NOTES ON THE NUCLEARITY OF RANDOM SUBALGEBRAS OF $\mathcal{O}_2$

BERNHARD BURGSTALLER

ABSTRACT. There exists a separable exact C^*-algebra A which contains all separable exact C^*-algebras as subalgebras, and for each norm-dense measure μ on A and independent μ-distributed random elements $x_1, x_2, \ldots$ we have $\lim_{n\to\infty} \mathbb{P}(C^*(x_1,\ldots,x_n) \text{ is nuclear}) = 0$. Further, there exists a norm-dense non-atomic probability measure μ on the Cuntz algebra $\mathcal{O}_2$ such that for an independent sequence $x_1, x_2, \ldots$ of μ-distributed random elements x_i we have $\liminf_{n\to\infty} \mathbb{P}(C^*(x_1,\ldots,x_n) \text{ is nuclear}) = 0$. We introduce the notion of the stochastic rank for a unital C^*-algebra and prove that the stochastic rank of $C([0,1]^d)$ is d.

1. Introduction

Every separable exact C^*-algebra A can be embedded as a subalgebra of the Cuntz-algebra $\mathcal{O}_2$ [C] by the deep work [KP]. On the other hand, each subalgebra of $\mathcal{O}_2$ is exact since $\mathcal{O}_2$ is nuclear and hence exact, and exactness passes to subalgebras. Therefore, if one has given n independent random elements $x_1, \ldots, x_n$ with values in $\mathcal{O}_2$, then the probability that the C^*-subalgebra $C^*(x_1,\ldots,x_n)$ of $\mathcal{O}_2$ generated by $x_1,\ldots,x_n$ is nuclear may be interpreted as the probability that an exact C^*-algebra is nuclear.

The non-nuclear exact C^*-algebra $\mathcal{O}_2 \otimes C^*_\lambda(\mathbb{F}_2)$ also contains every separable exact C^*-algebra as a subalgebra, where $\mathbb{F}_2$ denotes the free group of two generators. We show that for any sequence $x_1, x_2, \ldots$ of identically distributed norm-dense random elements x_i in $\mathcal{O}_2 \otimes C^*_\lambda(\mathbb{F}_2)$ we have

$$\lim_{n\to\infty} \mathbb{P}(C^*(x_1,\ldots,x_n) \text{ is nuclear}) = 0$$

(Theorem 4.3). Further, there exists a norm-dense probability measure μ on $\mathcal{O}_2$ such that for independent μ-distributed random elements $x_1, x_2, \ldots$

1991 *Mathematics Subject Classification.* 46L05, 60B11.

Key words and phrases. random C^*-algebra, nuclear, Cuntz algebra.

The author was supported by the Austrian Schrödinger stipend J2471-N12.

we have

$$\liminf_{n\to\infty} \mathbb{P}(C^*(x_1,\ldots,x_n) \text{ is nuclear}) = 0$$

(Theorem 4.6). However, μ is degenerate in the sense that it has concentration on a proper subalgebra $A \subset \mathcal{O}_2$, i.e. $\mu(A) > 0$. That may not be regarded as a really "fair" measure. Non-degenerate measures on $\mathcal{O}_2$ would exist (Corollary 3.6).

A further interesting result is that $\mathcal{O}_2$ is the norm closure of the union of a certain increasing sequence of non-nuclear subalgebras of $\mathcal{O}_2$, see Corollary 4.5.

In Proposition 5.2 we show that there exists a norm-dense probability measure μ on $C([0,1]^d)$ such that for independent μ-distributed random elements $f_1,\ldots,f_n$ we have $C^*(1,f_1,\ldots,f_n) = C([0,1]^d)$ almost surely if $n = d+1$. For $n < d+1$ this does not hold, even if one would choose another norm-dense measure μ. That is why we say that the stochastic rank of $C([0,1]^d)$ is d.

We finally remark that whereas, for instance, there are many papers on random groups, random C^*-algebras has been hardly considered. Results in the latter direction appear in Haagerup and Thorbjørnsen [HT], Corollary 8.4, and in Burgstaller [B].

2. The measurability of nuclearity

Throughout we regard every normed space X as a measurable space by endowing it with the Borel structure of the norm-topology of X. We often loosely say that x is a random element in X, and mean precisely that x is a random element with values in X, that is, $x : \Omega \to X$ is a measurable function for some probability space $(\Omega, \mathcal{B}, \mathbb{P})$.

Definition 2.1. Let $x_i : \Omega \to A$ be random elements with values in a C^*-algebra A. Then both functions $\omega \mapsto C^*(x_1(\omega),\ldots,x_n(\omega))$ (the C^*-subalgebra of A generated by $x_1(\omega),\ldots,x_n(\omega)$) and $\omega \mapsto C^*(x_1(\omega),x_2(\omega),\ldots)$ are said to be *random C^*-algebras*, and we write $C^*(x_1,\ldots,x_n)$ and $C^*(x_1,x_2,\ldots)$, respectively, for them.

We remark that we do not require that the random C^*-algebra $C^*(x_1,\ldots,x_n)$ is a measurable function (in any sense); that is, $C^*(x_1,\ldots,x_n)$ need not be a random element in the usual sense. Rather we are interested in how likely it is that $C^*(x_1,\ldots,x_n)$ satisfies certain properties.

