DOOB-MEYER-DECOMPOSITION OF HILBERT SPACE VALUED FUNCTIONS

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Abstract. (1) We give here a new proof of the Doob-Meyer-decomposition which is rather quick and elementary. It is more general in some aspects, and weaker in other aspects if compared to other approaches. The functions are defined on a totally ordered set with image in a Hilbert space. (2) We also give a second variant of the Doob-Meyer-decomposition; it is more specialized. (3) We apply (1) by reproving the Doléans-measure in a special setting and demonstrate that the stochastic integral could be defined on more general totally ordered time scales then $\mathbb{R}$.

0. Introduction

In **chapter 1** we show a variant of the Doob-Meyer-decomposition. We decompose a certain class of Hilbert space $\mathcal{H}$ valued functions $f : \mathcal{R} \to \mathcal{H}$ on a totally ordered set $\mathcal{R}$ into a martingale and a predictable function. Positive, $L^2(\Omega)$- valued submartingales fall into this class. The kind of proof is new to our knowledge.

In **chapter 2** we show an analogous decomposition *in* $L^2(\mathcal{R}, \mu, \mathcal{H})$ of a certain class of functions $X \in L^2(\mathcal{R}, \mu, \mathcal{H})$, where $\mathcal{R}$ is the time scale, μ is a positive finite measure on it and $\mathcal{H}$ is Hilbert. Though we formulate $\mathcal{R}$ more generally as usual, one may think of $\mathcal{R}$ as $[0, 1]$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Q}$, $\mathbb{N}$, etc.

The proof given in chapter 2 is considerably more work than the proof given in chapter 1.

In **chapter 3** we apply the result of chapter 1 to obtain a proof of the Doléans-measure for a positive submartingale for a more general time scale $\mathcal{R}$ than $\mathbb{R}$.

We also demonstrate then, that a stochastic integral could be defined for this time scale.

1. Discrete Doob-Meyer-decompostion of $\mathcal{H}$-valued functions

We give here a rather quick Doob-Meyer-decomposition of a certain class of adapted functions on an arbitrary totally ordered set. Also compare [9], [11], [10], [4].

Let us be given a Hilbert space $\mathcal{H}$, a totally ordered set $\mathcal{R}$ and an increasing family $(P_t)_{t\in\mathcal{R}}$ of projections $P_t \in B(\mathcal{H})$, i.e. $\forall s, t \in \mathcal{R} :\ P_tP_t = P_t,\ P_t^* = P_t,\ \ s \le t \Rightarrow P_sP_t = P_tP_s = P_s$.

1991 *Mathematics Subject Classification.* 60G05, 60G44, 60G48.

Key words and phrases. Doob-Meyer, decomposition, Hilbert space, totally ordered set, Doleans measure, stochastic integral.

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The author is supported by the Austrian Research Foundation (FWF) project S8305.

With $(P_t)_{t\in\mathcal{R}}$ we associate a second increasing family of projections $(L_t)_{t\in\mathcal{R}}$ defined as

$$L_t = \bigvee_{s<t} P_s \qquad \text{i.e. the projection onto } \overline{\bigcup_{s<t} \mathrm{Im}(P_s)}.$$

We call a function $X : \mathcal{R} \to \mathcal{H}$ **adapted** if $P_t X_t = X_t$ for all $t \in \mathcal{R}$, **predictable** if $L_t X_t = X_t$ for all $t \in \mathcal{R}$, a **martingale** if $P_s X_t = X_s$ for all $s \le t$ in $\mathcal{R}$.

Lemma 1.1. Fundamental lemma of the Doob-Meyer-decomposition
Let $\mathcal{H}$ be a Hilbert space. Let $(P_k)_{k\ge 0}$ be an increasing sequence of projections $P_k \in B(\mathcal{H})$. Let $X : \mathbb{N} \to \mathcal{H}$ be an adapted function, i.e. $\forall k \ge 1 : P_k X_k = X_k$.

Then for the Doob-decomposition

$$A_n := \sum_{k=1}^n P_{k-1}(X_k - X_{k-1}), \qquad M_n := X_n - A_n \qquad (n \ge 0)$$

(thus $\forall n \ge 1 : X_n = M_n + A_n$ and $A_n = P_{n-1} A_n$ and $\forall k \le n : M_k = P_k M_n$) we have

$$\|A_n\|^2 = 2\,Re < X_n, A_n > - \sum_{k=1}^n q(k) \qquad \textit{and} \qquad \|M_n\|^2 = \|X_n\|^2 - \sum_{k=1}^n q(k),$$

where $q(k) := \|P_{k-1} X_k\|^2 - \|X_{k-1}\|^2$.

Proof. Abbreviate $\Delta X_k := X_k - X_{k-1}$.

$$\begin{aligned} \|A_n\|^2 &= < \sum_{k=1}^n P_{k-1}\Delta X_k, \sum_{i=1}^n P_{i-1}\Delta X_i > \\ &= 2\mathrm{Re} \sum_{1\le i<k\le n} < P_{k-1}\Delta X_k, P_{i-1}\Delta X_i > + \sum_{k=1}^n < P_{k-1}\Delta X_k, P_{k-1}\Delta X_k > . \end{aligned}$$

Now we look at the first sum

$$\begin{aligned} \beta &:= \sum_{1\le i<k\le n} < P_{k-1}\Delta X_k, P_{i-1}\Delta X_i > = \sum_{1\le i<k\le n} < \Delta X_k, P_{i-1}\Delta X_i > \\ &= \sum_{k=1}^n < \Delta X_k, \sum_{i=1}^{k-1} P_{i-1}\Delta X_i > \\ &= \sum_{k=1}^n (< X_k, A_{k-1} > - < X_{k-1}, A_{k-1} >) \\ &= \sum_{k=1}^n (< X_k, A_k - P_{k-1}\Delta X_k > - < X_{k-1}, A_{k-1} >) \\ &= \sum_{k-1}^n (< X_k, A_k > - < X_{k-1}, A_{k-1} >) - \sum_{k=1}^n < X_k, P_{k-1}\Delta X_k > \\ &= < X_n, A_n > - \sum_{k=1}^n < X_k, P_{k-1}\Delta X_k > . \end{aligned}$$

Thus we get

$$\begin{aligned} \|A_n\|^2 &= 2\mathrm{Re}\,\beta + \sum_{k=1}^n < P_k\Delta X_k, P_{k-1}\Delta X_k > = 2\mathrm{Re} < X_n, A_n > \\ &+ \sum_{k=1}^n (- < X_k, P_{k-1}\Delta X_k > - < P_{k-1}\Delta X_k, X_k > + < P_{k-1}\Delta X_k, \Delta X_k >). \end{aligned}$$

Expanding here the ΔX_k to $X_k - X_{k-1}$ we get

$$\begin{aligned} \|A_n\|^2 &= 2\mathrm{Re} < X_n, A_n > - \sum_{k=1}^n (< X_k, P_{k-1}X_k > - < P_{k-1}X_{k-1}, X_{k-1} >) \\ &= 2\mathrm{Re} < X_n, A_n > - \sum_{k=1}^n q(k). \end{aligned}$$

And immediately we get the result

$$\|M_n\|^2 =< X_n - A_n, X_n - A_n >= \|X_n\|^2 - \sum_{k-1}^n q(k). \ \square$$

Now consider as an example some probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_n)_{n\geq 1}$. If $X : \mathbb{N} \to L^2(\Omega, \mathcal{F}, P)$ is a positive submartingale, i.e. $0 \leq X_k \leq E(X_n|\mathcal{F}_k)$ for $k \leq n$, and setting P_n as the projection in $B(L^2(\Omega, \mathcal{F}, P))$ onto $L^2(\Omega, \mathcal{F}_n, P)$ and considering the fact $E(X_n|\mathcal{F}_k) = P_k X_n$ (e.g. [5] chapter 6), then $\|X_k\|^2 \leq \|P_k X_n\|^2$.

