THE K-THEORY OF CERTAIN C^*-ALGEBRAS ENDOWED WITH GAUGE ACTIONS

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Abstract. We compute the K-theory of the Toeplitz algebra of a finitely aligned higher rank graph and of a higher rank Exel–Laca algebra under condition (II). Actually we deduce these results from a slightly more general technical theorem for C^*-algebras endowed with gauge actions and fixed point algebra AF, among other requirements.

1. Introduction

The K-theory of a C^*-algebra is an interesting data as it provides quite nontrivial information in many cases. This has become evident once more since Kirchberg [11] and Phillips [13] has shown that purely infinite simple nuclear unital C^*-algebras are completely classified by their K-theory. Generalizations of Cuntz–Krieger algebras like graph C^*-algebras usually allow a crossed product representation and thus in certain cases a K-theory computation by the Pimsner–Voiculescu sequence or by spectral sequences. Many generalized Cuntz–Krieger algebras are also classifiable by the Kirchberg–Phillips theorem and so their K-theory is quite interesting. In this paper we aim to compute the K-theory of two classes of generalized Cuntz–Krieger algebras: the Toeplitz algebras of finitely aligned higher rank graphs by Raeburn, Sims and Yeend [15], and higher rank Exel–Laca algebras [4] by the author under a condition called (II). More precisely we will show the following two theorems.

Theorem 1.1. *Let Λ be a finitely aligned higher rank graph and $\mathcal{T}C^*(\Lambda)$ its associated Toplitz algebra with generators $(t_\lambda)_{\lambda\in\Lambda}$. Then $K_0(\mathcal{T}C^*(\Lambda)) = \bigoplus_{\Lambda^0}\mathbb{Z}$ under the isomorphism $[t_\nu] \longleftrightarrow 1_{\{\nu\}}$ for all $\nu\in\Lambda^0$, and $K_1(\mathcal{T}C^*(\Lambda))) = 0$.*

Theorem 1.2. *Let $(\mathcal{A},\mathbb{F},\mathbb{I})$ be the generators and relations of a higher rank Exel–Laca algebra $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$ and suppose that $(\mathcal{A},\mathbb{F},\mathbb{I})$ satisfies condition (II). Then*

$$K_0(\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}) \cong \operatorname{lin}_{\mathbb{Z}}\{Q_{a_1}\dots Q_{a_n}\in\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}} |\, a_i\in\mathcal{A}\}$$

under the isomorphism $[Q_{a_1}\dots Q_{a_n}] \longleftrightarrow Q_{a_1}\dots Q_{a_n}$ (where $Q_a = a^*a$), and $K_1(\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}) = 0$.

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Theorem 1.1 generalizes the K-theory computation of the Toplitz algebra of the Cuntz algebra $\mathcal{O}_n$ in [7]. Theorem 1.2 sharpens the result in [5] in that a further assumption (I) is no longer required. The rank two Cuntz–Krieger type algebras in [3] can be computed by Theorem 1.2, see Corollary 2.5.

The method how these results are obtained is a mixture of the approach in [16] by Raeburn and Szymański, and the approach in [5] by the author. We start with a C^*-algebra $\mathcal{X}$ which is endowed with a gauge action $\mathcal{X} : \mathbb{T}^d \to \mathrm{Aut}(\mathcal{X})$, and (by further assumptions) write the stable form of $\mathcal{X}$ as a crossed product of an AF-algebra A by $\mathbb{Z}^d$ by Takai's duality theorem. Then we compute this crossed product by successive application of the Pimsner–Voiculescu sequence. In this way we will show the technical Theorem 2.2 which gives a nice description of the K-theory of $\mathcal{X}$, under special assumptions for $\mathcal{X}$, and deduce the above K-theory results for the graph and higher rank Exel–Laca algebras from this theorem.

The paper is organized as follows. In Section 2 we state the technical Theorem 2.2 and deduce a slightly stronger version of Theorem 1.1 and Theorem 1.2 from it. The Sections 3 and 4 are then dedicated to the proof of the technical Theorem 2.2.

2. The main results

Assume that $\mathcal{X}$ is a C^*-algebra which is generated by a subset $\mathcal{A}$ (called the *alphabet*) with a finite partition $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \ldots \sqcup \mathcal{A}_d$. Assume that we are given a *gauge action* $\Gamma : \mathbb{T}^d \to \mathrm{Aut}(\mathcal{X})$ determined by $\Gamma_\lambda(a_i) = \lambda_i a_i$ for all $1 \le i \le d$, $\lambda = (\lambda_1, \ldots \lambda_d) \in \mathbb{T}^d$ and $a_i \in \mathcal{A}_i$. We write $\mathbb{X}$ for the $*$-subalgebra of $\mathcal{X}$ generated by $\mathcal{A}$, W for the set of *words*

$$W = \{ a_1 \ldots a_n \in \mathcal{X} \mid n \ge 1,\, a_i \in \mathcal{A} \cup \mathcal{A}^* \},$$

and $(e_i)_{1 \le i \le d}$ for the canonical basis in $\mathbb{C}^d$.

Lemma 2.1. *There exists a map (called the* balance function*)* $\mathrm{bal} : W \backslash \{0\} \to \mathbb{Z}^d$ *determined by* $\mathrm{bal}(a_i) = e_i$ *for all* $1 \le i \le d, a_i \in \mathcal{A}_i \backslash \{0\}$ *and by the rules* $\mathrm{bal}(xy) = \mathrm{bal}(x) + \mathrm{bal}(y)$ *and* $\mathrm{bal}(z^*) = -z$ *for all nonzero words* x, y, z *such that* $xy \neq 0$.

Proof. For $x \in W \backslash \{0\}$ and $\lambda \in \mathbb{T}^d$ let $\mu_{x,\lambda}$ be the unique scalar such that $\Gamma_\lambda(x) = \mu_{x,\lambda} x$. Then $f : W \backslash \{0\} \to \hat{\mathbb{T}}^d$ given by $f(x)(\lambda) = \mu_{x,\lambda}$ satisfies $f(xy) = f(x) f(y)$ and $f(z^*) = z^{-1}$. If $\sigma : \hat{\mathbb{T}}^d \to \mathbb{Z}^d$ is the natural isomorphism then σf is the balance function. □

We write W_n for the set of nonzero words w with balance $\mathrm{bal}(w) = n$ ($n \in \mathbb{Z}^d$), and call W_0 the *zero-balanced* words. The linear span of W_0 is a $*$-subalgebra of $\mathcal{X}$ denoted by $\mathbb{A}$. Suppose that the following property (B) holds:

(B) $\mathbb{A}$ is the inductively ordered union of a family $(\Upsilon_l)_{l\in L}$ of finite dimensional C^*-subalgebras $\Upsilon_l \subseteq \mathbb{A}$. (That is, $\forall l_1, l_2 : \exists l_3 : \Upsilon_{l_1} \cup \Upsilon_{l_2} \subseteq \Upsilon_{l_3}$, see [3].)

For each $l \in L$ fix a finite set D_l (of our choice) consisting of minimal mutually orthogonal projections in Υ_l generating a maximal abelian subalgebra of Υ_l, and write $D = \bigcup_{l\in L} D_l$. Suppose that we are provided with a subset $\mathcal{S} \subseteq W$ of words consisting of partial isometries with commuting range projections such that $D \subseteq P$, where we denote

$$P = \operatorname{lin}_{\mathbb{Z}}\{\, xx^* \in \mathcal{X} \mid x \in \mathcal{S} \,\},$$
$$Q = \operatorname{Alg}^*\{\, x^*x \in \mathcal{X} \mid x \in \mathcal{S} \,\}.$$

(Thereby $\operatorname{lin}_{\mathbb{Z}}$ denotes the $\mathbb{Z}$-linear span, and Alg^* the generated $*$-algebra.) Moreover assume that we are given a self-adjoint subset W' of W such that $\operatorname{lin}(W') = \mathbb{X}$ and the following two technical conditions hold.