Definition 2.2. Let P be a property of separable C^*-algebras. We say that P is *measurable* if $\{\,\omega \in \Omega \mid C^*(x_1(\omega), \ldots, x_n(\omega))$ satisfies property $P\,\}$ is a measurable set for all $n \geq 1$ and all random C^*-algebras $C^*(x_1, \ldots, x_n)$. If this also holds for all random C^*-algebras $C^*(x_1, x_2, \ldots)$ for an infinite sequence $x_1, x_2, \ldots$ of random elements, then P is said to be *countably measurable.*

In this section we want to prove that nuclearity is a countably measurable property. Let M_n denote the complex valued $n \times n$-matrices with canonical matrix units e_{ij}, and let H be a Hilbert space.

Lemma 2.3. *For $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in B(H)^n$ let $u_{y,x}$: $\mathrm{lin}(x_1, \ldots, x_n) \to \mathrm{lin}(y_1, \ldots, y_n)$ be the linear map determined by $u_{y,x}(x_i) = y_i$. Then the set $D_n = \{\,(x, y) \in B(H)^{2n} \mid u_{y,x}$ is well defined$\}$ is measurable, and the map $(x, y) \mapsto u_{y,x}$ (the image being endowed to the completely bounded norm $\|.\|_{cb}$) is continuous on D_n.*

Proof. D_n is a measurable set since it can be expressed as

$$D_n = \{\,(x, y) \in B(H)^{2n} \mid \exists M \in \mathbb{N} : \forall \lambda \in (\mathbb{Q}+i\mathbb{Q})^n : \Big\| \sum_{i=1}^{n} \lambda_i y_i \Big\| \leq M \Big\| \sum_{i=1}^{n} \lambda_i x_i \Big\| \,\}.$$

The continuity follows by an application of [P, 2.13]. □

Lemma 2.4. *Let A be separable C^*-algebra, $A^{(0)}$ be a countable dense subset of A, and $M_n^{(0)}$ be a countable dense subsets of M_n. Then A is nuclear if and only if for all $\varepsilon > 0, n \in \mathbb{N}$ and $a_1, ..., a_n \in A^{(0)}$ there exist $N \geq 1$ and complete contractions u and v,*

$$\mathrm{lin}(a_1, \ldots, a_n) \xrightarrow{u} M_N \xrightarrow{v} A,$$

such that $u(a_k) \in M_N^{(0)}$ and $v(e_{ij}) \in A^{(0)}$ for all k, i, j, and

$$\|vu(a_i) - a_i\| \leq \varepsilon \qquad \forall 1 \leq i \leq n.$$

Proof. If A is nuclear then the condition must hold by the completely contractive factorization property in [S] (or see [P, 11.5.(iv)]), and by using some perturbation argument (see Lemma 2.3). On the other hand assume that the condition holds. For $\varepsilon > 0$ and $\{a_1, \ldots, a_n\} \subseteq A^{(0)}$ choose complete contractions u and v as stated in the lemma. By the extension theorem of Wittstock [W] and Paulsen [Pa] we can extend u to a complete contraction $\hat{u} : A \to M_n = B(\ell_n^2)$. In this way we obtain a net $(\varepsilon, \{a_1, \ldots, a_n\}) \mapsto (\hat{u}, v)$, indexed by $(\varepsilon, \{a_1, \ldots, a_n\})$, which satisfies the completely contractive factorization property in [S]. □

Proposition 2.5. *Nuclearity is a countably measurable property.*

Proof. Let $A = C^*(x_1, x_2, \ldots)$ for random elements x_i in $B(H)$. Let $\mathcal{A} = \mathrm{Alg}^*_{\mathbb{Q}+i\mathbb{Q}}[X_1, X_2, \ldots]$ be the $*$-algebra over $\mathbb{Q} + i\mathbb{Q}$ consisting of all finite $*$-polynomials with scalar coefficients in $\mathbb{Q} + i\mathbb{Q}$ and with countable non-commuting variables set $\{X_1, X_2, \ldots\}$. For $f \in \mathcal{A}$ and $\omega \in \Omega$ we denote by $f(\omega)$ the element in $A(\omega) = C^*(x_1(\omega), x_2(\omega), \ldots) \subseteq B(H)$ which arises if we replace each occurrence of X_i by $x_i(\omega)$. Clearly the map $\omega \mapsto f(\omega)$ is measurable for fixed f. Similarly, if $f = (f_1, \ldots, f_n) \in \mathcal{A}^n$ then we let $f(\omega) = (f_1(\omega), \ldots, f_n(\omega)) \in A(\omega)^n$. Let E_N denote the set of matrix units e_{ij} of M_N. Let $u_{y,x}$ be defined as in Lemma 2.3.

Fix $\omega \in \Omega$. Notice that $\mathcal{A}(\omega) = \{ f(\omega) \mid f \in \mathcal{A} \}$ is a dense subset of $A(\omega)$. By Lemma 2.4 we have that $A(\omega)$ is nuclear if and only if

$$\forall k \in \mathbb{N} : \forall n \geq 1 : \forall f \in \mathcal{A}^n : \exists N \in \mathbb{N} : \exists s \in (M_N^{(0)})^n : \exists g \in \mathcal{A}^{N \times N} :$$

(1) $u_{s,f(\omega)}$ and $u_{g(\omega),E_N}$ are well defined, $\|u_{s,f(\omega)}\|_{cb} \leq 1$, $\|u_{g(\omega),E_N}\|_{cb} \leq 1$,

(2) $\|u_{g(\omega),E_N} u_{s,f(\omega)} f_i(\omega) - f_i(\omega)\| \leq 1/k$ for all $1 \leq i \leq n$.

