ and $\mathcal{O}_X\widehat{\otimes}W(R)$ whose sections over an affinoid open $U = \mathrm{Spa}(A, A^+) \subset X$ are given by

$$\begin{aligned}\Gamma(U, \mathcal{O}_X^+\widehat{\otimes}W(R)) &= A^+\widehat{\otimes}_{\mathbb{Z}_p}W(R)\\ \Gamma(U, \mathcal{O}_X\widehat{\otimes}W(R)) &= \big(A^+\widehat{\otimes}_{\mathbb{Z}_p}W(R)\big)[\tfrac{1}{p}].\end{aligned}$$

In the same fashion we can define sheaves of topological rings $\mathcal{O}_X^+\widehat{\otimes}W(\mathrm{Frac}\,R)$ and $\mathcal{O}_X\widehat{\otimes}W(\mathrm{Frac}\,R)$.

Let $\mathbf{A}^{[0,1)} = W[\![u]\!]$ and let $\mathbf{A}$ denote the $p$-adic completion of $W(\!(u)\!)$. Further let $\mathbf{B} = \mathbf{A}[1/p]$. There are embeddings of $\mathbf{A}^{[0,1)}$, $\mathbf{A}$ and $\mathbf{B}$ into $W(\mathrm{Frac}\,R)$ sending $u$ to the Teichmüller representative $[\underline{\pi}] \in W(R)$ of the element $\underline{\pi} = (\pi_n)_n \in R$. We write $\widetilde{\mathbf{A}}$ for the ring of integers in the maximal unramified extension $\widetilde{\mathbf{B}}$ of $\mathbf{B}$ inside $W(\mathrm{Frac}\,R)[1/p]$. Finally we set $\widetilde{\mathbf{A}}^{[0,1)} = \widetilde{\mathbf{A}} \cap W(R) \subset W(\mathrm{Frac}\,R)$. All these rings come along with a Frobenius endomorphism $\varphi$ which is induced by the canonical Frobenius on $W(\mathrm{Frac}\,R)$. Note that all these rings have a canonical topology induced from the one on $W(\mathrm{Frac}\,R)$.

We define sheafified versions of these rings as follows, compare [Hel13, 8.1]. Let $X$ be a reduced adic space locally of finite type over $\mathbb{Q}_p$. We define the sheaves $\mathscr{A}_X$, resp. $\widetilde{\mathscr{A}}_X$, resp. $\widetilde{\mathscr{A}}_X^{[0,1)}$ by specifying their sections on open affinoids $U = \mathrm{Spa}(A, A^+) \subset X$: we define $\Gamma(U, \mathscr{A}_X)$, resp. $\Gamma(U, \widetilde{\mathscr{A}}_X)$, resp. $\Gamma(U, \widetilde{\mathscr{A}}_X^{[0,1)})$ to be the closure of $A^+ \otimes_{\mathbb{Z}_p} \mathbf{A}$, resp. $A^+ \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{A}}$, resp. $A^+ \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{A}}^{[0,1)}$ in $\Gamma(U, \mathcal{O}_X^+\widehat{\otimes}W(\mathrm{Frac}\,R))$.

Further we consider the rational analogues $\mathscr{B}_X$ and $\widetilde{\mathscr{B}}_X$ of these sheaves given by inverting $p$ in $\mathscr{A}_X$ and $\widetilde{\mathscr{A}}_X$. We can consider these sheaves also on non-reduces spaces by locally embedding $X$ into a reduced space $Y$ and restricting the corresponding sheaves from $Y$ to $X$.

Finally we recall the construction of the sheaf $\mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}}$ from [Hel13, 8.1]. For a reduced adic space $X$ the map $\theta : W(R) \to \mathcal{O}_{\mathbb{C}_p}$ given by $[(x, x^{1/p}, x^{1/p^2}, \dots)] \mapsto x$ extends to an $\mathcal{O}_X^+$-linear map

$$\theta_X : \mathcal{O}_X^+\widehat{\otimes}W(R) \to \mathcal{O}_X^+\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}.$$

We define $\mathcal{O}_X^+\widehat{\otimes}A_{\mathrm{cris}}$ to be the $p$-adic completion of the divided power envelope of $\mathcal{O}_X^+\widehat{\otimes}W(R)$ with respect to the kernel of $\theta_X$. Finally we set

$$\begin{aligned}\mathcal{O}_X\widehat{\otimes}B^+_{\mathrm{cris}} &= \big(\mathcal{O}_X^+\widehat{\otimes}A_{\mathrm{cris}}\big)[1/p],\\ \mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}} &= \big(\mathcal{O}_X\widehat{\otimes}B^+_{\mathrm{cris}}\big)[1/t],\\ \mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}} &= \big(\mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}}\big)[\ell_u].\end{aligned}$$

Here $t = \log[(1, \varepsilon_1, \varepsilon_2, \dots)] \in B_{\mathrm{cris}}$ is the period of the cyclotomic character (where $(\varepsilon_i)$ is a compatible system of $p^i$-th roots of unity) and $\ell_u$ is an indeterminate thought of as a formal logarithm of $[\underline{\pi}]$.

**Remark 7.1.** The indeterminate $\ell_u$ considered here is the same indeterminate as in section 2.2.(b) and we identify both indeterminates. That is, the inclusion $\mathbf{B}^{[0,1)} \subset B^+_{\mathrm{cris}}$ given by $u \mapsto [\underline{\pi}]$ will be extended to $\mathbf{B}^{[0,1)}[\ell_u] \hookrightarrow B^+_{\mathrm{st}}$ by means of $\ell_u \mapsto \ell_u$ and similar for the sheafified versions.

On $\mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}}$ there is a canonical Frobenius $\varphi$ induced by the Frobenius on $\mathcal{O}_X^+\widehat{\otimes}W(R)$. This endomorphism extends to a morphism

$$\varphi : \mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}} \longrightarrow \mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}},$$

where $\varphi(\ell_u) = p\ell_u$. Further $N = \frac{d}{d\ell_u}$ defines an endomorphism of $\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}}$ which satisfies $N\varphi = p\varphi N$.

Finally the continuous $\mathscr{G}_K$-action on $\mathcal{O}_X^+\widehat{\otimes}W(R)$ extends to $\mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}}$ and we further extend this action to $\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}}$ by means of $\gamma \cdot \ell_u = \ell_u + c(\gamma)t$, where $c : \mathscr{G}_K \to \mathbb{Z}_p$ is defined by $\gamma(\pi_n) = \pi_n \cdot (\varepsilon_n)^{c(\gamma)}$ for all $n \geq 0$.

The following proposition summarizes the basic properties of (some of) these sheaves. It will allow us to define $\mathbb{Z}$-filtrations $\mathrm{Fil}^i(\mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}})$ resp. $\mathrm{Fil}^i(\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}})$ on $\mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}}$ resp. $\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}}$ as follows. Given $i\in\mathbb{Z}$ and a reduced adic space $X$, a section $f\in\Gamma(X,\mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}})$ lies in $\Gamma(X,\mathrm{Fil}^i(\mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}}))$, if $f(x)\in\mathrm{Fil}^i B_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}\kappa(x)$ for all rigid analytic points $x\in X$. Here $\mathrm{Fil}^i B_{\mathrm{cris}}$ is the usual filtration on $B_{\mathrm{cris}}$ induced by restricting the $t$-adic filtration on Fontaine's ring $B_{\mathrm{dR}}$ to $B_{\mathrm{cris}}$. A similar construction also applies to the filtration on $\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}}$. As above we extend our definition of the filtrations to all adic spaces locally of finite type over $\mathbb{Q}_p$ by locally embedding a non-reduced space into a reduced adic space and restricting the filtration along the ideal defining the closed immersion.

**Proposition 7.2.** *Let $X$ be an adic space locally of finite type over $\mathbb{Q}_p$.*
(i) *For a rigid analytic point $x\in X$, we have*

$$\widetilde{\mathscr{B}}_X\otimes\kappa(x)=\widetilde{\mathbf{B}}\otimes_{\mathbb{Q}_p}\kappa(x)$$
$$\big(\mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}}\big)\otimes\kappa(x)=B_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}\kappa(x)$$
$$\big(\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}}\big)\otimes\kappa(x)=B_{\mathrm{st}}\otimes_{\mathbb{Q}_p}\kappa(x),$$

*where the fiber $\mathcal{R}\otimes\kappa(x)$ of a sheaf $\mathcal{R}$ of $\mathcal{O}_X$-algebras at a point $x\in X$ is given by the quotient of the stalk $\mathcal{R}_x$ by the ideal $\mathfrak{m}_{X,x}\mathcal{R}_x$ generated by the maximal ideal $\mathfrak{m}_{X,x}\subset\mathcal{O}_{X,x}$.*
(ii) *Let $f:X\to Y$ be a finite morphism with $Y$ reduced. The natural maps*

$$\Gamma(X,\widetilde{\mathscr{B}}_X)\longrightarrow\prod\widetilde{\mathbf{B}}\otimes_{\mathbb{Q}_p}\Gamma(f^{-1}(y),\mathcal{O}_X)$$
$$\Gamma(X,\mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}})\longrightarrow\prod B_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}\Gamma(f^{-1}(y),\mathcal{O}_X)$$
$$\Gamma(X,\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}})\longrightarrow\prod B_{\mathrm{st}}\otimes_{\mathbb{Q}_p}\Gamma(f^{-1}(y),\mathcal{O}_X)$$

*are injective. Here the product runs over all rigid analytic points $y\in Y$.*
(iii) *Let $g\in\Gamma(X,\widetilde{\mathscr{B}}_X)$ such that*

$$g(y)\in\Gamma(f^{-1}(y),\mathcal{O}_X)\subset\widetilde{\mathbf{B}}\otimes_{\mathbb{Q}_p}\Gamma(f^{-1}(y),\mathcal{O}_X)$$

*for all rigid analytic points $y\in Y$. Then $g\in\mathcal{O}_X\subset\widetilde{\mathscr{B}}_X$. Especially $\widetilde{\mathscr{B}}_X^{\Phi=\mathrm{id}}=\mathcal{O}_X$. The same holds true for $\mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}}$ and $\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}}$. Especially*

$$\mathrm{Fil}^0\big(\mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}}\big)^{\Phi=\mathrm{id}}=\mathcal{O}_X\ \text{ and}$$
$$\mathrm{Fil}^0\big(\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}}\big)^{\Phi=\mathrm{id},N=0}=\mathcal{O}_X.$$

(iv) *Further one has*

$$(\mathcal{O}_X\hat{\otimes}B_{\mathrm{cris}})^{\mathscr{G}_K}=(\mathcal{O}_X\hat{\otimes}B_{\mathrm{st}})^{\mathscr{G}_K}=\mathcal{O}_X\otimes_{\mathbb{Q}_p}K_0$$

*Proof.* The statements concerning $\widetilde{\mathscr{B}}_X$ and $\mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}}$ are contained in [Hel13, Lemma 8.6, Corollary 8.8] resp. [Hel13, Lemma 8.11, Corollary 8.13]. To prove the statements concerning $\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}}$, note that for every open subset $U\subset X$, every element $f\in\Gamma(U,\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}})$ is a polynomial with coefficients in $\Gamma(U,\mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}})$. The statements follow from this. For the last two statements note that we can always choose a finite morphism $X\to Y$ to a reduced space $Y$ locally on $X$. □

**Definition 7.3.** Let $\mathscr{G}$ denote a topological group. A *family of $\mathscr{G}$-representations* on an adic space $X$ consists of a vector bundle $\mathcal{E}$ on $X$ together with an $\mathcal{O}_X$-linear action of the group $\mathscr{G}$ on $\mathcal{E}$ which is continuous for the topologies on the sections $\Gamma(-,\mathcal{E})$. This definition extends to the category of stacks on $\mathrm{Ad}^{\mathrm{lft}}_{\mathbb{Q}_p}$.

