THE UNIQUENESS OF CUNTZ-KRIEGER TYPE ALGEBRAS

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Abstract. We introduce a class of C^*-algebras which can be viewed as a generalization of the classical Cuntz-Krieger algebras. The feature is that it is a flexible "generators and relations"-concept. The main result is a canonical uniqueness theorem stating that the C^*-algebras of this class are uniquely determined by their generators and relations. We can show that rank one Cuntz-Krieger algebras with infinitely large transition matrices fall in this class, and this provides an alternative proof of Exel and Laca. Further we elaborate a subclass of rank two Cuntz-Krieger algebras inspired by shifts of finite type in dimension two, with an infinite set of generators and relations.

1. Introduction

Based on the work of Cuntz [C], Cuntz and Krieger introduced their class of now so-called Cuntz-Krieger algebras $\mathcal{O}_A$ in [CK]. $\mathcal{O}_A$ is defined as the C^*-algebra generated by $n \geq 2$ generators satisfying certain relations associated with some matrix $A \in \{0,1\}^{n \times n}$. The definition is well-defined, since Cuntz and Krieger showed that $\mathcal{O}_A$ is canonically uniquely determined.

In the present paper we generalize this uniqueness theorem to a large extent. The general idea is that one is given an arbitrary set of generators and relations and seeks to prove that these generators and relations determine exactly one C^*-algebra up to isomorphisms. To deal with this task, one may apply our main result, Theorem 3.3.

More precisely we use the following picture. One is given an abstract generator set $\mathcal{A}$ and considers its free non-unital $*$-algebra $\mathbb{F}$. With some set of relations R in the $*$-algebra $\mathbb{F}$ we can associate the smallest two-sided self-adjoint ideal $\mathbb{I} \subseteq \mathbb{F}$ such that the relations R become valid in the quotient $\mathbb{F}/\mathbb{I}$. Instead of considering the C^*-algebra generated by $\mathbb{F}/\mathbb{I}$ we suppose that we are provided with a $*$-homomorphism $\pi : \mathbb{F}/\mathbb{I} \to A$ into some C^*-algebra A.

Then Theorem 3.3 states that the norm-closure $\overline{\pi(\mathbb{F}/\mathbb{I})}$ of the image of π is canonically unique if π is faithful on a certain small subalgebra $\mathbb{A}$ sitting in $\mathbb{F}/\mathbb{I}$ and if the conditions (A), (B) and (C), which are conditions on $\mathbb{F}/\mathbb{I}$, are satisfied.

The algebraic nature of our approach and of the conditions (A), (B) and (C) distinguishes us from other approaches to generalized Cuntz-Krieger algebras like those using graphs

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[KP1, KP2, KP3] or groupoids [R, D], which tend to emphasize topology. Further Cuntz-Krieger generalizations appear in [Ru, Put, Pim, RS1, RS2].

We elaborate two subclasses of the general class.

In section 4 we prove the uniqueness of rank one Cuntz-Krieger algebras associated with infinitely large transition matrices (see Proposition 4.3). Thereby possible infinite sums of range projections which build the source projections are understood in the convergence of strong operator topology. Using the uniqueness theorem the proof is short and smooth compared to Exel and Laca's proof [EL].

In section 5 we outline a class of rank two Cuntz-Krieger algebras and show their uniqueness in Proposition 5.7. Each of these C^*-algebras is generated by two interacting rank one Cuntz-Krieger algebras with infinitely large transition matrices. The class is motivated by one-sided shifts of finite type in dimension two, and the relationship between the shifts and the rank two algebras is similar to the relationship of shifts of finite type in dimension one and the classical Cuntz-Krieger algebras. Our major task will be to explicitly compute the structure of the fixed point algebra of a typical gauge action. Afterwards it is not too difficult to prove the conditions claimed by Theorem 3.3. We think the verification of the uniqueness of a higher rank Cuntz-Krieger algebra with essentially infinitely many generators is a novelty.

Beside the uniqueness property, criteria for simplicity, nuclearity and crossed product representations of our algebras appear in Corollary 3.5 and Theorem 3.6, respectively.

2. The $*$-Algebras

Let $\mathcal{A}$ be an alphabet (of any cardinality) and denote by $\mathbb{F}$ the nonunital $*$-algebra which is freely generated by the alphabet $\mathcal{A}$. More precisely let $\mathbb{F}$ be the vector space over $\mathbb{C}$ with the set of formal words $\{ x_1 x_2 ... x_n \mid n \geq 1, x_i \in \mathcal{A} \cup \mathcal{A}^* \}$ as basis. Multiplication is defined by formally concatenating words. Involution by formally adjoining words and conjugating scalars.

Let $\mathbb{I} \subseteq \mathbb{F}$ be a selfadjoint two-sided ideal in $\mathbb{F}$ (in the sequel simply called "ideal"). Considering then $\mathbb{F}/\mathbb{I}$ the task of $\mathbb{I}$ is to bring relations among the generators $\mathcal{A}$ of $\mathbb{F}$. So $\mathbb{F}/\mathbb{I}$ will play the role of our reference $*$-algebra from which $*$-homomorphisms map into C^*-algebras.

Example 2.1. For the classical Cuntz algebras $\mathcal{O}_n$ [C] we may define $\mathcal{A} = \{I, S_1, S_2, ..., S_n\}$ and let $\mathbb{I} \subseteq \mathbb{F}$ be the ideal which is generated by $S_1 S_1^* + ... + S_n S_n^* - I$ (Cuntz-property), $a^* a - I$ (isometry) and $I^* - I$, $Ia - a$, $aI - a$ (unit) for all letters $a \in \mathcal{A}$. (Clearly we could also do without the unit I in the alphabet $\mathcal{A}$.) We obtain then a $*$-homomorphism $\pi : \mathbb{F}/\mathbb{I} \to \mathcal{O}_n$ such that $\pi(S_i)$ $(i = 1, ..., n)$ are the isometries generating $\mathcal{O}_n$.

In the sequel we will consider the algebra $\mathbb{F}/\mathbb{I}$. Instead of $a+\mathbb{I}$ ($a \in \mathcal{A}$), say, we just write a. We denote the words in $\mathbb{F}/\mathbb{I}$, consisting of the letters $\mathcal{A}$ and its adjoint letters $\mathcal{A}^*$, by

$$W = \{\, a_1 a_2 ... a_n \in \mathbb{F}/\mathbb{I} \mid n \geq 1, a_i \in \mathcal{A} \cup \mathcal{A}^* \,\}.$$

A balance function is a kind of homomorphism which maps nonzero words into a group. We make this precise in the following property which we shall assume in this section.

(A_0) There exists a group G and a map bal : $W \backslash \{0\} \to G$ (*balance function*) such that for all $X, Y, Z \in W$ with $XY \neq 0$, $Z \neq 0$

$$\mathrm{bal}(XY) = \mathrm{bal}(X)\mathrm{bal}(Y), \quad \mathrm{bal}(Z^*) = \mathrm{bal}(Z)^{-1}.$$

We denote by $W_g = \{\, X \in W\backslash\{0\} \mid \mathrm{bal}(X) = g \,\}$ the set of words with balance $g \in G$. The words W_e we call neutral-balanced (or zero-balanced if G is abelian).

Example 2.2. Consider the Cuntz algebra $\mathcal{O}_n$. Then a useful balance function is defined by $\mathrm{bal}(S_i) = 1_{\mathbb{Z}}$ ($i = 1, ..., n$, $G = \mathbb{Z}$) and $\mathrm{bal}(I) = 0$. A nonzero word is then zero-balanced iff it contains just as many letters S_i as adjoint letters S_j^*. Then the zero-balanced words generate an AF-algebra. This was an important aspect in the paper [C] of Cuntz, and it is a fundamental property for our proof of the uniqueness theorem too. In the general case which we shall outline in this article, we claim this property axiomatically in condition (B) below.

For a $*$-algebra A, we use the term *inductively ordered union of finite dimensional C^*-algebras* to express that for all finitely many elements $x_1, ..., x_n \in A$ there exists a finite dimensional C^*-algebra $B \subseteq A$, such that $x_1, ..., x_n \in B$. Clearly such an algebra A carries a unique C^*-norm and its completion is the C^*-direct limit $\varinjlim_{B \in \Gamma} B$ where Γ is the directed set of all finite dimensional C^*-algebras lying in A.

Given a balance function we denote by $\mathbb{A} = \mathrm{Alg}^*(W_e) = \mathrm{lin}(W_e)$ the $*$-algebra in $\mathbb{F}/\mathbb{I}$ generated by the neutral-balanced words. By $\mathbb{P} \subseteq \mathbb{A}$ we denote the set of nonzero projections (per def. selfadjoint and idempotent) in $\mathbb{A}$.

The following two properties (B) and (C) will play a central role.

(B) $\mathbb{A}$ is the inductively ordered union of finite dimensional C^*-algebras.

(C) For all nonneutral-balanced words $X \in W$ and all $E, E_1, E_2 \in \mathbb{P}$ there exist $P \leq E, P_1 \leq E_1, P_2 \leq E_2$ in $\mathbb{P}$ such that $PXP = 0$ and $P_1 X P_2 = 0$.

Clearly any $X \in \mathbb{F}/\mathbb{I}$ is representable as sum $X = \sum_{g \in G} X_g$ for certain $X_g \in \mathrm{lin}(W_g)$, $g \in G$. Taking the Cuntz algebra $\mathcal{O}_n$ as the archetypical reference, in [C] a projection Q is constructed which deletes the X_g for $g \neq e$ but without losing information about X_e. A precise formulation of this is given in (C_1) below. In the next Proposition 2.3 we shall show that the assumption (C) is "axiomatically" no restriction if we assume property (C_1) to be necessary for the construction of Cuntz-Krieger type algebras.

(C_1) For all $X = \sum_{g \in G} X_g$ ($X_g \in \mathrm{lin}(W_g)$) with $X_e = y_1 + ... + y_N$ for nonzero elements $y_i \in \mathbb{A}$, there exists a projection $Q \in \mathbb{A}$ such that $QXQ = QX_eQ$ and $Qy_i = y_iQ \neq 0$ ($\forall i = 1, ..., N$).

Proposition 2.3. *Assume* (A_0) *and* (B). *Then* (C) *and* (C_1) *are equivalent.*

Proof. We start with the easy implication (C_1) $\Rightarrow$ (C). Consider projections $E_1, E_2 \in \mathbb{P}$ and a nonneutral-balanced word $X \in W$. Then, using (C_1), we find a projection $Q \in \mathbb{P}$ such that $Q(X + E_1 + E_2)Q = Q(E_1 + E_2)Q$ and $QE_i = E_iQ \neq 0$ ($i = 1, 2$). Hence, $E_iQ \leq E_i$ and $(E_1Q)X(E_2Q) = 0$, and (C) is evident.

We divide the reverse implication (C) $\Rightarrow$ (C_1) into three steps.

Step 1: First of all we will show that for all mutually orthogonal projections $E_1, ..., E_N \in \mathbb{P}$ and all nonneutral-balanced $X \in W$ there exist $P_i \leq E_i$ in $\mathbb{P}$ ($i = 1, ..., N$) such that $(P_1 + ... + P_N)X(P_1 + ... + P_N) = 0$.

