ANNIHILATING PROBABILITY MEASURES UNDER CONSTRAINTS

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Abstract. Let $\mathbb{P}$ be a probability measure and $\mathcal{H} \subseteq L^\infty(\mathbb{P})$ be a linear subspace and $0 < c \leq 1 \leq C$ real constants. Then we give a relatively computable criterion whether or not there exists a $\mathcal{H}$-annihilating probability measure $\mathbb{Q} \sim \mathbb{P}$ equivalent to $\mathbb{P}$ with density $c \leq d\mathbb{Q}/d\mathbb{P} \leq C$. In fact we also prove a version where $L^\infty(\mathbb{P})$ is replaced by $C(K)$ for a compact Hausdorff space K.

1. Introduction

In the theory of finance there exists an import theorem which is well-known under "the fundamental theorem of asset pricing". Mathematically spoken the following question is a central building block for its solution. One has given a probability measure $\mathbb{P}$, a subspace $\mathcal{H} \subseteq L^\infty(\mathbb{P})$, say, and the question is whether there exists an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $\int f d\mathbb{Q} = 0$ for all $f \in \mathcal{H}$ ($\mathbb{Q}$ is $\mathcal{H}$-annihilating).

This question and its answers, in connection with stochastic processes, are well discussed and a heap of papers deals with them (samples are [1, 2, 3, 4, 5, 6, 8, 9]). However, we think a pure mathematical aspect of this building can be enriched, for we asked for $\mathcal{H}$-annihilating, equivalent probability measures with constraint on the density function $d\mathbb{Q}/d\mathbb{P}$. We obtained Theorem 1.1 below.

For a integrable function f let $I(f) = \int f d\mathbb{P}$. If $I(f) \leq 0$, then let $G(f)$ be the unique real number $\text{essinf} f \leq \delta \leq 0$ such that $\int f \vee \delta d\mathbb{P} = 0$.

Theorem 1.1. *Let $\mathbb{P}$ be probability measure, $\mathcal{H} \subseteq L^\infty(\mathbb{P})$ a linear subspace and $0 < c \leq 1 \leq C$ real constants. Then there exists a $\mathcal{H}$-annihilating equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $c \leq d\mathbb{Q}/d\mathbb{P} \leq C$ if and only if for all $f \in \mathcal{H}$ and all $\alpha \in [I(f), \text{esssup} f]$ one has*

$$0 \leq (C-1)\alpha + (c-1)G(f-\alpha) - CI(f).$$

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It is easy to check that the criterion is positive homogenous, i.e. the inequality is true for $f \in L^\infty(\mathbb{P})$ and all α iff it is valid for λf $(\lambda > 0)$ and all α. In consequence it is already sufficient to check the criterion just for all $f \in \mathcal{H} \cap S$, where S is the sphere of the unit ball, say.

The criterium is particularly useful if $L^\infty(\mathbb{P})$ has high dimension and $\mathcal{H}$ has low dimension. If $\mathcal{H}$ is finite dimensional then $\mathcal{H} \cap S$ is a $[\dim(\mathcal{H}) - 1]$-dimensional compact sphere. If the above inequality is then numerically testable for one function f and one value α then it is thinkable that the inequality is testable for all $f \in \mathcal{H} \cap S$ and the corresponding α's up to a certain precision.

In the following example we consider the case when the probability space Ω is finite. In this case the above theorem can ensure a root for real-valued matrix A such that the root satisfies certain constraints.

Example 1.2. Let A be a real valued $m \times n$-matrix. Let $a \leq b$ be positive real constants. Does there exist a solution $\xi \in \mathbb{R}^n$ of the equality $A\xi = 0$ such that $1 = |\xi| := \sum_{i=1}^n \xi_i$ and $a \leq \xi_i \leq b$ for all $i = 1, ..., n$? Let $a_1, ..., a_m$ be the rows of A. A solution ξ of the equality $A\xi = 0$ exists if and only if $< a_i, \xi >= 0$ for all $i = 1, ..., m$. Equivalent is, that $< f, \xi >= \sum_{i=1}^n f_i \xi_i = 0$ for all f of the vector space $\mathcal{H} := \text{span}\{a_1, ..., a_m\}$. Regarding $\Omega = \{1, ..., n\}$ as a measurable space we have to find a $\mathcal{H}$-annihilating probability measure ξ.