**Definition 7.4.** Let $X$ be an adic space locally of finite type over $\mathbb{Q}_p$.
(i) A *$\varphi$-module over $\mathscr{A}_X$* is an $\mathscr{A}_X$-module $M$ which is locally on $X$ free of finite rank over $\mathscr{A}_X$ together with an isomorphism $\Phi:\varphi^*M\xrightarrow{\sim}M$.
(ii) A *$\varphi$-module over $\mathscr{B}_X$* is an $\mathscr{B}_X$-module $M$ which is locally on $X$ free of finite rank over $\mathscr{B}_X$ together with an isomorphism $\Phi:\varphi^*M\xrightarrow{\sim}M$.
(iii) A $\varphi$-module $M$ over $\mathscr{B}_X$ is called *étale* if it is locally on $X$ of the form $N\otimes_{\mathscr{A}_X}\mathscr{B}_X$ for a $\varphi$-module $N$ over $\mathscr{A}_X$.

The following theorem summarizes results of [Hel13] which are needed in the sequel.

**Theorem 7.5.** *Let $X$ be a reduced adic space locally of finite type over $\mathbb{Q}_p$ and let $(\mathscr{N},\Phi)$ be an étale $\varphi$-module of rank $d$ over $\mathscr{B}_X$.*
(i) *The set*

$$X^{\mathrm{adm}} = \{x \in X \mid \dim_{\kappa(x)}((\mathscr{N} \otimes_{\mathscr{B}_X} \widetilde{\mathscr{B}}_X) \otimes \kappa(x))^{\Phi=\mathrm{id}} = d\} \subset X$$

*is an open subspace and*

$$\mathcal{V} = (\mathscr{N} \otimes \widetilde{\mathscr{B}}_X)^{\Phi=\mathrm{id}}$$

*is a family of $\mathscr{G}_{K_\infty}$-representations on $X^{\mathrm{adm}}$.*
(ii) *If $f: Y \to X$ is a morphism in $\mathrm{Ad}^{\mathrm{lft}}$ and if $(\mathscr{N}_Y,\Phi_Y)$ denotes the pullback of $(\mathscr{N},\Phi)$ along $f$, then $Y^{\mathrm{adm}} = f^{-1}(X^{\mathrm{adm}})$ and*

$$(\mathscr{N}_Y \otimes \widetilde{\mathscr{B}}_Y)^{\Phi=\mathrm{id}} = (f|_{Y^{\mathrm{adm}}})^*\mathcal{V}$$

*as families of $\mathscr{G}_{K_\infty}$-representations on $Y^{\mathrm{adm}}$.*
(iii) *If $(\mathfrak{M},\Phi)$ is a $\varphi$-module of finite height over $\mathscr{A}_X^{[0,1)}$ as in Definition 6.2 and $(\mathscr{N},\Phi) = (\mathfrak{M},\Phi) \otimes_{\mathscr{A}_X^{[0,1)}} \mathscr{B}_X$, then*

$$U = X^{\mathrm{adm}} = \{x \in X \mid \mathrm{rk}_{\kappa(x)^+} \mathrm{Hom}_{\mathscr{A}_X^{[0,1)}\otimes\kappa(x),\Phi}(\mathfrak{M}\otimes\kappa(x), \widetilde{\mathscr{A}}_X^{[0,1)} \otimes \kappa(x)) = d\}$$

*and*

$$\mathscr{H}om_{\mathscr{A}_U^{[0,1)},\Phi}(\mathfrak{M}|_U, \widetilde{\mathscr{A}}_U^{[0,1)}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathscr{H}om_{\mathscr{B}_U,\Phi}(\mathfrak{M}|_U \otimes_{\mathscr{A}_U^{[0,1)}} \mathscr{B}_U, \widetilde{\mathscr{B}}_U)$$

*as families of $\mathscr{G}_{K_\infty}$-representations on $U = X^{\mathrm{adm}}$.*

*Proof.* This is a summary of [Hel13, Proposition 8.20, Corollary 8.21, Proposition 8.22 and Proposition 8.23]. □

Given a cocharacter $\mu$ as in (2.5), the stack $\mathscr{H}_{\varphi,N,\preceq\mu}$ is the stack quotient of $P_{K_0,d} \times_{\mathrm{Spec}\,\mathbb{Q}_p} Q_{K,d,\mu}$ by the action of the reductive group $(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}$. Let us denote by $\mathscr{H}^{\mathrm{red}}_{\varphi,N,\preceq\mu}$ the quotient of the reduced subscheme underlying $P_{K_0,d} \times_{\mathrm{Spec}\,\mathbb{Q}_p} Q_{K,d,\preceq\mu}$ by the induced action of $(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}$. Recall that $P_{K_0,d} \times_{\mathrm{Spec}\,\mathbb{Q}_p} Q_{K,d,\mu}$ is reduced, hence this modification will not be necessary if we restrict to the case where the Hodge type is fixed by $\mu$.

**Corollary 7.6.** *There is an open substack $\mathscr{H}^{\mathrm{red,ad,adm}}_{\varphi,N,\preceq\mu} \subset \mathscr{H}^{\mathrm{red,ad,int}}_{\varphi,N,\preceq\mu}$ and a family $\mathcal{E}$ of $\mathscr{G}_{K_\infty}$-representations on $\mathscr{H}^{\mathrm{red,ad,adm}}_{\varphi,N,\preceq\mu}$ such that*

$$\mathcal{E} = (\underline{\mathcal{M}}(D,\Phi,N,\mathfrak{q}) \otimes_{\mathscr{B}_X^{[0,1)}} \widetilde{\mathscr{B}}_X)^{\Phi=\mathrm{id}},$$

*where $(D,\Phi,N,\mathfrak{q})$ denotes the restriction of the universal family on $\mathscr{H}^{\mathrm{red,ad}}_{\varphi,N,\preceq\mu}$.*
*This subspace is maximal in the following sense: If $X$ is a reduced adic space and if $\underline{D}'$ is a $(\varphi,N)$-module with Hodge-Pink lattice over $X$ with Hodge polygon bounded by $\mu$, then the induced map $f: X \to \mathscr{H}^{\mathrm{ad}}_{\varphi,N,\preceq\mu}$ factors over $\mathscr{H}^{\mathrm{red,ad,adm}}_{\varphi,N,\preceq\mu}$ if and only if $X = X^{\mathrm{adm}}$ with respect to the family*

$$\underline{\mathcal{M}}(\underline{D}') \otimes_{\mathscr{B}_X^{[0,1)}} \widetilde{\mathscr{B}}.$$

*In this case there is a canonical isomorphism of $\mathscr{G}_{K_\infty}$-representations*

$$f^*\mathcal{E} = (\underline{\mathcal{M}}(\underline{D}') \otimes_{\mathscr{B}_X^{[0,1)}} \widetilde{\mathscr{B}})^{\Phi=\mathrm{id}}.$$

*If $L$ is a finite extension of $E_\mu$, then $\mathscr{H}^{\mathrm{red,ad,adm}}_{\varphi,N,\preceq\mu}(L) = \mathscr{H}^{\mathrm{red,ad,int}}_{\varphi,N,\preceq\mu}(L)$*

*Proof.* Let us write $X_{\preceq\mu} = (P_{K_0,d} \times_{\mathrm{Spec}\,\mathbb{Q}_p} Q_{K,d,\preceq\mu})^{\mathrm{red,ad}}$ for the moment. Further we denote the pullback of the universal family of vector bundles on the open unit disc to $X_{\preceq\mu}$ by $(\mathcal{M},\Phi,N_\nabla^{\mathcal{M}}) = \underline{\mathcal{M}}(D,\Phi,N,\mathfrak{q})$. Locally on $X^{\mathrm{int}}_{\preceq\mu}$ there exists a $\varphi$-module of finite height $\mathfrak{M}$ inside $(\mathcal{M},\Phi)$, at least after a Tate twist. It follows that $\mathfrak{M} \otimes_{\mathscr{A}_X^{[0,1)}} \mathscr{A}$ is étale and we may apply the above theorem. Then $X^{\mathrm{adm}}_{\preceq\mu} \subset X_{\preceq\mu}$ is invariant under the action of $(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}$ and hence its quotient by this group is an open substack $\mathscr{H}^{\mathrm{red,ad,adm}}_{\varphi,N,\preceq\mu}\subset\mathscr{H}^{\mathrm{red,ad}}_{\varphi,N,\preceq\mu}$. Further

$$(\mathcal{M}\otimes_{\mathscr{B}^{[0,1)}_{X^{\mathrm{adm}}_\mu}}\widetilde{\mathscr{B}}_{X^{\mathrm{adm}}_\mu})^{\Phi=\mathrm{id}}$$

is a $(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}$-equivariant vector bundle with $\mathscr{G}_{K_\infty}$-action on $X^{\mathrm{adm}}_{\preceq\mu}$. Hence it defines a family of $\mathscr{G}_{K_\infty}$-representations on $\mathscr{H}^{\mathrm{red,ad,adm}}_{\varphi,N,\preceq\mu}$.