Indeed, by induction hypothesis assume that for a subset $\gamma \subseteq \{1, ..., N\}^2$ we have already found projections $P_1, ..., P_N \in \mathbb{P}$ such that $P_i \leq E_i$ for all $i = 1, ..., N$ and $P_iXP_j = 0$ for all $(i, j) \in \gamma$. Let (i_0, j_0) be in the complement of γ. Due to (C) we find $Q_{i_0} \leq P_{i_0}$ and $Q_{j_0} \leq P_{j_0}$ in $\mathbb{P}$ such that $Q_{i_0}XQ_{j_0} = 0$. Put $Q_i := P_i$ for the remaining $i = \{1, ..., N\} \backslash \{i_0, j_0\}$. Then $Q_i \leq P_i \leq E_i$ for all $i = 1, .., N$ and $Q_iXQ_j = 0$ for all $(i, j) \in \gamma \cup \{(i_0, j_0)\}$.

Step 2: What is said in step 1 holds even simultaneously for given nonneutral-balanced words $X_1, ..., X_M \in W$ (rather than just one X).

Indeed, by induction hypothesis assume that for the first m words $X_1, ..., X_m$ we have already found $P_i \leq E_i$ ($i = 1, ..., N$) in $\mathbb{P}$ such that $(P_1 + ... + P_N)X_k(P_1 + ... + P_N) = 0$ ($k = 1, ..., m$). Due to step 1 we find $Q_i \leq P_i$ in $\mathbb{P}$ such that $(Q_1 + ... + Q_N)X_{m+1}(Q_1 + ... + Q_N) = 0$. We have finished this step since $Q_i \leq E_i$ for all $i = 1, .., N$ and clearly $(Q_1 + ... + Q_N)X_k(Q_1 + ... + Q_N) = 0$ for all $k = 1, ..., m + 1$.

Step 3: This part has its roots in a proof demonstrated in [C]. We represent $X \in \mathbb{F}/\mathbb{I}$ as a finite sum $X = X_e + \sum_{m=1}^{M} \alpha_m x_m$ where $X_e \in \mathbb{A}$, $x_m \in W$ are nonneutral-balanced words and $\alpha_m \in \mathbb{C}$. Due to (B), X_e is within a finite dimensional C^*-algebra $A \subseteq \mathbb{A}$ which has generating matrix units $\{ e^s_{ij} \mid 1 \leq s \leq k, 1 \leq i, j \leq n_s \}$. Hence clearly $A \subseteq \mathrm{lin}(a_1, ..., a_L)$ for certain words $a_i \in W_e$. Consider the words $Y = \{ a_i x_m a_j \mid 1 \leq i, j \leq L, 1 \leq m \leq M \}$ which, if nonzero, have nonneutral balance due to (A_0).

According to step 2 we can choose projections $P_s \leq e^s_{11}$ ($s = 1, ..., k$) in $\mathbb{P}$ such that $P = P_1 + ... + P_k$ satisfies $PyP = 0$ for all $y \in Y$. We put

$$Q = \sum_{s=1}^{k} \sum_{i=1}^{n} e^s_{i1} P e^s_{1i}.$$

Then clearly $QXQ = QX_eQ$ and $Qe^s_{ij}Q \neq 0$ since

$$e^s_{1i}Qe^s_{ij}Qe^s_{j1} = e^s_{1i}e^s_{i1}Pe^s_{1j}e^s_{j1} = P_s \neq 0.$$

Moreover, we easily verify that $Qx = xQ$ for all $x \in A$ and consequently obtain an injective $*$-homomorphism $f : A \to \mathbb{A}$ such that $f(x) = QxQ$.

It is now clear that we have come to an end since Q yields the sought projection appearing in (C_1). □

As convenient let per def. $P \precsim Q$ for $P, Q \in \mathbb{P}$ in the algebra $\mathbb{A}$ if there exists a partial isometry $S \in \mathbb{A}$ ($SS^*S = S$) with support projection P and range projection less or equal Q.

The aim of the next lemma is to reduce the set $\mathbb{P}$ to a subset $\mathbb{P}_2 \subseteq \mathbb{P}$ for which property (C) must be tested. In all cases we know of one chooses the words among $\mathbb{P}$, i.e. $\mathbb{P}_2 := W_e \cap \mathbb{P}$.

Lemma 2.4. *Supposing* (A_0) *the following criterion* (C^*) *is sufficient for* (C).

(C^*) $\mathbb{P}_2 \subseteq \mathbb{P}$ *is a subset such that each* $Q \in \mathbb{P}$ *has a lower bound* $P \precsim Q$, $P \in \mathbb{P}_2$, *in the algebra* $\mathbb{A}$.

Moreover, for all nonneutral-balanced words $X \in W$ *and all* $E, E_1, E_2 \in \mathbb{P}_2$ *there exist* $P \leq E, P_1 \leq E_1, P_2 \leq E_2$ *in* $\mathbb{P}_2$ *such that* $PXP = 0$ *and, if* E_1 *and* E_2 *have no common lower bound in* $\mathbb{P}$ *w.r.t. the relation* $\precsim$ *in the algebra* $\mathbb{A}$, *also* $P_1XP_2 = 0$.

Proof. Step 1: First of all we notice that (C^*) holds simultaneously for several given nonneutral-balanced words $X_1, ..., X_M \in W$ (rather than just for one word X).

Indeed, by induction hypothesis assume that we have already found a projection $P \leq E$ in $\mathbb{P}_2$ such that $PX_iP = 0$ for all $i = 1, ..., m$. Then choosing $Q \leq P$ in $\mathbb{P}_2$ such that $QX_{m+1}Q = 0$ we clearly get $QX_iQ = 0$ for all $i = 1, ..., m+1$. We argue similarly if E_1, E_2 have no lower bound w.r.t. $\precsim$.

Step 2: We shall show that we can replace $\mathbb{P}_2$ by $\mathbb{P}$ in (C^*).

Indeed, let $E \in \mathbb{P}$ and $E_1, E_2 \in \mathbb{P}$ have no lower bound w.r.t. the relation $\precsim$ in $\mathbb{A}$. Then choose $F \precsim E, F_1 \precsim E_1, F_2 \precsim E_2$ in $\mathbb{P}_2$ and corresponding partial isometries $S, S_1, S_2 \in \mathbb{A}$ with support projections F, F_1, F_2, respectively, and range projections $SS^* \leq E, S_1S_1^* \leq E_1$ and $S_2S_2^* \leq E_2$.

Notice that both S^*XS and $S_1^*XS_2$ are linear combinations of nonneutral-balanced words. Thus, due to step 1, we find $P \leq F, P_1 \leq F_1$ and $P_2 \leq F_2$ in $\mathbb{P}_2$ such that $PS^*XSP = 0$ and $P_1S_1^*XS_2P_2 = 0$. Considering the nonzero projections $Q = SPS^*$, $Q_i = S_iP_iS_i^*$ ($i = 1, 2$) in $\mathbb{P}$, we get $QXQ = 0$, $Q_1XQ_2 = 0$ and $Q \leq E$, $Q_1 \leq E_1$, $Q_2 \leq E_2$, as desired.

Step 3: It remains to show that for nonzero E_1, E_2 with some lower bound E w.r.t. $\precsim$ one finds $P_1 \leq E_1, P_2 \leq E_2$ in $\mathbb{P}$ such that $P_1XP_2 = 0$. Indeed, let S_1, S_2 be partial isometries in $\mathbb{A}$ with support projections E and range projections $S_iS_i^* \leq E_i$ ($i = 1, 2$). Then applying step 1 (applied to $\mathbb{P}$ rather than to $\mathbb{P}_2$), one finds $P \leq E$ in $\mathbb{P}$ such that $PS_1^*XS_2P = 0$. Set $P_i = S_iPS_i^* \leq E_i$ ($i = 1, 2$) to have $P_1XP_2 = 0$. □

Notice that applying criterion (C*) to $\mathbb{P}_2 := \mathbb{P}$ yields a slightly easier to check, but still equivalent criterion for (C).

Proposition 2.5. *Assume* (A$_0$)*,* (B)*,* (C)*. Consider two C^*-algebras A_i and $*$-homomorphisms $\pi_i : \mathbb{F}/\mathbb{I} \to A_i$ $(i = 1, 2)$, the first one being faithful on $\mathbb{A}$. Then the map $\sigma : \mathrm{Im}(\pi_1) \to \mathrm{Im}(\pi_2) : \sigma\pi_1(X) = \pi_2(X)$ is a well-defined $*$-homomorphism. It is one-to-one if π_2 is faithful on $\mathbb{A}$ too.*

Proof. Let $X = \sum_{g\in G} X_g$, where $X_g \in \mathrm{lin}(W_g)$ $(g \in G)$ and set $Y = X^*X$. Applying (C$_1$) (see Proposition 2.3) we choose a projection $Q \in \mathbb{A}$ such that $QYQ = QY_eQ$ and $QY_eQ \neq 0$ for the case that $Y_e \neq 0$.

Now assume $\pi_1(X) = 0$. Then $\pi_1(QYQ) = 0$ and thus $QY_eQ = 0$ for π_1 is faithful on $\mathbb{A}$. Necessarily we have $Y_e = 0$ and so $0 = \pi_2(Y_e) = \sum_{g\in G} \pi_2(X_g^*X_g)$. Thus, for all $g \in G$, $\pi_2(X_g^*X_g) = 0$ (since it is positive) and we conclude $\pi_2(X) = 0$.

This proves that the map σ is well-defined. The reverse implication proves that it is an isomorphism. □

Lemma 2.6. *Assume* (A$_0$)*,* (B)*,* (C)*, let A be a C^*-algebra and $\pi : \mathbb{F}/\mathbb{I} \to A$ be a $*$-homomorphism which is faithful on $\mathbb{A}$. Then we have a positive linear contraction*

$$F : \overline{\pi(\mathbb{F}/\mathbb{I})} \to \overline{\pi(\mathbb{A})} : F\pi(X) = \pi(X_e),$$

where $X = \sum_{g\in G} X_g$, $X_g \in \mathrm{lin}(W_g)$.

Proof. Due to (B) we find a finite-dimensional C^*-algebra $A \subseteq \mathbb{A}$ such that $X_e \in A$. Then, we choose a projection $Q \in \mathbb{A}$ (see (C$_1$) and Proposition 2.3) such that $QXQ = QX_eQ$ and $Qe_i = e_iQ \neq 0$ for a linear basis (e_i) of A. The last fact shows that the map $A \to \mathbb{A}$, such that $Y \mapsto QYQ$, defines an injective $*$-homomorphism. Since π is faithful on $\mathbb{A}$, this also gives a corresponding injective $*$-homomorphism in the image $\pi(\mathbb{A})$. Hence

$$\|F\pi(X)\| = \|\pi(X_e)\| = \|\pi(Q)\pi(X_e)\pi(Q)\| = \|\pi(QXQ)\| \le \|\pi(X)\|.$$

This also proves that F is well-defined. □

3. The C^*-Algebras

With each word $X = a_1^{\epsilon_1} ... a_n^{\epsilon_n} \in \mathbb{F}$ $(a_i \in \mathcal{A}, \epsilon_i \in \{1, *\})$ in the free $*$-algebra $\mathbb{F}$ we associate a character $c_X : \mathbb{T}^{\mathcal{A}} \to \mathbb{T}$ defined by $c_X(\lambda) = \lambda_{a_1}^{\epsilon_1} ... \lambda_{a_n}^{\epsilon_n}$ for $\lambda = (\lambda_a)_{a\in\mathcal{A}} \in \mathbb{T}^{\mathcal{A}}$ and where $\lambda_{a_i}^* = \overline{\lambda_{a_i}}$ per definition. Now we assume the following property.