(a) For each $1 \le \xi \le d$ there exists a function $R_\xi : \{\, x \in W' \backslash \{0\} \mid \operatorname{bal}(x)_\xi = 0 \,\} \to \mathbb{C}$ which respects the involution operator, such that for all $\alpha_i \in \mathbb{C}$ and all $v_i, w_i \in W'$ satisfying $v_i w_i^* \neq 0$, $\operatorname{bal}(v_i w_i^*) = c$ is constant (i.e. c is independent from i), $\operatorname{bal}(v_i w_i^*)_\xi = 0$ and $\operatorname{bal}(v_i)_\xi \ge 0$ we have

$$x = \sum_{i=1}^n \alpha_i v_i w_i^* \in Q \quad \Rightarrow \quad x = \sum_{i=1}^n 1_{\{\operatorname{bal}(v_i)_\xi = 0\}} R_\xi(v_i) R_\xi(w_i^*) \alpha_i v_i w_i^*,$$

where it is sufficient that this only holds for the case $x = 0$ in case that $c \neq 0$.

(b) For all $\alpha_i \in \mathbb{C}$ and $v_i, w_i \in W' \cap W_0$ assume that

$$x = \sum_{i=1}^n \alpha_i v_i w_i^* \in Q \quad \Rightarrow \quad x = \sum_{i=1}^n 1_{\{v_i \in Q\}} 1_{\{w_i \in Q\}} \alpha_i v_i w_i^* \in Q.$$

Then we have the following result.

Theorem 2.2. *The identical embedding $\theta : C^*(Q) \to \mathcal{X}$ induces an isomorphism $K_0(\theta)$, and $K_1(\mathcal{X}) = 0$.*

We will postpone the proof of this theorem to Section 4 and give its applications to graph and higher rank Exel–Laca algebras at first.

Suppose that $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ are the generators and relations generating a higher rank Exel–Laca algebra $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$ as described in [4]. We will assume that the partition $V = \{\mathcal{A}_1, \dots, \mathcal{A}_d\}$ of $\mathcal{A}$ stated in Definition 2.2 of [4] is finite, $\mathcal{A}$ is non-degenerate in $\mathbb{F}/\mathbb{I}$, and $\mathbb{F}/\mathbb{I}$ is non-degenerate in $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$, i.e. we will regard $\mathcal{A} \subseteq \mathbb{F}/\mathbb{I} \subseteq \overline{\mathbb{F}/\mathbb{I}} = \mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$, see also [6] for more on this. (We remark that these restrictions are not essential but facilitate the presentation.) We set $\mathcal{X} = \mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$ (so $\mathbb{X} = \mathbb{F}/\mathbb{I}$), $W' = W$,

$$\mathcal{S} = \{ a_1 \dots a_n Q_{b_1} \dots Q_{b_m} \in W \mid n \geq 0,\, m \geq 1,\, a_i \in \mathcal{A} \},$$

$G = \{ ww^* \in W \mid w \in \mathcal{S} \}$, and $\mathbb{A}_0 = \mathrm{lin}\{ xx^* \in \mathcal{X} \mid x \in W \}$. By [4, 4.4], all words are partial isometries and $\mathbb{A}_0$ is a commutative $*$-algebra, and so in particular the range projections of $\mathcal{S}$ commute. By [4, 4.3], each word $x \in W$ allows a representation

$$x = a_1 \dots a_n Q_{g_1} \dots Q_{g_k} b_m^* \dots b_1^* \qquad (1)$$

for some integers $n, m \geq 0, k \geq 1$ and letters $a_i, b_i, g_i \in \mathcal{A}$. Note that $\mathrm{bal}(x) = \mathrm{bal}(a_1) + \dots + \mathrm{bal}(a_n) - \mathrm{bal}(b_m) - \dots \mathrm{bal}(b_1)$. By [4, 4.5], $xx^* \in G$, whence $\mathrm{lin}(G) = \mathbb{A}_0$. Since G is also closed under multiplications (indeed, $w_1 w_1^* w_2 w_2^* = xx^* \in G$ for $x = w_1 w_1^* w_2$ as $\mathbb{A}_0$ is abelian), by choosing common refinements in G we see that $P = \mathrm{lin}_{\mathbb{Z}}(G)$ contains the projection of $\mathbb{A}_0$. Since the system $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ satisfies the axioms (A), (B) and (D) of [2], there exists a choice for D which is a subset of $\mathbb{A}_0$ by [2, Proposition 3.3], and so in particular $D \subseteq P$. By [4, 4.5] we have

$$Q = \mathrm{Alg}^*\{ Q_a \in \mathcal{X} \mid a \in \mathcal{A} \}.$$

Next we recall the following condition (II) already introduced in [5]. It implies that the lines of the transition matrices must contain infinite or no zeros and ones, see [5, 2.3].

(II) For all $\mathcal{A}_i \in V$, all finite subsets $\mathcal{B} \subseteq \mathcal{A}_i$, and all $q \in Q$ we require that $q \leq \sum_{b \in \mathcal{B}} P_b$ implies $q = 0$.

The following lemma is a slight generalization of [5, Lemma 4.3] (both lemmas coincide for the case $c = 0$).

Lemma 2.3. *Assume that condition (II) holds. Let $1 \leq \xi \leq d$, let $c \in \mathbb{Z}^d$ such that $c_\xi = 0$, let $\mathcal{B} \subseteq \mathcal{A}_\xi$ be a finite subset, let $p = \sum_{b \in \mathcal{B}} P_b$, and let x be an element in*

$$\begin{aligned} C_\xi^c \;=\; & \mathrm{lin}\{ w \in W \,|\, \mathrm{bal}(w) = c,\, \exists n, m \geq 0,\, \exists k \geq 1,\, \exists g_i \in \mathcal{A},\, \exists a_i, b_i \in \mathcal{A} \backslash \mathcal{A}_\xi \\ & w = a_1 \dots a_n Q_{g_1} \dots Q_{g_k} b_m^* \dots b_1^* \}. \end{aligned}$$

Then $px = x$ implies $x = 0$.

Proof. Let $x \in C_\xi^c$. Then by representation (1) there exist $v_k \in C_\xi^0$ and words a_k, b_k in the letters of $\mathcal{A} \backslash \mathcal{A}_\xi$ with $\mathrm{bal}(a_k) = \max(c, 0)$ and $\mathrm{bal}(b_k) = |\min(c, 0)|$, such that the pairs (a_k, b_k) are distinct for different k's and $x = \sum_k a_k v_k b_k^*$. Assume that $px = x$. By [4, 4.7] there exists a finite subset $\mathcal{B}' \subseteq \mathcal{A}_\xi$ such that for $p' = \sum_{b\in\mathcal{B}'} P_b$ we have $r_k := a_k^* x b_k = a_k^* p x b_k = p' a_k^* x b_k = p' r_k$. Notice that $r_k = a_k^* a_k v_k b_k^* b_k$ by [4, 4.5], and thus $r_k \in C_\xi^0$ ([4, 4.1 and 4.5]). So we have obtained $p' r_k = r_k$ and thus $r_k = 0$ by [5, 4.3]. (The condition (I) of [5] is not required in [5, 4.3].) Hence we obtain $x = \sum_k a_k r_k b_k^* = 0$. ☐

Lemma 2.4. *Properties (a) and (b) hold if condition (II) holds.*

Proof. In order to prove condition (a) we define

$$R_\xi(x) = 1_{\{\neg(\exists a,b\in\mathcal{A}_\xi \quad x = P_a x P_b)\}}$$

for those $x \in W\backslash\{0\}$ such that $\mathrm{bal}(x)_\xi = 0$. Notice that $R_\xi(x^*) = R_\xi(x) = \overline{R_\xi(x)}$. Let $x = \sum_{i=1}^n \alpha_i v_i w_i^* \in Q$ where $v_i w_i^* \neq 0$, $\mathrm{bal}(v_i w_i^*) = c$ (c is independent from i), $\mathrm{bal}(v_i w_i^*)_\xi = 0$ and $\mathrm{bal}(v_i)_\xi \geq 0$. Notice that $\mathrm{bal}(v_i)_\xi = \mathrm{bal}(w_i)_\xi$. We claim that whenever, for fixed i and ξ,

$$1_{\{\mathrm{bal}(v_i)_\xi = 0\}} R_\xi(v_i) R_\xi(w_i) = 0,$$

then there exists a finite subset $F \subseteq \mathcal{A}_\xi$ such that

$$v_i w_i^* = \Big(\sum_{b\in F} P_b\Big) v_i w_i^* \Big(\sum_{b\in F} P_b\Big).$$