Let $\mathbb{P}(\{i\}) = \frac{1}{n}$. Then the constraint $a \leq \xi_i \leq b$ is equivalent to the constraint $an \leq \xi n \leq bn$ where $d\xi/d\mathbb{P} = \xi n$. Hence the desired ξ exists if and only if the condition given in Theorem 1.1 holds.

We also obtain a result like Theorem 1.1 for the space $C(K)$, where K is a compact Hausdorff space (Theorem 2.3). Moreover we give criterions for $\mathcal{H}$-annihilating, equivalent probability measures $\mathbb{Q} \sim \mathbb{P}$ and less restricting, or lets say, other constraints on the density function $d\mathbb{Q}/d\mathbb{P}$ in section 3. But all these criterions contain several quantors and are by far less computable then the criterion given in Theorem 1.1.

2. Annihilation in $C(K)$

In this section we prove the $C(K)$-version of Theorem 1.1. The version for $L^\infty(\mathbb{P})$ can then be easily deduced from the $C(K)$-version due to Gelfands representation theorem. See also the discussion in the next section.

In what follows K denotes a compact Hausdorff space. In this paper all function spaces are real-valued. A *state* on $C(K)$ is a positive linear functional $\varphi : C(K) \to \mathbb{R}$ with $\varphi(\mathbf{1}) = 1$.

Two functionals $\varphi, \psi \in C(K)^*$ satisfy per def. the relation $\varphi \le \psi$ if $\varphi(f) \le \psi(f)$ for all positive functions $f \in C(K)$. The maximum and minimum of $f \in C(K)$ is denoted by $\max(f)/\min(f)$; the unit of $C(K)$ by $\mathbf{1}$.

Proposition 2.1. *Let K be a compact Hausdorff space, φ a state on $C(K)$, $\mathcal{H} \subseteq C(K)$ a linear subspace and $c \le 1 \le C$ reals. Then there exists a $\mathcal{H}$-annihilating linear functional ψ on $C(K)$ with $\psi(\mathbf{1}) = 1$, $c\varphi \le \psi \le C\varphi$ if and only if for all $x \in \mathcal{H}$*

$$\varphi(x) \le \inf_{y \in C(K)} (C-1)\max[x + y - \varphi(x+y)\mathbf{1}] + (1-c)\max[y - \varphi(y)\mathbf{1}]. \tag{1}$$

Proof. We start with some preparations. Let $E = \ker(\varphi)$ and

$$P : C(K) \to E : Px = x - \varphi(x)\mathbf{1}.$$

Then we write $C(K)$ as direct sum $C(K) = E \oplus \mathbb{R}\mathbf{1}$ using the projection P, i.e. $x = Px + \varphi(x)\mathbf{1}$ in this coordinate system. Abbreviate

$$m : C(K) \to \mathbb{R} : m(x) = -\min(x).$$

If we replace y by $-y$ in the inequality (1), then it reads

$$\varphi(x) \le \inf_{y} (C-1)m(-P(x-y)) + (1-c)m(P(y)). \tag{2}$$

The only if part. Now assume that we have given the annihilating functional ψ with the stated properties. Note that clearly $x + m(x)\mathbf{1} \ge 0$ for any $x \in C(K)$. The following conditions (3) and (4) are seen to be equivalent by using a few elementary manipulations (note that $\varphi(\mathbf{1}) = \psi(\mathbf{1}) = 1$ and $\varphi(x) = 0$ for all $x \in E$).

$$\forall x \in E : c\varphi(x + m(x)\mathbf{1}) \le \psi(x + m(x)\mathbf{1}) \le C\varphi(x + m(x)\mathbf{1}) \tag{3}$$

$$\forall x \in E : -\psi(x) \le (1-c)m(x) \text{ and } -\psi(x) \le (C-1)m(-x) \tag{4}$$