The second statement is local on $X$ and hence, after locally choosing a basis of $D$, we can locally lift the morphism $f:X\to\mathscr{H}^{\mathrm{red,ad}}_{\varphi,N,\preceq\mu}$ to a morphism $f':X\to X_{\preceq\mu}$ such that the pullback of $(D,\Phi,N,\mathfrak{q})$ on $X_{\preceq\mu}$ along $f'$ is isomorphic to $\underline{D}'$. Now the claim follows from Theorem 7.5 (ii). ☐

## 8. The universal semi-stable representation

In this section we want to construct a semi-stable $\mathscr{G}_K$-representation out of the $\mathscr{G}_{K_\infty}$-representation on $\mathscr{H}^{\mathrm{red,ad,adm}}_{\varphi,N,\preceq\mu}$ from Corollary 7.6. This will be possible only on a part of $\mathscr{H}^{\mathrm{red,ad,adm}}_{\varphi,N,\preceq\mu}$. First of all we need to restrict to the open subspace where the Hodge polygon is constant. This can be seen as follows. Let $\mathcal{E}$ be a family of $\mathscr{G}_K$-representations on an adic space $X$. It follows from [BC08, § 4.1] that the (generalized) Hodge-Tate weights vary continuously on $X$. Namely, they are the eigenvalues of Sen's operator $\Theta_{\mathrm{Sen}}$ constructed in [BC08, before Remark 4.1.3]. The characteristic polynomial of $\Theta_{\mathrm{Sen}}$ has coefficients in $\mathcal{O}_X\otimes_{\mathbb{Q}_p}K$. However, as $\mathcal{E}$ is semi-stable the Hodge-Tate weights of $\mathcal{E}\otimes\kappa(x)$ are integers for all $x\in X$ by [BC08, Corollaire 5.1.2] and hence the Hodge-Tate weights and the Hodge polygon are locally constant on $X$.

Secondly, Kisin [Kis06, Theorem 0.1 and Corollary 1.3.15] showed that the universal étale $(\varphi,N_\nabla)$-module $\underline{\mathcal{M}}$ on $\mathscr{H}^{\mathrm{ad,int}}_{\varphi,N,\preceq\mu}$ from Corollary 6.7 can come from a semi-stable $\mathscr{G}_K$-representation only if the connection $\nabla$ has logarithmic singularities, which is equivalent to $N^{\mathcal{M}}_\nabla$ being holomorphic; see Remark 4.2(2). Therefore we have to restrict further to the closed subspace $\mathscr{H}^{\nabla}_{\varphi,N,\mu}\cap\mathscr{H}^{\mathrm{ad,adm}}_{\varphi,N,\mu}$ of $\mathscr{H}^{\mathrm{ad,adm}}_{\varphi,N,\mu}$ which is isomorphic to $\mathscr{D}^{\mathrm{ad,adm}}_{\varphi,N,\mu}$. Here $\mathscr{D}^{\mathrm{ad,adm}}_{\varphi,N,\mu}\subset\mathscr{D}^{\mathrm{ad}}_{\varphi,N,\mu}$ is the admissible locus with respect to the family defined in Remark 4.10.

**Definition 8.1.** Let $\mathcal{E}$ be a family of $\mathscr{G}_K$-representations of rank $d$ on an adic space $X$ locally of finite type over $\mathbb{Q}_p$.
(i) The family $\mathcal{E}$ is called *crystalline* if

$$D_{\mathrm{cris}}(\mathcal{E}):=\big(\mathcal{E}\otimes_{\mathcal{O}_X}(\mathcal{O}_X\widehat{\otimes}B_{\mathrm{cris}})\big)^{\mathscr{G}_K}$$

is locally on $X$ free of rank $d$ as an $\mathcal{O}_X\otimes_{\mathbb{Q}_p}K_0$-module and if the descending $\mathbb{Z}$-filtration $\mathrm{Fil}^\bullet D_{\mathrm{cris}}(\mathcal{E})_K$ of $D_{\mathrm{cris}}(\mathcal{E})_K$ has locally free subquotients.
(ii) The family $\mathcal{E}$ is called *semi-stable* if

$$D_{\mathrm{st}}(\mathcal{E}):=\big(\mathcal{E}\otimes_{\mathcal{O}_X}(\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}})\big)^{\mathscr{G}_K}$$

is locally on $X$ free of rank $d$ as an $\mathcal{O}_X\otimes_{\mathbb{Q}_p}K_0$-module and if the descending $\mathbb{Z}$-filtration $\mathrm{Fil}^\bullet D_{\mathrm{st}}(\mathcal{E})_K$ of $D_{\mathrm{st}}(\mathcal{E})_K$ has locally free subquotients.
(iii) Let $\mu$ be a cocharacter as in (2.5), let $E_\mu$ be its reflex field, and let $X$ be an adic space locally of finite type over $E_\mu$. We say that a crystalline, resp. semi-stable $\mathscr{G}_K$-representation $\mathcal{E}$ over $X$ has *constant Hodge polygon equal to* $\mu$ if the $K$-filtered $\varphi$-module $D_{\mathrm{cris}}(\mathcal{E})$, resp. $D_{\mathrm{st}}(\mathcal{E})$ over $X$ has this property.

It is obvious from the definition that $D_{\mathrm{st}}$ defines a functor from the category of semi-stable representations with constant Hodge polygon $\mu$ over an adic space $X$ to the category of $K$-filtered $(\varphi,N)$-modules over $X$ with constant Hodge polygon $\mu$ and similarly for crystalline representations.

**Remark 8.2.** As in [Hel13, Lemma 8.15] one easily sees that the property of being crystalline oder semi-stable is local for the fpqc-topology.

**Lemma 8.3.** (i) *Let $\mathcal{E}$ be a family of crystalline representations on $X$. Then the canonical map*

$$f:D_{\mathrm{cris}}(\mathcal{E})\otimes_{\mathcal{O}_X\otimes K_0}(\mathcal{O}_X\hat{\otimes}B_{\mathrm{cris}})\longrightarrow\mathcal{E}\otimes_{\mathcal{O}_X}(\mathcal{O}_X\hat{\otimes}B_{\mathrm{cris}})$$

*is an isomorphism. Further one has*

$$D_{\mathrm{cris}}(\mathcal{E}) \otimes \kappa(x) \cong D_{\mathrm{cris}}(\mathcal{E} \otimes \kappa(x))$$
$$(\mathrm{Fil}^i D_{\mathrm{cris}}(\mathcal{E})) \otimes \kappa(x) \cong \mathrm{Fil}^i D_{\mathrm{cris}}(\mathcal{E} \otimes \kappa(x))$$

*for all rigid analytic points $x \in X$. Especially the representations $\mathcal{E} \otimes \kappa(x)$ are crystalline for all rigid points $x \in X$.*
(ii) *Let $\mathcal{E}$ be a family of semi-stable representations on $X$. Then the canonical map*

$$f : D_{\mathrm{st}}(\mathcal{E}) \otimes_{\mathcal{O}_X \otimes K_0} (\mathcal{O}_X \hat{\otimes} B_{\mathrm{st}}) \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} (\mathcal{O}_X \hat{\otimes} B_{\mathrm{st}}) \tag{8.1}$$

*is an isomorphism. Further one has*

$$D_{\mathrm{st}}(\mathcal{E}) \otimes \kappa(x) \cong D_{\mathrm{st}}(\mathcal{E} \otimes \kappa(x))$$
$$(\mathrm{Fil}^i D_{\mathrm{st}}(\mathcal{E})) \otimes \kappa(x) \cong \mathrm{Fil}^i D_{\mathrm{st}}(\mathcal{E} \otimes \kappa(x))$$

*for all rigid analytic points $x \in X$. Especially the representations $\mathcal{E} \otimes \kappa(x)$ are semi-stable for all rigid points $x \in X$.*

*Proof.* We only prove the second part of the lemma. The claim $D_{\mathrm{st}}(\mathcal{E}) \otimes \kappa(x) \cong D_{\mathrm{st}}(\mathcal{E} \otimes \kappa(x))$ follows from [BC08, Théorème 6.3.2] and hence the representations $\mathcal{E} \otimes \kappa(x)$ are semi-stable. Further the map (8.1) is a morphism between free $\mathcal{O}_X \hat{\otimes} B_{\mathrm{st}}$-modules of the same rank and $D_{\mathrm{st}}(\mathcal{E}) \otimes \kappa(x) \cong D_{\mathrm{st}}(\mathcal{E} \otimes \kappa(x))$ implies that $f \otimes \kappa(x)$ is an isomorphism for all rigid analytic points $x \in X$. Now Proposition 7.2 implies that (8.1) is an isomorphism. It remains to prove $(\mathrm{Fil}^i D_{\mathrm{st}}(\mathcal{E})) \otimes \kappa(x) \cong \mathrm{Fil}^i D_{\mathrm{st}}(\mathcal{E} \otimes \kappa(x))$. This may be checked after base change to some finite extension $L$ of $K$ containing the Galois closure $\widetilde{K}$ of $K$, i.e. we change $X$ to $X_L$. Given an embedding $\psi : K \to L$ we write $D_\psi = D \otimes_{\mathcal{O}_X \otimes K_0, \mathrm{id} \otimes \psi} \mathcal{O}_X$ and we write $\mathrm{Fil}^\bullet D_\psi$ for the corresponding $\mathbb{Z}$-filtration of $D_\psi$. Then $\mathrm{Fil}^i D_\psi$ is locally on $X$ a direct summand of $\mathcal{O}_X$ and it suffices to prove that $\mathrm{Fil}^\bullet(D_\psi \otimes \kappa(x)) = (\mathrm{Fil}^\bullet D_\psi) \otimes \kappa(x)$.