To show this we start with the case when $1_{\{\mathrm{bal}(v_i)_\xi=0\}} = 1$ and $R_\xi(v_i) = 0$; say that $v_i = P_a v_i P_b$ for $a, b \in \mathcal{A}_\xi$. Choose for w_i a representation as in (1). Since $\mathrm{bal}(w_i)_\xi = 0$ there must be as many letters a_s in $\mathcal{A}_\xi$ as letters b_t in $\mathcal{A}_\xi$. If no letter a_s is in $\mathcal{A}_\xi$ then we get $v_i w_i^* = P_a v_i P_b w_i^* = P_a v_i w_i^* P_d = (P_a + P_d) v_i w_i^* (P_a + P_d)$ for some $d \in \mathcal{A}_\xi$ by [4, 4.1] and since $\mathbb{A}_0$ is abelian, and we put $F = \{a, d\}$. If some letter a_s is in $\mathcal{A}_\xi$, w.l.o.g. we may assume that $a_1, b_1 \in \mathcal{A}_\xi$ by 'permutation' of letters (see [4, Def. 2.2.(2)]), then $v_i w_i^* = P_a v_i P_b P_{a_1} w_i^* P_{b_1} = P_a v_i w_i^* P_{b_1} = (P_a + P_{b_1}) v_i w_i^* (P_a + P_{b_1})$ in case that $b = a_1$ (what is the case since $v_i w_i^* \neq 0$). In the other possible case, when $1_{\{\mathrm{bal}(v_i)_\xi=0\}} = 0$, and hence $\mathrm{bal}(v_i)_\xi = \mathrm{bal}(w_i)_\xi > 0$, we find $a, b \in \mathcal{A}_\xi$ such that $P_a v_i = v_i$ and $P_b w_i = w_i$ (we see this if we choose for v_i and w_i representations like in (1)), and we obtain $v_i w_i^* = P_a v_i w_i^* P_b = (P_a + P_b) v_i w_i^* (P_a + P_b)$ in this case. Therefore we have

$$\begin{aligned} g &:= \sum_{i=1}^n (1 - 1_{\{\mathrm{bal}(v_i)_\xi=0\}} R_\xi(v_i) R_\xi(w_i)) \alpha_i v_i w_i^* \\ &= p\Big(\sum_{i=1}^n (1 - 1_{\{\mathrm{bal}(v_i)_\xi=0\}} R_\xi(v_i) R_\xi(w_i)) \alpha_i v_i w_i^*\Big) p, \end{aligned}$$

where $p = \sum_{a\in\mathcal{B}} P_a$ for a suitable finite set $\mathcal{B} \subseteq \mathcal{A}_\xi$.

Suppose that $\mathrm{bal}(v_i)_\xi = 0$, and hence $\mathrm{bal}(w_i)_\xi = 0$ since $\mathrm{bal}(v_i w_i^*)_\xi = 0$. Suppose further that $R_\xi(v_i) = R_\xi(w_i) = 1$. Then choose for v_i a representation like in (1). It is clear that, since $\mathrm{bal}(v_i)_\xi = 0$, there must be as many letters a_s in $\mathcal{A}_\xi$ as letters b_t in $\mathcal{A}_\xi$. But since $R_\xi(v_i) = 1$, none of the letters a_s and b_t can be in $\mathcal{A}_\xi$. The same argument holds for w_i. Hence $v_i w_i^* \in C_\xi^c$ for the set C_ξ^c defined in Lemma 2.3. We thus obtain

$$\sum_{i=1}^n 1_{\{\mathrm{bal}(v_i)_\xi = 0\}} R_\xi(v_i) R_\xi(w_i) \alpha_i v_i w_i^* \in C_\xi^c.$$

Notice that trivially $Q \subseteq C_\xi^0$, and so $x \in C_\xi^0$. Hence we have proved that $g \in C_\xi^0$ in case that $c = 0$. But we have also proved that $g \in C_\xi^c$ in case that $c \neq 0$, where we can simply assume that $x = 0$ according to condition (a). Since $g = pg$ we obtain $g = 0$ by Lemma 2.3. This proves the condition (a).

Condition (b) follows from condition (a) as follows. Let $x = \sum_{i=1}^n \alpha_i v_i w_i^* \in Q$ where $v_i, w_i \in W_0$, and let $y = \sum_{i=1}^n 1_{\{v_i \in Q\}} 1_{\{w_i \in Q\}} \alpha_i v_i w_i^* \in Q$. Then a d-fold application of condition (a) shows that $x = \sum_{i=1}^n \prod_{\xi=1}^d R_\xi(v_i) R_\xi(w_i) \alpha_i v_i w_i^*$, and similarly we do so for y. By choosing for v_i a representation as in (1) we see that $\prod_{\xi=1}^d R_\xi(v_i) = 1$ implies $v_i \in Q$. This proves that $x = y$, which proves the claim (b). □

Hence Theorem 1.2 follows from Lemma 2.4 and Theorem 2.2. The following corollary follows immediately from the discussion in Section 6 of [5].

Corollary 2.5. *The rank two Cuntz–Krieger type algebras inspired by shifts of finite type in Section 5 of* [3] *satisfy condition (II), and thus their K-theory is given by Theorem 1.2.*

We will now come to graph algebras. Let (Λ, d) be a finitely aligned k-graph [15] and $\mathcal{T}C^*(\Lambda)$ its Toeplitz algebra, that is, $\mathcal{T}C^*(\Lambda) = C^*(\Lambda, \emptyset)$ [18] is the universal C^*-algebra generated by a Toeplitz–Cuntz–Krieger Λ-family $\{ t_\lambda^{\mathcal{T}} \mid \lambda \in \Lambda \}$ consisting of partial isometries satisfying the properties (TCK1)-(TCK3) [18] (or see [17, Chapter 3]). We remark that there exist more general Toeplitz algebras, see [9, 10, 14]. Let $\mathcal{X}$ be a quotient of $\mathcal{T}C^*(\Lambda)$ and denote by $\{ t_\lambda \in \mathcal{X} \mid \lambda \in \Lambda \}$ the canonical generators of $\mathcal{X}$. Define

$$\mathcal{A}_i = \{ t_\lambda \in \mathcal{X} \mid \lambda \in \Lambda^{e_i} \} \backslash \{0\}$$

for $1 \leq i \leq k$, and $\mathcal{A} = \mathcal{A}_1 \sqcup \ldots \sqcup \mathcal{A}_k$ (excluding possible empty sets $\mathcal{A}_i$). Assume that the gauge action $\Gamma : \mathbb{T}^k \to \mathrm{Aut}(\mathcal{X})$ exists. For instance, the gauge action exists if $\mathcal{X}$ is the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ (see for example [17, 3.2.1]), or if $\mathcal{X}$ is the higher rank graph C^*-algebra $C^*(\Lambda)$ [15], or if $\mathcal{X}$ is a relative Cuntz–Krieger algebra $C^*(\Lambda, \mathcal{E})$ [18].

Notice that $\mathrm{bal}(t_\lambda t_\mu^*) = d(\lambda) - d(\mu)$ where d denotes the degree map. It is well known that property (B) is satisfied (since (B) holds for the Toeplitz algebra), and that we may choose $D \subseteq P$ if we set $\mathcal{S} = \{ t_\lambda \in \mathcal{X} \mid \lambda \in \Lambda \}$ (see [17, 3.5.3]). Define $W' = \{ t_\lambda t_\mu^* \in \mathcal{X} \mid \lambda, \mu \in \Lambda \}$. Notice that $Q = \mathrm{lin}\{ t_\lambda \mid \lambda \in \Lambda^0 \}$. For a finite subset $E \subseteq \Lambda$, and $\lambda \in E$, we write

$$Q_\lambda^E = t_\lambda t_\lambda^* \prod_{\lambda\nu \in E,\, d(\nu) > 0} (t_\lambda t_\lambda^* - t_{\lambda\nu} t_{\lambda\nu}^*),$$

and introduce the following condition.