Now due to our assumptions for any $x \in \mathcal{H}$ we have $0 = \psi(Px) + \varphi(x)$. Hence we get the desired result, i.e. for any $y \in C(K)$

$$\varphi(x) = -\psi(Px) = -\psi(P(x-y)) - \psi(Py)$$

$$\le (C-1)m(-P(x-y)) + (1-c)m(P(y)).$$

The if part. Note that

$$x \mapsto \alpha(x) := (C-1)m(-x) \qquad \text{and} \qquad x \mapsto \beta(x) := (1-c)m(x)$$

are positive sublinear functionals on E, where the positivity follows from $\max(x) \geq \varphi(x) = 0$. Thus their "convex hull"

$$n(x) := \inf\{ \alpha(x-y) + \beta(y) \mid y \in E \} \qquad (x \in E) \tag{5}$$

is also sublinear on E. Now we have a well-defined function

$$l : P(\mathcal{H}) \to \mathbb{R} : l(Px) = \varphi(x), \tag{6}$$

since $Px = 0$ gives $x = \varphi(x)\mathbf{1} \in \mathcal{H}$ and $\varphi(x) \neq 0$ would then contradict the inequality (2) by setting $x = y = \mathbf{1}$. We use the inequality (1) and get for all $x \in \mathcal{H}$

$$l(P(x)) = \varphi(x) \leq n(Px).$$

Via Hahn-Banach we extend l to a linear map $\hat{l} : E \to \mathbb{R}$ with $\hat{l} \leq n$. Now the functional

$$\psi : C(K) \to \mathbb{R} : \psi(y) = \varphi(y) - \hat{l}(Py) \tag{7}$$

obviously annihilates $\mathcal{H}$. For $x \in E$ we get

$$-\psi(x) = \hat{l}(x) \leq n(x) \leq \alpha(x), \beta(x).$$

This is exactly (4) and we obtain (3). Since $c \leq 1 \leq C$, for real $\lambda \geq 0$ we can add to (3) the inequality $c\varphi(\lambda\mathbf{1}) \leq \psi(\lambda\mathbf{1}) \leq C\varphi(\lambda\mathbf{1})$. If we put $\lambda_2 = m(x) + \lambda$ we obtain

$$c\varphi(x + \lambda_2\mathbf{1}) \leq \psi(x + \lambda_2\mathbf{1}) \leq C\varphi(x + \lambda_2\mathbf{1}).$$

Since $x \in E$ and $\lambda_2 \geq m(x)$ were arbitrary, and each positive $y \in C(K)$ has the representation $y = x + \lambda_2\mathbf{1}$ with $x \in E$ and $\lambda_2 \geq m(x)$, we have shown that $c\varphi \leq \psi \leq C\varphi$. The boundedness of ψ clearly follows from the inequality $0 \leq \psi - c\varphi \in C(K)^*$. □

Proposition 2.2. *Let K be a compact Hausdorff space, φ a state on $C(K)$, $0 \leq c, C$ reals and $x \in C(K)$. Then the infimum*

$$I = \inf_{y \in C(K)} C \max[x + y - \varphi(x+y)\mathbf{1}] + c \max[y - \varphi(y)\mathbf{1}] \tag{8}$$

is attained by $y = -((x - \alpha\mathbf{1}) \vee \beta)$ for some reals $\alpha, \beta \in \mathbb{R}$ with $\varphi(x) \leq \alpha \leq \max(x)$ and $\beta \leq 0$ such that $\varphi(y) = 0$.

Moreover its value is

$$I = \min\left\{ C\alpha - c\beta \middle| \varphi(x) \leq \alpha \leq \max(x), \beta \leq 0, \varphi((x - \alpha\mathbf{1}) \vee \beta) = 0 \right\}$$
$$-C\varphi(x).$$

Proof. For the following computations we will need also discontinuous functions and therefore actually we will consider the larger space $\ell^\infty(K)$ (all bounded functions on K) instead of $C(K)$. For this reason we extend φ via Hahn-Banach to a state $\tilde{\varphi} \in \ell^\infty(K)$ ($\tilde{\varphi}$ is a state since $\tilde{\varphi}(\mathbf{1}) = \varphi(\mathbf{1}) = \|\varphi\| = \|\tilde{\varphi}\|$, what already characterizes positivity, see e.g. [7, 4.3.2]). Assume that we have shown the lemma for the case that $C(K)$ is replaced by $\ell^\infty(K)$ and φ is replaced by $\tilde{\varphi}$. Then it is clear that our aimed $C(K)$-version holds too, since the infimum in (8) is already attained by the continuous function $y = -[(x - \alpha\mathbf{1}) \vee \beta] \in C(K)$.

So we shall consider $\ell^\infty(K)$ with a given state φ on it. We put

$$E := \ker(\varphi) = \{\, y - \varphi(y)\mathbf{1} \mid y \in \ell^\infty(K) \,\}.$$

We decompose now all vectors in $\ell^\infty(K)$ into its positive and negative parts where for the positive part we use capital letters and for the negative part we use lower case letters.