Since the set $M_{k,n,f,N,s,g} = \{ \omega \in \Omega \mid \text{conditions } (1) - (2) \text{ hold} \}$ is measurable for all k, n, f, N, s, g by Lemma 2.3, we have proved that $\{ \omega \in \Omega \mid A(\omega) \text{ is nuclear} \}$ is measurable. □

3. Non-degenerate measures

In this section let X be a normed space. A measure μ on X is called *norm-dense* if $\mu(B) > 0$ for every non-empty open ball B. A random element x with values in X is called norm-dense if the distribution of x is norm-dense, that is, if $\mathbb{P}(x \in B) > 0$ for every nonempty open ball B. Recall that the *distribution* of x is the probability measure μ on X given by $\mu(A) = \mathbb{P}(x^{-1}(A)) = \mathbb{P}(x \in A)$ for all Borel sets A in X.

Definition 3.1. A measure μ on X is called *non-degenerate* if $\mu(H) = 0$ for all closed affine subspaces $H \neq X$ of X. That is, μ is non-degenerate if and only if $\mu(\varphi^{-1}(\lambda)) = 0$ for all nonzero continuous linear functionals $\varphi \in X^* \backslash \{0\}$ and all $\lambda \in \mathbb{C}$.

The claim in the last definition follows from the fact that any closed linear subspace $H \neq X$ can be annihilated by a nonzero continuous linear functional $\varphi \in X^*$ by the theorem of Hahn-Banach. A random element x in X is called *non-degenerate* if the distribution of x is non-degenerate. That is, x is non-degenerate if and only if $\mathbb{P}(\varphi(x) = \lambda) = \mathbb{P}(x \in \varphi^{-1}(\lambda)) = 0$

for all $\varphi \in X^*\backslash\{0\}$ and $\lambda \in \mathbb{C}$, if and only if the distribution of $\varphi(x)$ is non-atomic for all $\varphi \in X^*\backslash\{0\}$.

Lemma 3.2. *Let x, y be independent random elements in X and let x be non-atomic. Then $x + y$ is non-atomic.*

Proof. The probability that $x + y = c$ for a constant c is, by Fubini,

$$\mathbb{P}(x + y = c) = \int\int 1_{\{x=c-y\}} dxdy = \int 0dy = 0.$$

□

With the following lemma one can easily construct non-degenerate random measures in many cases.

Lemma 3.3. *Let $X_1, X_2, \ldots \subseteq X$ be linear subspaces of X. Let $\sum_{i\geq 1} X_i$ be dense in X. Let $x_1, x_2, \ldots$ be a sequence of independent non-degenerate (and norm-dense) random elements x_i in X_i. Then the series $y = \sum_{i\geq 1} x_i$, when it converges almost surely, is a non-degenerate (and norm-dense) random element in X, as well as in the completion $\overline{X}$.*

Proof. Let $\varphi \in X^*$ be nonzero. For at least one j we have $\varphi|_{X_j} \neq 0$. Consequently, the distribution of $\varphi(x_j)$ is non-atomic. Since $\varphi(y) = \varphi(x_j) + \sum_{i\neq j} \varphi(x_i)$ a.s., the distribution of $\varphi(y)$ is non-atomic by Lemma 3.2. Thus the sequence $\sum x_i$ is non-degenerate in X. The fact that $\varphi \in \overline{X}^*$ is zero if and only if $\varphi|_X = 0$, proves the claim for $\overline{X}$. Finally assume that all x_i's are norm-dense in X_i. Let $y = \sum_{i=1}^k y_i \in X$ for constant $y_i \in X_i$. Since $s_N = \sum_{i=N+1}^\infty \|x_i\| \to 0$ a.s. for $N \to \infty$, there is some $N \geq k$ such that $\mathbb{P}(s_N \leq \varepsilon) > 0$. Hence, by the norm-density of the x_i's and by independence we get

$$\begin{aligned} \mathbb{P}(\|y - \sum_{i=1}^\infty x_i\| \leq 2\varepsilon) &\geq \mathbb{P}(\|y - \sum_{i=1}^N x_i\| \leq \varepsilon \text{ and } \|s_N\| \leq \varepsilon) \\ &= \mathbb{P}(\|y - \sum_{i=1}^N x_i\| \leq \varepsilon)\mathbb{P}(\|s_N\| \leq \varepsilon) > 0. \end{aligned}$$

Thus the sequence $\sum x_i$ is norm-dense. □

The following corollaries of Lemma 3.3 show how one can construct non-degenerate random measures on direct limits, crossed products, and tensor products of C^*-algebras. Let $\sum_{i\in\mathbb{N}} \lambda_i$ be an absolutely convergent series in $\mathbb{C}$.

Corollary 3.4. *Let $A = \overline{\bigcup_{i\in\mathbb{N}} A_i}$ be a Banach space, where $A_1 \subseteq A_2 \subseteq \ldots$ are linear subspaces of A. Let $a_1, a_2, \ldots$ be a sequence of independent non-degenerate (and norm-dense) random elements a_i in A_i such that $\mathbb{E}(\|a_i\|) \le 1$. Then $a = \sum_{i\in\mathbb{N}} \lambda_i a_i$ is an almost surely absolutely convergent non-degenerate (and norm-dense) series in A.*

Corollary 3.5. *Let (A, G, α) be a C^*-dynamical system where G is a countable discrete group. Let $(a_g)_{g\in G}$ be a family of independent non-degenerate (and norm-dense) random elements a_g in A such that $\mathbb{E}(\|a_g\|) \le 1$. Let U_g be the unitaries in $\mathcal{M}(A \rtimes_\alpha G)$ generating the action α_g. Then $a = \sum_{g\in G} \lambda_g a_g U_g$ is an almost surely absolutely convergent non-degenerate (and norm-dense) series in $A \rtimes_\alpha G$.*