(NT) For all $1 \le i \le k$, all finite subsets $\mathcal{B} \subseteq \Lambda^{e_i}$, all finite subsets $E \subseteq \{ \nu \in \Lambda \mid d(\nu)_i = 0 \}$, and all $\lambda \in E$, the following implication holds:

$$Q_\lambda^E \le \sum_{\mu \in \mathcal{B}} t_\mu t_\mu^* \quad \Rightarrow \quad Q_\lambda^E = 0.$$

Note that the classical higher rank graph C^*-algebras $C^*(\Lambda)$ [12] does not satisfy condition (NT) (provided that $\Lambda \neq \Lambda^0$) since $Q_\nu^{\{\nu\}} = t_\nu = \sum_{\mu \in \Lambda^{e_i}(\nu)} t_\mu t_\mu^*$ for $\nu \in \Lambda^0$. We are not sure if there exist nonempty subsets $\mathcal{E} \subseteq \mathrm{FE}(\Lambda)$ such that $C^*(\Lambda, \mathcal{E})$ [18] satisfies (NT). However, the Toeplitz algebras satisfy (NT).

Lemma 2.6. *$\mathcal{T}C^*(\Lambda)$ satisfies property (NT).*

Proof. We can prove this by using the path-space representation $\pi : \mathcal{T}C^*(\Gamma) \to B(\ell^2(\Lambda^*))$, see [17, Section 3.7] for a description of it. Let $\mathcal{B}, E$ and λ be as in condition (NT). Regard λ as an element in Λ^*. Let $\delta_\lambda \in \ell^2(\Lambda^*)$. Then $\pi(Q_\lambda^E)\delta_\lambda = \delta_\lambda$ and $\pi(\sum_{\mu \in \mathcal{B}} t_\mu t_\mu^*)\delta_\lambda = 0$, which contradicts $Q_\lambda^E \le \sum_{\mu \in \mathcal{B}} t_\mu t_\mu^*$. □

Lemma 2.7. *Let condition (NT) hold. Let $1 \le \xi \le d$, let $c \in \mathbb{Z}^d$ such that $c_\xi = 0$, let $\mathcal{B} \subseteq \Lambda^{e_\xi}$ be a finite subset and $p = \sum_{\mu \in \mathcal{B}} t_\mu t_\mu^*$, and let x be an element in*

$$C_\xi^c = \mathrm{lin}\{ t_\lambda t_\mu^* \in \mathcal{X} \mid \lambda, \mu \in \Lambda,\, \mathrm{bal}(t_\lambda t_\mu^*) = c,\, d(\lambda)_\xi = d(\mu)_\xi = 0 \}.$$

Then $px = x$ implies $x = 0$.

Proof. We prove the lemma for the case $c = 0$. The case for general c can be deduced from the case $c = 0$ in a similar way we deduced Lemma 2.3 from [5, 4.3]. So let $c = 0$. Let $G_\xi = \{ \lambda \in \Lambda \mid d(\lambda)_\xi = 0 \}$. We choose a finite subset $E \subseteq G_\xi$ such that $x \in M_E^t = \mathrm{lin}\{ t_\lambda t_\mu^* \mid \lambda, \mu \in E,\, d(\lambda) = d(\mu) \}$. The latter linear space is embedded in a finite dimensional C^*-algebra $M_{\Pi E}^t$ which has a representation as a direct sum of matrix algebras with diagonal entries $Q_\lambda^{\Pi E}$ for $\lambda \in \Pi E$ (see [17, 3.5.3]). By [17, 3.4.7] we have $\Pi E \subseteq G_\xi$. If we assume that $px = x$ and $x \neq 0$, then it is easy to check that $pQ_\lambda^{\Pi E} = Q_\lambda^{\Pi E}$ for at least one nonzero diagonal entry $Q_\lambda^{\Pi E}$. But this contradicts condition (NT). □

Lemma 2.8. *Properties (a) and (b) hold if condition (NT) holds.*

Proof. We can prove this like Lemma 2.4 with minimal and obvious adaption. Let also $P_a := aa^* = t_\lambda t_\lambda^*$ for $a = t_\lambda \in \mathcal{A}_\xi$ and $\lambda \in \Lambda^{e_\xi}$, and define R_ξ like in Lemma 2.4; and so on. One essential difference is that we use the definition of C_ξ^c given in Lemma 2.7 rather than in Lemma 2.3, and we apply Lemma 2.7 rather than Lemma 2.3. Another difference is that we use W' rather than W (recall that the conditions (a) and (b) require everything only for W', but in Lemma 2.4 we had $W' = W$, why we proved everything for W there). □

Theorem 2.9. *Let Λ be a finitely aligned higher rank graph. Let $\mathcal{X}$ be a quotient of the Toeplitz–Cuntz–Krieger algebra $\mathcal{T}C^*(\Lambda)$ such that the canonical gauge action Γ exists on $\mathcal{X}$ (for instance when $\mathcal{X} = C^*(\Lambda)$ or $\mathcal{X} = C^*(\Lambda, \mathcal{E})$). Assume that condition (NT) holds (for instance when $\mathcal{X} = \mathcal{T}C^*(\Lambda)$). Then $K_0(\mathcal{X}) \cong \bigoplus_{\Lambda^0} \mathbb{Z}$ under the isomorphism $[t_\nu] \longleftrightarrow 1_{\{\nu\}}$ for all $\nu \in \Lambda^0$, and $K_1(\mathcal{X}) = 0$.*

Example 2.10. Let Λ be a row-finite graph without sources and write $C^*(\Lambda)$ as the quotient of $\mathcal{T}C^*(\Lambda)$ by the closed 2-sided ideal J generated by $J_0 = \{\, t_\nu - \sum_{\lambda \in \Lambda^n(\nu)} t_\lambda t_\lambda^* \mid \nu \in \Lambda^0,\, n \in \mathbb{N}_0^k \,\}$, and write $\iota : J \to \mathcal{T}C^*(\Lambda)$ for the canonical embedding. Assume that $\bigoplus_{\Lambda^0} \mathbb{Z}$ is the $\mathbb{Z}$-span of all elements $1_{\{\nu\}} - \sum_{\lambda \in \Lambda^n(\nu)} 1_{\{s(\lambda)\}}$ ($\nu \in \Lambda^0, n \geq 0$). Then, as $K_0(\iota)(J_0) = \bigoplus_{\Lambda^0} \mathbb{Z}$ is a projective $\mathbb{Z}$-module, we get $K_0(J) \cong K_1(C^*(\Lambda)) \oplus \bigoplus_{\Lambda^0} \mathbb{Z}$ and $K_1(J) \cong K_0(C^*(\Lambda))$ by the cyclic six-term exact sequence in K-theory. This gives an explicit description of $K_*(J)$ for the computed cases of $K_*(C^*(\Lambda))$ in Evans [8], or Allen, Pask and Sims [1].

3. The K-theory of some crossed products by $\mathbb{Z}^d$

This section is a slight generalization of a computation done in [5]. We will compute the K-theory of a crossed product $A \rtimes_\alpha \mathbb{Z}^d$ by a canonical d-fold application of the Pimsner–Voiculescu sequence under assumptions ensuring that the K_1-part $K_1(A \rtimes \mathbb{Z}^n)$ ($n \leq d$) remains zero in each step.

Suppose that A is a C^*-algebra which is endowed with a $\mathbb{Z}^d$-action α. Write $T = K_0(A)$, $\phi_i = K_0(\alpha_i)$ (where $\alpha_i = \alpha_{e_i}$) and $\phi_n = K_0(\alpha_n)$ for $1 \leq i \leq d, n \in \mathbb{Z}^d$. Assume that $K_1(A) = 0$ and that we are given a family of subgroups $(T_n)_{n \in \mathbb{Z}^d}$ of T such that

$$T = \sum_{m \in \mathbb{Z}^d} T_m, \qquad \phi_i(T_n) = T_{n+e_i}, \qquad T_{e_i}^\infty \cap T_0^\infty(i) = \{0\} \tag{2}$$

for all $1 \leq i \leq d$ and $n \in \mathbb{Z}^d$. Thereby denotes $T_n^N = \sum_{n\leq m\leq N} T_m$, $T_n^\infty = \sum_{n\leq m} T_m$, $T_n^N(i) = \sum_{n\leq m\leq N,\, m_i=n_i} T_m$ and $T_n^\infty(i) = \sum_{n\leq m,\, m_i=n_i} T_m$ for all $n, N \in \mathbb{Z}^d$ if $n \leq N$, and $T_n^N = T_n^N(i) = \{0\}$ if $n \leq N$ is not satisfied. Clearly, $\sum_{n\in I} T_n$ denotes the set of finite sums $\sum_n t_n$ with $t_n \in T_n$. We further put

$$f_i = \phi_i - \mathrm{Id}_T, \qquad Y_i = \mathrm{Im}(f_1) + \ldots + \mathrm{Im}(f_i) \subseteq T,$$

and $Y_0 = \{0\}$. Notice that $\{\phi_1, \ldots, \phi_d, f_1, \ldots, f_d, \mathrm{Id}_T\}$ is a commuting set in $\mathrm{End}(T)$. Notice that the restriction $\phi_i|_{T_n} : T_n \to T_{n+e_i}$ is a bijection (since ϕ_i is a bijection on T). Consequently, $\phi_n|_{T_m} : T_m \to T_{m+n}$ is a bijection for all $n, m \in \mathbb{Z}^d$. Also the next lemma is clear from the assumptions in (2).