Before we proceed with the proof, we notice the following facts which we will use in the sequel several times. First of all, the transition from $C(K)$ to $\ell^\infty(K)$ imposes the replacement of min and max by inf and sup (where $\sup(f) := \sup_{\omega\in K} f(\omega)$, $f \in \ell^\infty(K)$).

Next, if $f \in \ell^\infty(K)$, $\sup(f) \geq 0$ and $A \subseteq K$, then we have

$$\sup(f) = \sup(f1_A) \vee \sup(f1_{K\setminus A}). \tag{9}$$

So this formula is especially applicable for functions $Y - y \in E$ ($Y, y \geq 0$, $Yy = 0$) since we have

$$0 = \varphi(Y - y) \leq \sup(Y - y).$$

Thus, for example, $\sup(Y - y) = \sup(Y) \vee \sup(y) = \sup(Y)$. Good to know is also the easy identity

$$(fg) \vee 0 = f(g \vee 0) \tag{10}$$

for a positive function f and an arbitrary function g, and hence we will often omit the brackets. Finally, we use the notion

$$S(f) = 1_{\{\,\omega\in K \;\mid\; f(\omega)\neq 0\,\}}$$

for the carrier of f.

Going back to the proof, we replace the expression $x - \varphi(x)\mathbf{1} \in E$ by $X - x \in E$ in (8) ($X, x \geq 0$, $xX = 0$, the x in $x - \varphi(x)\mathbf{1}$ is not identic with the x in $X - x$ (!)) to get good notations. At the end of the proof we will abandon this substitution. Then, in order to get the value (8), we have to look for $Y - y \in E$ which minimizes the value

$$C \sup(X - x + Y - y) + c \sup(Y) \tag{11}$$

($Y, y \geq 0$, $Yy = 0$) and notice that all numbers occurring in (11) are positive.

Optimization step 1: Let $Y - y \in E$ be given. We are going to improve (11) by several manipulations on $Y - y$. Put

$$z = S(X)y.$$

We obtain

$$\begin{aligned}\sup(X - x + Y - y) &= \sup S(X+Y)(X - x + Y - y) \\ &= \sup(X - x + Y - z).\end{aligned}$$

Optimization step 2: We put

$$\alpha = 0 \vee \sup S(z)(X - z),$$

$$w = S(z)(X - \alpha\mathbf{1}) \vee 0.$$

Observing that

$$\sup S(z)(X - w) = \sup[S(z)(X - \alpha\mathbf{1}) - S(z)(X - \alpha\mathbf{1}) \vee 0 + \alpha S(z)] = \alpha \tag{12}$$

we proceed (we use here (9))

$$\begin{aligned}0 &\leq \sup(X - x + Y - z) \\ &= \sup S(z)(X - z) \vee \sup(\mathbf{1} - S(z))(X - x + Y) \\ &= \sup S(z)(X - w) \vee \sup(\mathbf{1} - S(z))(X - x + Y) \qquad (13) \\ &= \sup(X - x + Y - w). \qquad (14)\end{aligned}$$

Note that $w \leq z \leq y$ since

$$S(z)X - z = S(z)(X - z) \leq (\sup S(z)(X - z))\mathbf{1} \leq \alpha\mathbf{1}.$$

We put $W = \varphi(w)\varphi(y)^{-1}Y$ and thus $W \leq Y$ and $W - w \in E$ since $\varphi(W) = \varphi(w)$. It is then advantageous to replace $Y - y$ by $W - w$,

$$\begin{aligned}\sup(X - x + Y - y) &= \sup(X - x + Y - w) \qquad (15) \\ &\geq \sup(X - x + W - w) =: \gamma \geq 0. \qquad (16)\end{aligned}$$

Optimization step 3: We put

$$v = S(z)(X - \gamma\mathbf{1}) \vee 0.$$

One can calculate (by considering the two cases $X(\omega) \geq \gamma$, $X(\omega) < \gamma$, $\omega \in K$, say)

$$\sup S(z)(X - v) = \gamma \wedge \sup S(z)X. \tag{17}$$

Note that $\sup(X - x + Y - w) \geq \alpha$ due to (12), (13) and (14). Combining this with (15) and (16) yields $\alpha \geq \gamma$.