Let us write $i$ for the integer such that $\mathrm{Fil}^i(D_\psi \otimes \kappa(x)) = D_\psi \otimes \kappa(x)$ for all $x \in X$ and $\mathrm{Fil}^{i+1} D_\psi \otimes \kappa(x) \neq D$ for at least one $x \in X$. By Definition 8.1 it follows that $\dim_{\kappa(x)} \mathrm{Fil}^i(D_\psi \otimes \kappa(x))/\mathrm{Fil}^{i+1}(D_\psi \otimes \kappa(x))$ is independent of $x \in X$. As the filtration on $\mathcal{O}_X \hat{\otimes} B_{\mathrm{st}}$ is defined using the filtrations on $(\mathcal{O}_X \hat{\otimes} B_{\mathrm{st}}) \otimes \kappa(x) = B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} \kappa(x)$ for all rigid analytic points $x \in X$ we find that $f \in D_\psi \subset \mathcal{E} \otimes_{\mathcal{O}_X} (\mathcal{O}_X \hat{\otimes} B_{\mathrm{st}})$ lies in $\mathrm{Fil}^j(\mathcal{E} \otimes_{\mathcal{O}_X} (\mathcal{O}_X \hat{\otimes} B_{\mathrm{st}}))$ if and only if

$$f(x) \in \mathrm{Fil}^j \left((\mathcal{E} \otimes \kappa(x)) \otimes_{\kappa(x)} (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} \kappa(x))\right)$$

for all rigid analytic points $x$. It follows that $D = \cdots = \mathrm{Fil}^{i-1} D_\psi = \mathrm{Fil}^i D_\psi$. Further it follows that

$$(\mathrm{Fil}^i D_\psi / \mathrm{Fil}^{i+1} D_\psi) \otimes \kappa(x) \longrightarrow \mathrm{Fil}^i(D_\psi \otimes \kappa(x))/\mathrm{Fil}^{i+1}(D_\psi \otimes \kappa(x))$$

is surjective for all $x \in X$. However, the construction implies that this map is generically on $X$ an isomorphism and hence it is an isomorphism everywhere as both sides have the same dimension for all $x \in X$. It follows that $(\mathrm{Fil}^{i+1} D_\psi) \otimes \kappa(x) = \mathrm{Fil}^{i+1}(D_\psi \otimes \kappa(x))$ and the claim follows by descending induction. □

**Lemma 8.4.** *Let $\mathcal{E}$ be a family of $\mathcal{G}_K$-representations on an adic space $X$ locally of finite type over $\mathbb{Q}_p$. Then $\mathcal{E}$ is crystalline if and only it is semi-stable and the monodromy $N$ on $D_{\mathrm{st}}(\mathcal{E})$ vanishes. In this case we have $D_{\mathrm{st}}(\mathcal{E}) = D_{\mathrm{cris}}(\mathcal{E})$ as subobjects of $\mathcal{E} \otimes_{\mathcal{O}_X} (\mathcal{O}_X \hat{\otimes} B_{\mathrm{st}})$.*

*Proof.* Let us assume that $\mathcal{E}$ is crystalline. Then by the preceding lemma we obtain an isomorphism

$$D_{\mathrm{cris}}(\mathcal{E}) \otimes_{\mathcal{O}_X \otimes K_0} (\mathcal{O}_X \hat{\otimes} B_{\mathrm{cris}}) \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} (\mathcal{O}_X \hat{\otimes} B_{\mathrm{cris}})$$

compatible with the action of $\varphi$ and $\mathcal{G}_K$ (and compatible with filtrations). Tensorizing this isomorphism over $B_{\mathrm{cris}}$ with $B_{\mathrm{st}}$ and taking $N$ to be the induced monodromy action on both sides we obtain an isomorphism

$$D_{\mathrm{cris}}(\mathcal{E}) \otimes_{\mathcal{O}_X \otimes K_0} (\mathcal{O}_X \hat{\otimes} B_{\mathrm{st}}) \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} (\mathcal{O}_X \hat{\otimes} B_{\mathrm{st}})$$

compatible with the actions of $\varphi, N$ and $\mathcal{G}_K$ (and compatible with the filtrations). Taking $\mathcal{G}_K$-invariants on both sides we find that $D_{\mathrm{cris}}(\mathcal{E}) \cong D_{\mathrm{st}}(\mathcal{E})$ compatible with the actions of $\varphi$ and $N$ (and compatible with the filtrations).

Conversely assume that $\mathcal{E}$ is semi-stable with vanishing monodromy. As the isomorphism (8.1) is equivariant for the action of $N$ the claim follows after taking $N=0$ on both sides. $\square$

Similar to the functors $D_{\rm cris}(-)$ and $D_{\rm st}(-)$ we also consider their contravariant variants

$$\begin{aligned} D^*_{\rm cris} &: \mathcal{E} \longmapsto \mathscr{H}om_{\mathcal{O}_X[\mathscr{G}_K]}(\mathcal{E}, \mathcal{O}_X\widehat{\otimes}B_{\rm cris}) \text{ and} \\ D^*_{\rm st} &: \mathcal{E} \longmapsto \mathscr{H}om_{\mathcal{O}_X[\mathscr{G}_K]}(\mathcal{E}, \mathcal{O}_X\widehat{\otimes}B_{\rm st}). \end{aligned} \tag{8.2}$$

An easy calculation shows that $D_{\rm cris}(\mathcal{E}^\vee)\cong D^*_{\rm cris}(\mathcal{E})$ and similar for semi-stable representations.

**Lemma 8.5.** *Let $X$ be an adic space locally of finite type over $\mathbb{Q}_p$ and let $\mathcal{E}$, $\mathcal{E}_1$ and $\mathcal{E}_2$ be families of semi-stable representations.*
(i) *The canonical morphism*

$$\mathcal{E}\longrightarrow V_{\rm st}(D_{\rm st}(\mathcal{E})) = \mathrm{Fil}^0(D_{\rm st}(\mathcal{E})\otimes_{\mathcal{O}_X\otimes_{\mathbb{Q}_p}K_0}(\mathcal{O}_X\widehat{\otimes}B_{\rm st}))^{\varphi=\mathrm{id},N=0}$$

*is an isomorphism.*
(ii) *One has $\mathcal{E}_1\cong\mathcal{E}_2$ if and only if $D_{\rm st}(\mathcal{E}_1)\cong D_{\rm st}(\mathcal{E}_2)$.*

*Proof.* The first part follows from $\mathrm{Fil}^0(\mathcal{O}_X\widehat{\otimes}B_{\rm st})^{\varphi=\mathrm{id},N=0}=\mathcal{O}_X$ and the lemma above. The second part is an easy consequence. $\square$

**Proposition 8.6.** *Let $X$ be a reduced adic space locally of finite type over $E_\mu$ and let $\underline{D}=(D,\Phi,N,\mathcal{F}^\bullet)\in\mathscr{D}^{\rm an,adm}_{\varphi,N,\mu}(X)$. Then*

$$\mathscr{H}om_{\mathscr{A}_X^{[0,1)},\Phi}(\mathfrak{M},\widetilde{\mathscr{A}}_X^{[0,1)})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\to\mathscr{H}om_{\Phi,N,\mathrm{Fil}}(\underline{D},\mathcal{O}_X\widehat{\otimes}B_{\rm st}) =: \mathcal{E}$$

*is an isomorphism for every $\varphi$-module $(\mathfrak{M},\Phi)$ of finite height over $\mathscr{A}_X^{[0,1)}$ such that $\mathfrak{M}\otimes_{\mathscr{A}_X^{[0,1)}}\mathscr{B}_X^{[0,1)}=\underline{\mathcal{M}}(\underline{D})$. Further $\mathcal{E}$ is a vector bundle of rank $d$ on $X$ and the $\mathscr{G}_K$-action on $B_{\rm st}$ makes $\mathcal{E}$ into a family of semi-stable $\mathscr{G}_K$-representations such that there is a canonical isomorphism of filtered $(\varphi,N)$-modules*

$$D^*_{\rm st}(\mathcal{E})\cong(D,\Phi,N,\mathcal{F}^\bullet).$$

*Proof.* As in the proof of [Hel13, Proposition 8.24], the proof follows the one of [Kis06, Proposition 2.1.5]. If we write $\mathcal{M}=\mathfrak{M}\otimes_{\mathscr{A}_X^{[0,1)}}\mathscr{B}_X^{[0,1)}$, then the morphism

$$\mathscr{H}om_{\mathscr{A}_X^{[0,1)},\Phi}(\mathfrak{M},\widetilde{\mathscr{A}}_X^{[0,1)})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\longrightarrow\mathscr{H}om_{\Phi,N,\mathrm{Fil}}(\underline{D},\mathcal{O}_X\widehat{\otimes}B_{\rm st})$$

is given by the composition of injections

$$\begin{aligned}\mathscr{H}om_{\mathscr{A}_X^{[0,1)},\Phi}(\mathfrak{M},\widetilde{\mathscr{A}}_X^{[0,1)})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p &\hookrightarrow \mathscr{H}om_{\mathscr{A}_X^{[0,1)},\Phi}(\mathfrak{M},\mathcal{O}_X\widehat{\otimes}B_{\rm st})\\ &\hookrightarrow \mathscr{H}om_{\mathscr{B}_X^{[0,1)},\Phi}(\mathcal{M},\mathcal{O}_X\widehat{\otimes}B_{\rm st})\\ &\hookrightarrow \mathscr{H}om_{\mathscr{B}_X^{[0,1)},\Phi}(\underline{D}\otimes_{K_0}\mathscr{B}_X^{[0,1)},\mathcal{O}_X\widehat{\otimes}B_{\rm st})\\ &= \mathscr{H}om_\Phi(\underline{D},\mathcal{O}_X\widehat{\otimes}B_{\rm st}),\end{aligned}$$

compare [Kis06, Proposition 2.1.5] and [Hel13, Proposition 8.24] for the definition of these maps. The image of the above injections is contained in

$$\mathscr{H}om_{\Phi,N,\mathrm{Fil}}(\underline{D},\mathcal{O}_X\widehat{\otimes}B_{\rm st})=\mathcal{E},$$

because this is true at all rigid analytic points by [Kis06, Proposition 2.1.5] and $X$ is reduced. Now

$$\mathcal{E}'=\mathscr{H}om_{\mathscr{A}_X^{[0,1)},\Phi}(\mathfrak{M},\widetilde{\mathscr{A}}_X^{[0,1)})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$$

is by Theorem 7.5 a vector bundle of rank $d$ on $X$. We claim that the inclusion

$$\mathcal{E}'\otimes_{\mathcal{O}_X}(\mathcal{O}_X\widehat{\otimes}B_{\rm st})\subset\mathscr{H}om_{\mathcal{O}_X}(\underline{D},\mathcal{O}_X\widehat{\otimes}B_{\rm st}) \tag{8.3}$$

is an equality at all rigid analytic points $x \in X$. First one has

$$\mathcal{E}' \otimes \kappa(x) \cong \mathrm{Hom}_{\mathscr{A}_X^{[0,1)}\otimes\kappa(x),\Phi}(\mathfrak{M}\otimes\kappa(x), \widetilde{\mathscr{A}}_X^{[0,1)}\otimes\kappa(x))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$$

$$\mathscr{H}om_{\mathcal{O}_X}(\underline{D}, \mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}})\otimes\kappa(x) \cong \mathrm{Hom}_{\kappa(x)}(\underline{D}\otimes\kappa(x), B_{\mathrm{st}}\otimes_{\mathbb{Q}_p}\kappa(x)) = \mathrm{Hom}_{\mathbb{Q}_p}(\underline{D}\otimes\kappa(x), B_{\mathrm{st}}).$$