Corollary 3.6. *Let A and B be normed spaces. Let $a_1, a_2, \ldots$ be sequences of independent non-degenerate (and norm-dense) random elements a_i in A such that $\mathbb{E}(\|a_i\|) \le 1$. Let $b_1, b_2, \ldots$ be another, independent, sequence of independent identically distributed norm-dense random elements in B such that $\mathbb{E}(\|b_i\|) \le 1$. Let $c_1, c_2, \ldots$ be a deterministic sequence in the unit ball of B such that $\mathrm{lin}(c_1, c_2, \ldots)$ is dense in B. Then both $\sum_{i\in\mathbb{N}} \lambda_i a_i \otimes b_i$ and $\sum_{i\in\mathbb{N}} \lambda_i a_i \otimes c_i$ are almost surely absolutely convergent non-degenerate (and norm-dense) series in $\overline{A \otimes_\alpha B}$ for any cross norm α.*

Proof. The claim for the sequence $\sum_{i\in\mathbb{N}} \lambda_i a_i \otimes c_i$ follows from Lemma 3.3, since $\sum_{i\in\mathbb{N}} A \otimes c_i$ is a dense subspace in $\overline{A \otimes_\alpha B}$. To prove the claim for the sequence $\sum_{i\in\mathbb{N}} \lambda_i a_i \otimes b_i$, let the underlying probability space be given by $\Omega = \Omega_1 \times \Omega_2$ such that $a_i = a_i(\omega_1)$, and $b_i = b_i(\omega_2)$ for $\omega_i \in \Omega_i$. Note that the series $\sum_i \lambda_i a_i \otimes b_i$ is almost surely absolutely convergent. Since the set $\{b_1, b_2, \ldots\}$ is almost surely dense in B by the norm-density of the b_i's (see the proof of [B, Lemma 2.5]) the random series $\sum_{i\in\mathbb{N}} \lambda_i a_i \otimes b_i(\omega_2)$ is non-degenerate (and norm-dense) in $\overline{A \otimes_\alpha B}$ for almost all *fixed* $\omega_2 \in \Omega_2$ by Lemma 3.3. Then, for any $\varphi \in \overline{(A \otimes_\alpha B)}^* \backslash \{0\}$ and $c \in \mathbb{C}$ we get

$$\mathbb{P}(\varphi(\sum_i \lambda_i a_i \otimes b_i) = c) = \int\int 1_{\{\varphi(\sum_i \lambda_i a_i(\omega_1)\otimes b_i(\omega_2))=c\}} d\omega_1 d\omega_2 = 0$$

by Fubini. Hence the series $\sum_i \lambda_i a_i \otimes b_i$ is non-degenerate. (The norm-density follows also with Fubini.) $\square$

Corollary 3.7. *There exists a non-degenerate norm-dense probability measure on $\mathcal{O}_2$.*

Proof. By [C], $\mathcal{O}_2$ is isomorphic to $p(A \rtimes_\alpha \mathbb{Z})p$, where p is some projection in $A \rtimes_\alpha \mathbb{Z}$, and A is the inductive limit $\ldots \xrightarrow{\varphi} \mathbb{K} \xrightarrow{\varphi} \mathbb{K} \xrightarrow{\varphi} \mathbb{K} \xrightarrow{\varphi} \ldots$ for some monomorphism $\varphi : \mathbb{K} \to \mathbb{K}$, α is the shift along this limit, and $\mathbb{K}$ denotes the compacts on a separable Hilbert space. By the above corollaries we can thus construct a norm-dense non-degenerate random element x in $A \rtimes_\alpha \mathbb{Z}$. Then pxp is also a norm-dense non-degenerate random element in $p(A \rtimes_\alpha \mathbb{Z})p$, since $\mathbb{P}(\varphi(pxp) = c) = \mathbb{P}(\tilde{\varphi}(x) = c) = 0$ for $\tilde{\varphi}(y) := pyp$. □

The following lemma shows that non-degenerate measures generate non-atomic random C^*-algebras in a certain sense and to some extent.

Lemma 3.8. *Let $x_1, \ldots, x_n$ be independent non-degenerate random elements in a C^*-algebra A, and let $B \subset A$ be a proper subalgebra of A. Then $\mathbb{P}(C^*(x_1, \ldots, x_n) = B(\text{as sets!})) = 0$.*

Proof. We even have $\mathbb{P}(C^*(x_1, \ldots, x_n) \subseteq B) \leq \mathbb{P}(\varphi(x_1) = \ldots = \varphi(x_n) = 0) = 0$ for any continuous linear functional $\varphi \in A^* \backslash \{0\}$ absorbing B. □

4. The nuclearity of exact random C^*-algebras

In this section we will consider random C^*-subalgebras of $\mathcal{O}_2$ and $\mathcal{O}_2 \otimes C^*_\lambda(\mathbb{F}_2)$.