(A) There exists a closed subgroup $H \subseteq \mathbb{T}^{\mathcal{A}}$ such that the ideal $\mathbb{I}$ is invariant under the automorphisms $t_\lambda : \mathbb{F} \to \mathbb{F} : t_\lambda(a) = c_a(\lambda)a$ $(a \in \mathcal{A})$ for all $\lambda \in H$.

We have the rules $t_\lambda t_\mu = t_{\lambda\mu}$ and $t_{\lambda^{-1}} = t_\lambda^{-1}$ and we can also derive automorphisms $t_\lambda : \mathbb{F}/\mathbb{I} \to \mathbb{F}/\mathbb{I}$ (not very rigorously we use the same notation t_λ) for $\lambda \in H$.

In practice, the ideal $\mathbb{I}$ may be defined by several equations $e_i = f_i$ (thus $\mathbb{I}$ is generated by $e_i - f_i$) and one endeavors to solve $t_\lambda(e_i - f_i) \in \mathbb{I}$ for $\lambda \in \mathbb{T}^{\mathcal{A}}$, what is necessary and sufficient for invariance property (A). In other words, one has to solve $t_\lambda(e_i) = t_\lambda(f_i)$ in the algebra $\mathbb{F}/\mathbb{I}$. For example, for the Cuntz algebra $\mathcal{O}_2$ we have

$$t_\lambda(S_1 S_1^* + S_2 S_2^*) = \lambda_1 \overline{\lambda}_1 S_1 S_1^* + \lambda_2 \overline{\lambda}_2 S_2 S_2^* = S_1 S_1^* + S_2 S_2^* = \lambda_I I.$$

Hence, $H = \{\lambda_I = 1\}$ satisfies (A), but also $H = \{\lambda_1 = \lambda_2, \lambda_I = 1\}$, say.

Lemma 3.1. *Assume property* (A)*. Then the map* $\mathrm{bal} : W\backslash\{0\} \to \widehat{H}$*, such that* $\mathrm{bal}(X + \mathbb{I}) = c_X|H$ *for all words* $X \in \mathbb{F}$*, is a balance function.*

Proof. What we have to show is that the map is well defined. More precisely, if we have given two words X, Y in the free $*$-algebra $\mathbb{F}$ such that they coincide modulo $\mathbb{I}$, then $c_X|H$ should coincide with $c_Y|H$.

Indeed, assume $X - Y \in \mathbb{I}$ where we suppose that $X \notin \mathbb{I}$, since otherwise $X + \mathbb{I} = 0$. Then $t_\lambda(X - Y) = c_X(\lambda)X - c_Y(\lambda)Y \in \mathbb{I}$ for all $\lambda \in H$ due to (A), and clearly $c_Y(\lambda)X - c_Y(\lambda)Y \in \mathbb{I}$. Hence, $(c_X(\lambda) - c_Y(\lambda))X \in \mathbb{I}$ and thus $c_X(\lambda) = c_Y(\lambda)$ for all $\lambda \in H$ since $X \notin \mathbb{I}$. □

Lemma 3.2. *Assume* (A), (B), (C) *with respect to the balance function Lemma* 3.1*. Let* A *be a* C^**-algebra and let* $\pi : \mathbb{F}/\mathbb{I} \to A$ *be faithful on* $\mathbb{A}$*. Then the positive linear map* F *of Lemma* 2.6 *associated to the representation* $\pi_\infty : \mathbb{F}/\mathbb{I} \to \bigoplus_{\lambda \in H} A : \pi_\infty(X)(\lambda) = \pi t_\lambda(X)$ *rather than* π *(i.e.* $F\pi_\infty(X) = \pi_\infty(X_e)$*) is faithful.*

Proof. Notice that for $X = \sum_{k=1}^n \alpha_k X_k$, $X_k \in W$, $\alpha_k \in \mathbb{C}$, we have actually

$$\|\pi_\infty(X)\| = \sup_{\lambda \in H} \|\sum_{k=1}^n \pi t_\lambda(\alpha_k X_k)\| \le \sum_{k=1}^n \|\pi(\alpha_k X_k)\| < \infty.$$

Let $C = \overline{\mathrm{Im}(\pi_\infty)}$ and define automorphisms $\tilde{t}_\lambda : C \to C : \tilde{t}_\lambda \pi_\infty(X) = \pi_\infty t_\lambda(X)$ $(\lambda \in H)$. Notice that π_∞ is simply the shift operator, more precisely

$$\tilde{t}_\lambda\big(\oplus_{\mu \in H}\, \pi t_\mu(X)\big) = \oplus_{\mu \in H} \pi t_{\mu\lambda}(X),$$

and it is immediate that $\tilde{t}_\lambda$ is indeed a well-defined automorphism.

We have for $c \in \widehat{H}$ and the normalized Haar measure on H

$$\int_H c(\lambda) d\lambda = \begin{cases} 1 & \text{if } c = 1 \\ 0 & \text{if } c \neq 1. \end{cases} \tag{1}$$

Notice that for $X \in W$, $\lambda \in H$,

$$\pi_\infty t_\lambda(X) = c_X(\lambda)\pi_\infty(X) = \mathrm{bal}(X)(\lambda)\pi_\infty(X),$$

$\mathrm{bal}(X) \in \widehat{H}$, and thus for $X \in \mathbb{F}/\mathbb{I}$ we get with (1)

$$\int_H \tilde{t}_\lambda \pi_\infty(X) d\lambda = \int_H \pi_\infty t_\lambda(X) d\lambda = \pi_\infty(X_e) = F(\pi_\infty(X)).$$

Thus we see by continuity that $F(x) = \int_H \tilde{t}_\lambda(x) d\lambda$ for all $x \in C$. Now $t_\lambda(\pi_\infty(X))$ is continuous in λ for fixed $X \in \mathbb{F}/\mathbb{I}$ and hence $\tilde{t}_\lambda(x)$ is also continuous in λ for $x \in C$. Thus if $F(x^*x) = 0$, then we have $\tilde{t}_e(x^*x) = x^*x = 0$ for $x \in C$. ☐

Theorem 3.3. Uniqueness *Assume* (A), (B), (C) *with respect to the balance function Lemma* 3.1. *Let A_i be C^*-algebras and $\pi_i : \mathbb{F}/\mathbb{I} \to A_i$ be $*$-homomorphisms with dense images ($i = 1, 2$). If π_1 is faithful on $\mathbb{A}$ then we have a $*$-homomorphism $\sigma : A_1 \to A_2$, $\sigma\pi_1 = \pi_2$. It is a bijection if π_2 is faithful on $\mathbb{A}$ too.*

$$
\begin{array}{ccc}
\mathbb{F}/\mathbb{I} & \xrightarrow{\pi_1} & A_1 \\
 & \overset{\pi_2}{\searrow} & \downarrow \sigma \\
 & & A_2
\end{array}
\tag{2}
$$

Proof. We introduce a third representation $\pi : \mathbb{F}/\mathbb{I} \to A_1 \oplus A_2$ such that $\pi(X) = \pi_1(X) \oplus \pi_2(X)$ and let then π_∞ be the representation written in Lemma 3.2 corresponding to π. Let $C = \overline{\mathrm{Im}(\pi_\infty)}$ and note that $\pi_\infty : \mathbb{F}/\mathbb{I} \to C$ has dense image and is injective on $\mathbb{A}$. We shall prove the theorem for the modified case where π_1 is replaced by π_∞ and π_2 is replaced by π_i for $i = 1, 2$ in the diagram (2). Then the proof of the originally formulated theorem can be easily deduced from this.

Let $i = 1, 2$. The canonical $*$-homomorphism $\sigma : \mathrm{Im}(\pi_\infty) \to \mathrm{Im}(\pi_i)$ such that $\sigma\pi_\infty = \pi_i$ is obviously contractive (and therefore well-defined) and can continuously be extended to C.

Suppose now π_i is injective on $\mathbb{A}$ too. Let F_∞, F_i be the maps of Lemma 2.6 corresponding to π_∞, π_i. We have $\sigma F_\infty = F_i \sigma$ on $\mathrm{Im}(\pi_\infty)$ and thus on C. Moreover, we note that $\pi_\infty(\mathbb{A})$ carries a unique C^*-norm (AF-algebra), whence σ must be an isometry on $\overline{\pi_\infty(\mathbb{A})}$. Now if $x \in C_+$ and $\sigma(x) = 0$, then $\sigma F_\infty(x) = F_i \sigma(x) = 0$ and so $F_\infty(x) = 0$. Hence $x = 0$ by Lemma 3.2 and we have proved that σ is injective and hence bijective. ☐

We would like to remark that the proofs of Proposition 2.5, Lemma 2.6, Lemma 3.2 and Theorem 3.3 are imitated corresponding proofs in [C].

Definition 3.4. Let $\mathbb{F}$, $\mathbb{I}$ and H be as discussed and properties (A), (B), (C) be satisfied. If there exist a C^*-algebra A and a $*$-homomorphism $\pi : \mathbb{F}/\mathbb{I} \to A$ having dense image and being faithful on $\mathbb{A}$, then A is unique by Theorem 3.3 and denoted by $\mathcal{O}_{\mathbb{F},\mathbb{I},H}$.

Corollary 3.5. Simplicity *$\mathcal{O}_{\mathbb{F},\mathbb{I},H}$ is simple iff each nonzero C^*-representation $\pi : \mathbb{F}/\mathbb{I} \to A$ is faithful on $\mathbb{A}$. Hence, $\mathcal{O}_{\mathbb{F},\mathbb{I},H}$ is simple if $\mathbb{A}$ is a simple $*$-algebra.*

Proof. This is an immediate consequence of Theorem 3.3. One just considers the diagram (2) for $A_1 = \mathcal{O}_{\mathbb{F},\mathbb{I},H}$. (For the second assertion notice that if $\pi|\mathbb{A} = 0$ then $\pi = 0$ since $\pi(a)^*\pi(a) = 0$ for all $a \in \mathcal{A}$.) ☐

Theorem 3.6. Crossed Product and Nuclearity

Let $\mathcal{O}_{\mathbb{F},\mathbb{I},H}$ be unital, $\pi : \mathbb{F}/\mathbb{I} \to \mathcal{O}_{\mathbb{F},\mathbb{I},H}$ be a $*$-homomorphism which is faithful on $\mathbb{A}$ and $\mathcal{A} \cap \mathbb{I} = \emptyset$.