Finally the inclusion (8.3) is defined by canonical maps and hence the induced map

$$\mathrm{Hom}_{\mathscr{A}_X^{[0,1)}\otimes\kappa(x),\Phi}(\mathfrak{M}\otimes\kappa(x), \widetilde{\mathscr{A}}_X^{[0,1)}\otimes\kappa(x))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \longrightarrow \mathrm{Hom}_{\mathbb{Q}_p}(\underline{D}\otimes\kappa(x), B_{\mathrm{st}})$$

again is the canonical map and hence an isomorphism by [Kis06, Proposition 2.1.5]. After choosing bases, the inclusion (8.3) is given by a $d\times d$-matrix with entries in $\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}}$. This matrix is invertible at all rigid analytic points in $X$ and hence it is an isomorphism by Proposition 7.2. It follows that $\mathcal{E}'=\mathcal{E}$, as both sides are given by $\mathrm{Fil}^0(-)^{\Phi=\mathrm{id},N=0}$ on

$$\mathcal{E}'\otimes_{\mathcal{O}_X}(\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}}) = \mathscr{H}om_{\mathcal{O}_X}(\underline{D}, \mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}}),$$

where $\Phi$ and $N$ act on the left hand side by acting on $B_{\mathrm{st}}$ and the filtration on the left hand side is induced from the filtration on $B_{\mathrm{st}}$, i.e. $\mathcal{E}'$ is equipped with the trivial filtration and with the monodromy $N=0$. Here we use that $\mathrm{Fil}^0(\mathcal{O}_X\hat{\otimes}B_{\mathrm{st}})^{\Phi=\mathrm{id},N=0}=\mathcal{O}_X$ by Proposition 7.2. Finally we have $D^*_{\mathrm{st}}(\mathcal{E})\cong\underline{D}$ as the canonical pairing

$$(\mathcal{E}\otimes_{\mathcal{O}_X}(\mathcal{O}_X\hat{\otimes}B_{\mathrm{st}}))\otimes_{\mathcal{O}_X\otimes B_{\mathrm{st}}}(D\otimes_{\mathcal{O}_X\otimes K_0}(\mathcal{O}_X\hat{\otimes}B_{\mathrm{st}}))\longrightarrow\mathcal{O}_X\hat{\otimes}B_{\mathrm{st}}$$

given by $f\otimes d\mapsto f(d)$ is a $\mathscr{G}_K$-equivariant paring of free $\mathcal{O}_X\hat{\otimes}B_{\mathrm{st}}$-modules of the same rank which induces a perfect pairing at all rigid points. Hence, using Proposition 7.2 again, we see that the pairing itself is perfect and the claim follows. □

Recall that the stack $\mathscr{D}^{\mathrm{ad}}_{\varphi,N,\mu}$ is the quotient of the adic space $X_\mu$ associated to $P_{K_0,d}\times\mathrm{Flag}_{K,d,\mu}$ by the action of the group $(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}$ and consider the open subspace $X_\mu^{\mathrm{adm}}\subset X_\mu^{\mathrm{int}}\subset X_\mu$. This subset is stable under the action of $(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}$ and we write $\mathscr{D}^{\mathrm{ad,adm}}_{\varphi,N,\mu}$ for the quotient of $X_\mu^{\mathrm{adm}}$ by this action.

**Proposition 8.7.** *Let $\mu$ be a cocharacter as in (2.5) and let $E_\mu$ be its reflex field. Let $X$ be a reduced adic space locally of finite type over $E_\mu$ and let $\mathcal{E}$ be a family of semi-stable $\mathscr{G}_K$-representations on $X$ with constant Hodge polygon equal to $\mu$. Then the morphism*

$$X\longrightarrow\mathscr{D}^{\mathrm{ad}}_{\varphi,N,\mu}$$

*induced by the $K$-filtered $(\varphi,N)$-module $D_{\mathrm{st}}(\mathcal{E})$ factors over $\mathscr{D}^{\mathrm{ad,adm}}_{\varphi,N,\mu}$.*

*Proof.* Write $D_{\mathrm{st}}(\mathcal{E})=(D,\Phi,N,\mathcal{F}^\bullet)=\underline{D}$ for the associated filtered $(\varphi,N)$-module on $X$. After twisting with a high enough power of the cyclotomic character, we may assume that all Hodge weights $-\mu_{\psi,j}$ lie between $-h$ and $0$ for some $h\gg0$. Hence we may consider the inclusions

$$\begin{aligned}\mathscr{B}_X^{[0,1)}(\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}})^{\mathscr{G}_K}&\subset\mathcal{M}=\underline{\mathcal{M}}(\underline{D})\\&\subset\lambda^{-h}\mathscr{B}_X^{[0,1)}(\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}})^{\mathscr{G}_K}\subset\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}},\end{aligned}$$

where $\lambda$ is the element defined in (4.1). Here $\mathscr{B}_X^{[0,1)}(\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}})^{\mathscr{G}_K}$ is the sub-$\mathscr{B}_X^{[0,1)}$-module of $\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}}$ generated by $(\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}})^{\mathscr{G}_K}$ and so on.

Now the $\mathscr{G}_{K_\infty}$-action on $\mathcal{M}\subset\mathcal{E}\widehat{\otimes}_{\mathbb{Q}_p}B_{\mathrm{st}}$ is trivial, as $\mathscr{G}_{K_\infty}$ acts trivially on $\mathscr{B}_X^{[0,1)}$, hence especially on $\lambda$, and on $(\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}})^{\mathscr{G}_K}$. This claim now implies that

$$\mathcal{M}\otimes_{\mathscr{B}_X^{[0,1)}}\mathscr{B}_X^{[r,s]}\subset(\mathcal{E}\otimes_{\mathcal{O}_X}\widetilde{\mathscr{B}}_X^{[r,s]})^{\mathscr{G}_{K_\infty}}$$

for some $0<r<r^{1/p^2}\le s<1$ near the boundary and where

$$\widetilde{\mathscr{B}}_X^{[r,s]}=\widetilde{\mathscr{A}}_X^{[0,1)}\widehat{\otimes}_{\mathscr{A}_X^{[0,1)}}\mathscr{B}_X^{[r,s]}.$$

Now

$$\mathcal{M}\otimes_{\mathscr{B}_X^{[0,1)}}\widetilde{\mathscr{B}}_X^{[r,s]}=\mathcal{E}\otimes_{\mathcal{O}_X}\widetilde{\mathscr{B}}_X^{[r,s]},$$

as the cokernel vanishes at all rigid analytic points by an analogous argument as in the proof of Proposition 8.6 and as $X$ is reduced. Now the proof runs exactly as the proof of [Hel13] Proposition 8.27 and Proposition 8.28. □

**Theorem 8.8.** *There is a family $\mathcal{E}$ of semi-stable $\mathscr{G}_K$-representations on $\mathscr{D}^{\mathrm{ad,adm}}_{\varphi,N,\mu}$ such that $D_{\mathrm{st}}(\mathcal{E}) = (D,\Phi,N,\mathcal{F}^\bullet)$ is the universal family of filtered $(\varphi,N)$-modules on $\mathscr{D}^{\mathrm{ad,adm}}_{\varphi,N,\mu}$. This family is universal in the following sense: Let $X$ be an adic space locally of finite type over $E_\mu$ and let $\mathcal{E}'$ be a family of semi-stable $\mathscr{G}_K$-representations on $X$ with constant Hodge polygon equal to $\mu$. Then there is a unique morphism $f : X \to \mathscr{D}^{\mathrm{ad,adm}}_{\varphi,N,\mu}$ such that $\mathcal{E}' \cong f^*\mathcal{E}$ as families of $\mathscr{G}_K$-representations.*

*Proof.* The existence of the family $\mathcal{E}$ follows by applying Proposition 8.6 to the family $(\mathfrak{M},\Phi)$ of $\varphi$-modules of finite height over $\mathscr{A}^{[0,1)}$ on

$$Y = (P_{K_0,d} \times \mathrm{Flag}_{K,d,\mu})^{\mathrm{ad,adm}}.$$

This family exists locally on open subsets $U \subset Y$ and the semi-stable $\mathscr{G}_K$-representation on $U$ is canonically contained in $\mathscr{H}om_{\mathcal{O}_X\otimes K_0}(D, \mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}})$. Hence they glue together to give rise to the claimed family $\mathcal{E}$ on $Y$. By functoriality of $\mathscr{H}om_{\Phi,N,\mathrm{Fil}}(-, \mathcal{O}_X\widehat{\otimes}B_{\mathrm{st}})$ this vector bundle is equivariant for the action of $(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}$ and hence defines the desired family of semi-stable $\mathscr{G}_K$-representations on $\mathscr{D}^{\mathrm{ad,adm}}_{\varphi,N,\mu}$. Further the isomorphism $D_{\mathrm{st}}(\mathcal{E}) \cong (D,\Phi,N,\mathcal{F}^\bullet)$ on $Y$ is by construction equivariant under $(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})_{E_\mu}$ and hence descends to $\mathscr{D}^{\mathrm{ad,adm}}_{\varphi,N,\mu}$.

