SOME MULTIDIMENSIONAL CUNTZ ALGEBRAS

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Abstract. Following closely the article of J. Cuntz where the Cuntz algebra was initiated in 1977, we introduce a certain class of multidimensional Cuntz algebras generated by several sets of isometries (instead of one set in the classical case) which interact in some quasi-abelian way. Then we compute the K-theory for some of these algebras.

1. Introduction

The aim of this paper is to introduce "multidimensional" Cuntz algebras. The Cuntz algebra $\mathcal{O}_n$ ([C1]) can be considered as an one dimensional C^*-dynamical system, since in [C1] it is proved that the stabilization of $\mathcal{O}_n$ is the crossed product of an AF-algebra A with $\mathbb{Z}$, i.e. $\mathbb{K} \otimes \mathcal{O}_n \cong A \rtimes \mathbb{Z}$.

Our motivation for the construction of multidimensional Cuntz algebras was the quest for higher dimensional Cuntz-Krieger algebras [CK]. The Cuntz algebras are special cases of Cuntz-Krieger algebras. As demonstrated in [CK] the Cuntz-Krieger algebras are strongly correlated to dynamics on shift spaces. Focussed on the above crossed product $A \rtimes \mathbb{Z}$ we realized that we may have the chance to introduce multidimensional aspects in the Cuntz algebra (i.e. the occurrence of $A \rtimes \mathbb{Z}^d$; the integer d is the dimension) by the following approach.

Take d Cuntz algebras $\mathcal{O}_{n_1}, \mathcal{O}_{n_2}, \ldots, \mathcal{O}_{n_d}$ and specify how the elements of the different algebras interact; then define the multidimensional Cuntz algebra simply as the C^*-algebra generated by the union of these algebras $\mathcal{O}_{n_i}$.

More precisely, for a two-dimensional Cuntz algebra consider two sets of isometries $\{S_0, \ldots, S_{n-1}\}$ and $\{T_0, \ldots, T_{m-1}\}$ obeying the Cuntz property,

$$S_0 S_0^* + \cdots + S_{n-1} S_{n-1}^* = I \text{ and } T_0 T_0^* + \cdots + T_{m-1} T_{m-1}^* = I,$$

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and specify the products S_iT_j and $S_i^*T_j$. A complete abelian context of the form $S_iT_j = T_jS_i$ and $S_i^*T_j = T_jS_i^*$, as it appears in the algebra $\mathcal{O}_n \otimes \mathcal{O}_m$, is here possible.

On the other hand we do not expect that any complicated relations, if ever a representation on a Hilbert space exists, may produce the results we desire, for example that the generators and relations determine the C^*-algebra uniquely. Particularly, as long as we focus on $\mathbb{Z}^d$-actions rather than general group actions (in the last case things would be very likely extremely ticklish), we probably must consider abelian-like interactions between the isometries. Indeed, in this paper we consider only one type of interaction, namely

$$(*) \quad S_xT_y = T_XS_Y \text{ where } x + yn = X + Ym.$$

This simple but non-abelian relation is motivated by the following example. For the Hilbert space $\ell^2(\mathbb{Z})$ define $S_i(\delta_k) = \delta_{nk+i}$ and $T_j(\delta_k) = \delta_{km+j}$ ($i = 0, \ldots, n-1$, $j = 0, \ldots, m-1$, $k \in \mathbb{Z}$).

More generally we assign sets $n = \{n_1, n_2, \ldots, n_d\} \subseteq \mathbb{N}$ (d can also be infinite) of mutually relative prime integers n_i to a multidimensional Cuntz algebra $\mathcal{O}_n$ of dimension d. It is the C^*-algebra generated by the classical Cuntz algebras $\mathcal{O}_{n_i}$ with interactions $(*)$. As in the classical case the algebras $\mathcal{O}_n$ do not depend on a representation on a Hilbert space, see Theorem 2.8. This fact fills chapter 2 and the way how we prove this is mimicked a proof in [C1].

In chapter 3 we show that $\mathbb{K} \otimes \mathcal{O}_n \cong A \rtimes \mathbb{Z}^d$ for an AF-algebra A (Theorem 3.2), which is the isomorphism we have looked out for the definition of a multidimensional Cuntz algebra. The multidimensional Cuntz algebras are also nuclear, simple and purely infinite (Theorem 3.3). All these facts appear already in its one-dimensional form in [C1] and the proofs are similar. Moreover we compute a few K-groups, more or less for the dimension $d = 2$, where we have $K_0(\mathcal{O}_{\{n_1,n_2\}}) = \mathbb{Z}_{\gcd(n_1-1,n_2-1)}$, see Theorem 3.4.

In the winter semester 2000/2001 Klaus Schmidt kindly invited me to Vienna. He pointed out to me the question whether the classical Cuntz-Krieger algebras with known connection to dynamical systems could be generalized in such a way that they would have connection to *higher* dynamical systems. The present paper is my first approach to this question.

In 2003 at a conference in Sinaia, Romania, Valentin Deaconu kindly pointed out to me that this paper has strong connection with the works

[RS1] and [RS2] of Robertson and Steger. Actually he showed me that - under a suitable translation - the multidimensional Cuntz algebras are special cases of the algebras introduced in [RS2]. Afterwards, in 2003 at a conference in Bolzano, Italy, Tim Steger also kindly pointed this out to me. Robertson and Steger also computed the K-theory of their algebras in [RS3] up to rank two; and this intersects with the K-theory computations in 3.4. Nevertheless, we hope that the present paper has its own right due to a different point of view and approach to higher rank Cuntz-Krieger algebras. The work of Robertson and Steger led to the introduction of higher rank graph C^*-algebras in [KP]. On the other hand, the present work led us to the different approach in [B1], [B2] and [B3].

2. The algebras $\mathcal{O}_n$

In this section we introduce the higher dimensional Cuntz-algebras $\mathcal{O}_n$ and prove the uniqueness theorem 2.8.

Let $d \geq 1$, let $n_1, \ldots, n_d$ mutually relative prime integers $n_i \geq 2$, and put $n = (n_1, \ldots, n_d)$. For $1 \leq i \leq d$ and $0 \leq j \leq n_i - 1$ let $S_{i,j}$ be isometries acting on a Hilbert space H which satisfy the equations

$$S_{i,0}S_{i,0}^* + S_{i,1}S_{i,1}^* + \cdots + S_{i,n_i-1}S_{i,n_i-1}^* = I \tag{1}$$

and

$$S_{i,x}S_{j,y} = S_{j,X}S_{i,Y} \quad \text{whenever } x + yn_i = X + Yn_j \tag{2}$$

for all $1 \leq i \neq j \leq d$. It is well known that $S_{i,0}, S_{i,1}, \ldots, S_{i,n_i-1}$ generate the Cuntz-algebra $\mathcal{O}_{n_i}$ ([C1]). Observe that, since n_i and n_j are relative prime, we find for each pair $(x, y) \in \{0, \ldots, n_i - 1\} \times \{0, \ldots, n_j - 1\}$ exactly one pair $(X, Y) \in \{0, \ldots, n_j - 1\} \times \{0, \ldots, n_i - 1\}$ such that $S_{i,x}S_{j,y} = S_{j,X}S_{i,Y}$.