Thus $\sum_{k=1}^n q(k) \geq 0$, where q is defined as in the previous lemma. This estimate is then applicable on the next theorem. Slightly more generally, if X is a submartingale with $X \geq c$ uniformly for some constant c, then $X - c \geq 0$ is a positive submartingale, such that the next theorem is also applicable. (cf. also [9] Prop.1 and Th.2)

Theorem 1.2. Doob-Meyer-decomposition variant 1

Let $\mathcal{R}$ be a totally ordered set. Let $\mathcal{H}$ be a Hilbert space and $(P_t)_{t\in\mathcal{R}}$ an increasing family of projections in $B(\mathcal{H})$. Let $X : \mathcal{R} \to \mathcal{H}$ an adapted function. For all $t \in \mathcal{R}$ let exist a constant $C_t \in \mathcal{R}$ such that for all $n \geq 1$ and all $t_0 < ... < t_n = t$ in $\mathcal{R}$ we have

$$\sum_{k=1}^n \|P_{t_{k-1}} X_{t_k}\|^2 - \|X_{t_{k-1}}\|^2 \geq C_t. \tag{1}$$

Then we find adapted functions $M, A : \mathcal{R} \to \mathcal{H}$ such that $X = M + A$ and M is a martingale and A is predictable.

If $\mathcal{H} = L^2(\Omega)$ and X is a positive submartingale then A can be chosen increasing (i.e. $A_s \leq A_t$ for all $s \leq t$ in $\mathcal{R}$).

Proof. Consider the set of finite subsets of $\mathcal{R}$, i.e.

$$\phi := \{ u \subseteq \mathcal{R} \mid \mathrm{card}(u) < \infty \}.$$

Then consider the filter basis B consisting of all end pieces of ϕ, where we think of ϕ being ordered under the set inclusion, i.e.

$$B = \{ \alpha \subseteq \phi \mid \exists u_0 \in \phi : \ \alpha = \{ u \mid u \supseteq u_0 \} \}.$$

Complete this filter basis to an ultrafilter $\mathcal{U}$ on ϕ. For fixed $t \in \mathcal{R}$ set

$$A_t : \phi \to \mathcal{H} \ : \ A_t(u) = 0 \text{ for } t \notin u \quad \text{and}$$

$$A_t(u) = \sum_{k=1}^{n} P_{t_{k-1}}(X_{t_k} - X_{t_{k-1}}) \text{ for } u = \{t_0 < ... < t_n = t < t_{n+1} < ... < t_m\}$$

Set $M_t : \phi \to \mathcal{H} : M_t(u) = X_t - A_t(u)$. Using the fundamental lemma 1.1 we get $\|M_t(u)\|^2 = \|X_t\|^2$ for $t \notin u$, and for $t \in u$ we have

$$\|M_t(u)\|^2 = \|X_t\|^2 - \sum_{k=1}^{n}(\|P_{t_{k-1}}X_{t_k}\|^2 - \|X_{t_{k-1}}\|^2) \le \|X_t\|^2 - C_t.$$

Thus $\|A_t(u)\| = \|X_t - M_t(u)\| \le \|X_t\| + \sqrt{\|X_t\|^2 - C_t}$. Note that the unit ball is compact under the weak topology in a Hilbert space. So we obtain weak limits along the ultrafilter, i.e. set (here wlim denotes the weak limit)

$$M : \mathcal{R} \to \mathcal{H} : M_t = \mathrm{wlim}_{\mathcal{U}}\, M_t(u)$$

and analogously define A. Note that the limits M_t respectively A_t are indeed in the subspace $\mathrm{Im}(P_t)$ respectively $\mathrm{Im}(L_t)$, so M respectively A are adapted respectively predictable. Now we get

$$X_t = \mathrm{wlim}_{\mathcal{U}}\, X_t - A_t(u) + A_t(u) = \mathrm{wlim}_{\mathcal{U}}\, M_t(u) + \mathrm{wlim}_{\mathcal{U}}\, A_t(u) = M_t + A_t.$$

Further more fix $s \le t$ in $\mathcal{R}$. Note that for fixed $u \in \phi$ with $\{s, t\} \subseteq u$ we have $P_s M_t(u) = M_s(u)$. So we get

$$P_s M_t = P_s(\mathrm{wlim}_{\mathcal{U}}\, M_t(u)) = \mathrm{wlim}_{\mathcal{U}}\, P_s M_t(u) = \mathrm{wlim}_{\mathcal{U}}\, M_s(u) = M_s,$$

i.e. M is a martingale. Now let $h = \mathbb{C}$ and X be a submartingale, i.e. $P_s(X_t - X_s) \ge 0$ for all $s \le t$ in $\mathcal{R}$. So we have $A_t(u) - A_s(u) \ge 0$ for all $s \le t$ and all $u \supseteq \{s, t\}$. Thus for all $0 \le f \in L^2(\Omega)$ we get

$$< A_t - A_s, f >= \mathrm{wlim}_{\mathcal{U}} \ < A_t(u) - A_s(u), f > \ge 0,$$

i.e. $A_t - A_s \ge 0$. □

Examples. Note that not all processes $X = M + A$, M a martingale, A predictable, fulfill the sufficient condition of theorem 1.2. Indeed, e.g., set $\mathcal{H} = \mathbb{C}$, $\mathcal{R} = (0, 1)$ and $P_s = I$ for all $s \in (0, 1)$. Then all processes are predictable, but $X(s) := 1/s$ does not satisfy the condition in 1.2, indeed (1) becomes

$$\sum |X(t_k)|^2 - |X(t_{k-1})|^2 = |X(t_n)|^2 - |X(t_0)|^2 = 1/t_n^2 - 1/t_0^2 \to -\infty. \tag{2}$$

Further $X(s) := 1_{\mathbb{Q}}(s)$ fulfills the condition of 1.2 and shows that the resulting predictable function need not have bounded variation, as the martingale must be constant with respect to the given (constant) filtration.

2. Doob-Meyer-decomposition in $L^2(\mathcal{R}, \mu, \mathcal{H})$

Using the fundamental lemma 1.1, in this chapter we give a Doob-Meyer-decomposition of a certain class of adapted functions *in* the Hilbert space $L^2(\mathcal{R}, \mu, \mathcal{H})$, where the time scale $\mathcal{R}$ is endowed with a finite measure μ.

For a totally ordered set $\mathcal{R}$ the **order topology** is induced by the open sets $\{ s \in \mathcal{R} \mid t < s \}$ and $\{ s \in \mathcal{R} \mid s < t \}$ for all $t \in \mathcal{R}$. Thus these "unbounded" intervals together with all finite open intervals $(s,t) \subseteq \mathcal{R}$ form a basis of the topology. $t \in \mathcal{R}$ is a **successor** of $s \in \mathcal{R}$ if $s < t$ and the open interval (s,t) is empty. Here s is a **predecessor** of t.

We make $\mathcal{R}$ a measurable space by endowing it with the Borel-Algebra $\mathcal{B}$. Let us be given a positive measure μ on $\mathcal{R}$. In this chapter we will restrict ourself to the following conditions.

(I) $\quad$ **$\mathcal{R}$ is separable and has countable many successor elements**

(II) $\quad \mu([s,t]) < \infty \qquad \forall s < t \in \mathcal{R}.$

Note for example that any subset $\mathcal{R} \subseteq \mathbb{R}$ fulfills condition (I): indeed at first one constructs a countable subset $D \subseteq \mathcal{R}$ which is dense in $\mathcal{R}$ with respect to the trace topology of $\mathbb{R}$. Then D is also dense in the coarser order topology.

Also note that there exist separable totally ordered sets, which have not countable many successors (e.g. take $\mathbb{R} \times \{0,1\}$ with lexicographical order).

Maybe every $\mathcal{R}$ fulfilling (I) is a subset in $\mathbb{R}$. We have not put effort in that direction.

Assume $(I) - (II)$. We can assume that the dense countable subset $D \subseteq \mathcal{R}$ contains also all elements which are successors or predecessors, i.e.

$$\{ s \in \mathcal{R} \mid \exists x < s : (x,s) = \emptyset \text{ or } \exists x > s : (s,x) = \emptyset \} \ \subseteq \ D. \tag{3}$$

Thus we note that D is left and right dense, i.e.

$$\forall s < t \qquad (s,t] \cap D \neq \emptyset \quad \text{and} \quad [s,t) \cap D \neq \emptyset. \tag{4}$$

Furthermore each open subset $O \subseteq \mathcal{R}$ can be written as a countable union of open intervals; indeed if O does neither contain $\min(\mathcal{R})$ nor $\max(\mathcal{R})$ (if they exist), then O can be written as $O = \bigcup_{x \in O} (\alpha_x, \beta_x)$ where one chooses $\alpha_x, \beta_x \in D$ such that $\alpha_x < x < \beta_x$ and $(\alpha_x, \beta_x) \subseteq O$, so that O is obviously written as countable union of open intervals. If O contains min / max one has to add at most two open "unbounded" intervals.