Let $\widehat{H}_+ \subseteq \widehat{H}$ *be the monoid generated by* $\mathrm{bal}(\mathcal{A})$ *and assume a map* $S : \widehat{H}_+ \to \mathbb{F}/\mathbb{I}$, $S_g \in \mathrm{lin}(W_g)$, *which is multiplicative under* π ($\pi(S_{gh}) = \pi(S_g)\pi(S_h)$) *and such that* $\pi(S_g)$ *are isometries.*

Then there exists an AF*-algebra* A *(ind. limit of* nets*), an action* $\beta : \widehat{H} \to \mathrm{Aut}(A)$ *and a projection* $P \in A$ *such that* $\mathcal{O}_{\mathbb{F},\mathbb{I},H} \cong P(A \times_\beta \widehat{H})P$. *Hence* $\mathcal{O}_{\mathbb{F},\mathbb{I},H}$ *is nuclear.*

Proof. The proof goes through as the proof of [B], Theorem 3.2 (there $\widehat{H} \cong \mathbb{Z}^d$), which is, in fact, imitated a proof in [C]. To avoid misunderstandings we write explicitly

$$\widehat{H}_+ = \{\, \mathrm{bal}(a_1)\mathrm{bal}(a_2)...\mathrm{bal}(a_n) \in \widehat{H} \mid n \geq 0, a_i \in \mathcal{A} \,\}.$$

Note that $\widehat{H} = \widehat{H}_+\widehat{H}_+^{-1}$ since $\widehat{\mathbb{T}^{\mathcal{A}}}$ is algebraically generated by $\{\, c_a \mid a \in \mathcal{A} \,\}$, $\widehat{H} \cong \widehat{\mathbb{T}^{\mathcal{A}}}/H^\perp$ and $\mathrm{bal}(a) \cong c_a H^\perp$.

We shall switch to the easier notations $G := \widehat{H}$, $G_+ := \widehat{H}_+$ and write the group operation additively. So $G = G_+ - G_+$ and we make G to a directed set by regarding G_+ as the positive elements. G is directed since given two elements $\mathrm{bal}(X_1) - \mathrm{bal}(X_2)$ and $\mathrm{bal}(X_3) - \mathrm{bal}(X_4)$ in G, where $\mathrm{bal}(X_i) \in G_+$, one has in fact the upper bound $\mathrm{bal}(X_1X_3) = \mathrm{bal}(X_1) + \mathrm{bal}(X_3)$ by the multiplicativity of the balance function.

Step 1. Put $\mathcal{F} = \overline{\pi(\mathbb{A})}$. For each $g \in G_+$ define an injective $*$-homomorphism ($X \in \mathbb{A}$)

$$\alpha_g : \mathcal{F} \to \mathcal{F} : \alpha_g\pi(X) = \pi(S_g X S_g^*).$$

Let $A_g := \mathcal{F}$ for all $g \in G$ and consider the directed system of maps

$$\phi_{h,g} = \alpha_{h-g} : A_g \to A_h, \qquad g \leq h \in G.$$

Denoting its direct limit by $A = \varinjlim_{g\in G} A_g$ and the induced embeddings by $\phi_g : A_g \to A$ we get an action $\beta_g \in \mathrm{Aut}(A)$ by shifting along the direct limit, $\beta_g(\phi_h(x)) = \phi_{h-g}(x)$, $x \in \mathcal{F}$, $h, g \in G$. Consider the crossed product $A \rtimes_\beta G \subseteq B(\mathcal{H})$ and let $U : G \to B(\mathcal{H})$ be the homomorphism into the unitary group which fulfils $U_g x U_g^* = \beta_g(x)$ ($x \in A$).

Step 2. Let $P := \phi_0(I)$. We shall show that

$$C := P(A \rtimes_\beta G)P = \mathrm{C}^*(\phi_0(\mathcal{F}) \cup \{\, U_g P \mid g \in G_+ \,\}). \tag{3}$$

First notice that $PU_gPU_g^* = U_gPU_g^*$ and thus $PU_gP = U_gP$ ($g \geq 0$). So the inclusion $\supseteq$ is already obvious in (3). For the reverse inclusion $\subseteq$ notice that C is spanned by ($x \in \mathcal{F}, k \in G$, $g, h \in G_+$)

$$\begin{aligned} P\phi_k(x)U_{g-h}P &= PU_{-h}U_h\phi_k(x)U_{-h}U_gP \\ &= PU_{-h}P\phi_{k-h}(x)PU_gP \end{aligned}$$

and we finally just have to compute (choose $k - h \leq m \in G_+$)

$$P\phi_{k-h}(x)P = \phi_m(\alpha_m(I)\alpha_{m-(k-h)}(x)\alpha_m(I)) \tag{4}$$

$$= \phi_m\alpha_m(\pi(S_m)^*\alpha_{m-(k-h)}(x)\pi(S_m)) \in \phi_0(\mathcal{F}). \tag{5}$$

Step 3. We shall show now that $C = \mathcal{O}_{\mathbb{F},\mathbb{I},H}$. Let

$$\sigma : \mathcal{A} \to C : \sigma(a) = \phi_0(\pi((a+\mathbb{I})S^*_{\mathrm{bal}(a)}))U_{\mathrm{bal}(a)}P,$$

where $(a+\mathbb{I})S^*_{\mathrm{bal}(a)} \in \mathbb{A}$. As usual we shall optionally omit writing $+\mathbb{I}$ to the right of a.

From the definitions introduced in "Step 1" we deduce the identity

$$U_g \phi_0 \pi(X) U_g^* = \phi_0\pi(S_g X S_g^*) \qquad X \in \mathbb{A}, g \geq 0. \tag{6}$$

Then for $a, b \in \mathcal{A}$ we compute $(A := \mathrm{bal}(a), B := \mathrm{bal}(b))$

$$\begin{aligned} \sigma(a)\sigma(b) &= \phi_0\pi(aS_A^*)U_A P \;\; \phi_0\pi(bS_B^*)U_A^* U_A U_B P \\ &= \phi_0\pi(aS_A^*) \;\phi_0\pi(S_A b S_B^* S_A^*)\; U_A U_B P \\ &= \phi_0\pi(abS^*_{A+B})U_{A+B}P. \end{aligned}$$

By a similar computation we see by induction on the length $n \geq 1$ of a word $X = a_1^{\epsilon_1}...a_n^{\epsilon_n} \in \mathbb{F}$ $(a_i \in \mathcal{A},\ \epsilon_i \in \{1, *\})$ that

$$\sigma(a_1)^{\epsilon_1}\sigma(a_2)^{\epsilon_2}...\sigma(a_n)^{\epsilon_n} = PU_A^*\phi_0\pi(S_A(a_1^{\epsilon_1}a_2^{\epsilon_2}...a_n^{\epsilon_n} + \mathbb{I})S_B^*)U_B P \tag{7}$$

for any $A, B \in G_+$ with $A + \mathrm{bal}(X + \mathbb{I}) - B = 0$.

Having the identity (7) we see by the following argument that σ can be naturally expanded to a $*$-homomorphism $\sigma : \mathbb{F}/\mathbb{I} \to C$. Firstly notice that σ is well defined on the set of words W by the representation (7).

Secondly let F be the map introduced in Lemma 2.6 with respect to our π. Then $0 = X = \sum_{g\in G} X_g$ for $X_g \in \mathrm{lin}(W_g) \Rightarrow 0 = F\pi(X^*X) = \sum_{g\in G}\pi(X_g^*X_g) \Rightarrow \pi(X_g) = 0$ for all $g \in G \Rightarrow \sigma(X) = 0$ due to (7).

Moreover, σ is faithful on $\mathbb{A}$ since $\sigma(X) = \phi_0\pi(X)$ $(X \in \mathbb{A})$ by (7) and for $g \in G_+$ we have by (7) and (6)

$$\sigma(S_g) = \phi_0\pi(S_g S_g^*)U_g P = U_g P U_g^* U_g P = U_g P.$$

Hence, $\mathrm{Im}(\sigma)$ is dense in C and Theorem 3.3 yields $C = \mathcal{O}_{\mathbb{F},\mathbb{I},H}$ as desired.

Step 4. $\mathcal{O}_{\mathbb{F},\mathbb{I},H}$ is nuclear since $A \rtimes G$ is nuclear (see for example [RøS], 2.1.2) why $P(A \rtimes G)P$ is nuclear by C.P.A.P., see [CE], 3.1.(2'). □

Example 3.7. Consider the Cuntz-Krieger algebras $\mathcal{O}_A$ [CK] generated by n partial isometries $\mathcal{A} = \{S_1, ..., S_n\}$. We let $H = \{\lambda_1 = \lambda_2 = ... = \lambda_n, \lambda_i \in \mathbb{T}\} \cong \mathbb{T}$. Then we have $\widehat{H} \cong \mathbb{Z}$ and $\widehat{H}_+ = \mathbb{Z}_+$, since the balance function is given by $\mathrm{bal}(S_i) = 1_{\mathbb{Z}}$ $(i = 1, ..., n)$.

If we choose $t(i) \in \{1, ..., n\}$ for each $i = 1, ..., n$ such that $S_{t(i)}S_i \neq 0$ (the transition matrix must have no sink), then

$$S := \sum_{i=1}^n S_{t(i)} S_i S_i^*$$

is an isometry in $\mathrm{lin}(W_1)$ and hence S^k ($k \in \mathbb{Z}_+$) yield the isometries such that Theorem 3.6 is applicable. Thus $\mathcal{O}_A \cong P(A \times_\beta \mathbb{Z})P$. (Note the similarity of this result to [CK, 3.7, 3.8] but without using topological Markov chains.)

If one has a representation $\pi : \mathbb{F}/\mathbb{I} \to A$ which is not necessarily injective on $\mathbb{A}$ then one may just divide $\mathbb{F}$ by a larger ideal $\mathbb{J}$ to obtain injectivity. This is the next lemma.

Lemma 3.8. *Let* (A) *be satisfied and* $\pi : \mathbb{F}/\mathbb{I} \to A$ *be a* $*$*-homomorphism into a* C^**-algebra* A. *Let* $\mathbb{I} \subseteq \mathbb{J} \subseteq \mathbb{F}$ *be canonically defined by* $\mathbb{F}/\mathbb{J} \cong (\mathbb{F}/\mathbb{I})/\ker(\pi|\mathbb{A})$.

Then $\mathbb{J}$ *also satisfies property* (A) *with the same group* H *and the quotient map* $\tilde{\pi} : \mathbb{F}/\mathbb{J} \to A$ *of* π *is injective on the new* $\mathbb{A} \subseteq \mathbb{F}/\mathbb{J}$.

Proof. Firstly we show that $\mathbb{J}$ satisfies (A) for the same group H.

Let $X \in \mathbb{J}$ and $\lambda \in H$. Then X permits a representation $X = \sum_i \lambda_i a_i Y_i b_i$ for scalars $\lambda_i \in \mathbb{C}$ and elements $a_i, Y_i, b_i \in \mathbb{F}$ such that $Y_i + \mathbb{I} \in \ker(\pi|\mathbb{A})$, and where a_i, b_i might be identities (or omitted in the sum).