This is the setting we consider throughout, if we leave out of account that we will later on permit also infinitely many sets of isometries ($d = \infty$). We introduce the alphabet

$$\mathcal{A} = \{ (i, j) \mid i = 1, \ldots, d \text{ and } j = 0, \ldots, n_i - 1 \},$$

and let $V = \bigsqcup_{N \geq 0} \mathcal{A}^N$ be the set of words. The special subset of words

$$W = \{ ((i_1, j_1), \ldots, (i_N, j_N)) \in V \mid N \geq 0 \text{ and } i_1 \leq i_2 \leq \cdots \leq i_N \}$$

is the set of *ordered words*. Notice that $\mathcal{A}$ has a partition $\mathcal{A} = \mathcal{A}_1 \sqcup \cdots \sqcup \mathcal{A}_d$ where $\mathcal{A}_i = \{(i, 0), \ldots, (i, n_i - 1)\}$. For $\alpha = \alpha_1 \ldots \alpha_N \in V$ and $N_i =$

$\#\{ k \mid \alpha_k \in \mathcal{A}_i \}$ we define

$$|\alpha| := (N_1, N_2, \dots, N_d).$$

Notice that for a concatenated word $\alpha\beta$ we get $|\alpha\beta| = |\alpha| + |\beta|$. For a word α in V and $k = (k_1, \dots, k_d) \in \mathbb{Z}_+^d = \{0, 1, 2, 3, \dots\}^d$ we introduce the notations

$$\begin{aligned} S_\alpha &= S_{a_1} S_{a_2} \dots S_{a_N}, \\ S^k &= S_{1,0}^{k_1} S_{2,0}^{k_2} \dots S_{d,0}^{k_d}, \\ n^k &= n_1^{k_1} n_2^{k_2} \dots n_d^{k_d}. \end{aligned}$$

For the next point we start with an example: If we code the number 13 with respect to the basis $(2, 3, 3)$ then we get $13 = (1, 0, 2)_{(2,3,3)}$ (the most left digit is the least significant one). Analogously we can regard a word $\alpha = ((i_1, j_1), (i_2, j_2), \dots, (i_N, j_N)) \in V$ as the digit representation $(j_1, j_2, \dots, j_N)$ of an integer $Z(\alpha)$ with respect to the basis $(n_{i_1}, n_{i_2}, \dots, n_{i_N})$. More precisely we require that

$$Z(\alpha) = \sum_{k=1}^{N} j_k n_{i_1} n_{i_2} \dots n_{i_{k-1}}.$$

Observe that we have

$$0 \le Z(\alpha) < n_{i_1} n_{i_2} \dots n_{i_N} = n^{|\alpha|}$$

and

$$Z(\alpha\beta) = Z(\alpha) + Z(\beta) n^{|\alpha|}. \tag{3}$$

for all $\alpha, \beta \in V$. Observe that an ordered word $\alpha \in W$ is uniquely determined by $|\alpha|$ and $Z(\alpha)$, more precisely we have a bijection

$$t : W \to \{ (k, j) \mid k \in \mathbb{Z}_+^d, 0 \le j < n^k \} : t(\alpha) = (|\alpha|, Z(\alpha)). \tag{4}$$

Readers which are not familiar with the Cuntz algebras should observe the following. From the Cuntz property (1) we deduce the relations

$$S_{i,x}^* S_{i,y} = \delta_{x,y} I \qquad \forall 1 \le i \le d \, \forall 0 \le x, y \le n_i - 1.$$

An successive application of this formula shows that $S_\alpha^* S_\beta = I$ for all ordered words $\alpha, \beta \in W$ that satisfy $|\alpha| = |\beta|$. From the Cuntz-property (1) we can deduce the formula

$$\sum_{\alpha \in W, |\alpha| = k} S_\alpha S_\alpha^* = I \qquad \forall k \in \mathbb{Z}_+^d \tag{5}$$

by induction. In the introduction we have addressed that one must determine the product $S_{i,x}^* S_{j,y}$. But in our case the product is already determined by the $S_{i,x}S_{j,y}$-relations: we automatically have the permutation rules

$$S_{i,x}^* S_{j,y} = S_{j,X} S_{i,Y}^* \text{ whenever } x + Xn_i = y + Yn_j. \tag{6}$$

Indeed, by the permutation relation (2) we have

$$S_{i,x}^* S_{j,y} = S_{i,x}^* S_{j,y} \left(\sum_{k=0}^{n_i - 1} S_{i,k} S_{i,k}^* \right) = \sum_{k=0}^{n_i - 1} S_{i,x}^* S_{i,K_k} S_{j,Y_k} S_{i,k}^*,$$

where $y + kn_j = K_k + Y_k n_i$. Since n_i, n_j are relative prime, different k's yield different K_k's and only one summand does not vanish. So for each pair (x, y) there exists exactly one pair (X, Y) such that the identity in (6) holds.

We would like to point out that the proof of Theorem 2.8 - as presented in this section - is mimicked a proof in [C1].

Lemma 2.1. *Let $\alpha, \beta \in V$. Then*

$$S_\alpha = S_\beta \quad \textit{if and only if} \quad \alpha \equiv \beta,$$

i.e. if $|\alpha| = |\beta|$ and $Z(\alpha) = Z(\beta)$. There exists a unique $\gamma \in W$ with $\gamma \equiv \alpha$ and $S_\gamma = S_\alpha$.