We conclude that $\mathcal{B}$ is induced by the open intervals and that $\mathcal{R}$ is the countable union of intervals of the form $(s,t]$ and of $\{\min(\mathcal{R})\}$, if $\min(\mathcal{R})$ exists. Furthermore each single point $\{s\}$, being closed, is measurable. Thus $\mathcal{B}$ is induced by the algebra $\mathcal{A}$ of finite unions of sets $(s,t]$ $(s,t \in \mathcal{R})$ and $\{\min(\mathcal{R})\}$.

We conclude that the step functions of $\mathcal{A}$ are dense in $L^1(\mathcal{R}, \mu)$ due to a theorem of the theory of measure extension (e.g. [7] VI.6). Thus they are dense in $L^2(\mathcal{R}, \mu)$. (Indeed if ϕ is a step function of the algebra $\mathcal{A}$ and $A \in \mathcal{B}$ then $\|1_A - (\phi \wedge 1) \vee 0\|_{L^2}^2 \ \leq \ \|1_A - (\phi \wedge 1) \vee 0\|_{L^1} \ \leq \ \|1_A - \phi\|_{L^1}$ and the last number can be made arbitrarily small.)

Now let $\mathcal{H}$ be a Hilbert space and $(P_t)_{t\in\mathcal{R}}$ be an increasing family of projections in $B(\mathcal{H})$. For a function $X : \mathcal{R} \to \mathcal{H}$ to be **adapted**, **predictable** or a **martingale**, we take the definition of chapter 1.

We define the **space of predictable functions** as the closure in $L^2(\mathcal{R},\mu,\mathcal{H})$ of

$$\text{span}\,\{\, 1_{(s,t]}\,\xi \mid s,t\in\mathcal{R},\ \xi\in\text{Im}(P_s)\,\}.$$

In lemma 2.2 we will show that the space of predictable functions is exactly the set of predictable functions (μ-a.e.).

Recall the definition of the increasing family of projections $(L_t)_{t\in\mathcal{R}}$ in chapter 1.

$$L_t = \bigvee_{s<t} P_s \qquad \text{i.e. the projection onto } \overline{\bigcup_{s<t} \text{Im}(P_s)}.$$

Analogously we define a increasing family $(R_t)_{t\in\mathcal{R}}$ of projections by

$$R_t = \bigwedge_{s>t} P_s \qquad \text{i.e. the projection onto } \bigcap_{s>t} \text{Im}(P_s).$$

In our setting, R_t is explicitly computable by the following general

Remark 2.1. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{R}$ a totally ordered set and $(P_t)_{t\in\mathcal{R}}$ an decreasing family of projections in $B(\mathcal{H})$. Then the projection*

$$P = \bigwedge_{t\in\mathcal{R}} P_t \in B(\mathcal{H}) \qquad \textit{onto} \qquad \bigcap_{t\in\mathcal{R}} \text{Im}(P_t)$$

is the strong operator limit of the net $(P_t)_{t\in\mathcal{R}}$.

I.e. $\forall\xi\in\mathcal{H} : P\xi = \lim_{t\uparrow} P_t\xi$ (i.e. the limit of the net $(P_t\xi)_{t\in\mathcal{R}}$).

Proof. So fix $\xi\in\mathcal{H}$ and consider the uniformly norm bounded net $(P_t\xi)_{t\in\mathcal{R}}$. It exists a subnet $(P_i\xi)_{i\in I}$ converging weakly to $\eta\in\mathcal{H}$. For fixed $t\in\mathcal{R}$ the limit acts in $\text{Im}(P_t)$. So $\eta\in\text{Im}(P_t)$. Thus $\eta\in\bigcap_{t\in\mathcal{R}}\text{Im}(P_t)$ and therefore $P\eta=\eta$. Thus

$$P\xi = \text{wlim}_i\, PP_i\,\xi = P\,\text{wlim}_i\, P_i\xi = P\eta = \eta.$$

Now fix $\varepsilon>0$. Then we find $s\in\mathcal{R}$ such that $|<P_s\xi-\eta,\xi>|\le\varepsilon$. Thus

$$\begin{aligned}\forall t\ge s\ :\ \|(P_t-P)\xi\|^2 &= \|(P_t-P)(P_s-P)\xi\|^2 \le \|(P_s-P)\xi\|^2\\ &= <(P_s-P)\xi,(P_s-P)\xi> = <(P_s-P)\xi,\xi>\\ &= <P_s\xi-\eta,\xi> \le \varepsilon.\ \square\end{aligned}$$

Let $\mathcal{H}_s=\text{Im}(P_s)$. Then L_t is the projection onto $\overline{\bigcup_{s<t}\mathcal{H}_s} = ((\bigcup_{s<t}\mathcal{H}_s)^\perp)^\perp = (\bigcap_{s<t}\mathcal{H}_s^\perp)^\perp$. Thus $I-L_t$ projects onto $\bigcap_{s<t}\mathcal{H}_s^\perp$. Note that $I-P_s$ projects onto $\mathcal{H}_s^\perp$. Thus by remark 2.1 $I-L_t$ is the strong operator limit of the decreasing net $(I-P_s)_{s<t}\downarrow$, i.e. $(I-L_t)\eta=\lim_{s\uparrow t}(I-P_s)\eta = I\eta-\lim_{s\uparrow t}P_s\eta$. Thus

$$L_t\eta = \lim_{s\uparrow t} P_s\eta \qquad \forall\eta\in\mathcal{H}. \tag{5}$$

A function $f : \mathcal{R} \to Y$ that maps into a topological space Y call **right continuous** in $t \in \mathcal{R}$, if the restriction $f|\{s \geq t\}$ is continuous in t with respect to the trace topology on $\{s \geq t\}$.

An equivalent definition is, that for any net $(t_i) \subseteq \{s \geq t\}$ with limit t in the topology of $\mathcal{R}$, the net $f(t_i)$ has limit $f(t)$.

Note that if t has a successor t', then f is always right continuous in t, as t is isolated to the right.

Define the **right limit** of f in t as the limit of the net $(f(s))_{s>t}$, if it exists, and we write $\lim_{s\to t, s>t} f(s)$ or $\lim_{s\downarrow t} f(s)$ for this limit. It is straight forward to check that f is right continuous in t iff t has a successor or $\lim_{s\downarrow t} f(s) = f(t)$.

The family of projections $(P_t)_{t\in\mathcal{R}} \uparrow$ call **right continuous**, if it is right continuous with respect to the strong operator topology in $B(\mathcal{H})$. Or, equivalently, if $\forall \xi \in \mathcal{H}$ the function $t \to P_t\xi$ is right continuous. Also equivalent is the assertion

$$P_t = R_t \qquad \forall t \text{ which have no successor.} \tag{6}$$

We remark that given a family of projections $(P_t)_{t\in\mathcal{R}}$ we can construct a right continuous one, say $(\tilde{P}_t)_{t\in\mathcal{R}}$, by setting $\tilde{P}_t = R_t$ if t has no successor and $\tilde{P}_t = P_t$ otherwise.

Lemma 2.2. *Let $\mathcal{R}$ be totally ordered with measure μ and assume $(I) - (II)$. Let $\mathcal{H}$ be Hilbert and $(P_t)_{t\in\mathcal{R}}$ an increasing family of projections in $B(\mathcal{H})$. Then the projection $P \in B(L^2(\mathcal{R}, \mu, \mathcal{H}))$ onto the predictable function space is L, i.e.*

$$P(X)(t) = L_t X_t \qquad X \in L^2(\mathcal{R}, \mu, \mathcal{H}),\, t \in \mathcal{R}$$

Proof. Choose $D \subseteq \mathcal{R}$ countable, dense in $\mathcal{R}$ and such that (3) fulfilled. Let $(\alpha, \beta] \subseteq \mathcal{R}$ be a finite interval. Consider $X = 1_{(\alpha,\beta]}\xi \in L^2(\mathcal{R}, \mu, \mathcal{H})$, i.e. X is a simple step function.