Lemma 4.1. *Let μ be a norm-dense probability measure on a separable C^*-algebra A. Let $x_1, x_2, \ldots$ be an infinite sequence of independent μ-distributed random variables. Then $C^*(x_1, x_2, \ldots) = A$ almost surely.*

Proof. It is enough to observe that the set $\{x_1, x_2, \ldots\}$ is almost surely dense in A, see the proof of [B, Lemma 2.5]. □

Lemma 4.2. *Let P be a measurable property of separable C^*-algebras which passes to direct limits (valid examples for P are nuclearity (Proposition 2.5) and exactness ([B])). Let A be a separable C^*-algebra not satisfying property P. Let $x_1, x_2, \ldots$ be a sequence of independent identically distributed norm-dense random elements x_i in A. Then*

$$\lim_{n\to\infty} \mathbb{P}(C^*(x_1, \ldots, x_n) \text{ satisfies } P) = 0.$$

Proof. Let $B = C^*(x_1, x_2, \ldots)$ and $B_n = C^*(x_1, \ldots, x_n)$. Let $f_n = 1_{\{B_n \text{ satisfies } P\}}$. Fix $\omega \in \Omega$. Assume that $\lim_k f_{n_k}(\omega) = 1$ for some increasing sequence $(n_k)_k$ in $\mathbb{N}$. Then $B(\omega)$ satisfies property P since $B(\omega)$ is the direct limit of the sequence $B_{n_k}(\omega)$. But since $B = A$ a.s. by Lemma 4.1, and A does not satisfy property P, we conclude that $\lim_n f_n = 0$ pointwise a.s. By dominated convergence we also have convergence in L^1. That means that $\lim_n \int f_n d\mathbb{P} = 0$, or $\lim_n \mathbb{P}(B_n \text{ satisfies } P) = 0$. □

Let $\mathbb{F}_2$ denote the free group of two generators and recall that $E = C^*_\lambda(\mathbb{F}_2)$ is not nuclear but exact [Wa]. The next theorem is our first result which may be interpreted as some information about the probability that an exact C^*-algebra is nuclear.

Theorem 4.3. *Let E be a separable unital non-nuclear exact C^*-algebra, say $E = C^*_\lambda(\mathbb{F}_2)$. Then $\mathcal{O}_2 \otimes E$ is a non-nuclear exact C^*-algebra which contains, up to isomorphism, each separable exact C^*-algebra as a subalgebra.*

Further, for any norm-dense probability measure μ on $\mathcal{O}_2 \otimes E$ and independent μ-distributed random elements $x_1, x_2, \ldots$ we have

$$\lim_{n\to\infty} \mathbb{P}(C^*(x_1, \ldots, x_n) \text{ is nuclear}) = 0.$$

If $E = C^*_\lambda(\mathbb{F}_2)$ then μ can be chosen to be non-degenerate.

Proof. By the associativity of the minimal and maximal tensor product $\mathcal{O}_2 \otimes E$ is not nuclear. If A is any separable exact C^*-algebra then A embeds into $\mathcal{O}_2$ by [KP, Theorem 2.8], and so also into $\mathcal{O}_2 \otimes E$. The claim with the limit follows from Lemma 4.2. The last claim follows from the Corollaries 3.4-3.7. □

Proposition 4.4. *For any sequence $E_1, E_2, \ldots$ of nonzero separable unital exact C^*-algebras E_i there exist unital monomorphisms $\varphi_i : \mathcal{O}_2 \otimes E_i \to \mathcal{O}_2 \otimes E_{i+1}$ such that $\mathcal{O}_2$ is isomorphic to the direct limit $\mathcal{O}_2 \otimes E_1 \xrightarrow{\varphi_1} \mathcal{O}_2 \otimes E_2 \xrightarrow{\varphi_2} \ldots$. Moreover, $\varphi_i = \varphi$ for all i if $E_i = E$ for all i.*

Proof. Consider the sequence

$$\mathcal{O}_2 \xrightarrow{a_1} \mathcal{O}_2 \otimes E_1 \xrightarrow{b_1} \mathcal{O}_2 \xrightarrow{a_2} \mathcal{O}_2 \otimes E_2 \xrightarrow{b_2} \mathcal{O}_2 \xrightarrow{a_3} \ldots,$$

where a_i is the unital canonical embedding, and b_i is a unital embedding according to [KP, Theorem 2.8] (alternatively, one may take the the canonical embedding $b_i : \mathcal{O}_2 \otimes E_i \to \mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ ([R2])). Then the direct limit A of the above sequence is the direct limit of unital maps between copies of $\mathcal{O}_2$, and hence $A \cong \mathcal{O}_2$ by Rørdam [R1, Theorem 7.2]. On the other hand, A is the direct limit as claimed when we set $\varphi_i = a_{i+1} b_i$. □

The following corollary is of theoretical interest in its own.

Corollary 4.5. *The Cuntz algebra $\mathcal{O}_2$ is the norm closure of the union of a certain increasing sequence of non-nuclear subalgebras of $\mathcal{O}_2$.*

Proof. By Proposition 4.4 we may write $\mathcal{O}_2$ as the direct limit

$$\mathcal{O}_2 \otimes C^*_\lambda(\mathbb{F}_2) \xrightarrow{\varphi} \mathcal{O}_2 \otimes C^*_\lambda(\mathbb{F}_2) \xrightarrow{\varphi} \mathcal{O}_2 \otimes C^*_\lambda(\mathbb{F}_2) \xrightarrow{\varphi} \dots$$

for some injective homomorphism φ. Now recall that $\mathcal{O}_2 \otimes C^*_\lambda(\mathbb{F}_2)$ embeds into $\mathcal{O}_2$ by [KP, Theorem 2.8], and is non-nuclear since $\mathcal{O}_2$ is nuclear and $C^*_\lambda(\mathbb{F}_2)$ is non-nuclear. □

The next theorem is the second result which may be interpreted as some piece of information about the probability that an exact C^*-algebra is nuclear. Here we consider random subalgebras of $\mathcal{O}_2$.