Proof. At first we observe that if we "permute" two neighboring letters in the product S_α like in (2) then $|\alpha| = |\beta|$ and $Z(\alpha) = Z(\beta)$ where S_β is the modified word. Indeed, under one permutation we have a situation like $S_\alpha = S_{\alpha_1} S_{i,x} S_{j,y} S_{\alpha_2} = S_{\alpha_1} S_{j,X} S_{i,Y} S_{\alpha_2} = S_\beta$ where $\alpha = \alpha_1 (i, x)(j, y)\alpha_2$ and $\beta = \alpha_1 (j, X)(i, Y)\alpha_2$ for some $\alpha_1, \alpha_2 \in V$. Thus clearly $|\alpha| = |\beta|$ and formula (3) yields

$$Z(\alpha) = Z(\alpha_1) + xn^{|\alpha_1|} + yn^{|\alpha_1|} n_i + Z(\alpha_2) n^{|\alpha_1|} n_i n_j$$

$$= Z(\alpha_1) + Xn^{|\alpha_1|} + Yn^{|\alpha_1|} n_j + Z(\alpha_2) n^{|\alpha_1|} n_j n_i = Z(\beta).$$

Successively "permuting" neighboring letters in this way in the product S_α it is clear that we find some $\gamma \in W$, $\gamma \equiv \alpha$, such that $S_\gamma = S_\alpha$.

Now if $\alpha \equiv \beta$ then we find some $\alpha_2, \beta_2 \in W$ such that $S_{\alpha_2} = S_\alpha$, $S_{\beta_2} = S_\beta$ and $\alpha_2 \equiv \alpha \equiv \beta \equiv \beta_2$. Thus $\alpha_2 = \beta_2$ due to the bijection t (4), and $S_\alpha = S_\beta$.

The other direction will immediately follow from 2.3 below. □

Proposition 2.2. *For $k \in \mathbb{Z}_+^d$ let $W_k = \{ \alpha \in W \mid |\alpha| = k \}$ and define*

$$\mathcal{F}_k := C^*(\{ S_\alpha S_\beta^* \mid \alpha, \beta \in W_k \}).$$

Then $\mathcal{F}_k \cong M_{n^k}$ is a finite dimensional C^*-algebra and one has $\mathcal{F}_k \subseteq \mathcal{F}_l$ for $k \leq l$. Denote by $\mathcal{F}$ the AF*-algebra $\mathcal{F} = \overline{\bigcup_{k\in\mathbb{Z}_+^d} \mathcal{F}_k}$.*

Proof. It is immediate from the formula $S_\alpha S_\beta^* = \delta_{\alpha,\beta} I$ $(\alpha, \beta \in W_k)$ that $(S_\alpha S_\beta^*)_{\alpha,\beta}$ forms a selfadjoint system of matrix units generating $\mathcal{F}_k$. Hence $\mathcal{F}_k = M_{n^k}$ since $|W_k| = n^k$.

If $k \leq l \in \mathbb{Z}_+^d$ and $|\alpha| = |\beta| = k$ then with (5)

$$S_\alpha S_\beta^* = \sum_{|\gamma|=l-k} S_\alpha S_\gamma S_\gamma^* S_\beta^*$$

and since $|\alpha| + |\gamma| = l$ this sum is obviously in $\mathcal{F}_l$ by 2.1. □

In the sequel we call the set $\{ S_a \mid a \in \mathcal{A} \} \cup \{ S_a^* \mid a \in \mathcal{A} \}$ *the letters of* $S \cup S^*$. Moreover we use the notations $\mathrm{Alg}^*(S)$, respectively $C^*(S)$, for the $*$-algebra, respectively the C^*-algebra, generated in $B(\mathcal{H})$ by the isometries S.

Lemma 2.3. *Any product X in the letters of $S \cup S^*$, if $X \neq 0$, has a unique representation $X = S_\alpha S_\beta^*$ for some $\alpha, \beta \in W$.*

Proof. Successively making "permutations" as in (6), and applying $S_{i,k}^* S_{i,l} = \delta_{k,l} I$ for $(i,k),(i,j) \in \mathcal{A}$, it is clear that we can achieve $X = S_\alpha S_\beta^*$ for some $\alpha, \beta \in V$. Then due to 2.1, α, β can be chosen in W. Now assume $X = S_{\alpha_1} S_{\beta_1}^* = S_{\alpha_2} S_{\beta_2}^*$ for some $\alpha_i, \beta_i \in W$. Then for the range projection we have (recall (5))

$$XX^* = S_{\alpha_1} S_{\alpha_1}^* = \sum_{\gamma\in W, |\gamma|=|\alpha_2|} S_{\alpha_1} S_\gamma S_\gamma^* S_{\alpha_1}^*.$$

The latter expression is a sum of exactly $n^{|\alpha_2|}$ different matrix units in $\mathcal{F}_{|\alpha_1|+|\alpha_2|}$ since if we write $S_{\alpha_1} S_\gamma = S_x$ for some $x \in W$ with $x \equiv \alpha_1 \gamma$ (see 2.1), then (3) and (4) show that different γ give different x.

Analogously, $S_{\alpha_2} S_{\alpha_2}^*$ can be written as a sum of $n^{|\alpha_1|}$ distinct matrix units. Hence $n^{|\alpha_1|} = n^{|\alpha_2|}$ and thus $|\alpha_1| = |\alpha_2|$ since the n_j are relative prime. Thus we have $S_{\alpha_1}^* S_{\alpha_2} = \delta_{\alpha_1,\alpha_2} I$ and we conclude $\alpha_1 = \alpha_2$ since we assumed $S_{\alpha_1} S_{\alpha_1}^* = S_{\alpha_2} S_{\alpha_2}^*$. Considering X^* instead of X we also obtain $\beta_1 = \beta_2$. □

Definition. For a nonzero word $X = S_\alpha S_\beta^*$ $(\alpha, \beta \in W)$ we call the integer vector $\mathrm{bal}(X) = |\alpha| - |\beta| \in \mathbb{Z}^d$ the *balance* of X.

Note that due to the uniqueness in 2.3 the balance is a well defined function on all nonzero products X in the letters of $S \cup S^*$. Now for an arbitrary nonzero product $X = S_{a_1}^{\epsilon_1} S_{a_2}^{\epsilon_1} \dots S_{a_N}^{\epsilon_N}$ ($a_i \in \mathcal{A}$, $\epsilon_i \in \{1, *\}$) define the counting function

$$\phi(X) = \mathrm{bal}(S_{a_1}^{\epsilon_1}) + \mathrm{bal}(S_{a_2}^{\epsilon_2}) + \cdots + \mathrm{bal}(S_{a_N}^{\epsilon_N}).$$

Using only the permutation rules (2) and (6), and $S_{i,x}^* S_{i,y} = \delta_{x,y} I$, we can achieve that $X = S_\alpha S_\beta^*$ for some $\alpha, \beta \in W$. Since the application of this rules does not modify the function ϕ, we see that $\phi(X) = \phi(S_\alpha S_\beta^*) = \mathrm{bal}(X)$. Hence, for nonzero words X, Y with $XY \neq 0$ we have

$$\mathrm{bal}(XY) = \mathrm{bal}(X) + \mathrm{bal}(Y). \tag{7}$$

Lemma 2.4. *Let $M_1, \dots, M_N$ be words in the letters of $S \cup S^*$ with nonzero balance. Then there exists $\gamma \in W$ such that $S_\gamma^* M_i S_\gamma = 0$ for all $1 \leq i \leq N$.*