Now let $X$ be as above. The $K$-filtered $(\varphi,N)$-module $D_{\mathrm{st}}(\mathcal{E}')$ defines a morphism $f : X \to \mathscr{D}^{\mathrm{ad}}_{\varphi,N,\mu}$. This map factors over $\mathscr{D}^{\mathrm{ad,adm}}_{\varphi,N,\mu}$ by Proposition 8.7 as factoring over an open subspace may be check on the reduced space underlying $X$. Further we have isomorphisms $D_{\mathrm{st}}(\mathcal{E}') \cong f^*D_{\mathrm{st}}(\mathcal{E}) \cong D_{\mathrm{st}}(f^*\mathcal{E})$. Now the claim follows from Lemma 8.5. □

**Corollary 8.9.** *There is a family $\mathcal{E}$ of crystalline $\mathscr{G}_K$-representations on $\mathscr{D}^{\mathrm{ad,adm}}_{\varphi,\mu}$ such that $D_{\mathrm{crys}}(\mathcal{E}) = (D,\Phi,\mathcal{F}^\bullet)$ is the universal family of filtered $\varphi$-modules on $\mathscr{D}^{\mathrm{ad,adm}}_{\varphi,\mu}$. This family is universal in the following sense: Let $X$ be an adic space locally of finite type over $E_\mu$ and let $\mathcal{E}'$ be a family of crystalline $\mathscr{G}_K$-representations on $X$ with constant Hodge polygon $\mu$. Then there is a unique morphism $f : X \to \mathscr{D}^{\mathrm{ad,adm}}_{\varphi,\mu}$ such that $\mathcal{E}' \cong f^*\mathcal{E}$ as families of $\mathscr{G}_K$-representations.*

*Proof.* This is a direct consequence of Lemma 8.4. □

## 9. The morphism to the adjoint quotient

As in [Hel11, § 4] we consider the adjoint quotient $A/\mathfrak{S}_d$ where $A \subset \mathrm{GL}_{d,\mathbb{Q}_p}$ is the diagonal torus and $\mathfrak{S}_d$ is the finite Weyl group of $\mathrm{GL}_d$. Under the morphism $c : A \to \mathbb{A}^{d-1}_{\mathbb{Q}_p} \times_{\mathbb{Q}_p} \mathbb{G}_{m,\mathbb{Q}_p}$ which maps an element $g$ of $A$ to the coefficients $c_1,\dots,c_d$ of its characteristic polynomial $\chi_g = X^d + c_1X^{d-1} + \ldots + c_d$, the adjoint quotient $A/\mathfrak{S}_d$ is isomorphic to $\mathbb{A}^{d-1}_{\mathbb{Q}_p} \times_{\mathbb{Q}_p} \mathbb{G}_{m,\mathbb{Q}_p} = \mathrm{Spec}\,\mathbb{Q}_p[c_1,\dots,c_d,c_d^{-1}]$. Recall from [Hel11, § 4] that there is a morphism

$$\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0} \longrightarrow A/\mathfrak{S}_d \tag{9.1}$$

which is invariant under $\varphi$-conjugation on the source. It is defined on $R$-valued points by sending $b \in (\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0})(R) = \mathrm{GL}_d(R\otimes_{\mathbb{Q}_p}K_0)$ to the characteristic polynomial of $(b\cdot\varphi)^f = b\cdot\varphi(b)\cdot\ldots\cdot\varphi^{(f-1)}(b)$ where $f = [K_0 : \mathbb{Q}_p]$. This characteristic polynomial actually has coefficients in $R$, because it is invariant under $\varphi$, as can be seen from the formula $\varphi(b\cdot\varphi)^f = b^{-1}\cdot(b\cdot\varphi)^f\cdot\varphi^f(b) = b^{-1}\cdot(b\cdot\varphi)^f\cdot b$. Since $\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0}$ acts on itself by $\varphi$-conjugation via $(g,b) \mapsto g^{-1}b\,\varphi(g)$ and $(g^{-1}b\,\varphi(g)\,\varphi)^f = g^{-1}\cdot(b\cdot\varphi)^f\cdot g$ the map (9.1) is invariant under $\varphi$-conjugation.

Let $\mu$ be a cocharacter as in (2.5), let $E_\mu$ be its reflex field, and set $(A/\mathfrak{S}_d)_{E_\mu} := A/\mathfrak{S}_d \times_{\mathbb{Q}_p} E_\mu$. By projecting to $\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_{d,K_0}$ we may extend $\beta$ to morphisms

$$\begin{tikzcd}
P_{K_0,d} \times_{\mathbb{Q}_p} Q_{K,d,\preceq\mu} \arrow[r, "\widetilde{\alpha}"] \arrow[d] & (A/\mathfrak{S}_d)_{E_\mu} \arrow[d, equal] \\
\mathscr{H}_{\varphi,N,\preceq\mu} \arrow[r, "\alpha"] & (A/\mathfrak{S}_d)_{E_\mu} .
\end{tikzcd}$$

We further obtain morphisms to $(A/\mathfrak{S}_d)_{E_\mu}$ from the locally closed substacks $\mathscr{H}_{\varphi,\preceq\mu}$, $\mathscr{H}_{\varphi,N,\mu}$, $\mathscr{H}_{\varphi,\mu}$, $\mathscr{D}_{\varphi,N,\mu}$, and $\mathscr{D}_{\varphi,\mu}$, which we likewise denote by $\alpha$. Here we view $\mathscr{D}_{\varphi,N,\mu}$ and $\mathscr{D}_{\varphi,\mu}$ as substacks of $\mathscr{H}_{\varphi,N,\mu}$ via the zero section from Remark 2.7(3). We also consider the adification of these morphisms.

**Theorem 9.1.** *Let $\mu$ be a cocharacter as in (2.5), let $E_\mu$ be its reflex field and let $x \in (A/\mathfrak{S}_d)^{\mathrm{ad}}_{E_\mu}$. Then there exists an open subscheme $X$ of $\widetilde{\alpha}^{-1}(x)$ such that the weakly admissible locus in the fiber over $x$ is given by*

$$\widetilde{\alpha}^{-1}(x)^{\mathrm{wa}} \;=\; X^{\mathrm{ad}}.$$

*Proof.* This is similar to the proof of [Hel11, Theorem 4.1]. Let $x = (c_1, \dots, c_d) \in \kappa(x)^{d-1} \times \kappa(x)^\times$ and let $v_x$ denote the (multiplicative) valuation on $\kappa(x)$. First note that

$$c_d = \det{}_{\kappa(x)\otimes_{\mathbb{Q}_p}K_0}(b\cdot\varphi)^f = \det{}_{\kappa(x)}((b\cdot\varphi)^f)^{1/f}$$

and hence $\widetilde{\alpha}^{-1}(x)^{\mathrm{wa}} = \emptyset$ unless

$$v_x(c_d)^{-1/f}\cdot v_x(p)^{\frac{1}{ef}\sum_{\psi,j}\mu_{\psi,j}} \;=\; \lambda(\underline{D}) \;=\; 1.$$

In the following we will assume that this condition is satisfied. We now revert to the notation of the proof of Theorem 5.6. In particular we consider the projective $P_{K_0,d}$-schemes $Z_i$, the global sections $f_i \in \Gamma(Z_i, \mathcal{O}_{Z_i})$, the functions $h_i$, the closed subsets

$$Y_{i,m} = \{y \in Z_i^{\mathrm{ad}} \times Q^{\mathrm{ad}}_{K,d,\preceq\mu} \mid h_i(y) \geq m\}$$

and the proper projections $\mathrm{pr}_{i,m} : Y_{i,m} \to P_{K_0,d} \times_{\mathbb{Q}_p} Q_{K,d,\preceq\mu}$. This time

$$S_{i,m} = \{y = (g_y, N_y, U_y, \mathfrak{q}_y) \in Y_{i,m} \times_{(P_{K_0,d}\times Q_{K,d,\preceq\mu})} \widetilde{\alpha}^{-1}(x) \mid v_y(f_i(g_y, U_y)) > v_y(p)^{f^2 m}\}$$

is a union of connected components of $Y_{i,m} \times_{(P_{K_0,d}\times Q_{K,d,\preceq\mu})} \widetilde{\alpha}^{-1}(x)$, hence a closed subscheme and not just a closed adic subspace. This can be seen as follows: Let $\lambda_1, \dots, \lambda_d$ denote the zeros of the polynomial

$$X^d + c_1 X^{d-1} + \dots + c_{d-1}X + c_d.$$

Then every possible value of the $f_i$ is a product of some of the $\lambda_i$ and hence $f_i$ can take only finitely many values. As in the proof of Theorem 5.6

$$\widetilde{\alpha}^{-1}(x)^{\mathrm{wa}} \;=\; \widetilde{\alpha}^{-1}(x) \smallsetminus \bigcup_{i,m} \mathrm{pr}_{i,m}(S_{i,m}),$$

where the union runs over $1 \leq i \leq d-1$ and $m \in \mathbb{Z}$. So $\widetilde{\alpha}^{-1}(x)^{\mathrm{wa}}$ is an open subscheme of $\widetilde{\alpha}^{-1}(x)$. □

**Corollary 9.2.** *Let $x \in (A/\mathfrak{S}_d)^{\mathrm{ad}}_E$ and consider the 2-fiber product*

$$\begin{array}{ccc} \alpha^{-1}(x)^{\mathrm{wa}} & \longrightarrow & \mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\preceq\mu} \\ \downarrow & & \downarrow{\scriptstyle\alpha} \\ x & \longrightarrow & (A/W)^{\mathrm{ad}}_E. \end{array}$$

*Then there exists an Artin stack in schemes $\mathfrak{A}$ over the field $\kappa(x)$ which is an open substack of $\alpha^{-1}(x)$, such that $\alpha^{-1}(x)^{\mathrm{wa}} = \mathfrak{A}^{\mathrm{ad}}$. The same is true for $\mathscr{H}_{\varphi,\preceq\mu}$, $\mathscr{H}_{\varphi,N,\mu}$, $\mathscr{H}_{\varphi,\mu}$, $\mathscr{D}_{\varphi,N,\mu}$, and $\mathscr{D}_{\varphi,\mu}$.*

*Proof.* This is an immediate consequence of Theorem 9.1 and the proof of Corollary 5.7. □

We also determine the image of the weakly admissible locus in the adjoint quotient.

**Theorem 9.3.** *The image of $\mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\preceq\mu}$ (and $\mathscr{H}^{\mathrm{ad,wa}}_{\varphi,\preceq\mu}$, $\mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\mu}$, $\mathscr{H}^{\mathrm{ad,wa}}_{\varphi,\mu}$, $\mathscr{D}^{\mathrm{ad,wa}}_{\varphi,N,\mu}$, and $\mathscr{D}^{\mathrm{ad,wa}}_{\varphi,\mu}$) under the morphism(s) $\alpha$ is equal to the affinoid subdomain*

(9.2)

$$\Big\{ c = (c_1, \dots, c_d) \in (A/\mathfrak{S}_d)^{\mathrm{ad}}_{E_\mu} \;\Big|\; v_c(c_i) \leq v_c(p)^{\frac{1}{e}\sum_\psi(\mu_{\psi,d} + \ldots + \mu_{\psi,d+1-i})} \text{ with equality for } i = d \Big\},$$

*where $v_c$ is the (multiplicative) valuation of the adic point $c$ with $v_c(p) < 1$.*

**Remark 9.4.** (1) The subset described in (9.2) is really an affinoid subdomain. Indeed the adjoint quotient $(A/\mathfrak{S}_d)^{\mathrm{ad}}_{E_\mu}$ is (admissibly) covered by the (admissible) open affinoid rigid spaces (resp. adic spaces) $X_M = \{c = (c_1, \dots, c_d) \in (A/\mathfrak{S}_d)^{\mathrm{ad}}_{E_\mu} \mid v_c(c_i) \le p^M, \text{ for all } i \text{ and } v_c(v_d) \ge -p^M\}$ and the subspace (9.2) is easily seen to be a Laurent subdomain of each of these $X_M$ for $M \gg 0$.