Lemma 3.1. *We have $T_{n+e_i}^\infty \cap T_n^\infty(i) = \{0\}$ for all $1 \leq i \leq d$ and $n \in \mathbb{Z}^d$.*

Lemma 3.2. *Let $1 \leq i \leq d$, $n \in \mathbb{Z}^d$ and $x \in T_n^\infty$. Then there exist unique $y, z \in T_n^\infty$ and unique $a, b \in T_n^\infty(i)$ such that $x = \phi_i(y) + a$ and $x = f_i(z) + b$.*

Proof. Step 1. We show the 'ϕ_i-representation'. Since $x \in T_n^\infty$ we have $x \in T_n^N$ for some $n \leq N \in \mathbb{Z}^d$. Then x allows a representation $x = y + a$ where $y \in T_{n+e_i}^N$ and $a \in T_n^N(i)$. Since $\phi_i(T_m) = T_{m+e_i}$, there exists a $y_1 \in T_n^{N-e_i}$ such that $\phi_i(y_1) = y$. Hence $x = \phi_i(y_1) + a$. The uniqueness of y and a follows from Lemma 3.1.

Step 2. We next proof the second representation. Let $x \in T_n^N$ and write $x = \phi_i(y_1) + a_1$ for some $y_1 \in T_n^{N-e_i}$ and $a_1 \in T_n^N(i)$ as before. Then $x = f_i(y_1) + y_1 + a_1$. If $N - e_i \geq n$, then we can similarly decompose y_1, i.e. we choose $y_2 \in T_n^{N-e_i-e_i}$ and $a_2 \in T_n^N(i)$ such that $y_1 = \phi_i(y_2) + a_2$. Hence $x = f_i(y_1 + y_2) + y_2 + a_2 + a_1$. We proceed in this way as long as $y_k \in T_n^{N-e_i-\ldots-e_i} = \{0\}$, which shows the existence of the 'f_i-representation'. We claim that $x \in T_n^\infty$ and $f_i(x) \in T_n^N$ imply $T_n^{N-e_i}$. Assume that $x \notin T_n^{N-e_i}$. Then $x \in T_n^m \backslash T_n^{m-e_i}$ for some $m \geq N$. Then $\phi_i(x) - x = f_i(x) \in T_n^N \subseteq T_n^m$, and so $\phi_i(x) \in T_n^m$. As in Step 1, we may write $\phi_i(x) = \phi_i(y_1) + a$ for $y_1 \in T_n^{m-e_i}$ and $a \in T_n^m(i)$. Hence $a = 0$ and $x = y_1 \in T_n^{m-e_i}$ by the uniqueness of the 'ϕ_i-representation', which is a contradiction. Now, to prove the uniqueness of the 'f_i-representation', assume that $f_i(x) + a = 0$ for $x \in T_n^N$ and $a \in T_n^N(i)$. We have $f_i(x) \in T_n^N(i) = T_n^m$ for $m = N(1 - e_i) + ne_i$. Hence $x \in T_n^{m-e_i} = \{0\}$ by the above claim. □

Lemma 3.3. *We have $f_i^{-1}(Y_{i-1}) \subseteq Y_{i-1}$ for all $1 \leq i \leq d$.*

Proof. Using the decomposition of Lemma 3.2, one may almost copy the proof of Lemma 5.2 in [5] under the following dictionary: $T = K_0(A)$, $T_0 = B_0$, $T_N^\infty = K_0(A_N)$, $T_N^\infty(i)$ being the elements of $K_0(A_N)$ with i-degree zero in $K_0(A_N)$. □

Lemma 3.4. *We have* $Y_d \cap T_0 = \{0\}$.

Proof. Using the decomposition of Lemma 3.2, one may almost copy the proof of Lemma 5.3 in [5] under the dictionary already stated in the proof of the previous lemma. □

Proposition 3.5. *Let* $\varphi_A : A \to A \rtimes_\alpha \mathbb{Z}^d$ *be the canonical embedding. Then the restriction map* $K_0(\varphi_A)|_{T_0} : T_0 \to K_0(A \rtimes_\alpha \mathbb{Z}^d)$ *is an isomorphism, and* $K_1(A \rtimes_\alpha \mathbb{Z}^d) = 0$.

Proof. It follows from Lemma 3.3 and the proof of [5, Lemma 3.4] that $K_1(A \rtimes_\alpha \mathbb{Z}^d) = 0$ and $g : K_0(A)/Y_d \to K_0(A \rtimes \mathbb{Z}^d)$ given by $g(x + Y_d) = K_0(\varphi_A)(x)$ is an isomorphism. Since $T = \sum_n T_n = \sum_n \phi_n(T_0)$, and $\phi_i(x) \equiv x \mod Y_d$, the quotient map $s : T_0 \to T/Y_d$ is surjective. Further s is injective by Lemma 3.4. This proves the proposition since $K_0(\varphi_A)|_{T_0} = gs$ is an isomorphism. □

4. Proof of Theorem 2.2

To prove Theorem 2.2 we set $A = \mathcal{X} \rtimes_\Gamma \mathbb{T}^d$, write the stable from of $\mathcal{X}$ as $\mathcal{X} \otimes \mathbb{K}(L^2(\mathbb{T}^d)) \cong A \rtimes_{\hat{\Gamma}} \mathbb{Z}^d$ by Takai's duality theorem and apply Proposition 3.5 to this crossed product. At first we will analyze A. Write U_λ ($\lambda \in \mathbb{T}^d$) for the unitaries in $\mathcal{M}(A)$ that generate the action Γ. Define $X_n = \int_{\mathbb{T}^d} \lambda^n U_\lambda d\lambda$ for all $n \in \mathbb{Z}^d$ (integration in $\mathcal{M}(A)$ and $d\lambda$ being the Haar measure). Notice that $(X_n)_{n \in \mathbb{Z}^d}$ is a family of mutually orthogonal projections in $\mathcal{M}(A)$ and

$$X_n w = w X_{n+\mathrm{bal}(w)} = X_n w X_{n+\mathrm{bal}(w)} \tag{3}$$

for all words $w \in W \backslash \{0\}$ and all $n \in \mathbb{Z}^d$. The dual action $\hat{\Gamma}$ is given by

$$\hat{\Gamma}_m(w X_n) = \int_{\mathbb{T}^d} w \lambda^n \lambda^m U_\lambda d\lambda = w X_{n+m} \tag{4}$$

for all $w \in W$ and $n, m \in \mathbb{Z}^d$. By an application of the Stone-Weierstrass theorem we get $A = \overline{A}_0$ (norm closure), where

$$A_0 = \mathrm{lin}\{\, X_n w \in A \mid n \in \mathbb{Z}^d,\, w \in W \,\}.$$

Observe the simple identities $\overline{\mathbb{A}} \cong X_n \overline{\mathbb{A}} X_n = \overline{X_n \mathbb{A} X_n} = X_n A X_n$ for all $n \in \mathbb{Z}^d$.

Property (B) has now the following consequence.

Proposition 4.1. *The dense $*$-subalgebra $A_0 \subseteq A$ is an inductively ordered union of finite dimensional C^*-algebras.*

Proof. There exists a canonical isomorphism $g : H \to \mathbb{T}^d$ where

$$H = \{\, (\lambda_a)_{a \in \mathcal{A}} \in \mathbb{T}^{\mathcal{A}} \mid \lambda_a = \lambda_b \text{ whenever } a, b \in \mathcal{A}_i \text{ for some } 1 \le i \le d \,\}. \tag{5}$$

Then $\mathcal{X} \rtimes_\Gamma \mathbb{T}^d$ and $\mathcal{X} \rtimes_\theta H$ are isomorphic by equivariance for $\theta = \Gamma g$. We can then copy the proof of [6, 4.1] which passes without modification and which exactly proves the claim. ☐

Notice that by Proposition 4.1, $\mathcal{X}$ is the corner of a crossed product of an AF-algebra with an abelian group and hence nuclear.