Proof. By induction hypothesis assume that we have already found a word $\gamma \in V$ such that $S_\gamma^* M_i S_\gamma = 0$ for $i = 1, \dots, m$. Since the product $S_\gamma^* M_{m+1} S_\gamma = S_\alpha S_\beta^*$ (for certain $\alpha, \beta \in W$) is either zero or has nonzero balance (see (7)), it cannot be the identity I. Thus it is rather easy to choose some $a \in \mathcal{A}$ such that $S_a^* S_\alpha S_\beta^* S_a = 0$. Hence $S_a^* S_\gamma^* M_i S_\gamma S_a = 0$ for all $i = 1, \dots, m+1$. □

Proposition 2.5. *Let $X \in \mathrm{Alg}^*(S)$. Then X can be written as finite sum*

$$X = \sum_{k \in \mathbb{Z}^d} X_k, \tag{8}$$

where X_k is a linear combination of words with balance k.

We have $\|X_k\| \leq \|X\|$ for all $k \in \mathbb{Z}^d$, and thus in particular $X = 0$ if and only if $X_k = 0$ for all $k \in \mathbb{Z}^d$.

Proof. It follows from 2.3 that X can be represented in this form. It is enough to show that $\|X_0\| \leq \|X\|$. Indeed, if this is proved in general, then given $g_+ - g_- \in \mathbb{Z}_+^d - \mathbb{Z}_+^d$ we simply consider $Y = S^{g_-} X S^{*g_+}$ instead of X and by $\mathrm{bal}(S^{g_-} X_h S^{*g_+}) = h - g$ we get

$$\|X_g\| = \|S^{*g_-} Y_0 S^{g_+}\| \leq \|Y_0\| \leq \|Y\| = \|S^{*g_-} X S^{g_+}\| \leq \|X\|.$$

Now to prove the case $g = 0$ we choose $k \in \mathbb{Z}_+^d$ large enough such that $X_0 \in \mathcal{F}_k$. By 2.4 we find a projection $P = S_\gamma S_\gamma^*$ such that $P S_\alpha^* X_k S_\beta P = 0$ for all $k \neq 0$ and all $\alpha, \beta \in W_k$. Defining $Q = \sum_{\alpha \in W_k} S_\alpha P S_\alpha^*$ we get $QXQ = QX_0Q$. Note that Q is a projection and commutes with the elements of $\mathcal{F}_k$, whence the map

$$\sigma : \mathcal{F}_k \to \mathrm{Alg}^*(S) : \sigma(x) = QxQ$$

is a *-homomorphism. The image $\sigma(\mathcal{F}_k)$ has the same dimension as $\mathcal{F}_k$ since $QS_\alpha S_\beta^* Q = S_\alpha P S_\beta^*$ forms a selfadjoint system of matrix units when α, β runs through W_k. Thus $\sigma : \mathcal{F}_k \to \sigma(\mathcal{F}_k)$ is a *-isomorphism and we get $\|X_0\| = \|QX_0Q\| = \|QXQ\| \leq \|X\|$. □

Corollary 2.6. *For a second set of isometries $\hat{S}$ acting on $\hat{\mathcal{H}}$, the canonical map $\sigma : \mathrm{Alg}^*(S) \to \mathrm{Alg}^*(\hat{S}) : \sigma(S_{i,j}) = \hat{S}_{i,j}$ is a well-defined *-isomorphism.*

Proof. Firstly observe that the restriction $\sigma|\mathcal{F}_k$ is well defined (since the matrix units are necessarily linearly independent). Thus, if $0 = X = \sum_{k \in \mathbb{Z}^d} X_k \in \mathrm{Alg}^*(S)$ (using the representation in 2.5) then $X_k = 0$ for all $k = k_+ - k_- \in \mathbb{Z}_+^d - \mathbb{Z}_+^d$ and

$$\sigma(X_k) = \sigma(S^{*k_-})\sigma(S^{k_-} X_k S^{*k_+})\sigma(S^{k_+}) = 0.$$

Hence $\sigma(X) = 0$. □

Proposition 2.7. *Let C be the universal C^*-algebra (*[Bl]*) generated by $\mathrm{Alg}^*(S)$. Let $F_0 : C \to \mathcal{F}$ be the continuous extension of the linear map $\mathrm{Alg}^*(S) \to \mathcal{F} : X \mapsto X_0$ (2.5). Let $x \in C$. Then $x = 0$ if $F_0(x^*x) = 0$.*

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{T}^d$ where $\mathbb{T}$ is the torus. We define operators $\hat{S}_{i,j} := \lambda_i S_{i,j}$ for all $1 \leq i \leq d$ and $1 \leq j \leq n_i$. These form a second, parallel set $\hat{S}$ of generating isometries, and we have a canonical *-isomorphism $\sigma_\lambda : \mathrm{Alg}^*(S) \to \mathrm{Alg}^*(\hat{S})$ (2.6). Denote by C the universal C^*-algebra generated by $\mathrm{Alg}^*(S)$, and let $\tilde{\sigma}_\lambda : C \to B(\mathcal{H})$ be the continuous extension of σ_λ. Using the representation of 2.5 we have

$$\tilde{\sigma}_\lambda(X) = \sum_{k \in \mathbb{Z}^d} \lambda^k X_k$$

for $X \in \mathrm{Alg}^*(S)$ and $\lambda^k := \lambda_1^{k_1} \ldots \lambda_d^{k_d}$. It is immediate from this formula that $\mathbb{T}^d \to B(\mathcal{H}) : \lambda \mapsto \tilde{\sigma}_\lambda(X)$ is continuous, whence it is also continuous for $X \in C$. Moreover, for the Haar measure $d\lambda$ we have $F_0(x) = \int_{\mathbb{T}^d} \tilde{\sigma}_\lambda(x) d\lambda$ for all $x \in C$. Thus, if $F_0(x^*x) = 0$ then $0 = \tilde{\sigma}_0(x^*x) = x^*x$, since $\tilde{\sigma}_\lambda(x^*x) \geq 0$ for all λ. □

Theorem 2.8. *If we have given two sets of isometries S and $\hat{S}$ satisfying the rules we introduced in the beginning of this chapter, then $C^*(S) \cong C^*(\hat{S})$ via the canonical map in 2.6.*