Theorem 4.6. *There exists a norm-dense non-atomic (but degenerate) probability measure μ on $\mathcal{O}_2$ such that for an independent sequence $x_1, x_2, \dots$ of μ-distributed random elements x_i we have*

$$\liminf_{n\to\infty} \mathbb{P}(C^*(x_1, \dots, x_n) \text{ is nuclear}) = 0.$$

Proof. By 4.4, $\mathcal{O}_2$ can be written as the inductive limit $A_1 \xrightarrow{\varphi} A_2 \xrightarrow{\varphi} A_3 \xrightarrow{\varphi} \dots$ where $A_n = \mathcal{O}_2 \otimes C^*_\lambda(\mathbb{F}_2)$ for all $n \in \mathbb{N}$, and where φ is an injective homomorphism. Let $\Psi_n : A_n \to \mathcal{O}_2$ be the canonical embedding such that $\overline{\bigcup_{n\in\mathbb{N}} \Psi_n(A_n)} = \mathcal{O}_2$. Let α be a random variable with values in $\mathbb{N}$ such that $\mathbb{P}(\alpha = n) = \lambda_n > 0$ for all $n \in \mathbb{N}$. Let a be a norm-dense non-atomic random element in $\mathcal{O}_2 \otimes C^*_\lambda(\mathbb{F}_2)$. Let $x = \Psi_\alpha(a)$, and notice that x is a norm-dense and non-atomic (since $\mathbb{P}(\Psi_\alpha(a) = d) \le \mathbb{P}(a \in \bigcup_{k\in\mathbb{N}} \Psi_k^{-1}(\{d\})) = 0$) random element in $\mathcal{O}_2$. Let $a_1, a_2, \dots$ be a sequence of independent a-distributed random elements a_i (that means, each a_i should have the same distribution as a). Let $\alpha_1, \alpha_2, \dots$ be a sequence of independent α-distributed random elements α_i. Then the sequence $x_1 = \Psi_{\alpha_1}(a_1), x_2 = \Psi_{\alpha_2}(a_2), \dots$ forms a sequence of independent x-distributed random elements.

We need to adjust the values $\lambda_1, \lambda_2, \dots$ to get the right distribution for x. To achieve this, we use an inductive argument. Let $1 = \varepsilon_1 > \varepsilon_2 > \dots > 0$ be a sequence converging to zero.

$k \Rightarrow (k+1)$: Let $n_1 < n_2 < \dots < n_k \in \mathbb{N}$ and suitable $\lambda_1, \dots, \lambda_k$ already be chosen such that for arbitrary $\lambda_{k+1}, \lambda_{k+2}, \dots$ satisfying $\sum_{i=1}^\infty \lambda_i = 1$ one has

$$\mathbb{P}(C^*(x_1, \dots, x_{n_k}) \text{ is nuclear}) \le 3\varepsilon_k.$$

Let β be the random variable in $\mathbb{N}$ given by $\mathbb{P}(\beta = i) = \lambda_i$ for $1 \le i \le k$, $\mathbb{P}(\beta = k+1) = 1 - (\lambda_1 + \ldots + \lambda_k)$, and $\mathbb{P}(\beta > k+1) = 0$. Let $\beta_1, \beta_2, \ldots$ be a sequence of independent β-distributed random variables. Let y_i be the norm-dense random element $y_i = \Psi_{\beta_i}(a_i)$ in $\Psi_{k+1}(A_{k+1}) \cong \mathcal{O}_2 \otimes C^*_\lambda(\mathbb{F}_2)$. By Theorem 4.3 we can choose $n_{k+1} > n_k$ such that

$$\mathbb{P}(C^*(y_1, \ldots, y_{n_{k+1}}) \text{ is nuclear}) \le \varepsilon_{k+1}.$$

Let $\overline{\alpha} = (\alpha_1, \ldots, \alpha_{n_{k+1}})$ and $\overline{\beta} = (\beta_1, \ldots, \beta_{n_{k+1}})$. We choose $0 < \lambda_{k+1} < 1 - (\lambda_1 + \ldots + \lambda_k)$ such that $\lambda_1 + \ldots + \lambda_{k+1}$ is close enough at 1 (at least closer than ε_{k+1}), such that for arbitrary $\lambda_{k+2}, \lambda_{k+3}, \ldots$ satisfying $\sum_{i=1}^\infty \lambda_i = 1$ the distribution of $\overline{\alpha}$ is so close to the distribution of $\overline{\beta}$, namely $|\mathbb{P}(\alpha = n) - \mathbb{P}(\beta = n)| = |\mathbb{P}(\alpha_i = n) - \mathbb{P}(\beta_i = n)| \le \delta$ for all $n \in \mathbb{N}$ for a very small fixed δ, such that the following estimates hold.

$$\begin{aligned}
& \mathbb{P}(C^*(x_1, \ldots, x_{n_{k+1}}) \text{ is nuclear}) \\
\le\ & \mathbb{P}(C^*(x_1, \ldots, x_{n_{k+1}}) \text{ is nuclear } | \, \alpha_1, \ldots, \alpha_{n_{k+1}} \le k+1) + \varepsilon_{n_{k+1}} \\
=\ & \sum_{s \in \{1,\ldots,k+1\}^{n_{k+1}}} \mathbb{P}(\overline{\alpha} = s) \mathbb{P}(C^*(x_1, \ldots, x_{n_{k+1}}) \text{ is nuclear } | \overline{\alpha} = s) + \varepsilon_{n_{k+1}} \\
\le\ & \sum_{s \in \{1,\ldots,k+1\}^{n_{k+1}}} (n_{k+1}\delta + \mathbb{P}(\overline{\beta} = s)) \mathbb{P}(C^*(y_1, \ldots, y_{n_{k+1}}) \text{ is nuclear } | \overline{\beta} = s) + \varepsilon_{n_{k+1}} \\
\le\ & (k+1)^{n_{k+1}} n_{k+1}\delta + \mathbb{P}(C^*(y_1, \ldots, y_{n_{k+1}}) \text{ is nuclear}) + \varepsilon_{k+1} \le 3\varepsilon_{k+1}.
\end{aligned}$$

This ends the inductive step.