Let $D' = (\alpha, \beta) \cap D = \{d_1, d_2, d_3, ...\}$. For fixed $n \geq 1$ order the set $\{d_1, ..., d_n\}$ and obtain, lets say, $\{\alpha =: e_0 < e_1 < ... < e_n < e_{n+1} := \beta\}$. Set

$$X_n = \sum_{k=1}^{n+1} 1_{(e_{k-1},e_k]} P_{e_{k-1}}\xi.$$

Obviously X_n is predictable. Now let $t \in (\alpha, \beta]$.

If t is the successor of an element s then $L_t = P_s$ and $\{s, t\} \subseteq \{\alpha, \beta\} \cup D'$. Thus for some m we have $\lim_n X_n(t) = X_m(t) = P_s\xi = L_t X(t)$.

If t is not a successor then $(s, t) \cap D' \neq \emptyset \ \forall s < t$. Thus recalling (5)

$$\lim_n X_n(t) = \lim_{s\uparrow t, s\in D'} P_s\xi = \lim_{s\uparrow t, s\in\mathcal{R}} P_s\xi = L_t\xi = L_t X(t).$$

Summing up we have $\lim_n X_n(t) = L_t X(t)$ for all $t \in \mathcal{R}$, thus LX is μ-measurable as a pointwise limit of step functions.

If X is a linear combination of step functions like above then $\|LX\|^2 = \int \|L_t X(t)\|^2 d\mu(t) \leq \|X\|^2$. Due to the assumptions $(I) - (II)$ this step functions are dense in L^2 and we can continuously extend the map L on $L^2(\mathcal{R}, \mu, \mathcal{H})$.

LX is predictable for X step, thus $\mathrm{Im}(L)$ remains in the closed space of predictable functions. On the other hand obviously L does not alter a predictable step function $1_{(s,t]}\xi$,

$\xi \in \mathrm{Im}(P_s)$. Thus $\mathrm{Im}(L)$ is exactly the space of predictable processes. Obviously L is idempotent and $L^* = L$, i.e.

$$< Lf, g >= \int < L_t f(t), g(t) > d\mu(t) = \int < f(t), L_t g(t) > d\mu(t) =< f, Lg > .\square$$

Lemma 2.3. *Let $\mathcal{R}$ be totally ordered with measure μ and assume $(I) - (II)$. Let $\mathcal{H}$ be Hilbert and $(P_t)_{t\in\mathcal{R}}$ be an increasing, right continuous family of projections in $B(\mathcal{H})$. Then the projection $P \in B(L^2(\mathcal{R}, \mu, \mathcal{H}))$ onto the adapted functions is*

$$P(X)(t) = P_t X_t \qquad X \in L^2(\mathcal{R}, \mu, \mathcal{H}),\ t \in \mathcal{R}. \tag{7}$$

Proof. We show that one can analogously to lemma 2.2 define a function

$$R \in B(L^2(\mathcal{R}, \mu, \mathcal{H})) \qquad R(X)(t) = R_t X_t.$$

The problem is the μ-measurability of the expression. To avoid double work we reverse the order of $\mathcal{R}$ and obtain the order relation $\tilde{<}$. Transform the projections P_t to $\tilde{P}_t := I - P_t \uparrow$. Now lemma 2.2 shows that $\tilde{L}_t : L^2 \to L^2$ is defined. But

$$\tilde{L}_t(X)(t) = \lim_{s \tilde{<} t} \tilde{P}_s X_t = \lim_{s>t} (I - P_s) X_t = X_t - R_t X_t.$$

Thus $R = I_{L^2(\mathcal{R},\mu,\mathcal{H})} - \tilde{L}$, thus $R \in B(L^2(\mathcal{R}, \mu, \mathcal{H}))$. Now let $S \subseteq \mathcal{R}$ the (countable) set of elements which have a successor. Then using (6) we have $R_t = P_t\ \forall t \in S^C$. Thus for step functions $X = 1_{(\alpha,\beta]}\xi$ ($\alpha < \beta \in \mathcal{R}, \xi \in \mathcal{H}$) we get that

$$Y : \mathcal{R} \to \mathcal{H}\ :\ Y(t) = P_t X_t = 1_{S\cap(\alpha,\beta]}(t)\, P_t \xi + 1_{S^C\cap(\alpha,\beta]}(t)\, R_t \xi$$

is μ-measurable. Thus the map (7) is well behaved for step functions X and can be extended continuously on $L^2(\mathcal{R}, \mu, \mathcal{H})$. It is now easy to check that (7) is the projection onto the adapted functions. $\square$

In the next theorem we show an integrable version of theorem 1.2. It gives a decomposition in $L^2(\mathcal{R}, \mu, \mathcal{H})$ and we now use convergence in that space rather than in $\mathcal{H}$ for each point of time. The proof is divided into steps and we urgently remark that the length of step 1a and step 1b, where we estimate the norm of some step function Y_{α_n} ((9) and (11)), is mainly due to the generality of the measure μ; finiteness is our only restriction. E.g. if $\mathcal{R} = (0, 1)$ and μ the Lebesgue measure the estimation is simple. Or e.g. if we assume the function X in L^∞, the proof of step 1a/1b reduces to an immediate and trivial estimate ($\|Y_{\alpha_n}\|^2_{L^2(\mathcal{R},\mu,\mathcal{H})} \le \mu(\mathcal{R})\, \|X\|^2_{L^\infty(\mathcal{R},\mu,\mathcal{H})}$, compare (9), (11)).

Theorem 2.4. Doob-Meyer-decomposition variant 2

Let $\mathcal{R}$ be a totally ordered set with finite measure μ and assume (I). Let $\mathcal{H}$ be a Hilbert space and $(P_t)_{t\in\mathcal{R}}$ an increasing, right continuous family of projections in $B(\mathcal{H})$. Let $X \in \mathcal{L}^2(\mathcal{R}, \mu, \mathcal{H})$ be adapted and right continuous w.r.t. the weak topology in $\mathcal{H}$ and let exist a constant $C \in \mathbb{R}$ such that $\forall n \ge 1\ \forall t_0 < \dots < t_n \in \mathcal{R}$

$$\sum_{k=1}^{n} \|P_{t_{k-1}} X_{t_k}\|^2 - \|X_{t_{k-1}}\|^2 \ge C.$$

Then we find $M, A \in \mathcal{L}^2(\mathcal{R}, \mu, \mathcal{H})$ such that $X = M + A$ μ-a.e., such that M is a right continuous martingale μ-a.e. and such that A is predictable μ-a.e.

If $\mathcal{H} = L^2(\Omega)$ and X is a positive submartingale then A can be chosen increasing, i.e. $A_s \le A_t$ for all $s \le t$ in $\mathcal{R}\backslash Z$ for some μ-null set Z.

Proof. **STEP 1a.** Consider the function

$$f : \mathcal{R} \to \mathbb{R} : f(s) = \mu(\{\, t \in \mathcal{R} \mid t \le s \,\}).$$

Notice $f \uparrow$. Fix $\alpha < \beta$ in $\mathcal{R}$ and consider $I = f([\alpha, \beta])$. Let exist $u_0 < u_1 < u_2 < u_3 \in \mathbb{R}$ such that

$$[f(\alpha), f(\beta)] = [u_0, u_1] \cup (u_1, u_2] \cup (u_2, u_3].$$

Now assume

$$(u_1, u_2] \cap I = \emptyset \qquad \text{and let} \qquad J := \{\, s \in \mathcal{R} \mid f(s) > u_2 \,\}.$$

Our aim here is to show (8) below. Notice $J \neq \emptyset$ and $J^C \neq \emptyset$. Consider the dense countable subset $D \subseteq \mathcal{R}$ fulfilling (3) and also recall (4). At first let us assume $\nexists \inf J$ in $\mathcal{R}$. Then neither J nor J^C can be consist of a single point. Thus for $x \neq y$ in J we have $[x, y) \cap D \neq \emptyset$, thus $J \cap D \neq \emptyset$. Analogously $J^C \cap D \neq \emptyset$. Set $A = \bigcap_{s \in J^C \cap D,\, t \in J \cap D} (s, t]$. D is countable, thus

$$\mu(A) = \inf_{s \in J^C \cap D,\, t \in J \cap D} f(t) - f(s) \ge u_2 - u_1 > 0.$$

But A is empty: assume $x \in A \cap J$ then $x > y \in J$ for some y, thus $[y, x) \cap D \neq \emptyset$ thus $x \notin A \cap J$, contradiction. Similarly $A \cap J^C = \emptyset$, otherwise $\exists \max J^C$ and thus $\exists \inf J$. Summing up $A = \emptyset$ but $\mu(A) > 0$, thus the assumption $\nexists \inf J$ was wrong.