Proof. Let C be the universal C^*-algebra generated by $A = \mathrm{Alg}^*(S)$. There exists a canonical *-homomorphism $\pi : C \to C^*(S)$ by 2.6. Note that $\pi|\mathcal{F}$ is a *-isomorphism since $\mathcal{F} \cap A$ (2.2) carries a unique C^*-norm. Let $F_0 : C \to \mathcal{F}$ be as in 2.7, and $F_0' : C^*(S) \to \mathcal{F}$ be the analogous continuous map. Then $\pi F_0 = F_0' \pi$ (since this is obviously true on A). Now if $\pi(x^* x) = 0$ then $F_0'(\pi(x^* x)) = \pi(F_0(x^* x)) = 0$ and hence $F_0(x^* x) = 0$, whence 2.7 yields $x = 0$. □

Definition. We denote the C^*-algebra $C^*(S)$ introduced in this section by $\mathcal{O}_n$. For infinite sets $n \subseteq \mathbb{N} \backslash \{1\}$ consisting of mutually relative prime numbers we let $\mathcal{O}_n = \overline{\bigcup_{m \subset n} \mathcal{O}_m}$ where m runs through the finite subsets of n.

Notice that the C^*-algebra $\mathcal{O}_n$ is uniquely determined up to $*$-isomorphisms by Theorem 2.8.

3. Some properties of $\mathcal{O}_n$

In this section we proof the following facts for higher dimensional Cuntz-algebras $\mathcal{O}_n$. They are simple unital separable nuclear und purely infinite C^*-algebras (so-called Kirchberg algebras); the stabilized algebra $\mathbb{K} \otimes \mathcal{O}_n$ can be represented as a crossed product of an AF-algebra A by an $\mathbb{Z}^d$-action α. Last but not least we will calculate the K-theory of $\mathcal{O}_n$ for certain n in Theorem 3.4.

In the definition of $\mathcal{O}_n$ we have also permitted that $\mathcal{O}_n$ is generated by infinitely many sets of isometries $\{S_{i,0}, \ldots, S_{i,n_i - 1}\}$, $i \in \mathbb{N}$ and $n = \{n_1, n_2, n_3, \ldots\}$. For this reason, when the dimension d is infinite, we extend our notation to $\mathbb{Z}^\infty = \{k \in \Pi_{i=1}^\infty \mathbb{Z} \mid k \text{ has finite carrier}\}$ and adopt also the definition $\mathcal{F} = \overline{\cup_{k \in \mathbb{Z}_+^d} \mathcal{F}_k}$ for $d = \infty$. The Algebra $\mathrm{Alg}^*(S)$ denotes the $*$-algebra which is algebraically generated by the finitely or infinitely many, depending on the dimension d, isometries S_a $(a \in \mathcal{A})$.

In the next results we shall include the infinite dimensional Cuntz algebras, apart from the cases where we say something else (only 3.4).

By $\mathbb{K}$ denote the compact operators on an infinite-dimensional separable Hilbert space.

Lemma 3.1. *For $k \leq l \in \mathbb{Z}_+^d$ let $f_{l,k} : \mathcal{F}_k \to \mathcal{F}_l : S_\alpha S_\beta^* \mapsto S^{k-l} S_\alpha S_\beta^* S^{*k-l}$. Then for the direct limit associated to $(f_{l,k})$ we have $\varinjlim_{k \in \mathbb{Z}_+^d} \mathcal{F}_k \cong \mathbb{K}$.*

Proof. Consider a Hilbert space h with normal base $(e_i)_{i \in \mathbb{Z}_+}$. Let $k \leq l \in \mathbb{Z}_+^d$ and let $\alpha, \beta \in W$ with $|\alpha| = |\beta| = k$. For $S_\alpha S_\beta^* \in \mathcal{F}_k$ let $\pi_k(S_\alpha S_\beta^*)$ be the one-dimensional operator in $B(h)$ that maps $e_{Z(\beta)} \mapsto e_{Z(\alpha)}$. Then

$\pi_l f_{l,k}\pi_k^{-1} : \pi_k(\mathcal{F}_k) \to \pi_l(\mathcal{F}_l)$ is just the identic embedding. Since the closure of $\bigcup_k \pi_k(\mathcal{F}_k)$ is $\mathbb{K}$, the claim follows. $\square$

Theorem 3.2. (a) *There exists an* AF*-algebra* $A \cong \mathbb{K} \otimes \mathcal{F}$*, a projection* $P \in A$*, and an action* $\beta : \mathbb{Z}^d \to \mathrm{Aut}(A)$ *such that* $\mathcal{O}_n \cong P(A \rtimes_\beta \mathbb{Z}^d)P$ *and* $\mathbb{K} \otimes \mathcal{O}_n \cong A \rtimes_\beta \mathbb{Z}^d$*.*

(b) $\mathcal{O}_n$ *is nuclear.*

Proof. (a) We follow here [C1, chapter 2], adapted to the multidimensional case. For fixed $k \in \mathbb{Z}^d_+$ consider the injective *-homomorphism

$$\alpha_k : \mathcal{F} \to \mathcal{F} : x \mapsto S^k x S^{*k}.$$

For each $k \in \mathbb{Z}^d$ let A_k be a copy of $\mathcal{F}$. The mappings $\alpha_{l,k} := \alpha_{l-k} : A_k \to A_l$ ($k \le l \in \mathbb{Z}^d$) generate a direct limit

$$A := \varinjlim_{k \in \mathbb{Z}^d} A_k, \tag{9}$$

and associated with it is a *-homomorphisms $\phi_k : A_k \to A$ satisfying $\phi_k \alpha_{k-l} = \phi_l$ ($l \le k \in \mathbb{Z}^d$).

Let $g \in \mathbb{Z}^d$. Then we define the "shifting" β_g on A, $\beta_g : A \to A$, such that $\beta_g \phi_k(x) = \phi_{k-g}(x)$ for $x \in \mathcal{F}$.

Obviously we have constructed an action $\beta : \mathbb{Z}^d \to \mathrm{Aut}(A)$, and we can consider the crossed product $B := A \rtimes_\beta \mathbb{Z}^d$. We find a faithful representation of B on a Hilbert space h such that $\beta_g(x) = U_g x U_g^*$ for some group representation $U : \mathbb{Z}^d \to B(h)$ into the unitary group of $B(h)$. Now B is the closure of finite sums of the form ($X_k \in A$)

$$X = \sum_{k \in \mathbb{Z}^d} X_k U_k = \sum_{k \in \mathbb{Z}^d} U_{-k_-} U_{k_-} X_k U_{-k_-} U_{k_+} = \sum_{k \in \mathbb{Z}^d} U_{k_-}^* \tilde{X}_k U_{k_+}, \tag{10}$$

where $\tilde{X}_k = \beta_{k_-}(X_k)$ and $k = k_+ - k_- \in \mathbb{Z}^d_+ - \mathbb{Z}^d_+$.