In general we have $t_\lambda(Y) = c_Y(\lambda)Y = Y$ for all neutral-balanced words $Y \in \mathbb{F}$ since $c_Y|H = \mathrm{bal}(Y + \mathbb{I}) \equiv 1$. Therefore we obtain $t_\lambda(Y) = Y \in \mathbb{J}$ for all $Y + \mathbb{I} \in \ker(\pi|\mathbb{A}) \subseteq \mathbb{A}$. Consequently we obtain $t_\lambda(X) \in \mathbb{J}$ by the above sum representation.

To check the faithfulness of $\tilde{\pi}$ on the new $\mathbb{A} \subseteq \mathbb{F}/\mathbb{J}$ one must know what the new $*$-algebra $\mathbb{A}$ looks like. At this point we notice the following. Due to the definition in Lemma 3.1 the new balance function on the nonzero words $W \subseteq \mathbb{F}/\mathbb{J}$ essentially coincides with the old balance function, i.e. we have $\mathrm{bal}(X+\mathbb{I}) = \mathrm{bal}(X+\mathbb{J})$ for all words $X \in \mathbb{F}$, $X \notin \mathbb{J}$. Therefore the new $*$-algebra $\mathbb{A} \subseteq \mathbb{F}/\mathbb{J}$ essentially coincides with the old $\mathbb{A} \subseteq \mathbb{F}/\mathbb{I}$. Per definition we have

$$\mathbb{A} = \mathrm{lin}\{\, X + \mathbb{J} \in \mathbb{F}/\mathbb{J} \mid X \in \mathbb{F} \text{ is a word}, \mathrm{bal}(X + \mathbb{J}) = 0 \,\} \subseteq \mathbb{F}/\mathbb{J}$$

and one easily shows that $\tilde{\pi}$ is faithful on this new $\mathbb{A} \subseteq \mathbb{F}/\mathbb{J}$. ☐

4. Cuntz-Krieger Algebras of Rank One

In this section we shall prove the uniqueness of the classical or rank one Cuntz-Krieger algebras. However, we permit arbitrarily many generators, and this corresponds to infinitely large transition matrices. It will be useful to consider such rank one Cuntz-Krieger algebras for the next section where two rank one Cuntz-Krieger algebras with infinitely many generators will interact. The concepts of this section (rank one Cuntz-Krieger algebras) and the next section (rank two Cuntz-Krieger algebras) are very similar, too, and it is advantageous to start with the easier rank one case.

Moreover this gives an alternative and shorter proof of the uniqueness theorem of Exel and Laca [EL, Corollary 13.2]. After the main result, Proposition 4.3, we compare the Exel-Laca algebras with ours in detail and reproduce the Exel-Laca uniqueness theorem in Corollary 4.5.

Let $\mathcal{A}$ be a set of any cardinality and let $\mathbb{F}$ be the non-unital involutive algebra generated freely by the alphabet $\{ S_a \mid a \in \mathcal{A} \}$ ($\leftrightarrow \mathcal{A}$; we simply prefer it here to write S_a instead of a, but this is very convenient). We let $\mathbb{I}$ be the smallest ideal in $\mathbb{F}$ such that the following properties hold in $\mathbb{F}/\mathbb{I}$ (once again we shall simply write S_a instead of $S_a + \mathbb{I}$).

All letters S_a are partial isometries, more precisely we assume $S_a S_a^* S_a = S_a$ for all $a \in \mathcal{A}$; the support and range projections we denote by $P_a := S_a S_a^*$ and $Q_a := S_a^* S_a$ respectively. The range projections are orthogonal, i.e. $P_a P_b = \delta_{a,b} P_a$ for all $a, b \in \mathcal{A}$, the support projections commute, i.e $Q_a Q_b = Q_b Q_a$ for all $a, b \in \mathcal{A}$, and there exists a map $s : \mathcal{A} \times \mathcal{A} \to \{0, 1\}$ such that

$$Q_a P_b = s(a, b) P_b$$

for all $a, b \in \mathcal{A}$.

We define the subgroup H appearing in property (A) as the diagonal

$$H = \{ (\lambda_a)_{a \in \mathcal{A}} \in \mathbb{T}^{\mathcal{A}} \mid \forall a, b \in \mathcal{A} : \lambda_a = \lambda_b \} \cong \mathbb{T}.$$

Then the balance function defined in Lemma 3.1 maps into $\widehat{H} \cong \mathbb{Z}$ and using this group isomorphism we get $\mathrm{bal}(S_a) = 1_{\mathbb{Z}}$ for all $a \in \mathcal{A}$.

The subsequent notations will be used. In accordance to the previous section, W denotes the set of all words consisting of letters S_a and S_b^*; the empty word is not allowed. We denote by $W_{\mathcal{B}}$ the set of all words in the letters $\mathcal{B} \subseteq \mathcal{A}$; the empty word *is* here allowed. For $\alpha = (a_1, a_2, ..., a_n) \in W_{\mathcal{B}}$ we write $S_\alpha := S_{a_1} S_{a_2} ... S_{a_n}$ and $\ell(\alpha) := n$. If the word α is empty, then $\ell(\alpha) = 0$ and $S_\alpha := I$ for an imaginary identity I. More precisely we shall use I only in expressions where it is not really necessary, for example the expressions IQ $(:= Q)$, or $(I - P)Q$ $(:= Q - PQ)$. So, although we do not assume an identity I, we introduce it to simplify notations.

A further notation is $Q_\alpha := Q_{a_1} Q_{a_2} ... Q_{a_n}$ $(\alpha = (a_1, ..., a_n) \in W_{\mathcal{A}})$. We should not confuse Q_α with the support projection of $S_\alpha \neq 0$, which is $S_\alpha^* S_\alpha = S_{a_n}^* S_{a_n} = Q_{a_n}$. Beside, also notice the relation $S_\alpha^* S_\beta = 0$ if $\ell(\alpha) = \ell(\beta)$ but $\alpha \neq \beta$.

Lemma 4.1. (I) *Each nonzero word $X \in W$ has a representation*

$$X = S_\alpha Q_\gamma S_\beta^*, \quad \alpha, \beta, \gamma \in W_{\mathcal{A}}, \ell(\alpha) \geq 0, \ell(\beta) \geq 0, \ell(\gamma) \geq 1.$$

(II) *Every word is a partial isometry and their range and support projections,*

$$\{ S_\alpha Q_\gamma S_\alpha^* \mid \alpha, \gamma \in W_A, \ell(\gamma) \geq 1 \} \cup \{0\},$$

generate a commutative $*$-algebra.

(III) *Every nonempty finite subset $\mathcal{B} \subseteq \mathcal{A}$ and integer $N \geq 0$ define a finite dimensional C^*-algebra*

$$\mathcal{F}_{\mathcal{B},N} := \mathrm{Alg}\{ S_\alpha Q_\gamma S_\beta^* \mid \alpha, \beta, \gamma \in W_{\mathcal{B}}, 0 \leq \ell(\alpha) = \ell(\beta) \leq N, \ell(\gamma) \geq 1 \}$$

in $\mathbb{A}$, and $\mathbb{A}$ is the inductively ordered union of all these C^*-algebras.

(IV) *The algebra $\mathcal{F}_{\mathcal{B},N}$ is representable as the direct sum*

$$\mathcal{F}_{\mathcal{B},N} = \bigoplus_{i=0}^{N} \bigoplus_{L\in\mathbb{Q}(\mathcal{B})} \mathcal{M}_{i,L} \tag{8}$$

of simple matrix algebras ($0 \le i < N$)

$$\mathcal{M}_{i,L} = \mathrm{Alg}\{\, S_\alpha(I-P)LS_\beta^* \mid \alpha,\beta \in W_{\mathcal{B}} \text{ and } \ell(\alpha)=\ell(\beta)=i \,\},$$

$$\mathcal{M}_{N,L} = \mathrm{Alg}\{\, S_\alpha L S_\beta^* \mid \alpha,\beta \in W_{\mathcal{B}} \text{ and } \ell(\alpha)=\ell(\beta)=N \,\},$$

where

$$\mathbb{Q}(\mathcal{B}) = \Big\{ \prod_{b\in\mathcal{B}} X_b \;\Big|\; \forall b\in\mathcal{B}:\ X_b = Q_b \text{ or } X_b = I-Q_b, \exists c\in\mathcal{B}: X_c = Q_c \Big\},$$

$$P = \sum_{b\in\mathcal{B}} P_b.$$

Proof. (I) This can be easily seen by induction on the length $\ell(X)$. Just note that $S_a = S_aQ_a$ for all $a\in\mathcal{A}$ and $S_a^*Q_b = S_a^*$ or zero ($a,b\in\mathcal{A}$).

(II) Also straightforward.

(III) It follows from the representation (IV) that $\mathcal{F}_{\mathcal{B},N}$ is a finite-dimensional C^*-algebra. Due to the representations (I), $\mathbb{A}$ is obviously the union of these C^*-algebras.

(IV) For $0 \le \ell(\alpha) = \ell(\beta) < N$ we write

$$\begin{aligned} S_\alpha Q_\gamma S_\beta^* &= S_\alpha(\sum_{b\in\mathcal{B}} S_bS_b^*)Q_\gamma S_\beta^* + S_\alpha(I - \sum_{b\in\mathcal{B}} S_bS_b^*)Q_\gamma S_\beta^* \\ &= \sum_{b\in\mathcal{B}} S_\alpha S_b Q_b S_b^* S_\beta^* + S_\alpha(I-P)Q_\gamma S_\beta^*. \end{aligned}$$

We leave the right summand $S_\alpha(I-P)Q_\gamma S_\beta^*$ in this form. The words in the left sum are decomposed again in this vein. One will stop when all words have either the form $S_\alpha Q_\gamma S_\beta$ for $\ell(\alpha)=\ell(\beta)=N$ or the form $S_\alpha(I-P)Q_\gamma S_\beta^*$ for $\ell(\alpha)=\ell(\beta)<N$.

Next we proceed by splitting ($b\in\mathcal{B}$)

$$S_\alpha(I-P)Q_\gamma S_\beta^* = S_\alpha(I-P)Q_\gamma(I-Q_b)S_\beta^* + S_\alpha(I-P)Q_\gamma Q_b S_\beta^*,$$

and similarly we split $S_\alpha Q_\gamma S_\beta^*$ if $\ell(\alpha)=N$. Then we split the summands on the right side of this vein with another $b\in\mathcal{B}$. One will stop when this procedure has been completed for all $b\in\mathcal{B}$. Each element of $\mathcal{F}_{\mathcal{B},N}$ is now represented as a linear combination of elements in $\mathcal{M}_{i,L}$ ($0\le i\le N$, $L\in\mathbb{Q}(\mathcal{B})$).