Notice that we get $\sum_{i=1}^\infty \lambda_i = 1$ at the end. The distribution μ of x is the desired distribution. This proves the claim. □

5. Stochastic rank

We have seen in Lemma 3.8 that a non-degenerate measure induces random C^*-algebras which are non-atomic on proper subalgebras of the carrier algebra A. Nothing is said, however, about the probability that the random C^*-algebra is the whole carrier algebra A. The probability may even be one, which creates a trivial situation. This consideration motivates the following definition.

Definition 5.1. The *stochastic rank* of a unital C^*-algebra A is the smallest integer n such that there exists a norm-dense probability measure μ on A such that $n+1$ independent μ-distributed random elements $a_1, \ldots, a_{n+1}$ and 1 almost surely generate A, i.e. $C^*(1, a_1, \ldots, a_{n+1}) = A$ almost surely. If no such n exists then the stochastic rank is said to be infinite.

The notion 'rank', and the incorporation of the unit 1 in the above definition, is based on the following observation.

Proposition 5.2. *The stochastic rank of* $C([0,1]^d)$ *(complex valued continuous functions) is* d*.*

Proof. Assume the stochastic rank were smaller than d. Then there would exist a norm-dense probability measure μ on $C([0,1]^d)$ such that for independent μ-distributed random elements $\zeta_1,\dots,\zeta_d$ we have $C^*(1,\zeta_1,\dots,\zeta_d) = C([0,1]^d)$ almost surely. Choose any continuous function $f:[0,1]^d \to \mathbb{R}^{2d}$ for which there exists a closed ball $V \subseteq \mathbb{R}^d$ centered in the origin and with radius $\delta > 0$, and two points $a,b \in [0,1]^d$ satisfying $a+V, b+V \subseteq [0,1]^d$, $f(a+y) = (y,0) \in \mathbb{R}^d \times \mathbb{R}^d$, and $f(b+y) = (0,y) \in \mathbb{R}^d \times \mathbb{R}^d$ for all $y \in V$. [1] If $s,t : [0,1]^d \to \mathbb{R}^d$ are continuous functions with supnorm smaller then $\delta/2$, and if $g:[0,1]^d \to \mathbb{R}^{2d}$ is given by $g(x) = f(x) + (s(x),t(x))$, then the equation $g(a+x) = (x+s(x),t(x)) = g(b+y) = (s(y),y+t(y))$ has a solution $(x,y) \in V^2$, since it is equalent to the fixed point equation $(x,y) = (s(y)-s(x),t(x)-t(y))$, which has a solution by the fixed point theorem of Brauer.

Thus for all h in the ball

$$H = \{\, g \in C([0,1]^d,\mathbb{R}^{2d}) \mid \|f-g\| \le \delta/2 \,\}$$

we have $h(a+x_h) = h(b+y_h)$ for some $x_h,y_h \in V$ depending on h. We regard now H as a subset of $C([0,1],\mathbb{C}^d)$ in the canoncial way. Let $\tilde h_k : [0,1]^d \to \mathbb{C}$ be the k-th complex valued coordinate of h. Then

$$C^*(1,\tilde h_1,\dots,\tilde h_d) \subseteq \{\, g \in C([0,1]^d) \mid g(a+x_h) = g(b+y_h) \,\} \ne C([0,1]^d)$$

for all $h \in H$. Since μ is a norm-dense measure, we must have $\mathbb{P}((\zeta_1,\dots,\zeta_d) \in H) > 0$. This is a contradiction to the assumption.

We next show that the stochastic rank is smaller or equal than d. We will only give a heuristic argument at the end. Let μ be a norm-dense probability measure on $C([0,1]^d)$. Let $f_1,\dots,f_{d+1}$ be independent μ-distributed random variables, and let $f = (f_1,\dots,f_{d+1})$. If $f(s) \ne f(t)$ for all $s \ne t$, then it follows from the Stone-Weierstrass theorem that $C^*(1,f_1,\dots,f_{d+1}) = C([0,1]^d)$. Thus, our aim will be to choose μ in such a way that for almost all f we have $f(s) \ne f(t)$ for all $s \ne t$.