Let $P \in A$ be the unit of $A_0 \subseteq A \subseteq B$ (in the sequel we will often regard A_0 as a subset of A via the injection $\phi_0 : A_0 \to A$; so more precisely $P = \phi_0(I)$ where I is the unit of A_0). Note that $U_g P U_g^* = \beta_g(\phi_0(I)) = \phi_{-g}(I) \in A_0 \subseteq A$ for $g \in \mathbb{Z}^d_+$. Hence we have $P U_g P U_g^* = U_g P U_g^*$ and thus $P U_g P = U_g P$ for all $g \ge 0$. With this rule we compute with (10)

$$PXP = \sum_{k \in \mathbb{Z}^d} (U_{k_-} P)^* P \tilde{X}_k P U_{k_+} P. \tag{11}$$

The middle term $P\tilde{X}_k P$ is here in A_0. Indeed, consider some $g \in \mathbb{Z}^d$, $x \in A_g$ and let $m \in \mathbb{Z}^d_+$ be such that $g \le m$. Then we have

$$P\phi_g(x)P = \phi_0(I)\phi_g(x)\phi_0(I) = \phi_m \alpha_m(I) \phi_m \alpha_{m-g}(x) \phi_m \alpha_m(I)$$

$$= \phi_m(S^m S^{*m} \alpha_{m-g}(x) S^m S^{*m}) = \phi_m \alpha_m(S^{*m} \alpha_{m-g}(x) S^m) \in A_0 \subseteq A.$$

Since all expressions like (11) are dense in in the C^*-algebra $C = PBP$, C is generated by A_0 and by the elements $U_g P$ where g runs through $\mathbb{Z}_+^d$.

Let $1 \leq i \leq d, 0 \leq k \leq n_i - 1$. We put

$$\hat{S}_{i,k} := \phi_0(S_{i,k} S_{i,0}^*) U_{\delta_i} P$$

where $\delta_i \in \mathbb{Z}^d$ is the characteristic function of $\{i\}$. In the sequel we will omit ϕ_0 in the notations once again.

Note that $U_{\delta_i} x U_{\delta_i}^* = S_{i,0} x S_{i,0}^*$ for all $x \in A_0$; in particular $U_{\delta_i} P U_{\delta_i}^* = S_{i,0} S_{i,0}^*$. Then it is straightforward to calculate that $\hat{S}_{i,k}^* \hat{S}_{i,k} = P$, $\sum_{k=0}^{n_i-1} \hat{S}_{i,k} \hat{S}_{i,k}^* = P$, and by induction, that $\hat{S}_{\gamma_1} \hat{S}_{\gamma_2}^* = S_{\gamma_1} S_{\gamma_2}^*$ for all $\gamma_1, \gamma_2 \in W$ with $|\gamma_1| = |\gamma_2|$. The last fact shows that $A_0 \cong \mathcal{F}$ is contained in the C^*-algebra generated by the letters of $\hat{S}$.

To sum up, the isometries $\hat{S} \in C$ also satisfy the properties assumed in the previous chapter, where P is the unit. To proof rule (2) we consider $\hat{S}_{i,a} \hat{S}_{j,b} \hat{S}_{1,0}^* \hat{S}_{1,0}^* = S_{i,a} S_{j,b} S_{1,0}^* S_{1,0}^*$. By using $\hat{S}_{i,0} = U_{\delta_i} P$ (and thus $U_{\delta_i + \delta_j} P = \hat{S}_{i,0} \hat{S}_{j,0}$ etc.) we have

$$C^*(\hat{S}) \subseteq C^*(A_0, U_g P) \subseteq C^*(\hat{S}).$$

With 2.8 one gets $\mathcal{O}_n \cong C^*(A_0, U_g P) = P(A \rtimes_\beta \mathbb{Z}^d) P = PBP$.

The units $P_k := \phi_k(I) \in A_k \subseteq A \subseteq B$ (I is the unit of $\mathcal{F}$) of A_k form an increasing net and approximating unit for B. Hence we can write $B = \overline{\bigcup_{k \in \mathbb{Z}_+^d} P_k B P_k}$. The relation between P_k and P is, using (5),

$$P_k = \phi_k\Big(\sum_{\alpha \in W, |\alpha| = k} S_\alpha S_\alpha^* \Big) = \sum_{\alpha \in W, |\alpha| = k} e_{k,\alpha}^* P e_{k,\alpha}, \tag{12}$$

where $e_{k,\alpha} := \phi_k(S_\alpha S^{*k}) \in A_k$. We have a *-isomorphisms

$$T_k : \mathcal{F}_k \otimes PBP \to P_k B P_k : T_k(S_\alpha S_\beta^* \otimes z) = e_{k,\alpha} z e_{k,\beta}^*.$$

The surjectivity of T_k follows obviously from the above formula (12). Thus we get

$$B \cong \varinjlim_{k \in \mathbb{Z}_+^d} P_k B P_k \cong \varinjlim_{k \in \mathbb{Z}_+^d} \mathcal{F}_k \otimes PBP \cong \varinjlim_{k \in \mathbb{Z}_+^d} \mathcal{F}_k \otimes \mathcal{O}_n.$$

To analyze this isomorphism we compute ($g \in \mathbb{Z}_+^d$)

$$\begin{aligned} T_k(S_\alpha S_\beta^* \otimes z) &= e_{k,\alpha} z e_{k,\beta} \\ &= \phi_{k+g}(S^g S_\alpha S^{*k+g}) z \phi_{k+g}(S^g S_\beta S^{*k+g}) \\ &= T_{k+g}(S^g S_\alpha S_\beta^* S^{*g} \otimes z). \end{aligned}$$

Therefore, by Lemma 3.1 we obtain $A \rtimes_\beta \mathbb{Z}^d = B \cong \mathbb{K} \otimes \mathcal{O}_n$. Since $T_k(\mathcal{F}_k \otimes PA_0P) = A_k$ we obtain $A = \overline{\cup_k A_k} \cong \mathbb{K} \otimes \mathcal{F}$ by restriction of the isomorphism $B \cong \mathbb{K} \otimes \mathcal{O}_n$.