On the other hand, we easily see that $\mathcal{M}_{i,L}\subseteq\mathcal{F}_{\mathcal{B},N}$ and that they are mutually orthogonal. Moreover, each $\mathcal{M}_{i,L}$ is a simple matrix algebra since if we write $X_\alpha = S_\alpha(I-P)L$ then the algebra $\mathcal{M}_{i,L}$ ($i<N$) is per definition generated by the self-adjoint system of matrix units

$$\Gamma = \{\, X_\alpha X_\beta^* \mid \alpha,\beta\in W_{\mathcal{B}},\ \ell(\alpha)=\ell(\beta)=i, X_\alpha\neq 0, X_\beta\neq 0 \,\}.$$

Each of these matrix units is in fact nonzero, since assuming $X_\alpha X_\beta^* = 0$ would yield the contradiction $0 = S_\alpha^* X_\alpha X_\beta^* S_\beta = (I - P)L$. Therefore, $\mathcal{M}_{i,L}$ is isomorphic to the simple matrix algebra $M_k(\mathbb{C})$ where $k^2 = \mathrm{card}(\Gamma)$. □

We are now considering a representation $\pi : \mathbb{F}/\mathbb{I} \to B(\mathcal{H})$ such that $\pi(P_a) \neq 0$ for all $a \in \mathcal{A}$ and such that the strong operator sum $P = \sum_{b\in\mathcal{A}} \pi(P_b)$ is a unit for $\pi(\mathbb{F}/\mathbb{I})$.

Equivalent properties to the last one are $\pi(Q_a) \le P$ for all $a \in \mathcal{A}$, or the strong operator sum representations

$$\pi(Q_a) = \sum_{b\in\mathcal{A}} s(a,b)\pi(P_b) \qquad \forall a \in \mathcal{A}. \tag{9}$$

Indeed if P is a unit then $\pi(Q_a) = \pi(Q_a)P = P\pi(Q_a) \le P$ for all $a \in \mathcal{A}$. This in turn yields

$$\pi(Q_a) = \pi(Q_a)P = \sum_{b\in\mathcal{A}} \pi(Q_a)\pi(P_b) = \sum_{b\in\mathcal{A}} s(a,b)\pi(P_b).$$

Finally having (9), we similarly verify that $\pi(S_a)P = \pi(S_a)\pi(Q_a)P = \pi(S_a)\pi(Q_a) = \pi(S_a)$ and analogously that $P\pi(S_a) = \pi(S_a)$. This in turn shows that P is a unit for $\pi(\mathbb{F}/\mathbb{I})$.

Let us also notice that such a representation π can be easily constructed for each transition matrix s. Choose an arbitrary set Γ, huge enough such that $\mathrm{card}(\Gamma) = \mathrm{card}(\mathcal{A} \times \Gamma)$. Let $\mathcal{H} := \ell^2(\mathcal{A} \times \Gamma)$. Then for $a \in \mathcal{A}$ choose any partial isometry $\widetilde{S}_a \in B(\mathcal{H})$ with range space $\ell^2(\{a\} \times \Gamma) \subseteq \mathcal{H}$ and support space $\ell^2(\mathcal{B} \times \Gamma) \subseteq \mathcal{H}$ where $\mathcal{B} := \{ b \in \mathcal{A} \mid s(a,b) = 1 \}$. Clearly the range and support spaces are isomorphic Hilbert spaces and we have no problem in finding such a partial isometry $\widetilde{S}_a$. Letting $\pi(S_a) := \widetilde{S}_a$ we have a desired representation.

We are now passing from the ideal $\mathbb{I}$ to the larger ideal $\mathbb{J}$ as done in Lemma 3.8 such that

$$\tilde{\pi} : \mathbb{F}/\mathbb{J} \to B(\mathcal{H}) : \tilde{\pi}(X + \mathbb{J}) = \pi(X + \mathbb{I}) \tag{10}$$

is injective on $\mathbb{A}$. (If nothing else is mentioned, all notations, $\mathbb{A}$ etc., are now related to the ideal $\mathbb{J}$.) Thereby notice that the new balance function with respect to $\mathbb{F}/\mathbb{J}$ essentially coincides with the original balance function since the closed subgroup H of $\mathbb{T}^{\mathcal{A}}$ is the same, i.e. we have $\mathrm{bal}(X + \mathbb{J}) = \mathrm{bal}(X + \mathbb{J})$ for words $X \in \mathbb{F}$ per definition in Lemma 3.1. Therefore, the new algebra $\mathbb{A} \subseteq \mathbb{F}/\mathbb{J}$ is generated by the same words $X \in \mathbb{F}$ as the former algebra $\mathbb{A} \subseteq \mathbb{F}/\mathbb{I}$.

That elements can collapse to zero is structurally the most important fact for us to observe when passing to the larger ideal $\mathbb{J}$.

Thus the representations in Lemma 4.1 also remain valid for the new reference algebra $\mathbb{F}/\mathbb{J}$. Thereby we should be careful with the sum (8). However, it remains a sum of simple matrix algebras since each of the simple factors $\mathcal{M}_{i,L}$ can only either vanish completely or survive unaltered the transition $\mathbb{F}/\mathbb{I} \to \mathbb{F}/\mathbb{J}$. Therefore the representation (8) shows that property (B) is satisfied for the $*$-algebra $\mathbb{F}/\mathbb{J}$ too.

Another possibility to see property (B) is to realize that one gets a C^*-norm on the finite dimensional $*$-algebras $\mathcal{F}_{\mathcal{B},N} \subseteq \mathbb{F}/\mathbb{J}$ from the image of the faithful map $\tilde{\pi}$.

For a word $A = (a_1, ..., a_n) \in W_{\mathcal{A}}$ we remark that $S_A \neq 0 \Leftrightarrow S_A^* S_A \neq 0 \Leftrightarrow S_{a_n} \neq 0$ and $s(a_i, a_{i+1}) = 1$ for all $i = 1, ..., n-1$. Consequently the following property (Two Paths) depends on the transition matrix s only since all letters $S_a \in \mathbb{F}/\mathbb{J}$ $(a \in \mathcal{A})$ are nonzero due to $\tilde{\pi}(S_a S_a^*) \neq 0$.

(Two Paths) The transition matrix s satisfies: for all $a \in \mathcal{A}$ there exist nonempty words $X, Y \in W_{\mathcal{A}}$ such that $S_a S_X \neq 0$, $S_a S_Y \neq 0$ but $S_X^* S_Y = 0$.

Lemma 4.2. *Property* (C) *holds in* $\mathbb{F}/\mathbb{J}$ *if* s *satisfies property* (Two Paths).

Proof. We will check criterion (C*) of Lemma 2.4 under the set of projections

$$\mathbb{P}_2 := \{\, S_\alpha S_\alpha^* \mid \alpha \in W_{\mathcal{A}}, \ell(\alpha) \geq 1 \,\} \backslash \{0\}.$$

Step 1: We will firstly check that $\mathbb{P}_2 \precsim \mathbb{P}$ in $\mathbb{A}$. If $E \in \mathbb{P}$ then we have $p \precsim E$ for some minimal projection p sitting on the diagonal of some matrix algebra $\mathcal{M}_{i,L}$ by Lemma 4.1. So either $p = S_\alpha (I-P) L S_\alpha^*$ or $p = S_\alpha L S_\alpha^*$ and we choose $a \in \mathcal{A}$ such that $P_a \leq (I-P)L$ or $P_a \leq L$, respectively. (Note that such a choice P_a is possible since this holds in the image of $\tilde{\pi}$ and $\tilde{\pi}$ if faithful on $\mathbb{A}$.) Then $P := S_\alpha P_a S_\alpha^* \in \mathbb{P}_2$ satisfies $P \leq p \precsim E$.

Step 2: Preparing the next step we will show that for all $A \in W_{\mathcal{A}}$ with $S_A \neq 0$ we find $X \in W_{\mathcal{A}}$ such that $S_A S_X \neq 0$ and $S_X^* S_A S_X = 0$.

Indeed, by property (Two Paths) we can choose $X, Y \in W_{\mathcal{A}}$ such that $S_A S_X \neq 0$, $S_A S_Y \neq 0$ but $S_Y^* S_X = 0$. Consider the case that $S_X^* S_A S_X \neq 0$. Then X must be periodic with A, i.e. be of the form $X = AAA....$ Since $X \neq Y$, Y can not be periodic with A, i.e. $Y \neq AAA....$ Hence $S_Y^* S_A S_Y = 0$ and Y is our choice.

Step 3: We will proceed verifying criterion (C*). Consider a nonzero-balanced word $U = S_\alpha Q_\gamma S_\beta^* \in W$, where we assume without loss of generality (otherwise consider U^*) that $\ell(\alpha) \geq \ell(\beta)$. Let $E \in \mathbb{P}_2$. Then we find a smaller projection $S_C S_C^* \leq E$ in $\mathbb{P}_2$ with $\ell(C) > \ell(\alpha)$ by property (Two Paths). Thus, we have $S_C^* U S_C = 0$ or $S_C^* U S_C = S_A$ for some $A \in W_{\mathcal{A}}$. We have completed the first case ($P := S_C S_C^*$) and in the second case we use the result of "Step 2" and choose S_X such that $S_A S_X \neq 0$ but $S_X^* S_A S_X = 0$. This yields $0 \neq P := S_C S_X S_X^* S_C^* \leq E$ and $PUP = 0$.

Step 4: Completing the verification of (C*) we will proceed similarly as in "Step 3". Let $E_1, E_2 \in \mathbb{P}_2$ and U be a nonzero-balanced word. Choose nonempty $C, D \in W_{\mathcal{A}}$ with suitable length such that $S_C S_C^* \leq E_1$, $S_D S_D^* \leq E_2$ and $S_C^* U S_D$ is either zero or equal to $S_C^* U S_D = Q_a$ for some $a \in \mathcal{A}$. In the latter case we choose $X, Y \in W_{\mathcal{A}}$ such that $Q_a S_X \neq 0$, $Q_a S_Y \neq 0$ but $S_Y^* S_X = 0$. Then $0 \neq P_1 := S_C S_Y S_Y^* S_C^* \leq E_1$, $S_D S_X S_X^* S_D^* \leq E_2$ and $P_1 U P_2 = 0$ as desired. □

Proposition 4.3. Uniqueness of rank one Cuntz-Krieger algebras

Consider two sets of partial isometries $(\widetilde{S}_a)_{a\in\mathcal{A}} \subseteq B(\widetilde{\mathcal{H}})$, $(\widehat{S}_a)_{a\in\mathcal{A}} \subseteq B(\widehat{\mathcal{H}})$ *both satisfying* $\widetilde{Q}_a = \sum_{b\in\mathcal{A}} s(a,b)\widetilde{P}_b$ *and* $\widehat{Q}_a = \sum_{b\in\mathcal{A}} s(a,b)\widehat{P}_b$*, respectively, for all* $a \in \mathcal{A}$*, where the sum is in the strong operator topology. Denote by* $\widetilde{A}$ *and* $\widehat{A}$ *their generated* C^**-algebras, respectively. Suppose that* $\widetilde{S}_a \neq 0$ *for all* $a \in \mathcal{A}$ *and* s *is such that property* (Two Paths) *holds (for example when* (I) *in* [CK]*, p254, holds). Then there exists a* $*$*-homomorphism* $\sigma : \widetilde{A} \to \widehat{A}$, $\sigma(\widetilde{S}_a) = \widehat{S}_a$ *which is one-to-one if* $\widehat{S}_a \neq 0$ *for all* $a \in \mathcal{A}$ *too.*

Proof. We have two representations $\pi_1 : \mathbb{F}/\mathbb{I} \to \widetilde{A}$ and $\pi_2 : \mathbb{F}/\mathbb{I} \to \widehat{A}$ such that $\pi_1(S_a) = \widetilde{S}_a$ and $\pi_2(S_a) = \widehat{S}_a$ for all $a \in \mathcal{A}$. Let $J \subseteq \mathbb{F}$ be the self-adjoint two-sided ideal generated by $\ker(\pi_1|\mathbb{A}) \subseteq \mathbb{F}/\mathbb{I}$ and $\mathbb{J} \subseteq \mathbb{F}$ be the ideal canonically defined by $(\mathbb{F}/\mathbb{I})/J \cong \mathbb{F}/\mathbb{J}$. Let $\tilde{\pi}_1 : \mathbb{F}/\mathbb{J} \to A$ be the $*$-homomorphism canonically associated to π_1 and which is faithful on the new $\mathbb{A} \subseteq \mathbb{F}/\mathbb{J}$, see (10) and the discussion there.