So let $x = \inf J$. Assume $x \in J^C$. Then $|J| > 1$ thus $J \cap D \neq \emptyset$. Consider $A = \bigcap_{t \in J \cap D} (x, t]$. $A = \emptyset$, as $x = \max J^C$. But $\mu(A) = \inf f(t) - f(x) \ge u_2 - u_1 > 0$. Contradiction, thus $x \in J$.

If $J^C = \{y\}$ then $\mu(\{x\}) = \mu((y, x]) = f(x) - \sup f(J^C)$. Otherwise

$$f(x) - \sup f(J^C) = \inf\{\, f(x) - f(s) \mid s \in J^C \,\} = \inf_{s \in J^C} \mu((s, x]) \ge \mu(\{x\})$$

$$= \mu(\{x\}) = \mu(\bigcap_{s \in J^C \cap D} (s, x]) = f(x) - \sup f(J^C \cap D) \ge f(x) - \sup f(J^C).$$

Additionally we remark $f(x) = \min(I \cap (u_2, u_3])$. Summing up we have shown

$$\exists x \in \mathcal{R} \qquad x = \min J \quad \text{and} \quad \mu(\{x\}) = f(x) - \sup(I \cap [u_0, u_1]). \tag{8}$$

STEP 1b. For $\alpha = \{\alpha_0 < ... < \alpha_N\} \subseteq \mathcal{R}$ set

$$Y_\alpha = \sum_{k=1}^{N} 1_{(\alpha_{k-1}, \alpha_k]} X_{\alpha_k} \in L^2(\mathcal{R}, \mu, \mathcal{H}). \tag{9}$$

Note that $\|X(s)\|^2 + C \le \|P_s X(t)\|^2 \le \|X(t)\|^2$ for all $s \le t$ in $\mathcal{R}$. So we get

$$\|Y_\alpha\|^2 - \|X\|^2 \le \int_{(\alpha_0, \alpha_N]} \|Y_\alpha(s)\|^2 - \|X(s)\|^2 ds$$

$$= \sum_{k=1}^{N} \int_{(\alpha_{k-1},\alpha_k]} \|X(\alpha_k)\|^2 - \|X(s)\|^2 ds$$

$$\leq \sum_{k=1}^{N} \int_{(\alpha_{k-1},\alpha_k]} \|X(\alpha_k)\|^2 - \|X(\alpha_{k-1})\|^2 - C\, ds$$

$$\leq \sum_{k=1}^{N} (f(\alpha_k) - f(\alpha_{k-1}))\,(\|X(\alpha_k)\|^2 - \|X(\alpha_{k-1})\|^2) \ + \ \mu(\mathcal{R})|C|. \tag{10}$$

Now decompose equidistantly the interval $[f(\alpha_0), f(\alpha_N)] = [u_0, u_1] \cup (u_1, u_2] \cup ... \cup (u_{n-1}, u_n]$, i.e. for some $\varepsilon \geq 0$, $u_k - u_{k-1} = \varepsilon$ for all $1 \leq k \leq n$. Construct a finite set $\beta \subseteq \mathcal{R}$ with $\alpha \subseteq \beta$ by adding points x_k to α for each k such that $f(x_k) \in (u_{k-1}, u_k]$, if possible. If for some k the set $f([\alpha_0, \alpha_N]) \cap (u_{k-1}, u_k]$ is empty then add the point x with nonzero point mass and $f(x) = \min(f([\alpha_0, \alpha_N]) \cap (u_k, u_n])$ that we have found in 'step 1a'. Let $\beta = \{\alpha_0 = \beta_0 < \beta_1 < ... < \beta_M = \alpha_N\}$. Set

$$\gamma_1 = \{\, 1 \leq k \leq M \mid f(\beta_k) - f(\beta_{k-1}) \leq 2\varepsilon \,\}.$$

Set $\gamma_2 = \{1, ..., M\} \backslash \gamma_1$. For all $k \in \gamma_2$, β_k must be one of the special points with nonzero point mass and taking into account the result of 'step 1a' about the value of this mass we get

$$\mu(\{\beta_k\}) + \varepsilon \geq f(\beta_k) - f(\beta_{k-1}) > 2\varepsilon.$$

Abbreviate

$$g(k) := \|X(\beta_k)\|^2 - \|X(\beta_{k-1})\|^2.$$

Set $\gamma_{1+} = \{\, k \in \gamma_1 \mid g(k) \geq 0 \,\}$ and $\gamma_{1-} = \gamma_1 \backslash \gamma_{1+}$. Analogously represent $\gamma_2 = \gamma_{2+} \cup \gamma_{2-}$. We are ready to estimate (continuing (10))

$$\|Y_\beta\|^2 - \|X\|^2 \leq \sum_{k \in \gamma_{1+}} 2\varepsilon\, g(k) + \sum_{k \in \gamma_{2+}} (\mu(\{\beta_k\}) + \varepsilon)\, g(k) + \sum_{k \in \gamma_{2-}} 2\varepsilon\, g(k) \ + \ \mu(\mathcal{R})|C|$$

$$= \ 2\varepsilon \sum_{k=1}^{M} g(k) \ - \ 2\varepsilon \sum_{k \in \gamma_{1-}} g(k) + \sum_{k \in \gamma_{2+}} (\mu(\{\beta_k\}) - \varepsilon)\, g(k) \ + \ \mu(\mathcal{R})|C|$$

$$\leq \ 2\varepsilon(\|X(\beta_M)\|^2 - \|X(\beta_0)\|^2) \ - \ 2\varepsilon \sum_{k \in \gamma_{1-}} C + \sum_{k \in \gamma_{2+}} \mu(\{\beta_k\}) \|X(\beta_k)\|^2 \ + \ \mu(\mathcal{R})|C|$$

$$\leq \ 2\varepsilon \|X(\alpha_N)\|^2 \ + \ 2\varepsilon\, |\gamma_{1-}|\, |C| \ + \ \|X\|^2_{L^2(\mathcal{R},\mathcal{H})} \ + \ \mu(\mathcal{R})|C|.$$

Recall we have chosen an equidistant decomposition into n intervals of $[f(\alpha_0), f(\alpha_N)]$. Notice that $\varepsilon \leq \mu(\mathcal{R})/n$. Choose n big enough such that $2\varepsilon \|X(\alpha_N)\|^2 \leq 1$ and that $N \leq n$. Recall the situation when we had filled each interval $(u_{k-1}, u_k]$ to obtain β. There chose maximal only one new β_i for each interval. Thus $M \leq N + n \leq 2n$. Thus $2\varepsilon\, |\gamma_{1-}|\, |C| \leq 2\frac{\mu(\mathcal{R})}{n} 2n|C| = 4\mu(\mathcal{R})|C|$.

Thus for fixed decomposition α of the interval $[\alpha_0, \alpha_N]$ we find a finer $\beta \supseteq \alpha$ one such that

$$\|Y_\beta\|^2 - \|X\|^2 \leq 1 + 4\mu(\mathcal{R})|C| + \|X\|^2 + \mu(\mathcal{R})|C| =: \delta.$$

Now choose intervals $(a_1, b_1] \subseteq (a_2, b_2] \subseteq ...$ such that $\bigcup_{k=1}^{\infty}(a_k, b_k] = \mathcal{R}$. (We assume here $\mathcal{R}$ has no minimum. A slight addition by taking also $\{\min\}$ into account would assimilate the general case.) Choose a dense countable subset $D \subseteq \mathcal{R}$ fulfilling (3). $D = \{d_1, d_2, ...\}$. Define inductively: Start with $\alpha_1 := \{a_1, b_1\}$. If $\alpha_{n-1} \subseteq \mathcal{R}$ is a decomposition of $(a_{n-1}, b_{n-1}]$ then add a_n, b_n and $(a_n, b_n] \cap \{d_1, ...d_n\}$ to α_{n-1}. Choose a finer decomposition α_n of this new α_{n-1} such that $\|Y_{\alpha_n}\|^2 - \|X\|^2 \leq \delta$. Summing up we get

$$\forall n \geq 1 \quad \|Y_{\alpha_n}\|^2 \leq \|X\|^2 + \delta \qquad \text{and} \qquad D \subseteq \bigcup_{n=1}^{\infty} \alpha_n \,, \qquad \alpha_1 \subseteq \alpha_2 \subseteq ... \tag{11}$$