(b) $B = A \rtimes_\beta \mathbb{Z}^d$ is nuclear since A is nuclear (as any AF-algebra) and $\mathbb{Z}^d$ is amenable (as any abelian group), see for instance [RøS, 2.1.2] (the crossed product of a nuclear C^*-algebra with an amenable group is nuclear). Hence $\mathcal{O}_n \cong PBP$ is nuclear. This follows for example by using the completely positive approximation property which characterizes nuclearity [CE, 3.1.(2')]. □

Theorem 3.3. *For any $x \in \mathcal{O}_n \backslash \{0\}$ there exists $a, b \in \mathcal{O}_n$ such that $axb = I$. Hence $\mathcal{O}_n$ is an unital Kirchberg algebra (i.e. purely infinite, simple, nuclear and separable).*

Proof. The proof of [C1, 1.13] can almost be copied. Hence we give only a very brief description.

Let $x \in \mathcal{O}_n$ with $x \neq 0$, so $F_0(x^*x) > 0$ (2.7). Choose a finite subset $m \subseteq n$ and $Y \in \mathrm{Alg}^*(S) \cap \mathcal{O}_m$ such that $\|x^*x - Y\| \leq \varepsilon$.

Using the properties of the projection Q appearing in the proof of 2.5 (applied to the algebra $\mathcal{O}_m$) one finds $A \in \mathcal{O}_m$ such that $AYA^* = I$.

Since $x^*x \approx Y$ and $\|A\| \approx 1$, by an appropriate estimate one gets $\|Ax^*xA - I\| \leq 2\varepsilon$ which shows that Ax^*xA is invertible, and the theorem is proved. □

Note that due to the last theorem each nonzero ideal of $\mathcal{O}_n$ contains I and hence is $\mathcal{O}_n$ itself. Thus $\mathcal{O}_n$ is simple. For the definition of a purely infinite algebra we refer, for example, to the book [RøS, Def. 4.1.2].

Finally, we will compute some K-groups. It seems to us that the following proof, for $\mathcal{O}_{\{n_1,n_2\}}$ say, could also be copied without difference for the algebra $\mathcal{O}_{n_1} \otimes \mathcal{O}_{n_2}$. Hence would expect the same result for this tensor product. Indeed, it is evident that $K_*(\mathcal{O}_{n_1} \otimes \mathcal{O}_{n_2}) = 0 = K_*(\mathcal{O}_{\{n_1,n_2\}})$ ([C3, 2.3]) in case that $\gcd(n_1 - 1, n_2 - 1) = 1$. Of course such an observation has a strong consequence: recall that by the Kirchberg and Phillips classification theory for Kirchberg algebras ([K] and [P]) we have $\mathcal{O}_n = \mathcal{O}_m$ if and only if $K_*(\mathcal{O}_n) = K_*(\mathcal{O}_m)$.

Theorem 3.4. *Let $g = \gcd(n_1 - 1, n_2 - 1)$. If the dimension d is finite and $\gcd(n_1 n_2 \ldots n_d, n_1 - 1) = 1$, then the K_0- and K_1-groups of $\mathcal{O}_n$ are finite.*

If $g = 1$ then $K_0(\mathcal{O}_n) = K_1(\mathcal{O}_n) = 0$.

If $d = 2$ then we have $K_0(\mathcal{O}_{\{n_1,n_2\}}) = \mathbb{Z}_g$, and moreover $K_1(\mathcal{O}_{\{n_1,n_2\}}) = \mathbb{Z}_g$ if additionally $\gcd(n_2, n_1 - 1) = 1$.

Proof. We proof this by successively using the six term exact sequence of Pimsner and Voiculescu [PV], also compare [Co, page 49] for a non-unital version. The first step of the proof we are going to show we have learned in [C2, Prop. 3.1].

Due to Theorem 3.2 we have $K_*(\mathcal{O}_n) = K_*(A \rtimes_\beta \mathbb{Z}^d)$ and $K_*(A) = K_*(\mathcal{F})$. Recall from 2.2 that $\mathcal{F}$ is the direct limit of $\mathcal{F}_k \cong M_{n^k}$ ($k \in \mathbb{Z}_+^d$), which carries over to the K-groups. Choose a minimal projection $S_\alpha S_\alpha^*$ in $\mathcal{F}_k$ (hence $[S_\alpha S_\alpha^*]$ generates $K_0(\mathcal{F}_k) = \mathbb{Z}$). Then ($i = 1, \ldots, d$)

$$[S_\alpha S_\alpha^*] = \sum_{j=0}^{n_i - 1} [S_\alpha S_{i,j} S_{i,j}^* S_\alpha^*] = \sum_{j=0}^{n_i - 1} [Q_j],$$

where the Q_j are (mutually orthogonal, 2.1) minimal projections in $\mathcal{F}_{k+\delta_i}$. Thus for the inclusion map $\psi_{k+\delta_i,k} : \mathcal{F}_k \to \mathcal{F}_{k+\delta_i}$, the associated K_0-group map is given by the injections $(\psi_{k+\delta_i,k})_* : \mathbb{Z} \to \mathbb{Z} : x \mapsto n_i x$. Hence we have

$$K_0(A) = K_0(\mathcal{F}) = \bigcup_{k \in \mathbb{Z}_+^d} \tau_k K_0(\mathcal{F}_k)$$

for canonical injective homomorphisms $\tau_k : K_0(\mathcal{F}_k) \to K_0(A)$.

Recall how we have defined the action β acting on A (see the proof of 3.2). A was the direct limit of C^*-algebras A_k which are isomorphic to $\mathcal{F}$. For the injective embeddings $\phi_k : A_k \to A$ induced by the direct limit construction we had $\beta_g(\phi_k(z)) = \phi_{k-g}(z)$, so β simply shifts $A_k \to A_{k-g}$. Thus we have

$$\beta_{-\delta_i}(\phi_0(S_{i,0} z S_{i,0}^*)) = \phi_{\delta_i}(S_{i,0} z S_{i,0}^*) = \phi_0(z).$$

In the sequel we will identify A_0 with $\phi(A_0)$ and so we will omit ϕ_0.