Let N be a random element in $\mathbb{N}$ such that $\mathbb{P}(N=n) > 0$ for all $n \in \mathbb{N}$. Let $(\alpha_n)_{n\in\mathbb{Z}^d}$ be a family of independent $N(0,1)$-normal distributed real

[1] I am indebted to Tilman Bauer who told me this example for f.

valued random variables α_n. Let $p : [0,1]^d \to \mathbb{R}$ be the random polynomial given by $p(x) = \sum_{n=0}^{N^d} \alpha_n x^n$. Let q be a further, independent, random polynomial with the same distribution as p. We let μ be the distribution of $p + iq$. Then clearly μ is a norm dense probability measure on $C([0,1]^d)$. We have now fixed μ. We now regard the function f from above as a continuous function $f : [0,1]^d \to \mathbb{R}^{2d+2}$. In other words, f is the random element $f(s) = (p_1(s), p_2(s), \ldots, p_{2d+2}(s))$, where $p_1, \ldots, p_{2d+2}$ are independent random polynomials, and where each p_i is distributed like p. If $d = 1$, then f is a 1-dimensional, random polynomial, curve in $\mathbb{R}^4$, and it is suggestive that for almost all f, the curve f will not intersect itself in any point. If $d = 2$, then f is a 2-dimensional, random polynomial, surface in $\mathbb{R}^6$, and again we believe that for almost all f, the surface f will not intersect itself in any point almost surely. And so on. (A heuristically argument is that the equation $f(s) = f(t)$ consists of a system of $2d + 2$ equations, but with only $2d$ free parameters, namely s and t.) Hence we obtained $f(s) \neq f(t)$ for all $s \neq t$ almost surely as required. □

Proposition 5.3. *The stochastic rank of* $\tilde{\mathbb{K}} = \mathbb{K} \oplus \mathbb{C}$ *is zero.*

Proof. Let $\alpha, (X_{ij})_{i,j\in\mathbb{N}}$, respectively $(\beta_i)_{i\in\mathbb{N}}$, independent identically normal distributed complex, respectively real, valued random variables. Let J be the random element in $B(\ell^2(\mathbb{N}))$ with matrix representation $J_{ij} = 2^{-i}2^{-j}X_{ij}$ and $J_{ji} = \overline{J}_{ij}$ for all $i \leq j$. Note that J is a.s. a Hilbert-Schmidt operator and hence compact. By Gram-Schmidt's orthogonalization of an i.i.d norm-dense sequence in $H = \ell^2(\mathbb{N})$ we get a random unitary operator U in $B(H)$ as described in Section 2 of [B]. We also choose a random diagonal operator D in $B(H)$ by putting $D_{nn} = 2^{-n}\beta_n$ on the diagonal of D for all $n \in \mathbb{N}$, and setting all other entries of D to zero. Let P_n denote the projection onto $\ell^2(\{1\ldots,n\})$. For any deterministic unitary operator V, deterministic diagonal operator $E \in \mathbb{K}_{sa}$, $\varepsilon > 0$ and $n \in \mathbb{N}$ we have $\mathbb{P}(\|(U-V)P_n\| \leq \varepsilon) > 0$. An elementary estimate thus shows that $\mathbb{P}(\|UDU^* - VEV^*\| \leq \varepsilon) > 0$, in other words, $R = UDU^*$ is norm-dense in $\mathbb{K}_{sa}$. Set $Z = R + iJ + \alpha 1$ and $A = C^*(Z, 1)$. We claim that $A = \tilde{\mathbb{K}}$ a.s. Clearly $R, J \in A$. Regarding D as a continuous real valued function on the one point compactification $\mathbb{N} \sqcup \{\infty\}$ shows that $e_{ii} \in C^*(D, 1)$ for all $i \geq 1$ a.s. by the Stone-Weierstrass theorem. Since U^*JU has a.s. everywhere nonzero matrix entries, for all $i, j \in \mathbb{N}$ there is a nonzero $\lambda \in \mathbb{C}$ such that $Ue_{ij}U^* = \lambda U e_{ii} U^* J U e_{jj} U^* \in A$. Hence $A = \tilde{\mathbb{K}}$ a.s. □

Acknowledgement. I thank Joachim Cuntz for his invitation and hospitality at the Institute of Mathematics in Münster.

References

[B] Burgstaller, B., *The exactness of certain randomized C^*-algebras.* preprint.

[C] Cuntz, J., *Simple C^*-Algebras Generated by Isometries.* Commun. math. Phys. **57**, 173–185 (1977).

[HT] Haagerup, U., Thorbjørnsen, S., *A new application of random matrices:* $\mathrm{Ext}(C^*_{red}(F_2))$ *is not a group.* Ann. Math. **162**, 711–775 (2005).

[KP] Kirchberg, E., Phillips, N. C., *Embedding of exact C^*-algebras in the Cuntz algebra $\mathcal{O}_2$.* J. reine angew. Math. **525**, 17–53 (2000).

[Pa] Paulsen, V. I. *Completely bounded maps on C^*-algebras and invariant operator ranges.* Proc. Amer. Math. Soc. **86**, 91–96 (1982).

[P] Pisier, G. *Introduction to operator space theory.* London Mathematical Society Lecture Note Series, volume **294**, Cambridge University Press, Cambridge, 2003.

[R1] Rørdam, M. *Classification of inductive limits of Cuntz algebras.* J. reine angew. Math. **440**, 175–200 (1993).

[R2] Rørdam, M. *A short proof of Elliott's theorem: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.* C. R. Math. Rep. Acad. Sci. Canada **16**, 31–36 (1994).

[S] Smith, R. R. *Completely contractive factorizations of C^*-algebras.* J. Funct. Anal. **64**, 330–337 (1985).

[Wa] Wassermann, S. *On tensor products of certain group C^*-algebras.* J. Funct. Anal. **23**, 239–254 (1976).

[W] Wittstock, G. *Ein operatorwertiger Hahn-Banach Satz.* J. Funct. Anal. **40**, 127–150 (1981).

Institute of Mathematics, University of Münster, Einsteinstrasse 62, 48149 Münster, Germany

E-mail address: bernhardburgstaller@yahoo.de