STEP 2. Fix $n \geq 1$. For our $\alpha_n = \{t_0 < ... < t_N\}$ consider the Doob-Meyer-decomposition to the process $X(t_0), ..., X(t_N)$ (see lemma 1.1) and we obtain $A(t_k)$ and $M(t_k)$ for $0 \leq k \leq N$, where M is the martingale part and A is the predictable part, i.e. $A(t_k) \in \mathrm{Im}(P_{t_{k-1}})$. Define adapted processes $M_n, A_n \in \mathcal{K} := L^2(\mathcal{R}, \mu, \mathcal{H})$

$$A_n = \sum_{k=1}^{N} 1_{(t_{k-1}, t_k]} A(t_k) \qquad \text{and}$$

$$M_n(s) = P_s M(t_N) \qquad \forall s \in \mathcal{R}.$$

We remark that due to lemma 2.3 M_n is the projection of $1_{\mathcal{R}} M(t_N)$ onto the adapted functions, thus indeed $M_n \in \mathcal{K}$. Further note that $M_n(t_k) = P_{t_k} M(t_N) = M(t_k)$. Thus due to the fundamental lemma 1.1 and our assumption

$$\forall t_{k-1} < s \leq t_k \quad \|M_n(s)\|^2 \leq \|M_n(t_k)\|^2 \leq \|X(t_k)\|^2 - C = \|Y_{\alpha_n}(s)\|^2 - C \tag{12}$$

$$\forall s \geq t_N \quad \|M_n(s)\|^2 = \|M(t_N)\|^2 \leq \|X(t_N)\|^2 - C \leq \|X(s)\|^2 - 2C$$

We estimate (also recall $\|Y_{\alpha_n}\|^2 \leq \|X\|^2 + \delta$; note $M_n(t_0) = X(t_0)$)

$$\|M_n\|^2 = \int_{\mathcal{R}} \|M_n(s)\|^2 ds$$

$$\leq \mu(\mathcal{R}) \|X(t_0)\|^2 + \int_{t_0}^{t_N} \|Y_{\alpha_n}(s)\|^2 - C \, ds + \int_{s > t_N} \|X(s)\|^2 - 2C \, ds$$

$$\leq \mu(\mathcal{R})(\|X(a_1)\|^2 - C) + \|X\|^2 + \delta + \mu(\mathcal{R}) \, |C| + \|X\|^2 + 2\mu(\mathcal{R}) \, |C|$$

Analogously we get $\forall t_{k-1} < s \leq t_k$ (recall (12))

$$\|A_n(s)\| = \|A(t_k)\| = \|X(t_k) - M(t_k)\| \leq \|Y_{\alpha_n}(s)\| + \sqrt{\|Y_{\alpha_n}(s)\|^2 - C} \leq 2\|Y_{\alpha_n}(s)\| + \sqrt{|C|}.$$

Thus

$$\|A_n\| \leq 2\sqrt{\|X\| + \delta} + \sqrt{\mu(\mathcal{R})}\sqrt{|C|}.$$

So $(A_n, M_n)_{n \geq 1}$ defines a bounded sequences in $\mathcal{K} \oplus \mathcal{K}$ and we obtain a weakly convergent subsequence $(A_{n_k}, M_{n_k})_{k \geq 1}$, as it is well known for Hilbert spaces. The limit be (A, M).

STEP 3. We show that $X = M + A$. Consider a finite interval $\beta \subseteq \mathcal{R}$. Choose m such that $\beta \subseteq \alpha_n$ for all $n \geq m$. Let $t(n, s) = \min\{\, t \in \alpha_n \mid s \leq t \,\}$. Let $\xi \in \mathcal{H}$. We compute

$$< M_n + A_n - X, 1_\beta \xi >= \int_\beta < M_n(s) + A_n(s) - X(s), \xi > ds$$

$$= \int_\beta < P_s M_n(t(n,s)) + P_s A_n(t(n,s)) - X(s), \xi > ds$$

$$= \int_\beta < P_s X(t(n,s)) - X(s), \xi > ds$$

Recall that $\alpha_1 \subseteq \alpha_2 \subseteq ...$ and $D \subseteq \bigcup_{k \geq 1} \alpha_k$. We have

$$| < P_s X(t(n,s)) - X(s), \xi > | \leq \big(\|X(t(n,s))\| + \|X(s)\| \big) \|\xi\|$$

$$\leq \big(\|X(t(m,s))\| - C + \|X(s)\| \big) \|\xi\| = \big(\|Y_{\alpha_m}(s)\| - C + \|X(s)\| \big) \|\xi\|.$$

This function is in $L^1(\mathcal{R}, \mu)$. We therefore can apply dominated convergence to get

$$< M + A - X, 1_\beta \xi >= \lim_k < M_{n_k} + A_{n_k} - X, 1_\beta \xi >$$

$$= \int_\beta \lim_k < X(t(n_k, s)) - X(s), P_s \xi > ds = 0,$$

for the last equality using the weakly right continuity of X. We have shown that $X = M + A$ μ-a.e.

Now note that M_n is a martingale and A_n is predictable for all $n \geq 1$. So the weak limits M and A must be a martingale respectively predictable as the space of martingales respectively space of predictable processes is norm-closed. Indeed the space of predictable functions is closed by definition. The $\mathcal{L}^2$-space of martingales define as

$$m = \{ M \in \mathcal{L}^2(\mathcal{R}, \mu, \mathcal{H}) \mid \exists \text{ nullset } Z \subseteq \mathcal{R} \ : \ M|_{\mathcal{R}-Z} \text{ is a martingale} \}.$$

Its L^2-space is closed: let a sequence $M_n \to M$ in L^2-seminorm where $M_n \in m$, $M \in \mathcal{L}^2$. Then $M_{n_k}(s) \to M(s)$ for a subsequence for almost all $s \in \mathcal{R}$. Thus clearly $P_s M(t) = P_s \lim_k M_{n_k}(t) = \lim_k P_s M_{n_k}(t) = M(s)$ for all $s, t \in \mathcal{R} - Z$ for some null set Z, so $M \in m$.

We extend the 'carrier' $\Gamma \subseteq \mathcal{R}$, $\mu(\Gamma^C) = 0$, of M, where it is a martingale, by setting for all $s \in \mathcal{R}$, $M_s := P_s M_t$ for some $t \geq s$ (axiom of choice) if such t exists. The new carrier Γ_2 is so big, such that $\forall x \notin \Gamma_2 : x > \Gamma_2$. If $\exists \max(\Gamma_2)$ then we can (obviously) extend Γ_2 such that $\Gamma_2 = \mathcal{R}$. Anyway, M is now right continuous on Γ_2: fix $s \in \Gamma_2$, let $S > s$, $S \in \Gamma_2$. Then using the r.c. of the family of projections, $\lim_{t \downarrow s} M_t = \lim_{t \downarrow s} P_t M_S = P_s M_S = M_s$.

For the last statement, when X is a submartingale and the family of projections $(P_t)_{t \in \mathcal{R}}$ is (naturally) induced by a filtration $(\mathcal{F}_t)_{t \in \mathcal{R}}$, then A_n is increasing $\forall n \geq 1$. Recall that every norm-closed, convex set is closed under the weak topology. So A is increasing as the weak limit acts in the norm-closed (convex) cone of all increasing processes. Here the increasing processes define analogously as the space m and argue analogously for its closeness. □

Remark. Again (as we also remarked below theorem 1.2) theorem 2.4 does not decompose all decomposable processes. Indeed, e.g., one could set $\mathcal{H} = \mathbb{C}$, $\mathcal{R} = (0,1)$, μ the Lebesgue measure, $P_s = I$ for $s \in (0,1)$, and $X(s) := s^{-1/4}$ (please compare (2)).

3. Application of the discrete Doob-Meyer-decomposition: Doleans-measure and integral associated to a martingale on a totally ordered set

For this chapter also compare [3] (Doléans-measure); and [6] or e.g. [8], [1] for stochastic integration. Also cf. [2] for the time scale to be $\mathbb{R}^2$.

In this chapter we are given a totally ordered set $\mathcal{R}$, a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $(\mathcal{F}_t)_{t\in\mathcal{R}}$, i.e. an increasing family of σ-algebras in $\mathcal{F}$.