Corollary 4.2. *$\mathcal{X}$ is nuclear.*

Lemma 4.3. *The $*$-algebra A_0 is the inductively ordered union of a certain family $(M)_{M\in\Omega}$ of finite dimensional C^*-subalgebras $M \subseteq A_0$ such that the following properties hold.*

(a) $X_n M \subseteq M$ and $M X_n \subseteq M$ for all $n \in \mathbb{Z}^d$.

(b) $M_n = X_n M X_n$ is a C^*-subalgebra of M, and $M_n = \{0\}$ for all but finitely many n.

(c) Each M_n has a representation $M_n = M_{n,1} \oplus \ldots \oplus M_{n,\tau_n}$, where each $M_{n,j}$ is isomorphic to a simple matrix algebra $M_{\ell(n,j)}(\mathbb{C})$ for some integer $\ell(n,j)$.

(d) The subalgebras $M_{n,j} \subseteq M$ are mutually orthogonal for distinct pairs (n,j).

(e) M has a representation $M = N_1 \oplus \ldots \oplus N_K$ where each N_i is a simple C^*-algebra. More precisely, for each $1 \le i \le K$ there exist a sequence $(n_1, \ldots, n_k)$ of distinct n_t *and a sequence $(j_1, \ldots, j_k)$ such that we have a commutative diagram*

$$\begin{array}{ccc}
M_{n_1,j_1} \oplus \ldots \oplus M_{n_k,j_k} & \xrightarrow{\quad\iota\quad} & N_i \\
\updownarrow & & \updownarrow \\
M_{\ell(n_1,j_1)}(\mathbb{C}) \oplus \ldots \oplus M_{\ell(n_k,j_k)}(\mathbb{C}) & \longrightarrow & M_{\ell(n_1,j_1)+\ldots+\ell(n_k,j_k)}(\mathbb{C})
\end{array}$$

where ι is the identical embedding, the left vertical arrow is by the isomorphisms of point (c), the right vertical arrow is another isomorphism, and the bottom arrow is the natural embedding.

Proof. By Proposition 4.1 and a similar argument as in [2, 3.2], it is easy to check that A_0 is the inductively ordered union of a family $(M)_{M\in\Omega}$ of finite dimensional C^*-subalgebras M such that each M is the linear span of words $X_n w$, more precisely M has the shape

$$M = \operatorname{lin}\{X_{g_1} w_1 X_{h_1}, \ldots, X_{g_\nu} w_\nu X_{h_\nu}\} \tag{6}$$

for certain $w_i \in W$ and $g_i, h_i \in \mathbb{Z}^d$. Then (a) is clear from the structure of M. Point (b) follows from (a). Since M_n is finite dimensional, points (c) and (d) are obvious.

(e) It follows from the proof [19, I.11.2] that if M is a finite dimensional C^*-algebra, and $C_1, \ldots, C_K \in M$ are the mutually orthogonal minimal projections of the center $\mathcal{C}(M)$ of M then $N_i = C_i M$ are simple matrix algebras and $M = N_1 \oplus \ldots \oplus N_K$. Furthermore, if for each $1 \le i \le K$ there exists a finite subset $\mathcal{P}_i \subseteq M$ consisting of mutually orthogonal minimal projections in M such that $C_i = \sum_{p \in \mathcal{P}_i} p$ (that means that $\mathrm{lin}(\mathcal{P}_i)$ is a maximal abelian subalgebra of $C_i M$), then there exists an isomorphism $\varphi : M \to M_{t_1}(\mathbb{C}) \oplus \ldots \oplus M_{t_K}(\mathbb{C})$ mapping N_i onto the i-th factor $M_{t_i}(\mathbb{C})$ and mapping $\mathcal{P}_i$ onto the diagonal entries $(e_{ss})_{1 \le s \le t_i}$ of $M_{t_i}(\mathbb{C})$. Our aim is thus to compute the C_i's and specify sets $\mathcal{P}_i$. The point (e) will then become clear.

Let $z \in \mathcal{C}(M)$. Set $G = \{g_1, \ldots, g_\nu, h_1, \ldots, h_\nu\}$. Notice that we have $h_i = g_i + \mathrm{bal}(w_i)$ by (3) (provided that $X_{g_i} w_i X_{h_i} \neq 0$). Hence $M_n \subseteq X_n \mathbb{A} X_n$ by the representation (6). Notice that the natural map $\mathbb{A} \to X_n \mathbb{A} X_n$ is a $*$-isomorphism. By (6) we have $z = \sum_{i=1}^{\nu} \lambda_i X_{g_i} w_i X_{h_i} = \sum_{a,b \in G} X_a y_{a,b} X_b$ for certain $y_{a,b} \in \mathrm{lin}(W_{b-a})$ such that $X_a y_{a,b} X_b \in M$. Let I be the unit of M. Since $I X_a y_{a,b} X_b I = X_a y_{a,b} X_b$ we get $X_a y_{a,b} X_b = X_a I X_a y_{a,b} X_b I X_b = (X_a I X_a) z (X_b I X_b) = (X_a I X_a)(X_b I X_b) z = 0$ for $a \neq b$. Hence $z = \sum_{a \in G} X_a y_{a,a} X_a$. Since z commutes with $M_g = X_g M X_g$, $X_g y_{g,g} X_g$ is in the center of M_g. It is clear that $X_g y_{g,g} X_g = \sum_{k \in \gamma_g} 1_{M_{g,k}}$ for some subset $\gamma_g \subseteq \{1, \ldots, \tau_g\}$, where we use the notations in (c) of the lemma. Hence z is the finite sum

$$z = \sum_{a \in G} \sum_{k \in \gamma_a} 1_{M_{a,k}}.$$

Let $P_{n,k} = \{p_{n,k,1}, \ldots, p_{n,k,\ell(n,k)}\} \subseteq M_{n,k}$ be a finite set of minimal mutually orthogonal projections in $M_{n,k}$ such that $\mathrm{lin}(P_{n,k})$ is a maximal abelian subalgebra of $M_{n,k}$. (That is, $P_{n,k,j}$ corresponds to the diagonal entry $e_{j,j}$ of $M_{\ell(n,k)}(\mathbb{C})$.) Each $p \in P_{n,k}$ is a minimal projection in M as $q \in M \backslash \{0\}$ and $q \le p \le X_n$ imply $q = X_n q X_n \in M_n$, and so $q = p$ as p is minimal in M_n. So we can write z as the orthogonal sum of minimal projections as follows.

$$z = \sum_{a \in G} \sum_{k \in \gamma_a} 1_{M_{a,k}} = \sum_{a \in G} \sum_{k \in \gamma_a} \sum_{s=1}^{\ell(n,k)} p_{a,k,s}. \tag{7}$$

Since I is in the center, it follows from this formula for $z = I$ that I is the sum of the minimal projections $\mathcal{P} = \{\, p_{a,k,s} \mid a \in G,\ 1 \le k \le \tau_a, 1 \le s \le \ell(a,k) \,\}$ in M. The linear span $\mathrm{lin}(\mathcal{P})$ is thus a maximal abelian subalgebra of M.

It remains to get a description of all minimal projections $C_1, \dots, C_K$ in $\mathcal{C}(M)$. Choose for C_i a representation like in (7). Then we claim that $|\gamma_a| \le 1$, which proves that the n_t stated in point (e) of the lemma are really distinct. Indeed, otherwise, since C_iM is a simple matrix algebra, there would exist a partial isometry $s \in C_iM$ connecting $p_{a,m_1,1} = ss^*$ with $p_{a,m_2,1} = s^*s$ for some $m_1, m_2 \in \gamma_a$. But since $s = X_a s X_a \in M_a$, this contradicts the fact that $p_{a,m_1,1}$ and $p_{a,m_2,1}$ are not connectable in M_a. The claim (e) is thus proved. □

Corollary 4.4. *One has* $K_0(A) = \mathrm{lin}_{\mathbb{Z}}\{\, [X_n d X_n] \in K_0(A) \mid n \in \mathbb{Z}^d,\ d \in D \,\}$.