In order to also get a canonical map $\tilde{\pi}_2 : \mathbb{F}/\mathbb{J} \to \widehat{A}$ we have to show that $\ker(\pi_1|\mathbb{A}) \subseteq \ker(\pi_2)$. But this is clear since considering a matrix entry $e = S_\alpha(I-P)QS_\beta^* \in \mathbb{F}/\mathbb{I}$ according to Lemma 4.1.(IV), it is easy to see that $\pi_1(e) = 0 \Rightarrow \pi_2(e) = 0$ (and vice versa $\Leftarrow$ if $\pi_2(\widehat{S}_a) \neq 0$ for all a).

Further recall that $\mathbb{F}/\mathbb{J}$ satisfies property (B) as discussed above. By Lemma 10 the ideal $\mathbb{J}$ satisfies property (A) for the group H too. By Lemma 4.2 we have property (C).

Therefore we can finish the proof by applying Theorem 3.3 to $\mathbb{F}$, $\mathbb{J}$ and H. $\square$

In the remaining part of this section we compare our result with the uniqueness theorem of Exel and Laca in [EL], Corollary 13.2. This was suggested by the referee and we will, in fact, reveal interesting aspects.

Our proposition above and the uniqueness theorem of Exel and Laca differ in two points. The first point is that we claim that each of the source projections Q_a can be written as the strong operator sum of range projections P_a. Exel and Laca manage their algebras without strong operator topology, however, they claim some extra relations among the generators.

Now notice the following. In the proof of Proposition 4.3 we went over from the ideal $\mathbb{I}$ to the larger ideal $\mathbb{J}$ and then applied the uniqueness theorem of the last section to the algebra $\mathbb{F}/\mathbb{J}$. Therefore we can formulate the uniqueness without any strong operator topology, simply by considering representations $\pi : \mathbb{F}/\mathbb{J} \to A$ into C^*-algebras A (cf. Corollary 4.5). However, the costs are the larger amount of relations, represented by the larger ideal $\mathbb{J}$.

In Lemma 4.4 below, and in the discussion that follows, we will show that a set of relations determining $\mathbb{J}$ coincide exactly with the relations claimed by Exel and Laca. Thus the Cuntz-Krieger algebras we considered here coincide with the definition of the Exel-Laca algebras.

The second point is interesting. To get the uniqueness property, Exel and Laca claim in their paper that the directed graph with vertices $\mathcal{A}$, and arrows $a \to b$ iff $s(a,b) = 1$ (iff $S_aS_b \neq 0$), associated to the transition matrix s, must not have a terminal circuit. A terminal circuit is a loop in the graph without exit, cf. [EL], Defintion 12.1.

However, our condition (Two Paths) is logically stronger, or in other words, our result is weaker than the one of Exel and Laca. Consider for example $\mathcal{A} = \mathbb{N}$ and the transition matrix s associated to the relations $Q_1 = P_2$, $Q_2 = P_3$, $Q_3 = P_4$,... The graph has no terminal circuit, but starting from any vertex $n \in \mathbb{N}$ you can only continue with $n \to n+1 \to n+2$ etc. Therefore it does not satisfy the condition (Two Paths) which simply says that you can always find two different continuations.

This aspect of our theory does not seem to be repairable. The projections P_n $(n \in \mathbb{N})$ are *minimal* in $\mathbb{A}$ ($S_n S_{n+1} ... S_{n+m} S_{n+m}^* ... S_{n+1}^* S_n^* = P_n$ are the only nonzero zero-balanced words and thus $\mathbb{A} = \mathrm{lin}((P_n)_{n \in \mathbb{N}})$) and then you are without chance: given a nonzero-balanced letter S_a and projections $E_1 := P_a$ and $E_2 := Q_a$ you cannot find nonzero projections $P_1 \le E_1$ and $P_2 \le E_2$ such that $P_1 S_a P_2 = 0$, as claimed in condition (C).

However, we can deduce the uniqueness theorem of Exel and Laca from our Proposition 4.3 by a simple trick. You consider modifications t of the transition matrix s such that the uniqueness theorem is applicable and such that s and t coincide on a certain local place. Having the uniqueness locally you can extend it globally and get the Exel-Laca result stated in Corollary 4.6.

This shows that there seems to exist a further uniqueness potential than the concept of this paper covers, namely possibly by such extensions as just described (a net of local homomorphisms extends to a global homomorphism).

In the following Lemma, for $\mathbb{Q}(\mathcal{C})$ we use the definition in Lemma 4.1.

Lemma 4.4. *The ideal* $\mathbb{J}$*, per definition generated by* $\ker(\pi|\mathbb{A})$*, is already generated by* $\ker(\pi|\mathbb{B})$*, where*

$$\mathbb{B} := \{ (I-P)L \in \mathbb{F}/\mathbb{I} \mid \mathcal{B}, \mathcal{C} \subseteq \mathcal{A} \textit{ finite subsets}, \mathcal{C} \neq \emptyset, L \in \mathbb{Q}(\mathcal{C}), P = \sum_{b \in \mathcal{B}} P_b \}.$$

Proof. Let $X \in \ker(\pi|\mathbb{A})$ and choose a matrix representation $X = \sum_{i,j,s} \lambda_{ij}^s e_{ij}^s \in \bigoplus_s M_{n_s}$ in $\mathbb{F}/\mathbb{I}$ as computed in Lemma 4.1.(IV). Since $\pi(X) = 0$ we have

$$\pi(e_{ii}^s X e_{ji}^s) = \pi(\lambda_{ij}^s e_{ii}^s) = \pi(\lambda_{ij}^s S_\alpha (I-P) L S_\alpha^*) = 0.$$

Consequently $\pi(S_\alpha^*)\pi(\lambda_{ij}^s S_\alpha (I-P) L S_\alpha^*)\pi(S_\alpha) = \lambda_{ij}^s \pi((I-P)L) = 0$ (we suppose that $e_{ii}^s \neq 0$ and thus $S_\alpha \neq 0$). Thus either $\lambda_{ij}^s = 0$ or $(I-P)L \in \ker(\pi|\mathbb{B})$. Since we can write $\lambda_{ij}^s e_{ij}^s = \lambda_{ij}^s e_{ii}^s e_{ij}^s$, this shows that X is in the ideal generated by $\ker(\pi|\mathbb{B})$. □

With regard to the above lemma, consider $(I-P)L \in \mathbb{B}$ where $P = \sum_{b \in \mathcal{B}} P_b$ for a finite set $\mathcal{B} \subseteq \mathcal{A}$, and where

$$L = Q_{x_1} ... Q_{x_n} (I - Q_{y_1}) ... (I - Q_{y_m})$$

for certain $x_i, y_i \in \mathcal{A}$. Then $L P_a = s(x, y, a) P_a$ where

$$s(x, y, a) := s(x_1, a) ... s(x_n, a)(1 - s(y_1, a)) .... (1 - s(y_m, a)).$$

Now $(I-P)L \in \ker(\pi|\mathbb{B}) \Leftrightarrow \pi(L) = \pi(LP) \Leftrightarrow \pi(L) \le \pi(P) \Leftrightarrow \pi(LP_b) \le \pi(PP_b)$ for all $b \in \mathcal{A} \Leftrightarrow s(x,y,b) = 0$ for all $b \in \mathcal{A}\backslash\mathcal{B}$, and in this case we have

$$(I-P)L = L - \sum_{b\in\mathcal{B}} s(x,y,b)P_b.$$

These elements forming the ideal $\mathbb{J}$ coincide exactly with the postulated equations (1.3) on page 121 of the Exel-Laca paper [EL].

Therefore our extra relations coming from $\mathbb{J}$ coincide with the extra relations introduced by Exel and Laca.

Corollary 4.5. *Consider two representations $\pi_i : \mathbb{F}/\mathbb{J} \to A_i$ with dense images in C^*-algebras A_i $(i = 1,2)$. Assume property* (Two Paths) *and $\pi_1(S_a) \neq 0$ for all $a \in \mathcal{A}$. Then there exists a canonical $*$-homomorphism $\sigma : A_1 \to A_2$ such that $\sigma\pi_1(S_a) = \pi_2(S_a)$.*

Proof. This follows by Theorem 3.3 applied to $\mathbb{F}$, $\mathbb{J}$ and H and recall that we did this at the end of the proof of Proposition 4.3. The only open question is whether π_1 is indeed faithful on $\mathbb{A} \subseteq \mathbb{F}/\mathbb{J}$, as claimed by Theorem 3.3.

To this end let $\pi_1' : \mathbb{F}/\mathbb{I} \to A_1$ be such $\pi_1'(X+\mathbb{I}) = \pi_1(X+\mathbb{J})$ for $X \in \mathbb{F}$. Let $X+\mathbb{J} \in \mathbb{A} \in \mathbb{F}/\mathbb{J}$ and $\pi_1(X+\mathbb{J}) = 0$. Then $X+\mathbb{I} \in \mathbb{A} \subseteq \mathbb{F}/\mathbb{I}$ and $\pi_1'(X+\mathbb{I}) = 0$. Then by the same argument as in the proof of Lemma 4.4 we obtain that $X+\mathbb{I}$ is in the ideal generated by $\ker(\pi_1'|\mathbb{B})$.

Thus, in order to obtain $X+\mathbb{J} = 0$, we should just show that $\ker(\pi_1'|\mathbb{B}) \subseteq \ker(\pi|\mathbb{B})$. If $\pi_1'((I-P)L) = 0$, then $\pi_1'(P_b(I-P)L)) = 0$ for all $b \in \mathcal{A}$. Consequently $P_b(I-P)L = 0$ since $\pi_1'(P_b) \neq 0$ excludes the only other alternative $P_b(I-P)L = P_b$.

Thus $\pi((I-P)L) = \sum_{b\in\mathcal{A}} \pi(P_b(I-P)L) = 0$ where the sum is converging in the strong operator topology. □

Corollary 4.6. *The statements of Proposition 4.3 or Corollary 4.5 remain valid if one replaces the condition* (Two Paths) *by the weaker condition that the graph associated to the transition matrix s has no terminal circuit.*

Proof. We replace all partial isometries $\widetilde{S}_a \in B(\widetilde{\mathcal{H}})$ for $a \in \mathcal{A}$ by the partial isometries $\widetilde{T}_a := \bigoplus_{k=1}^\infty \widetilde{S}_a \in B(\bigoplus_{k=1}^\infty \widetilde{\mathcal{H}})$ and consider the set $(\widetilde{T}_a)_{a\in\mathcal{A}}$ rather than the set $(\widetilde{S}_a)_{a\in\mathcal{A}}$. This creates no problem since their generated C^*-algebras coincide canonically. The reason for this change is that we can now be sure that each range projection $\widetilde{P}_a := \widetilde{T}_a\widetilde{T}_a^*$ projects onto a *infinite* dimensional Hilbert space.