For the definition of right continuity and right limit, we refer to the definitions in chapter 2.

A totally ordered set $\mathcal{R}$ let us call **sequentially right complete** if every countable subset in $\mathcal{R}$ bounded from below has an infimum in $\mathcal{R}$, i.e.

$$\forall M \subseteq \mathcal{R} : \forall x \in \mathcal{R} : M \text{ countable } \wedge\, x \leq M \to \exists \inf M.$$

A function $f : \mathcal{R} \to Y$ into a topological space Y call **sequentially right continuous** if f is right continuous in all points of the set

$$\mathcal{S} := \{\, s \in \mathcal{R} \mid \exists C \subseteq \mathcal{R} :\ C \text{ countable },\ s < C,\ s = \inf C \,\}. \tag{13}$$

The **predictable σ-algebra** $\mathcal{P}$ is defined as the σ-algebra in $\mathcal{R}\times\Omega$, generated by the sets

$$\{\, (s,t] \times C \mid s,t \in \mathcal{R},\ s < t,\ C \in \mathcal{F}_s \,\}.$$

Theorem 3.1. Doléans-measure

Let $\mathcal{R}$ be a totally ordered, sequentially right complete set. Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $(\mathcal{F}_t)_{t\in\mathcal{R}}$. Let $X : \mathcal{R} \to L^2(\Omega)$ a positive submartingale, such that $t \to E(M_t)$ is sequentially right continuous. Then

$$\mu(C \times (s,t]) = E(1_C\,(X_t - X_s)),$$

for $C \in \mathcal{F}_s$ and $s < t$ in $\mathcal{R}$, defines a measure on the predictable σ-algebra $\mathcal{P}$.

Proof. Consider the set algebra which consists of finite unions of elementary rectangles of the form $C \times (s,t]$ for $C \in \mathcal{F}_s$ and $s < t$ in $\mathcal{R}$. Due to the extension theorem of Hahn (e.g. [7] VI.7) it remains to show that μ is σ-additive on this algebra.

For a disjoint union $\bigcup_{k=1}^N r_k$ of elementary rectangles r_k the additively defined μ, i.e. $\mu(\bigcup_{k=1}^N r_k) = \sum_{k=1}^N \mu(r_k)$, is well defined. We skip the well-known and easy proof.

We now show that μ is σ-additive. Let

$$\bigcup_{k=1}^{\infty} C_k \times (s_k, t_k] = C \times (S,T]$$

a disjoint decomposition where $C_k \in \mathcal{F}_{s_k}$, $C \in \mathcal{F}_S$ and $s_k < t_k$, $S < T$ in $\mathcal{R}$.

We apply the Doob-Meyer-decomposition theorem 1.2 to the positive submartingale X on the interval $[S,T]$, i.e. we obtain adapted functions $M, A : [S,T] \to L^2(\Omega)$ such that $X = M + A$ on $[S,T]$, such that M is a martingale and such that A is increasing. Thus we can simplify the definition of μ: $(s < t,\ D \in \mathcal{F}_s)$

$$E(1_D(X_t - X_s)) = \int 1_D\, E(M_t + A_t - M_s - A_s|\mathcal{F}_s)dP = \int 1_D(A_t - A_s)dP.$$

Thus we have

$$\mu(D \times (s,t]) = \int 1_D(A_t - A_s)dP. \tag{14}$$

$E(X)$ was assumed to be sequentially right continuous thus

$$\lim_{t \downarrow s} \int |A_t - A_s| dP = \lim_{t \downarrow s} \int X_t - X_s dP = 0.$$

for all points $s \in \mathcal{S}$ defined by (13). Thus A is sequentially $L^1(\Omega)$-right continuous.

Now we reduce our 'time' scale to the countable index set

$$I := \{S, T, s_1, t_1, s_2, t_2, s_3, t_3, ...\}.$$

As $A_s \le A_t$ for all $s \le t$ we find a null set $Z \subseteq \Omega$ such that $A_s(\omega) \le A_t(\omega)$ $\forall \omega \in \Omega - Z$, $\forall s \le t \in I$. Fix $i \in I$. Assume $i = \inf\{\, s \in I \mid s > i \,\}$. Then $\inf_{s>i, s \in I} \|A_s - A_i\|_{L^1(\Omega)} = 0$ as A is L^1-right continuous in i. Thus $\lim_n \|A_{v_n} - A_i\|_{L^1(\Omega)} = 0$ for a sequence $v_n > i$, $v_n \in I$. Thus we find a subsequence (u_n) of (v_n) and a null set $Y \subseteq \Omega$ such that $\lim_n A_{u_n}(\omega) - A_i(\omega) = 0$ $\forall \omega \in \Omega - Y$. Especially $\forall \omega \in \Omega - (Y \cup Z)$ we get

$$0 \le \inf_{s>i, s\in I} A_s(\omega) - A_i(\omega) \le \inf_n A_{u_n}(\omega) - A_i(\omega) = 0.$$

As I is countable we find a null set $Z_2 \supseteq Z$ such that

$$\forall i \in I : \forall \omega \in \Omega - Z_2 : i = \inf\{\, s \in I \mid s > i \,\} \Rightarrow \inf_{s>i, s \in I} A_s(\omega) - A_i(\omega) = 0. \tag{15}$$

Now with monotone convergence and recalling (14) we get

$$\sum_{k=1}^{\infty} \mu(C_k \times (s_k, t_k]) = \int \sum_{k=1}^{\infty} 1_{C_k}(\omega) \left(A_{t_k}(\omega) - A_{s_k}(\omega) \right) d\omega. \tag{16}$$

For fixed $\omega \in C - Z_2$ we consider the right sum. Abbreviate $A(s) := A_s(\omega)$. We drop all summands with $1_{C_k}(\omega) = 0$ and obtain a subsequence $(u_k, v_k)_k$ of $(s_k, t_k)_k$ such that $\bigcup_k (u_k, v_k] = (S, T]$ is a disjoint partition. Thus

$$\sum_{k=1}^{\infty} 1_{C_k}(\omega) \big(A(t_k) - A(s_k) \big) = \sum_{k=1}^{\infty} A(v_k) - A(u_k)$$

$$= \sum_k \lambda\big(\left(A(u_k), A(v_k) \right] \big) = \lambda\big(\bigcup_k \left(A(u_k), A(v_k) \right] \big) = \lambda(l),$$

λ being the Lebesgue measure, $l \subseteq \mathbb{R}$ being obviously defined.

Assume $l \ne (A(S), A(T)]$, then $\exists x \in (A(S), A(T)] \backslash l$. Set

$$J = \{u_1, v_1, u_2, v_2, ...\} \subseteq I \qquad \text{and} \qquad U = \{\, i \in J \mid A(i) \ge x \,\}.$$

Note $U \ne \emptyset$ as $T \in U$. $\exists \gamma = \inf U$ using the sequentially completeness of $\mathcal{R}$. $\gamma \in [S, T]$. At first look at the case $\gamma > S$. Then we find $n \ge 1$ with $\gamma \in (u_n, v_n]$. Then $v_n \le U$. Thus $\gamma = v_n \in J$. Especially $u_n \notin U$. Thus $A(u_n) < x$. Anyway $\gamma \in J$. First case: $\gamma \in U$. Then $A(\gamma) \ge x$. Thus $\gamma \ne S$ as $A(S) < x$. But then $x \in (A(u_n), A(v_n)]$. Contradiction. Second case: $\gamma \notin U$. Then $U \subseteq \{ i \in I \mid i > \gamma \} =: U_2$. Thus $\gamma \leq \inf U_2 \leq \inf U = \gamma$. And note if $\gamma < i \in I$ then $\exists u \in U : u \in (\gamma, i)$, so $A(i) \geq A(u)$. Thus recalling (15)

$$0 = \inf_{i>\gamma, i\in I} A(i) - A(\gamma) = \inf_{u\in U} A(u) - A(\gamma) \geq x - A(\gamma).$$

Thus $\gamma \in U$, contradiction. We conclude

$$\sum_{k=1}^{\infty} 1_{C_k}(\omega)(At_k(\omega) - A_{s_k}(\omega)) = 1_C(\omega)(A(T) - A(S)) \qquad \forall \omega \in \Omega - Z_2.$$

Thus $\sum_{k=1}^{\infty} \mu(C_k \times (s_k, t_k]) = \mu(C \times (S, T])$ with (16) □

With the filtration we can naturally associate an increasing family of projections $(P_t)_{t\in\mathcal{R}}$, $P_t \in B(L^2(\Omega, \mathcal{F}, P))$, where P_t projects onto $L^2(\Omega, \mathcal{F}_t, P)$. For the definition of a right continuous such family, we refer to chapter 2.

