Aufgabe 1:
Prove that the closed unit ball of $L^1([0,1])$ has no extreme points. Deduce from here that $L^1([0,1])$ is not a dual space.

Solution:
Let $f \in L^1([0,1])$ be any element of the (closed) unit ball. We will show that f is not an extreme point. We may suppose without loss of generality that $\|f\|_1 = 1$. Consider the function

$$g(x) = \int_0^x |f(t)|dt, \qquad x \in [0,1].$$

By Lebesgue's dominated convergence theorem, g is continuous. Note that $g(0) = 0$ and $g(1) = 1$. By the intermediate value theorem, choose $0 < y < 1$ such that $g(y) = 1/2$, and set

$$f_1 = 2f1_{[0,y]}, \quad f_2 = 2f1_{[y,1]}.$$

Then f_1 and f_2 belong to the unit ball, $f_1 \neq f \neq f_2$ and $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$, which shows that f is not an extreme point.

If $L^1([0,1])$ were a dual space, it's unit ball would be weak* compact. By the Krein-Milman theorem, the unit ball would be the weak* closure of the convex hull of the extreme points. By the above, there are no extreme points, so the unit ball would be empty, which is clearly absurd. Therefore, $L^1([0,1])$ is not a dual space.

Aufgabe 2:
Let A be a C*-algebra. Show that the extreme points of $\{x \in A \mid 0 \leq x \leq 1\}$ are precisely the projections in A.

Solution:
Set $B = \{x \in A \mid 0 \leq x \leq 1\}$. Let $x \in B$. Then $2x - x^2 \in B$ (make a drawing) and $x = \frac{1}{2}x^2 + \frac{1}{2}(2x - x^2)$. If x is an extreme point of B, then we have $x = x^2$ so that x is a projection.

Conversely, suppose $p \in A$ is a projection, $0 < \lambda < 1$, and $p = \lambda a + (1-\lambda)b$ for some $a, b \in B$. Then $\lambda a \leq p$, and hence $0 \leq (1-p)a(1-p) \leq 0$. It follows that $a^{1/2}(1-p) = 0$ and $a(1-p) = 0$. So $a = ap = pa = pap$. We have $a \leq 1$ so that $a = pap \leq p$. Similarly, $b \leq p$. Suppose $a < p$. Then

$$p = \lambda a + (1-\lambda)b < \lambda p + (1-\lambda)p = p,$$

a contradiction. So $a = p$. Then also, $b = p$.

Aufgabe 3:
Let X be a compact Hausdorff space. Assume $C(X)$ is (*-isomorphic to) a von Neumann algebra. Show that X must be totally disconnected (see Blatt 4).

Solution:
The projections in $C(X)$ correspond to indicator functions 1_E of sets E that are open and closed. Since the projections span a norm dense subset of $C(X)$, we deduce by Urysohn's lemma that the projections in $C(X)$ separate points. If $Y \subseteq X$ is a subset containing two points $x \neq y$, we have shown that there is a (continuous) projection $p \in C(X)$ such that $p(x) = 1$ and $p(y) = 0$. Thus, X is totally disconnected.