Thus $v \leq w$ and we put $V = \varphi(v)\varphi(w)^{-1}W$. It is then advantageous to replace $W - w$ by $V - v \in E$. Indeed, since $v \leq w \leq y$ we have

$$S(z)v = v, \quad S(z)w = w, \tag{18}$$

and

$$\gamma = \sup(X - x + W + w) \geq \sup((\mathbf{1} - S(z))(X - x + W)$$
$$\geq \sup((\mathbf{1} - S(z))(X - x + V)$$

yields therefore with (15), (16) and (17)

$$\sup(X - x + Y - y)$$
$$\geq \gamma \geq \sup S(z)(X - v) \vee \sup(\mathbf{1} - S(z))(X - x + V) \tag{19}$$
$$= \sup(X - x + V - v).$$

Since (note (18) and (16))

$$(\mathbf{1} - S(z))(X - x) \leq (\mathbf{1} - S(z))(X - x + W - w) \leq \gamma\mathbf{1}$$

we obtain (note also (10) and $S(z)x = 0$)

$$(X - x - \gamma\mathbf{1}) \vee 0 = S(z)(X - x - \gamma\mathbf{1}) \vee 0 = v. \tag{20}$$

If $\gamma \leq \sup S(z)X$ then, due to (19) and (17), we obtain the equality

$$\gamma = \sup(X - x + V - v). \tag{21}$$

Thereby v is defined by (20) and

$$\gamma \leq \sup(X). \tag{22}$$

The same facts we obtain for the contrary case $\gamma > \sup S(z)X$, if we just replace γ by $\gamma_2 := \sup(X)$. Indeed in this case we have $v = V = 0$ and $v = 0$ is expressed by $v = (X - x - \gamma_2\mathbf{1}) \vee 0$ like in (20), and we get $\gamma_2 = \sup(X - x) = \sup(X - x + V - v)$ like in (21).

Optimization step 4: It became clear that we can restrict us to the following case in the optimization problem (11): We have a scalar

$$0 \leq \alpha \leq \sup(X) \tag{23}$$

such that

$$y = (X - x - \alpha \mathbf{1}) \vee 0, \tag{24}$$

$$\sup(X - x + Y - y) = \alpha \tag{25}$$

(see (20), (21) and (22)). We are going to optimize Y. We have, inserting (24),

$$0 \geq X - x + Y - y - \alpha \mathbf{1} = (X - x - \alpha \mathbf{1}) \wedge 0 + Y. \tag{26}$$

Therefore we obtain the following necessary condition on Y:

$$0 \leq Y \leq -((X - x - \alpha \mathbf{1}) \wedge 0). \tag{27}$$

On the other hand, if some Y satisfies (27), then obviously $Yy = 0$, and equality (25) holds further, since using the equality in (26) and (24) and $\alpha \leq \sup(X)$ we get

$$0 \geq \sup(X - x + Y - y - \alpha \mathbf{1}) \geq \sup(X - x - \alpha \mathbf{1} - y) = 0.$$

Thus we can feel free to choose any Y satisfying (27) without touching the left summand in (11) and we just have to optimize $\sup(Y)$.

Now, for the only further necessary condition on Y (beside (27)) is $\varphi(Y) = \varphi(y)$, it is clear that the best choice is to simply cut the righthanded function in (27) function, i.e. to choose a $\beta \leq 0$ and put

$$Y = (-\beta) \wedge (-((X - x - \alpha \mathbf{1}) \wedge 0)) = -((X - x - \alpha \mathbf{1}) \wedge 0 \vee \beta). \tag{28}$$

To be precise such β exists since for arbitrary $z \in A$ the function $\mathbb{R} \to \mathbb{R} : \beta \mapsto \varphi(z \vee \beta)$ is continuous and hence the necessary condition

$$0 = \varphi(y - Y) = \varphi((X - x - \alpha \mathbf{1}) \vee \beta) \tag{29}$$

has a solution for some $\beta \leq 0$ (recall (23)). So

$$Y - y = -((X - x - \alpha \mathbf{1}) \vee \beta) \tag{30}$$

is now of the desired form and (11) becomes $C\alpha - c\beta$ (due to (25) and (28); alternatively one simply inserts (30) in (11)).