Lemma 3.2. *Let $\mathcal{R}$ be a totally ordered set and $(\mathcal{F}_s)_{s\in\mathcal{R}}$ a filtration. Fix $t \in \mathcal{R}$. If the filtration is right continuous in t then the family $(P_s)_{s\in\mathcal{R}}$ is right continuous in t.*

If the filtration is augmented, i.e. each $\mathcal{F}_s$ contains the P-null-sets of $\mathcal{F}$, then we can reverse this conclusion.

Proof. Remark: if t has a successor element s we always say the filtration is right continuous in t as t is isolated to the right in the sense that $[t, s) = \{t\}$. Thus in that case both the filtration and the projections are automatically right continuous. Thus assume t has no successor. View $\mathcal{H}_s = L^2(\Omega, \mathcal{F}_s)$ as a closed subspace in $\mathcal{H} = L^2(\Omega, \mathcal{F})$. Consider the isometrically identic embedding $i : L^2(\Omega, \bigcap_{s>t} \mathcal{F}_s) \to \bigcap_{s>t} L^2(\Omega, \mathcal{F}_s)$. i is surjective: indeed, we can assume the filtration is augmented as it does not change the L^2-spaces; let $f \in \mathcal{H}$ be a $\mathcal{F}$-measurable representant of the right space. $[f] \in L^2(\mathcal{F}_s)$ for $s > t$, thus f is $\mathcal{F}_s$-measurable, thus f is $\bigcap_{s>t} \mathcal{F}_s$ measurable, thus $[f] \in L^2(\Omega, \bigcap_{s>t} \mathcal{F}_s)$.

Recall R_t is the projection onto $\bigcap_{s>t} \mathcal{H}_s$. Thus we get the filtration is right continuous in $t \iff \mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s \implies \mathcal{H}_t = \bigcap_{s>t} \mathcal{H}_s \iff P_t = R_t$ (please also compare 2.1 and (6)) The only right arrow can be reversed if the filtration is augmented. □

Lemma 3.3. *Let $\mathcal{R}$ be a totally ordered, sequentially right complete set. Let $(\Omega, \mathcal{F}, P)$ be a probability space with sequentially right continuous filtration $(\mathcal{F}_t)_{t\in\mathcal{R}}$. Let h be a Hilbert space and $M : \mathcal{R} \to L^2(\Omega, h)$ be a martingale. Then*

$$\mu(C \times (s, t]) = \|1_C (M_t - M_s)\|^2_{L^2(\Omega,h)},$$

for $C \in \mathcal{F}_s$ and $s < t$ in $\mathcal{R}$, defines a measure on the predictable σ-algebra $\mathcal{P}$.

Proof. Please note that we have the well-known (and easy) identity

$$\|1_C (M_t - M_s)\|^2_{L^2(\Omega,h)} = \|1_C M_t\|^2 - \|1_C M_s\|^2 = E(1_C(X_t - X_s)),$$

where $X : \mathcal{R} \to L^1(\Omega)$ and $X_t(\omega) = \|M_t(\omega)\|^2_h$ and X is a positive submartingale.

We note that if the filtration is right continuous in a point t, a martingale automatically is L^2-right continuous there. Indeed lemma 3.2 shows that for some fixed $T > t$ in $\mathcal{R}$ we have $\lim_{s\downarrow t} M_s = \lim_{s\downarrow t} P_s M_T = P_t M_T = M_t$.

Thus if we assume that $M : \mathcal{R} \to L^4(\Omega, h)$, then we can directly apply theorem 3.1 on $X : \mathcal{R} \to L^2(\Omega)$.

For the general case we show that μ is σ-additive. Let $\bigcup_{k=1}^{\infty} C_k \times (s_k, t_k] = C \times (S, T]$ a disjoint decomposition where $C_k \in \mathcal{F}_{s_k}$, $C \in \mathcal{F}_S$ and $s_k < t_k$, $S < T$ in $\mathcal{R}$. Consider the projections $Q_k : L^2(\Omega, h) \to L^2(\Omega, h) : Q_k(\xi) = 1_{C_k}(P_{t_k} - P_{s_k})\xi$, where $k \geq 1$. Consider the projection $Q : L^2(\Omega, h) \to L^2(\Omega, h) : Q(\xi) = 1_C(P_T - P_S)\xi$. Note that the Q_k are pairwise orthogonal (i.e. $Q_k Q_i = 0$ for $k \neq i$) and all Q_k are smaller than Q (i.e. $Q_k Q = Q_k$).

For μ to be σ-additive we have to show $\sum_{k=1}^{\infty} \mu(C_k \times (s_k, t_k]) = \mu(C \times (S, T])$. Equivalently $\sum_{k=1}^{\infty} \|Q_k M_T\|^2 = \|Q M_T\|^2$. Equivalently $0 = \|Q M_T\|^2 - \sum_{k=1}^{\infty} \|Q_k M_T\|^2 = \|(Q - \sum_{k=1}^{\infty} Q_k) M_T\|^2$, where the convergence of $\sum_k Q_k$ is respect the strong-operator-topology (i.e. pointwise; remark 2.1).

If we can show that equation for $M_T \in L^\infty(\Omega, h)$ then we are done as L^∞ is dense in L^2. Indeed, let $\eta_n \in L^\infty(\Omega, h)$ form a sequence such that $\|\eta_n - M_T\|_{L^2} \to 0$ for $n \to \infty$. Then $\|(Q - \sum_{k=1}^{\infty} Q_k) M_T\|^2 = \|(Q - \sum_{k=1}^{\infty} Q_k)(\lim_n \eta_n)\|^2 = \lim_n \|(Q - \sum_{k=1}^{\infty} Q_k)\eta_n\|^2 = 0$.

So let $M_T \in L^\infty(\Omega, h)$. Then $M_t \in L^\infty(\Omega, h)$ for all $t \leq T$ by a standard argument using the averaging theorem. Thus, again, theorem 3.1 supplies the desired σ-additivity. $\square$

Lemma 3.4. Stochastic Integral

Consider the assumptions of lemma 3.3. Then define a stochastic integral

$$I : L^2(\mathcal{R} \times \Omega, \mathcal{P}, \mu, h) \to L^2(\Omega)$$

via ($\xi \in h$, $\omega \in \Omega$, $s < t$ in $\mathcal{R}$, $C \in \mathcal{F}_s$)

$$I(\,1_{C \times (s,t]}\,\xi\,)(\omega) \;= <\xi, 1_C(\omega)\big(M_t(\omega) - M_s(\omega)\big)>_h$$

I is a contraction. If $h = \mathbb{C}$ then I is a isometry.

Proof. This is well-known straightforward stuff, by checking that for $X \in L^2(\mathcal{R} \times \Omega, \mathcal{P}, \mu, h)$ a step function, i.e. $X = \sum_{n=1}^{N} X_n$ where the X_n are pairwise disjoint, simple step functions on a predictable rectangle,

$$\|I(X)\|_{L^2(\Omega)}^2 = \sum_{n=1}^{N} \|I(X_n)\|_{L^2(\Omega)}^2 \leq \|X\|_{L^2(\mathcal{R} \times \Omega, \mathcal{P}, \mu, h)}^2 . \square$$

Similarly as the last lemma one checks

Lemma 3.5. *Consider the assumptions of lemma 3.3. Then a second kind of integral ($\omega \in \Omega$, $s < t$ in $\mathcal{R}$, $C \in \mathcal{F}_s$)*

$$\mathcal{I} : L^2(\mathcal{R} \times \Omega, \mathcal{P}, \mu) \to L^2(\Omega, \mathcal{H}) \;:\; \mathcal{I}(1_{C \times (s,t]}) = 1_C\,(M_t - M_s)$$

can be defined. $\mathcal{I}$ is a isometry. Especially $A \mapsto \mathcal{I}(1_A)$, $A \in \mathcal{A}$, defines a vectorial measure on the algebra $\mathcal{A} = \{\, p \in \mathcal{P} \mid \mu(p) < \infty \,\}$ with quadratic variation μ.

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