(2) The morphisms $\alpha$ forget the Hodge-Pink lattice $\mathfrak{q}$ (or the $K$-filtration $\mathcal{F}^\bullet$) and in general their fibers contain infinitely many weakly admissible points.

(3) Like in [Hel11, Proposition 5.2] the affinoid subdomain of Theorem 9.3 can be described as the *closed Newton stratum* of the coweight $(-\frac{1}{e}\sum_\psi \mu_{\psi,d} \ge \dots \ge -\frac{1}{e}\sum_\psi \mu_{\psi,1})$ of $A$. By this we mean that the $\overline{\mathbb{Q}}_p$-valued points (i.e. the rigid analytic points) of 9.2 coincide with the points of the corresponding Newton stratum in the sense of Kottwitz. In [Hel11] the claim is made for all points of the corresponding Berkovich space. In the set up of adic spaces we can not rely in Kottwitz definition of a Newton stratum for all points of the adic space, as the valuations are not necessarily rank one valuations, i.e. the value group is not necessarily a subgroup of the real numbers. Especially the Newton strata do not cover the adic space $(A/\mathfrak{S}_d)^{\mathrm{ad}}$.

(4) Our affinoid subdomain equals $\mathfrak{S}_d\backslash\mathbf{T}'_\xi$ from [BS, Corollary 2.5], where $\xi$ is associated with the cocharacter $\tilde{\xi} := \big(-\mu - (0,1,\dots,d-1)\big)_{\mathrm{dom}} \in X_*(\widetilde{T})$. In this way our theorem generalizes [BS, Proposition 3.2].

Before we prove the theorem we note the following

**Lemma 9.5.** *Set $l_i := \frac{1}{ef}\sum_\psi(\mu_{\psi,d} + \dots + \mu_{\psi,d+1-i})$. Then $l_i$ equals the number $l_i$ defined in* [Hel11, Formula (5.2) on p. 988]. *If $\underline{D} = (D, \Phi, N, \mathfrak{q})$ is a $(\varphi, N)$-module with Hodge-Pink lattice over a field $L \supset E_\mu$ whose Hodge polygon is bounded by $\mu$ and if $\underline{D}' = \big(D', \Phi|_{\varphi^*D'}, N|_{D'}, \mathfrak{q} \cap D' \otimes_{L\otimes K_0} \mathbb{B}_L\big) \subset \underline{D}$ for a free $L \otimes_{\mathbb{Q}_p} K_0$-submodule $D' \subset D$ of rank $i$ which is stable under $\Phi$ and $N$, then $t_H(\underline{D}') \ge l_i$.*

*Proof.* The number $l_i$ in [Hel11, (5.2)] was defined as follows. Write $\{\mu_{\psi,1}, \dots, \mu_{\psi,d}\} = \{x_{\psi,1}, \dots, x_{\psi,r}\}$ with $x_{\psi,j} > x_{\psi,j+1}$. Let $n_{\psi,j} := \max\{k : \mu_{\psi,k} \ge x_{\psi,j}\}$. In particular $n_{\psi,r} = d$ and $\mu_{\psi,n_{\psi,j}} \ge x_{\psi,j}$. For $0 \le i \le d$ let $m_{\psi,j}(i) := \max\{0, n_{\psi,j} + i - d\}$. So $m_{\psi,j}(0) = 0$ for all $j$ and $m_{\psi,r}(i) = i$. It follows that $n_{\psi,j} \ge d - i$ if and only if $\mu_{\psi,d-i} \ge x_{\psi,j}$. Now $l_i$ was defined in [Hel11, (5.2)] as

$$l_i = \frac{1}{ef}\sum_\psi\left(\sum_{j=1}^{r-1}(x_{\psi,j} - x_{\psi,j+1})m_{\psi,j}(i) + x_{\psi,r}m_{\psi,r}(i)\right).$$

We compute $l_{i+1} - l_i = \frac{1}{ef}\sum_\psi\Big(\sum_{j=1}^{r-1}(x_{\psi,j} - x_{\psi,j+1})\big(m_{\psi,j}(i+1) - m_{\psi,j}(i)\big) + x_{\psi,r}\Big)$. The difference $m_{\psi,j}(i+1) - m_{\psi,j}(i)$ is 1 if $n_{\psi,j} + i - d \ge 0$, that is if $x_{\psi,j} \le \mu_{\psi,d-i}$. Else $m_{\psi,j}(i+1) - m_{\psi,j}(i)$ is 0. Therefore $l_{i+1} - l_i = \frac{1}{ef}\sum_\psi \mu_{\psi,d-i}$ and $l_0 = 0$ implies that $l_i = \frac{1}{ef}\sum_\psi(\mu_{\psi,d} + \dots + \mu_{\psi,d+1-i})$.

To prove the second assertion let $s \in \operatorname{Spec} L \otimes_{E_\mu} \widetilde{K}$ be a point and let $\mu' = \mu_{\underline{D}}(s)$ be the Hodge polygon of $\underline{D}$ at $s$. Then $\mu_{\psi,d} + \dots + \mu_{\psi,d+1-i} \le \mu'_{\psi,d} + \dots + \mu'_{\psi,d+1-i}$ for all $\psi$ and all $i$ by Proposition 2.12(b). We let $\mathfrak{p}_\psi$ be the $\psi$-component of $s^*\mathfrak{p} := s^*D \otimes_{\kappa(s)\otimes K_0} \mathbb{B}^+_{\kappa(s)}$ and $\mathfrak{p}'_\psi$ be the $\psi$-component of $s^*\mathfrak{p}' := s^*D' \otimes_{\kappa(s)\otimes K_0} \mathbb{B}^+_{\kappa(s)}$. By definition of the Hodge polygon, see Construction 2.9, we can choose a $\kappa(s)[\![t]\!]$-basis $(v_{\psi,1}, \dots, v_{\psi,d})$ of $\mathfrak{p}_\psi$ such that $(t^{-\mu'_{\psi,1}} v_{\psi,1}, \dots, t^{-\mu'_{\psi,d}} v_{\psi,d})$ is a $\kappa(s)[\![t]\!]$-basis of the $\psi$-component $\mathfrak{q}_\psi$ of $s^*\mathfrak{q}$. Since $\dim_{\kappa(s)(\!(t)\!)}\big(\mathfrak{p}'_\psi[\frac{1}{t}] \cap \langle v_{\psi,1}, \dots, v_{\psi,n}\rangle_{\kappa(s)(\!(t)\!)}\big) \ge n + i - d$ for all $n$, we can find a $\kappa(s)[\![t]\!]$-basis $(v'_{\psi,1}, \dots, v'_{\psi,i})$ of $\mathfrak{p}'_\psi$ with $v'_{\psi,j} \in \langle v_{\psi,1}, \dots, v_{\psi,d+j-i}\rangle_{\kappa(s)[\![t]\!]}$. Namely, for each $j$ we let $\bar{v}'_j$ be an element of $\big(\mathfrak{p}'_\psi \cap \langle v_{\psi,1}, \dots, v_{\psi,d+j-i}\rangle_{\kappa(s)[\![t]\!]}\big)/\langle v'_{\psi,1}, \dots, v'_{\psi,j-1}\rangle_{\kappa(s)[\![t]\!]}$ which generates a non-zero saturated $\kappa(s)[\![t]\!]$-submodule, and we let $v'_{\psi,j} \in \mathfrak{p}'_\psi \cap \langle v_{\psi,1}, \dots, v_{\psi,d+j-i}\rangle_{\kappa(s)[\![t]\!]}$ be a lift of $\bar{v}'_{\psi,j}$. Then $(v'_{\psi,1}, \dots, v'_{\psi,j})$ is linearly independent over $\kappa(s)(\!(t)\!)$ and generates a saturated $\kappa(s)[\![t]\!]$-submodule of $\mathfrak{p}'_\psi$. Using this basis we see that $t^{-\mu'_{\psi,d+j-i}} \cdot v'_{\psi,j} \in \mathfrak{q}_\psi \cap \mathfrak{p}'_\psi[\frac{1}{t}]$. This implies that $t_H(\underline{D}') \ge \frac{1}{ef}\sum_\psi \mu'_{\psi,d} + \dots + \mu'_{\psi,d+1-i} \ge l_i$. □

*Proof of Theorem 9.3.* We consider the embedding of $\mathscr{D}_{\varphi,\mu}$ into $\mathscr{H}_{\varphi,\mu}$ via the zero section. Under this section $\mathscr{D}^{\mathrm{ad,wa}}_{\varphi,\mu}$ is contained in $\mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\preceq\mu}$, $\mathscr{H}^{\mathrm{ad,wa}}_{\varphi,\preceq\mu}$, $\mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\mu}$, $\mathscr{H}^{\mathrm{ad,wa}}_{\varphi,\mu}$, and $\mathscr{D}^{\mathrm{ad,wa}}_{\varphi,N,\mu}$ by Lemma 5.3 or Remark 5.8. Conversely they are all contained in $\mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\preceq\mu}$. Moreover, these inclusions are compatible with the morphisms $\alpha$ to $(A/\mathfrak{S}_d)_{E_\mu}$.

We first claim that the affinoid subdomain is contained in the image of the weakly admissible locus for all these stacks. By the above it suffices to prove the claim for $\mathscr{D}_{\varphi,\mu}^{\mathrm{ad,wa}}$. In this case the claim follows from [Hel11, Theorem 5.5 and Proposition 5.2] using Lemma 9.5. Note that in loc. cit. only Berkovich's analytic points are treated, but the given argument works verbatim also for adic points.