Final step: Therefore (11) attains its infimum by (recall (29))

$$\inf \left\{ C\alpha - c\beta \,\middle|\, 0 \leq \alpha \leq \sup(X - x), \beta \leq 0, \varphi((X - x - \alpha \mathbf{1}) \vee \beta) = 0 \right\}.$$

It is clear that the infimum is attained here, as (α, β) can be chosen from a compact set. More detailed $(\alpha, \beta) \in [0, \sup(X - x)] \times [\inf(X - x), 0] \cap \ker(f)$ where f is the continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} : f(\alpha, \beta) = \varphi((x - \alpha \mathbf{1}) \vee \beta)$.

Last but not least we finish the proof by abandoning the "substitution" $x-\varphi(x)\mathbf{1} \equiv X-x$ of the beginning and easily obtain the announced value for I. □

The combination of propositions 2.1 and 2.2 yields

Theorem 2.3. *Let K be a compact Hausdorff space, φ a state on $C(K)$, $\mathcal{H} \subseteq C(K)$ a linear subspace and $c \leq 1 \leq C$ reals. Then there exists a $\mathcal{H}$-annihilating functional $\psi \in C(K)^*$ with $c\varphi \leq \psi \leq C\varphi$ and $\psi(\mathbf{1}) = 1$, if and only if*

$$\forall f \in \mathcal{H} : \varphi(f) \leq C^{-1} \min(\alpha(C-1) + \beta(c-1)), \tag{31}$$

where the minimum is taken over all $\alpha, \beta \in \mathbb{R}$ with $\varphi(f) \leq \alpha \leq \sup(f)$ and $\beta \leq 0$ such that $\varphi((f-\alpha) \vee \beta) = 0$.

The next corollary is the $C(K)$-version of Theorem 1.1. Let us recall some definitions before. A functional $\varphi \in C(K)^*$ is said to be *normal* if for all nets in the unit ball $(f_i)_{i\in I} \subseteq B_{C(K)}$ converging to zero ($\forall x \in K : f_i(x) \to 0$) we have $\varphi(f_i) \to 0$.

A state φ is *faithful* if $\varphi(f) > 0$ for all $f > 0$.

Corollary 2.4. *Let φ be a faithful normal state on $C(K)$ and $0 < c \leq 1 \leq C$. Then there exists a $\mathcal{H}$-annihilating faithful normal state ψ with $c\varphi \leq \psi \leq C\varphi$ iff criterion* (31) *is fulfilled.*

3. Annihilation in $L^\infty(\mathbb{P})$

We review now our last results under the light of probability measures. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then we have an isometric isomorphism $L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \cong C(K)$ via Gelfand's transformation and where K is stonean ([10, I.4.4] or [7, 5.2.1]).

Alternatively, all proofs of the previous section would also go through for $L^\infty(\mathbb{P})$ if one just replaces $\max(f)$ by $\operatorname{ess\,sup}(f)$. Thereby one can skip a transition like from $C(K)$ to $\ell^\infty(K)$ in the proof of Proposition 2.2, for we have all necessary projections in $L^\infty(\mathbb{P})$ and so all operations, which have been needed, act in $L^\infty(\mathbb{P})$ anyway. So actually the Gelfand representation is not really necessary.

Now if we consider the functional

$$\varphi : L^\infty(\mathbb{P}) \to \mathbb{R} : \varphi(f) = \int f d\mathbb{P} \tag{32}$$

and apply Theorem 2.3 for $0 < c \leq 1 \leq C$, then we directly obtain our main result Theorem 1.1. This is pretty clear: A functional ψ with $c\varphi \leq \psi \leq C\varphi$ and $\psi(1_\Omega) = 1$ defines an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ via

$$\mathbb{Q}(A) = \psi(1_A) \qquad A \in \mathcal{F}. \tag{33}$$

And the estimate $c\varphi \leq \psi \leq C\varphi$ is equivalent to $c \leq d\mathbb{Q}/d\mathbb{P} \leq C$.

In the next result we weaken the constraint on $\mathbb{Q}$ to the one sided estimate $d\mathbb{Q}/d\mathbb{P} \leq C$. But the criterion is abruptly more difficult to decide for it contains more quantors.