Proof. By the structure result of Lemma 4.3,

$$K_0(A) = \mathrm{lin}_{\mathbb{Z}}\{\, [p] \mid n \in \mathbb{Z}^d,\ p \in X_n \mathbb{A} X_n \text{ is a projection} \,\}$$

(notice that $X_n A_0 X_n = X_n \mathbb{A} X_n$). On the other hand, $K_0(X_n A X_n) = \mathrm{lin}_{\mathbb{Z}}\{\, [X_n q X_n] \mid q \in D \,\}$ by property (B). To get the claim just notice that $[p] = [j_n(p)] = K_0(j_n)([p])$ for $p \in X_n \mathbb{A} X_n$ and the identity embedding $j_n : X_n A X_n \to A$. □

Lemma 4.5. *Let $B \subseteq B(H)$ be a unital C^*-algebra satisfying the cancelation property, $p \in B(H)$ a projection with $pB, Bp \subseteq B$, and $\varphi : pBp \to B$ the identical embedding. Then $K_0(\varphi)$ is injective.*

Proof. Let $x \in K_0(pBp)$. Then there exist $a, b \in M_n(B) \cong M_n \otimes B$ such that $x = [p_n a p_n] - [p_n b p_n]$, where $p_n = 1_n \otimes p$. If $K_0(\varphi)(x) = 0$ then there exists $t \in M_n(B)$ such that $p_n a p_n = tt^* \sim t^*t = p_n b p_n$ by the cancelation property of B. Hence $tt^*, t^*t \le p_n$ and thus $t = p_n t p_n \in M_n(pBp)$, and so $x = 0$. □

Corollary 4.6. *Let $\varphi_0 : X_0 A X_0 \to A$ be the identical embedding. Then $K_0(\varphi_0)$ is injective.*

Proof. We write A as the inductive limit of a family $(M)_{M\in\Omega}$ of finite dimensional C^*-algebras M as described in Lemma 4.3. Then $X_0 A X_0$ is the inductive limit of the family $(X_0 M X_0)_{M\in\Omega}$. By Lemma 4.5 the maps $K_0(\varphi_M)$ are injective for the identical embeddings $\varphi_M : X_0 M X_0 \to M$. It is now evident that $K_0(\varphi_0)$ is injective. □

Our next aim is to to apply Proposition 3.5 to the crossed product $A \rtimes_\alpha \mathbb{Z}^d$, where $\alpha = K_0(\hat{\Gamma})$. To this end we have to define subgroups $(T_n)_{n\in\mathbb{Z}^d}$ of $T = K_0(A)$ which satisfy the conditions (2). By Corollary 4.4, $D \subseteq P$, and the fact that $\{\, ww^* \mid w \in \mathcal{S} \,\}$ consists of commuting projections,

$$T = \mathrm{lin}_{\mathbb{Z}}\{\, [X_n ww^* X_n] \in T \mid n \in \mathbb{Z}^d,\ w \in \mathcal{S} \,\}.$$

Since $[X_n w w^* X_n] = [w^* X_n X_n w] = [X_{n+\mathrm{bal}(w)} w^* w X_{n+\mathrm{bal}(w)}]$ for all $w \in \mathcal{S}$,

$$T = \mathrm{lin}_{\mathbb{Z}}\{ [X_n w^* w X_n] \in T \mid n \in \mathbb{Z}^d,\, w \in \mathcal{S} \}.$$

Define

$$T_n = \mathrm{lin}_{\mathbb{Z}}\{ [X_n q X_n] \in T \mid q \in Q,\, q \text{ is a projection} \}.$$

Then we have $T = \sum_{n \in \mathbb{Z}^d} T_n$ and (see (4))

$$\phi_i([X_n q X_n]) = K_0(\hat{\Gamma}_i)([X_n q X_n]) = [\hat{\Gamma}_i(X_n q X_n)] = [X_{n+e_i} q X_{n+e_i}],$$

whence $\phi_i(T_n) = T_{n+e_i}$. It remains to show the following.

Proposition 4.7. *We have $T^\infty_{e_\xi} \cap T^\infty_0(\xi) = \{0\}$ for all $1 \le \xi \le d$.*

Proof. Let

$$A'_0 = \mathrm{Alg}^*\{ X_n w X_m \in A \mid n, m \in \mathbb{Z}^d_+,\, w \in W \}$$

and A' its norm closure in A. Let $(M)_{M \in \Omega}$ be as in Lemma 4.3. Then A'_0 is the union of the family $(Y_n M Y_n)_{n \in \mathbb{Z}^d_+, M \in \Omega}$ of finite dimensional subalgebras $Y_n M Y_n \subseteq A'_0$, where $Y_n = \sum_{0 \le k \le n} X_k$. Hence, by a similar argument as in Corollary 4.6 we can prove that the identical embedding $\varphi : A' \to A$ yields an injection $K_0(\varphi)$. Therefore we can and will identify $K_0(A')$ with a subset of T. Let $q \in T^\infty_0(\xi) \subseteq K_0(A')$ and $p \in T^\infty_{e_\xi} \subseteq K_0(A')$ such that $q = p$ holds in T. In particular, $p = q$ also holds in $K_0(A')$. To prove the proposition we must show that $q = 0$.

We write $q = [q_1] - [q_2]$ and $p = [p_1] - [p_2]$ for projections $q_1, q_2, p_1, p_2 \in M_\infty(A')$. By the definitions of $T^\infty_{e_\xi}$, $T^\infty_0(\xi)$ and T_n we can choose q_i and p_i in such a way that q_i and p_i are only nonzero on the diagonal, such that the diagonal entries of q_i consist only of elements of the form $X_n r X_n$ where $r \in Q$ is a projection and $n \ge 0$ and $n_\xi = 0$, and the diagonal entries of p_i consist only of elements of the form $X_n r X_n$ where $r \in Q$ is a projection and $n \ge 0$ and $n_\xi > 0$.

Since p_i, q_i lie in $M_\infty(F)$ for some finite dimensional C^*-subalgebra $F = Y_\mu M Y_\mu \subseteq A'_0$, which can be chosen large enough such that $q = p$ not only holds in $K_0(A')$ but also in $K_0(F)$, and since F has the cancelation property, there exists $t \in M_\infty(F)$ such that

$$q_1 \oplus p_2 = t t^* \sim t^* t = p_1 \oplus q_2. \tag{8}$$

Let $(e_{i,j})_{i,j \in \mathbb{Z}^d_+}$ be the canonical matrix units of M_∞. We have a representation

$$t = \sum_{i,j \in \mathbb{Z}^d_+} \Big(\sum_{n,m \in \mathbb{Z}^d_+} \sum_{\alpha \in \mathbb{N}} \lambda^{i,j}_{n,m,\alpha} X_n t^{i,j}_{n,m,\alpha} X_m \Big) \otimes e_{i,j} \quad \in F \otimes M_\infty, \tag{9}$$

where $\lambda^{i,j}_{n,m,\alpha} \in \mathbb{C}$ (almost all of which are zero), and $t^{i,j}_{n,m,\alpha} \in W'$ with $\mathrm{bal}(t^{i,j}_{n,m,\alpha}) = m - n$. For the case that a word $t^{i,j}_{n,m,\alpha}$ with balance $m-n$ should not exist we put $t^{i,j}_{n,m,\alpha} = 0$. Then we get

$$\begin{aligned} tt^* &= \sum_{i,j,n,m,\alpha} \sum_{a,b,f,g,\beta} \left(\lambda^{i,j}_{n,m,\alpha} X_n t^{i,j}_{n,m,\alpha} X_m \otimes e_{i,j}\right) \times \\ &\quad \times \left(\overline{\lambda^{a,b}_{f,g,\beta}} X_g (t^{a,b}_{f,g,\beta})^* X_f \otimes e_{b,a}\right) \\ &= \sum_{i,j,n,m,\alpha} \sum_{a,f,\beta} \lambda^{i,j}_{n,m,\alpha} \overline{\lambda^{a,j}_{f,m,\beta}} X_n t^{i,j}_{n,m,\alpha} (t^{a,j}_{f,m,\beta})^* X_f \otimes e_{i,a} \qquad (10) \\ &= \sum_{(n,i),(f,a) \in (\mathbb{Z}^d_+)^2} \zeta_{(n,i),(f,a)}, \end{aligned}$$

where

$$\begin{aligned} \zeta_{(n,i),(f,a)} &= (X_n \otimes e_{i,i}) tt^* (X_f \otimes e_{a,a}) \\ &= \sum_{j,m,\alpha,\beta} \lambda^{i,j}_{n,m,\alpha} \overline{\lambda^{a,j}_{f,m,\beta}} X_n t^{i,j}_{n,m,\alpha} (t^{a,j}_{f,m,\beta})^* X_f \otimes e_{i,a}. \qquad (11) \end{aligned}$$