Conversely let $c = (c_1, \ldots, c_d)$ be an $L$-valued point of $(A/\mathfrak{S}_d)^{\mathrm{ad}}_{E_\mu}$ which lies in the image of the weakly admissible locus of one of these stacks. By the above it lies in the image of $\mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\preceq\mu}$. So let $\underline{D} \in \mathscr{H}^{\mathrm{ad,wa}}_{\varphi,N,\preceq\mu}(L')$ for a field extension $L'/L$, such that $\underline{D}$ maps to $c$. By extending the field $L'$ further we may assume that $K_0 \subset L'$ and that $X^d + c_1X^{d-1} + \ldots + c_d = \prod_{j=1}^d (X - \lambda_j)$ splits into linear factors with $\lambda_j \in L'$. We claim that $v_{L'}(\prod_{j\in I} \lambda_j) \le v_{L'}(p)^{fl_i}$ for all subsets $I \subset \{1, \ldots, d\}$ of cardinality $i$. By Lemma 9.5 this implies that $c$ lies in our affinoid subdomain.

To prove the claim we use Remark 2.4. Then $X^d + c_1X^{d-1} + \ldots + c_d$ is the characteristic polynomial of the $L'$-endomorphism $(\Phi^f)_0$ of $D_0$ and $t_N(\underline{D}) = v_{L'}(\det_{L'}(\Phi^f)_0)^{1/f}$. We write $(\Phi^f)_0$ in Jordan canonical form and observe that $N_0$ maps the generalized eigenspace of $(\Phi^f)_0$ with eigenvalue $\lambda_j$ into the one with eigenvalue $p^{-f}\lambda_j$. If $I \subset \{1, \ldots, d\}$ is a subset with cardinality $i$ this allows us to find an $i$-dimensional $L'$-subspace $D'_0 \subset D_0$ which is stable under $(\Phi^f)_0$ and $N_0$ such that the eigenvalues of $(\Phi^f)_0$ on $D'_0$ are of the form $(p^{-n_j}\lambda_j : j \in I)$ for suitable $n_j \in \mathbb{Z}_{\ge 0}$. We let $D' \subset D$ be the $(\varphi, N)$-submodule corresponding to $D'_0 \subset D_0$ under Remark 2.4. Then

$$v_{L'}\big(\prod_{j\in I} \lambda_j\big) \;\le\; v_{L'}\big(\prod_{j\in I} p^{-n_j}\lambda_j\big) \;=\; v_{L'}\big(\det_{L'}(\Phi^f)_0|_{D'_0}\big) \;=\; t_N(\underline{D}')^f \;\le\; v_{L'}(p)^{f\,t_H(\underline{D}')} \;\le\; v_{L'}(p)^{fl_i}$$

by the weak admissibility of $\underline{D}$ and by Lemma 9.5. This proves the theorem. □

## References


[AK80] A. Altman, S. Kleiman: *Compactifying the Picard scheme*, Adv. in Math. **35** (1980), no. 1, 50–112.

[BL94] A. Beauville, Y. Laszlo: *Conformal blocks and generalized theta functions*, Comm. Math. Phys. **164** (1994), no. 2, 385–419; also available as arXiv:alg-geom/9309003.

[BL93] S. Bosch, W. Lütkebohmert: *Formal and Rigid Geometry II. Flattening Techniques*, Math. Ann. **296** (1993), 403–429.

[BD] A. Beilinson, V. Drinfeld: *Quantization of Hitchin's integrable system and Hecke eigensheaves*, preprint on http://www.math.uchicago.edu/∼mitya/langlands.html.

[BC08] L. Berger, P. Colmez: *Familles de représentations de de Rham et monodromie p-adique*, in Représentations $p$-adiques de groupes $p$-adiques I: représentations galoisiennes et $(\varphi, \Gamma)$-modules, Astérisque **319** (2008), 303–337; also available at http://www.math.jussieu.fr/∼colmez.

[BS] C. Breuil, P. Schneider, *First steps towards p-adic Langlands functoriality*, J. reine angew. Math. **610** (2007), 149–180; also available as arXiv:math.NT/0603.5499.

[Bou70] N. Bourbaki: *Algèbre, Chapitres 1 à 3*, Hermann, Paris 1970.

[BC09] O. Brinon, B. Conrad: *p-adic Hodge theory*, notes from the CMI summer school 2009; also available at http://math.stanford.edu/∼conrad.

[BGR84] S. Bosch, U. Güntzer, R. Remmert: *Non-Archimedean Analysis*, Grundlehren **261**, Springer-Verlag, Berlin etc. 1984.

[BLR90] S. Bosch, W. Lütkebohmert, M. Raynaud: *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **21**, Springer-Verlag, Berlin, 1990.

[CF00] P. Colmez, J.-M. Fontaine: *Construction des représentations p-adiques semi-stables*, Invent. Math. **140** (2000), no. 1, 1–43.

[DOR10] J-F. Dat, S. Orlik, M. Rapoport: *Period domains over finite and p-adic fields*, Cambridge Tracts in Math. **183**, Cambridge University Press 2010.

[Eis95] D. Eisenbud: *Commutative Algebra with a View Toward Algebraic Geometry*, GTM **150**, Springer-Verlag, Berlin etc. 1995.

[FF] L. Fargues, J.-M. Fontaine: *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, preprint, 2010, available at http://www.math.jussieu.fr/∼fargues.

[FGA] A. Grothendieck: *Fondements de la Géométrie Algébrique*, Extraits du Séminaire Bourbaki 1957–1962, Secrétariat mathématique, Paris 1962; also available at http://www.numdam.org/numdam-bin/recherche?au=Grothendieck.

[EGA] A. Grothendieck: *Élements de Géométrie Algébrique*, Publ. Math. IHES **4**, **8**, **11**, **17**, **20**, **24**, **28**, **32**, Bures-Sur-Yvette, 1960–1967; see also Grundlehren **166**, Springer-Verlag, Berlin etc. 1971; also available at http://www.numdam.org/numdam-bin/recherche?au=Grothendieck.

[GL11] A. Genestier, V. Lafforgue: Théorie de Fontaine en égales charactéristiques, *Ann. Sci. École Norm. Supér.* **44** (2011), no. 2, 263–360; also available at http://www.math.jussieu.fr/∼vlafforg/.

[GL12] A. Genestier, V. Lafforgue: *Structures de Hodge-Pink pour les $\varphi/\mathfrak{S}$-modules de Breuil et Kisin*, Compos. Math. **148** (2012), no. 3, 751–789; also available a http://www.math.jussieu.fr/~vlafforg/.

[Har11] U. Hartl: *Period Spaces for Hodge Structures in Equal Characteristic*, Annals of Math. **173**, n. 3 (2011), 1241–1358; also available as arXiv:math.NT/0511686.

[HV11] U. Hartl, E. Viehmann: *The Newton stratification on deformations of local $G$-shtukas*, J. reine angew. Math. (Crelle) **656** (2011), 87–129; also available as arXiv:0810.0821.

[Hel11] E. Hellmann: *On families of weakly admissible filtered $\varphi$-modules and the adjoint quotient of* $\mathrm{GL}_d$, Documenta Math. **16** (2011), 969–991, also available as arXiv:1102.0119.

[Hel13] E. Hellmann: *On arithmetic families of filtered $\varphi$-modules and crystalline representations*, to appear in J. Inst. Math. Jussieu; also available as arXiv:1010.4577.

[Hub93] R. Huber: *Continuous valuations*, Math. Z. **212** (1993), 445–447.

[Hub94] R. Huber: *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), 513–551.

[Hub96] R. Huber: *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Math., E30, Friedr. Vieweg & Sohn, Braunschweig, 1996.

[Ked08] K. Kedlaya: *Slope filtrations for relative Frobenius*, in Représentations $p$-adiques I: représentations galoisiennes et $(\varphi, \Gamma)$-modules, Asterisque **319** (2008), 259–301; also available as arXiv:math/0609272.

[KL10] K. Kedlaya, R. Liu: *On families of $(\varphi, \Gamma)$-modules*, Algebra and Number Theory **4** (2010), no. 7, 943–967; also available as arXiv:0812.0112.

[Kis06] M. Kisin: *Crystalline representations and $F$-crystals*, in Algebraic geometry and number theory, pp. 459–496, Progress in Math. 253, Birkhäuser Boston, Boston MA, 2006; also available at www.math.harvard.edu/~kisin.

[Kis08] M. Kisin: *Potentially semi-stable deformation rings*, J. of the American Math. Soc. **21** (2008) no. 2, 513–546; also available at www.math.harvard.edu/~kisin.

[KPX12] K. Kedlaya, J. Pottharst, L. Xiao: *Cohomology of arithmetic families of $(\varphi, \Gamma)$-modules*, preprint 2012, available as arXiv:1203.5718

[LS97] Y. Laszlo, C. Sorger: *The line bundles on the moduli of parabolic $G$-bundles over curves and their sections*, Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 4, 499–525, also available as arXiv:alg-geom/9507002.

[LM00] G. Laumon, L. Moret-Bailly: *Champs algébriques*, Ergebnisse **39**, Springer-Verlag, Berlin etc. 2000.

[Laz62] M. Lazard: *Les zéros des fonctions analytiques d'une variable sur un corps valué complet*, Inst. Hautes Études Sci. Publ. Math. **14** (1962), 47–75; also available at http://www.numdam.org/item?id=PMIHES_1962__14__47_0.

[Liu] R. Liu: *Slope filtrations in families*, to appear in J. Inst. Math. Jussieu; also available as arXiv:0809.0331.

[Mum99] D. Mumford: *The Red book of Varieties and Schemes*, second edition, Lecture Notes in Math. **1358**, Springer-Verlag 1999.

[Pin97] R. Pink: *Hodge Structures over Function Fields*, Preprint 1997; available at http://www.math.ethz.ch/~pinkri/.

[PR09] G. Pappas, M. Rapoport: *$\Phi$-modules and coefficient spaces*, Moscow Math. J. **9** no. 3 (2009), 625–664, also available as arXiv:0811.1170.

[Ser79] J.-P. Serre: *Local fields*, GTM **67**, Springer-Verlag, New York-Berlin, 1979.

[Var04] Y. Varshavsky: *Moduli spaces of principal $F$-bundles*, Selecta Math. (N.S.) **10** (2004), no. 1, 131–166, also available as arXiv:math/0205130.



Urs Hartl
Universität Münster
Mathematisches Institut
Einsteinstr. 62
D – 48149 Münster
Germany
www.math.uni-muenster.de/u/urs.hartl/

Eugen Hellmann
Universität Bonn
Mathematisches Institut
Endenicher Allee 60
D – 53115 Bonn
Germany
E-mail: hellmann@math.uni-bonn.de