Corollary 3.1. *Let $\mathbb{P}$ be probability measure, $\mathcal{H} \subseteq L^\infty(\mathbb{P})$ a linear subspace and $1 \leq C$. Then there exists a $\mathcal{H}$-annihilating, equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ with density $d\mathbb{Q}/d\mathbb{P} \leq C$, if and only if for each measurable set A with $\mathbb{P}(A) > 0$ there exists a smaller measurable set $B \subseteq A$ with $\mathbb{P}(B) > 0$ and a real $\gamma > 0$ such that*

$$\forall f \in \mathcal{H} \oplus [1_B - \gamma \mathbf{1}] : \int f d\mathbb{P} \leq \min \frac{1}{C}(\alpha(C-1) - \beta),$$

where the minimum is taken over all $\alpha, \beta \in \mathbb{R}$ with $\int f d\mathbb{P} \leq \alpha \leq \sup f$ and $\beta \leq 0$ such that

$$\int (f - \alpha) \vee \beta d\mathbb{P} = 0.$$

This can be proved by using an argument of Kreps [8] and Yan [11]. We skip the proof.

In the next corollary we have no restriction on the density function $d\mathbb{Q}/d\mathbb{P}$. This corollary should just point out, that the previous versions under constraint, are "dense" in a certain sense within the cases where we have no restriction on $d\mathbb{Q}/d\mathbb{P}$.

Corollary 3.2. *Let $\varepsilon > 0$. Then the following conditions are equivalent.*

(1) *There exists a $\mathcal{H}$-annihilating probability measure $\mathbb{Q} \sim \mathbb{P}$.*
(2) *There exists a probability measure $\mathbb{R} \sim \mathbb{P}$ and reals $0 < c \leq 1 \leq C$ such that $\|d\mathbb{R}/d\mathbb{P} - \mathbf{1}\|_{L^1(\mathbb{P})} \leq \varepsilon$ and the criterion of Theorem 1.1 is satisfied for $\mathbb{R}$ (instead of $\mathbb{P}$).*
(3) *For all measurable sets $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ there exist a $\mathbb{P}$-absolute continuous probability measure $\mathbb{R}$ with $\mathbb{R}(A) > 0$ and reals $0 < c \leq 1 \leq C$ such that the criterion of Theorem 1.1 is satisfied for $\mathbb{R}$ (instead of $\mathbb{P}$).*

We omit its proof.

In the next Proposition we give a necessary and sufficient condition for the existence of an annihilating probability measure $\mathbb{Q}$ under the constraint $d\mathbb{Q}/d\mathbb{P} \in L^q(\mathbb{P})$. However, one easily realizes that the condition is in the best case good to falsify the existence of $\mathbb{Q}$.

Proposition 3.3. *Let $\mathbb{P}$ be a probability space, $\mathcal{H} \subseteq L^\infty(\mathbb{P})$ a linear subspace and $0 < c \leq 1 \leq C$, $1 < q \leq \infty$ reals. We define p by $1/p + 1/q = 1$. Then there exists a $\mathcal{H}$-annihilating, equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ with the following constraints on the density function.*

(1)
$$d\mathbb{Q}/d\mathbb{P} \in L^q(\mathbb{P}) \qquad c \leq d\mathbb{Q}/d\mathbb{P},$$

if and only if there exists a real $D \geq 0$ *such that*
$\forall x \in \mathcal{H} \forall y \in L^\infty(\mathbb{P})$

$$\int x d\mathbb{P} \leq D \Big\| x + y - \int (x+y) d\mathbb{P} \mathbf{1} \Big\|_{L^p(\mathbb{P})} + (1-c) \sup \Big(y - \int y d\mathbb{P} \mathbf{1} \Big).$$

(2)
$$d\mathbb{Q}/d\mathbb{P} \in L^q(\mathbb{P}),$$

if and only if for each nonzero projection $f \in L^\infty(\mathbb{P})$ *there exist a nonzero projection* $e \leq f$ *and reals* $\gamma > 0$, $D \geq 0$ *such that* $\forall x \in \mathcal{H} \oplus [e - \gamma \mathbf{1}] \forall y \in L^\infty(\mathbb{P})$

$$\int x d\mathbb{P} \leq D \Big\| x + y - \int (x+y) d\mathbb{P} \mathbf{1} \Big\|_{L^p(\mathbb{P})} + \sup \Big(y - \int y d\mathbb{P} \mathbf{1} \Big).$$

We omit the proof. It is similar to the proof of Proposition 2.1 on the one hand, and by using the method of Kreps and Yan on the other hand.

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