A closure look at the proof of 3.2, where we have shown the existence of an isomorphism $\pi : A \to \mathbb{K} \otimes \mathcal{F}$, yields that there must exist a projection $e_{11} (\cong S^k S^{*k} \in \mathcal{F}_k) \in \mathbb{K}$ such that

$$\pi | A_0 : A_0 \to \mathbb{K} \otimes \mathcal{F} : \pi(w) = e_{11} \otimes w.$$

Hence for $w \in \mathcal{F}$ we have

$$\beta_{-\delta_i}(e_{11} \otimes S_{i,0} w S_{i,0}^*) = e_{11} \otimes w.$$

Notice that $(\pi|A_0)_*$ yields an isomorphism $(\pi|A_0)_* : K_0(\mathcal{F}) \to K_0(\mathbb{K} \otimes \mathcal{F})$. Using this isomorphism the map $(\beta_{-\delta_i})_*$ is completely described by the identity

$$(\beta_{-\delta_i})_*([S_{i,0} w S_{i,0}^*]) = [w] \qquad \forall w \in \mathcal{F}.$$

Thus $(\beta_{-\delta_i})_*$ simply shifts

$$1_{\mathbb{Z}} = \tau_{k+\delta_i}[S_{i,0} S_\alpha S_\alpha^* S_{i,0}^*] \in \tau_{k+\delta_i} K_0(\mathcal{F}_{k+\delta_i})$$

to

$$1_{\mathbb{Z}} = \tau_k [S_\alpha S_\alpha^*] \in \tau_k K_0(\mathcal{F}_k).$$

Hence we have ($x \in K_0(\mathcal{F}_k)$)

$$(\beta_{-\delta_i})_*(\tau_k x) = \tau_{k-\delta_i} x = n_i(\tau_k x).$$

We have a canonical isomorphism

$$A \rtimes_\beta \mathbb{Z}^d \cong A \rtimes_{\beta_1} \mathbb{Z} \rtimes_{\beta_2} \mathbb{Z} \cdots \rtimes_{\beta_d} \mathbb{Z},$$

where $\beta_1 \cong \beta_{\delta_1}$ acts on A, $\beta_2 \cong \beta_{\delta_2}$ acts on $A \rtimes_{\beta_1} \mathbb{Z}$, and so on.

Thus by the exact sequence [PV, 2.4] we obtain

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\mathrm{id} - (\beta_1^{-1})_*} & K_0(A) & \xrightarrow{\iota_*} & K_0(A \rtimes_{\beta_1} \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_{\beta_1} \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow{\mathrm{id} - (\beta_1^{-1})_*} & K_1(A) \end{array}$$

Since $K_1(A) = 0$, we have

$$\begin{aligned} K_0(A \rtimes_{\beta_1} \mathbb{Z}) &= K_0(A)/\mathrm{Im}(\mathrm{id} - (\beta_1^{-1})_*), \\ K_1(A \rtimes_{\beta_1} \mathbb{Z}) &= \ker(\mathrm{id} - (\beta_1^{-1})_*). \end{aligned}$$

By $\mathrm{id} - (\beta_1^{-1})_*(x) = (1 - n_1)x$ it follows that

$$G := K_0(A \rtimes_{\beta_1} \mathbb{Z}) = K_0(A)/(n_1 - 1)K_0(A)$$

and $K_1(A \rtimes_{\beta_1} \mathbb{Z}) = 0$.

Applying the Pimsner-Voiculescu exact sequence once again we get

$$\begin{array}{ccccc} K_0(A \rtimes_{\beta_1} \mathbb{Z}) & \xrightarrow{\mathrm{id} - (\beta_2^{-1})_*} & K_0(A \rtimes_{\beta_1} \mathbb{Z}) & \longrightarrow & K_0(A \rtimes_{\beta_1} \mathbb{Z} \rtimes_{\beta_2} \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_{\beta_1} \mathbb{Z} \rtimes_{\beta_2} \mathbb{Z}) & & \longleftarrow 0 \longleftarrow & & 0 \end{array}$$

Let $x \in K_0(A)$. Then we have

$$(\beta_2^{-1})_* \iota_*(x) = (\beta_2^{-1}\iota)_*(x) = (\iota\beta_{-\delta_2})_*(x) = \iota_*(\beta_{-\delta_2})_*(x) = n_2\iota_*(x).$$

Hence $f := \mathrm{id} - (\beta_2^{-1})_* \in \hom(\mathrm{G})$ maps x to $(1-n_2)x$ by the surjectivity of ι_* in the first PV-diagram. Hence

$$K_0(A \rtimes_{\beta_1} \mathbb{Z} \rtimes_{\beta_2} \mathbb{Z}) = G/(n_2-1)G$$

$$= K_0(A)/((n_1-1)K_0(A) + (n_2-1)K_0(A)) = K_0(A)/gK_0(A) =: H$$

where $g = \gcd(n_1-1, n_2-1)$.

In the sequel we will regard $K_0(\mathcal{F}_k) \cong \mathbb{Z}$ as subsets of $K_0(A)$ (so we omit the τ_k map).

Let $x \in K_0(\mathcal{F}_k)$. Then we find some $y \in K_0(\mathcal{F}_0)$ and $a \in K_0(\mathcal{F}_k)$ such that

$$(\psi_{k,0})_*(y) + ga = n^k y + ga = x,$$

since we assume $\gcd(n_1 n_2 \ldots n_d, g) = 1$ (what is automatically satisfied if $d = 2$ or if $g = 1$).

This shows that the canonical map $\sigma : K_0(\mathcal{F}_0) \to H$ is surjective, and we obtain

$$H = \mathbb{Z}/\{\, y \in \mathbb{Z} \mid \exists k \in \mathbb{Z}_+^d, a \in \mathbb{Z} : n^k y + ga = 0 \,\},$$

and thus $H = \mathbb{Z}_g$.

If even $\gcd(n_1 \ldots n_d, n_1 - 1) = 1$, then by the same argument we have just used, we get $G = \mathbb{Z}_{n_1-1}$. Hence in this case it follows from the second PV-diagram that $K_1(A \rtimes_{\beta_1} \mathbb{Z} \rtimes_{\beta_2} \mathbb{Z}) = \ker(f) = \mathbb{Z}_g$ (since $f(x) = (1-n_2)x$; $f \in \hom(\mathbb{Z}_{n_1-1})$).

Moreover, in this case all groups appearing in the last PV-diagram are finite, and successively applying the PV-sequence this property carries over until $K_*(\mathcal{O}_n)$.

Last but not least assume $g = 1$. Let $x \in K_0(\mathcal{F}_l)$ be a representant of $\tilde{x} \in G$. Then $f(\tilde{x}) = (1-n_2)\tilde{x} = 0$ iff there exists some $k \in \mathbb{Z}_+^d$ and $a \in K_0(\mathcal{F}_{l+k})$ such that

$$n^k(1-n_2)x + (n_1-1)a = 0.$$

Since $g = 1$, $n^k x$ must be a multiple of $(n_1 - 1)$ and thus $\tilde{x} = 0$.

So in this case $K_1(A \rtimes_{\beta_1} \mathbb{Z} \rtimes_{\beta_2} \mathbb{Z}) = \ker(f) = 0$. Since then both K-groups are zero, this property carries over to $K_*(\mathcal{O}_n) = 0$ by successively applying the PV-sequence. □

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