Since $\zeta_{(n,i),(f,a)} = 0$ for $(n,i) \neq (f,a)$ by (8), we can restrict the sum (10) to the constraints $i = a$ and $n = f$ and obtain

$$tt^* = \sum_{i,j,n,m,\alpha,\beta} \lambda^{i,j}_{n,m,\alpha} \overline{\lambda^{i,j}_{n,m,\beta}} X_n t^{i,j}_{n,m,\alpha} (t^{i,j}_{n,m,\beta})^* X_n \otimes e_{i,i}. \qquad (12)$$

Since $tt^* = q_1 \oplus p_2$ we get

$$q_1 = \sum_{i,j,n,m,\alpha,\beta} 1_{\{n_\xi = 0\}} \lambda^{i,j}_{n,m,\alpha} \overline{\lambda^{i,j}_{n,m,\beta}} X_n t^{i,j}_{n,m,\alpha} (t^{i,j}_{n,m,\beta})^* X_n \otimes e_{i,i}. \qquad (13)$$

Notice that either $t^{i,j}_{n,m,\alpha} (t^{i,j}_{n,m,\beta})^* = 0$ or $\mathrm{bal}(t^{i,j}_{n,m,\alpha})_\xi = m_\xi \geq 0$ everywhere in the sum (13). Hence, by considering those partial sums in (13) where n and i are fixed, we get

$$\begin{aligned} q_1 &= \sum_{i,j,n,m,\alpha,\beta} 1_{\{n_\xi = m_\xi = 0\}} R(t^{i,j}_{n,m,\alpha}) \overline{R(t^{i,j}_{n,m,\beta})} \qquad (14) \\ &\qquad \lambda^{i,j}_{n,m,\alpha} \overline{\lambda^{i,j}_{n,m,\beta}} X_n t^{i,j}_{n,m,\alpha} (t^{i,j}_{n,m,\beta})^* X_n \otimes e_{i,i} \end{aligned}$$

by condition (a) for $c = 0$. We define $\tilde{t}$ exactly like t in (9) with the only difference that we replace $\lambda^{i,j}_{n,m,\alpha}$ by

$$\tilde{\lambda}^{i,j}_{n,m,\alpha} = 1_{\{n_\xi = m_\xi = 0\}} R(t^{i,j}_{n,m,\alpha}) \lambda^{i,j}_{n,m,\alpha}.$$

We want to analyze

$$\tilde{\zeta}_{(n,i),(f,a)} = (X_n \otimes e_{i,i}) \tilde{t}\tilde{t}^* (X_f \otimes e_{a,a}).$$

By the definition of $\tilde{\lambda}$ we get $\tilde{\zeta}_{(n,i),(f,a)} = 0$ (confer (11)) in case that $n_\xi > 0$ or $f_\xi > 0$. If $(n,i) \neq (f,a)$ and $n_\xi = f_\xi = 0$, then we have $0 = \zeta_{(n,i),(f,a)}$, and we obtain from the formula (11) and the fact that $\mathrm{bal}(t^{i,j}_{n,m,\alpha})_\xi = m_\xi \geq 0$, that $0 = \zeta_{(n,i),(f,a)} = \tilde{\zeta}_{(n,i),(f,a)}$ by an application of property (a) for $c = f - n$. Recalling (14) we observe that we have proved that

$$q_1 = \sum_{(n,i),(f,a) \in (\mathbb{Z}^d_+)^2} \tilde{\zeta}_{(n,i),(f,a)} = \tilde{t}\tilde{t}^*.$$

Repeating the above computations with $t' := t^*$ rather than t (adjoining equation (9) we see that it is enough to replace $\lambda^{i,j}_{n,m,\alpha}$ by $\overline{\lambda^{j,i}_{m,n,\alpha}}$ and $t^{i,j}_{n,m,\alpha}$ by $(t^{j,i}_{m,n,\alpha})^*$ in the definition of t) it turns out that $q_2 = \tilde{t'}\tilde{t'}^* = \tilde{t}^*\tilde{t}$ since $\tilde{t'} = \tilde{t}^*$ (everything passes as above, since the only essential property we required for $\lambda^{i,j}_{n,m,\alpha}$ and $t^{i,j}_{n,m,\alpha} \in W'$, namely $\mathrm{bal}(t^{i,j}_{n,m,\alpha}) = m - n$, does also hold for the new coefficients $\overline{\lambda^{j,i}_{m,n,\alpha}}$ and $(t^{j,i}_{m,n,\alpha})^* \in W'$, namely $\mathrm{bal}((t^{j,i}_{m,n,\alpha})^*) = m - n$). This proves that $[q_1] = [q_2]$, and thus $q = 0$. □

Proposition 4.8. *$K_0(\varphi_Q)$ is injective for the embedding $\varphi_Q : C^*(Q) \to X_0 A X_0$ given by $\varphi_Q(q) = X_0 q X_0$.*

Proof. Since $Q \subseteq \mathbb{A}$, and $\mathbb{A}$ is the inductively ordered union of finite dimensional C^*-subalgebras, the same holds for Q and $X_0 \mathbb{A} X_0$. Hence any element of $K_0(C^*(Q))$ allows a representation $q = [q_1] - [q_2]$ for projections $q_1, q_2 \in M_\infty(Q)$. Assume that $K_0(\varphi_Q)(q) = 0$. Notice that $(\varphi_Q)_\infty(q_i) \in M_\infty(X_0 \mathbb{A} X_0)$, and therefore there exists a finite dimensional C^*-subalgebra $F \subseteq X_0 \mathbb{A} X_0$ and some $t \in M_\infty(F)$ such that $(\varphi_Q)_\infty(q_1) = tt^* \sim t^*t = (\varphi_Q)_\infty(q_2)$.

Since we will next have similar computations as in the proof of Proposition 4.7, we will save space and refer to some formulas appearing there. We choose for t a representation as in (9) (notice that $F \subseteq A'_0$). In particular we have $n = m = 0$ everywhere in the representation (9) of t, and we have $\mathrm{bal}(t^{i,j}_{n,m,\alpha}) = m - n = 0$. In the representation (10) (fixing i and a, whereas $n = m = f = 0$ anyway) we can replace $\lambda^{i,j}_{n,m,\alpha}$ by $\tilde{\lambda}^{i,j}_{n,m,\alpha} = 1_{\{t^{i,j}_{n,m,\alpha} \in Q\}} \lambda^{i,j}_{n,m,\alpha}$ by property (b). Thus we get $(\varphi_Q)_\infty(q_1) = tt^* = \tilde{t}\tilde{t}^*$ for $\tilde{t}$ being defined like t with the only difference that $\lambda^{i,j}_{n,m,\alpha}$ is replaced by $\tilde{\lambda}^{i,j}_{n,m,\alpha}$. Similarly we get $(\varphi_Q)_\infty(q_2) = t^*t = \tilde{t}^*\tilde{t}$. Since $\tilde{t} \in M_\infty(X_0 Q X_0)$ we obtain $[q_1] = [q_2]$. □

Proof of Theorem 2.2. Clearly, $\mathbb{A}$ and $Q \subseteq \mathbb{A}$ are the inductively ordered union of their finite dimensional subalgebras. Thus $\{ [q] \mid q \in Q \text{ is a projection} \}$ generates $K_0(C^*(Q))$. By Proposition 4.8 and Corollary 4.6, $K_0(\varphi_0\varphi_Q)$ is injective and has image T_0. By Proposition 3.5 (justified by Proposition 4.1 and Proposition 4.7), $K_0(\theta')$ is an isomorphism for $\theta' =$

$\varphi_A\varphi_0\varphi_Q$, and $K_1(A \rtimes_{\hat{\Gamma}} \mathbb{Z}^d) = 0$. A slight analysis of the Takai duality map (exploiting that Q is invariant under Γ) shows that $K_0(\theta)$ is also an isomorphism. □

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