We go over from $(\widehat{S}_a)_{a\in\mathcal{A}}$ to $(\widehat{T}_a)_{a\in\mathcal{A}}$ in the same vein.

To get the uniqueness statement it is enough to prove that for each finite set $\mathcal{B} \subseteq \mathcal{A}$ we have a canonical $*$-homomorphism (and contraction or isometry) $\sigma_\mathcal{B} : C^*((\widetilde{T}_a)_{a\in\mathcal{B}}) \to C^*((\widehat{T}_a)_{a\in\mathcal{B}})$ such that $\sigma_\mathcal{B}(\widetilde{T}_a) = \widehat{T}_a$ for all $a \in \mathcal{B}$.

For this we replace for all $a \in \mathcal{A}\backslash\mathcal{B}$ those partial isometries $\widetilde{T}_a$, which have the property that the support projection $\widetilde{Q}_a$ is the range projection $\widetilde{P}_b$ for some $b \in \mathcal{A}$, i.e. $\widetilde{Q}_a = \widetilde{P}_b$, by a partial isometry $\widetilde{U}_a$ with the same range projection $\widetilde{U}_a\widetilde{U}_a^* = \widetilde{P}_a$ but with the larger support projection $\widetilde{U}_a^*\widetilde{U}_a = \widetilde{P}_a + \widetilde{P}_b$. Thereby notice that necessarily $a \neq b$ since $a = b$ would obviously yield a terminal circuit in s.

All others $\widetilde{U}_a$ which have not been defined so far, we put $\widetilde{U}_a := \widetilde{T}_a$. In particular we have $\widetilde{U}_a = \widetilde{T}_a$ for all $a \in \mathcal{B}$.

Completely analogously we replace the set $(\widehat{T}_a)_{a\in\mathcal{A}}$ by a set $(\widehat{U}_a)_{a\in\mathcal{A}}$, i.e. $\widehat{T}_a$ is replaced when $a \notin \mathcal{B}$ and the row $s(a,.)$ of the transition matrix contains exactly one 1. Thereby we must be careful when $\widehat{T}_a = 0$. But we are consistent, for in this case $\widehat{P}_a = \widehat{P}_b = 0$ and $\widehat{U}_a$ with support $\widehat{P}_a + \widehat{P}_b$ is simply the zero operator too, and therefore defined.

We have achieved that both sets of partial isometries $(\widetilde{U}_a)_{a\in\mathcal{A}}$ and $(\widehat{U}_a)_{a\in\mathcal{A}}$ correspond to a common transition matrix t which satisfies the condition (Two Paths). Indeed if $a \in \mathcal{A}\backslash\mathcal{B}$, then we always find different $b, c \in \mathcal{A}$ such that $S_aS_b \neq 0$ and $S_aS_c \neq 0$. If $a \in \mathcal{B}$ then we either find an efferent path starting at a and ending in $\mathcal{A}\backslash\mathcal{B}$ and we have finished, or we stay in $\mathcal{B}$ and the no terminal circuit property ensures two different paths starting at a.

By Proposition 4.3 we obtain a $*$-homomorphism $\sigma : C^*((\widetilde{U}_a)_{a\in\mathcal{A}}) \to C^*((\widehat{U}_a)_{a\in\mathcal{A}})$ such that $\sigma(\widetilde{U}_a) = \widehat{U}_a$ for all $a \in \mathcal{A}$. Therefore restricting σ to $C^*((\widetilde{U}_a)_{a\in\mathcal{B}}) = C^*((\widetilde{T}_a)_{a\in\mathcal{B}})$ yields the desired $*$-homomorphism $\sigma_{\mathcal{B}}$ and we have come to an end. □

5. Rank-Two Examples Coming from a Shift Space

In this section we introduce and prove the uniqueness of a class of rank two Cuntz-Krieger algebras which are formally generated by two rank one Cuntz-Krieger algebras which interact in a way which we shall call "permutation rules". Here, each of the rank one Cuntz-Krieger algebras has infinitely many generators.

The algebras are motivated by representations on a Hilbert space associated with shifts of finite type in dimension two. The shift spaces are somewhat modified since we allow finitely many local failures with respect to a finite set of allowed words. This avoids some difficulties, or at least makes things easier.

However, this restriction can be circumvented by arbitrarily choosing the transition matrices $s^{(1)}$ and $s^{(2)}$ associated with the rank one algebras. We will allow any transition matrices, but in these cases we are lacking evident representations on a Hilbert space. Otherwise the transition matrices are determined by the shifts and we are provided with representations.

Let Ω be a *finite* alphabet, $\Omega^{\mathbb{N}^2}$ be the one-sided full shift in dimension two, and $s : \Omega^4 \to \{0,1\}$ be a test function whether a square with four entries is allowed. With the test function s we can associate a shift of finite type. However, we will consider a modified space where finitely many "failures" are allowed. More precisely let

$$J := \Big\{ x \in \Omega^{\mathbb{N}^2} \Big| s\Big(x(n,m), x(n+1,m), x(n,m+1), x(n+1,m+1)\Big) = 1$$

$$\text{for all but finitely many pairs } (n,m) \in \mathbb{N}^2 \Big\}.$$

Let $\mathcal{A}^{(i)} \subseteq \Omega^{\mathbb{N}}$ $(i = 1,2)$ be two subsets of the one-sided one-dimensional full shift such that they are stable with respect to shifting, i.e. if $a = (a_1, a_2, a_3, ...) \in \mathcal{A}^{(i)}$ then suppose that both $(a_2, a_3, a_4, ...) \in \mathcal{A}^{(i)}$ and $(a_0, a_1, a_2, ...) \in \mathcal{A}^{(i)}$ for all $a_0 \in \Omega$. We shall denote the shift operation by

$$\sigma : \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}} : \sigma(a_1 a_2 a_3 ...) = a_2 a_3 a_4 ... \qquad (a_i \in \Omega).$$

With these two alphabets $\mathcal{A}^{(i)}$ two sets of partial isometries acting on $\ell^2(J)$ are associated as follows.

Let $a = (a_1, a_2, a_3, ...) \in \mathcal{A}^{(1)}$ $(a_j \in \Omega)$, $x \in J$ and $y := \begin{pmatrix} x \\ a_1 a_2 a_3 ... \end{pmatrix} \in \Omega^{\mathbb{N}^2}$. (The x is shifted one step upwards and the blank line is filled with $a_1 a_2 a_3 ....$) Then the partial isometry $S_a \in B(\ell^2(J))$ is defined by $S_a(\delta_x) = \delta_y$ if $y \in J$ and $S_a(\delta_x) = 0$ otherwise.

Similarly we define partial isometries T_b for $b = (b_1, b_2, b_3, ...) \in \mathcal{A}^{(2)}$ $(b_j \in \Omega)$. Let $y := \begin{pmatrix} . & \\ b_2 & \\ b_1 & x \end{pmatrix}$. Then we put $T_b(\delta_x) = y$ for $y \in J$ and $T(\delta_x) = 0$ otherwise.

We shall now leave the representation on a Hilbert space and pass to the free algebra. The above representation was the motivation and what follows is the setting for the sequel.

Let $\mathcal{A}^{(i)} \subseteq \Omega^{\mathbb{N}}$ $(i = 1,2)$ be two subsets of the one-sided full shift and suppose they are stable under shifting, i.e. $\sigma^{\pm 1}(\mathcal{A}^{(i)}) \subseteq \mathcal{A}^{(i)}$. More precisely this means that if $a = (a_1 a_2 ...) \in \mathcal{A}^{(i)}$ then both $\sigma(a) = (a_2 a_3 ...)$ and $(x a_1 a_2 ...)$ are in $\mathcal{A}^{(i)}$ for all $x \in \Omega$. Let $\mathcal{A} = \mathcal{A}^{(1)} \uplus \mathcal{A}^{(2)}$ be the disjoint union which shall form our alphabet. Then $\mathbb{F}$ is the non-unital free $*$-algebra generated by $\mathcal{A}$. However, we use other notations and think $\mathbb{F}$ is generated by $\{ S_a^{(i)} \mid 1 \le i \le 2, a \in \mathcal{A}^{(i)} \} \cong \mathcal{A}$. Each alphabet $\mathcal{A}^{(i)}$ will correspond to a Cuntz-Krieger algebra of rank 1.

Let $\mathbb{I}$ be the smallest ideal in $\mathbb{F}$ such that the following properties hold in $\mathbb{F}/\mathbb{I}$.

All letters are partial isometries with range projections $P_a^{(i)} := S_a^{(i)} S_a^{(i)^*}$ and support projections $Q_a^{(i)} := S_a^{(i)^*} S_a^{(i)}$. Assume maps ("transition matrices") $s^{(i)} : \mathcal{A}^{(i)} \times \mathcal{A}^{(i)} \to \{0,1\}$ $(i = 1,2)$ such that for all $a, b \in \mathcal{A}^{(i)}$,

$$P_a^{(i)} P_b^{(i)} = \delta_{a,b} P_a^{(i)}, \quad P_a^{(i)} Q_b^{(i)} = s_{a,b}^{(i)} P_a^{(i)}.$$

Moreover assume $Q_a^{(i)}$ and $Q_b^{(i)}$ commute for all $a, b \in \mathcal{A}^{(i)}$. In the sequel we shall also often write $S_a := S_a^{(1)}$ and $T_b := S_b^{(2)}$ $(a \in \mathcal{A}^{(1)}, b \in \mathcal{A}^{(2)})$. Let the interaction between the rank one Cuntz-Krieger algebras be regulated by $((a_1 a_2 a_3 ...) \in \mathcal{A}^{(1)}, (b_1 b_2 b_3 ...) \in \mathcal{A}^{(2)})$

$$T_{(b_1 b_2 b_3 ...)} S_{(a_1 a_2 a_3 ...)} = S_{(b_1 a_1 a_2 a_3 ...)} T_{(b_2 b_3 b_4 ...)}$$

$$T^*_{(b_1 b_2 b_3 ...)} S_{(a_1 a_2 a_3 ...)} = \begin{cases} 0 & \text{if } b_1 \neq a_1 \\ S_{(a_2 a_3 a_4 ...)} T^*_{(b_2 b_3 b_4 ...)} & \text{if } b_1 = a_1. \end{cases}$$

We sometimes refer to the above interactions between T_b and S_a as the *permutation rules*. Observe that by symmetry they hold just so if we exchange the letters T and S.

Property (A) is satisfied for the product of two diagonals

$$H := \{ \lambda \in \mathbb{T}^{\mathcal{A}^{(1)} \uplus \mathcal{A}^{(2)}} \mid \forall i = 1,2 : \forall a, b \in \mathcal{A}^{(i)} : \lambda_a^{(i)} = \lambda_b^{(i)} \} \cong \mathbb{T} \times \mathbb{T}.$$

The balance function defined in Lemma 3.1 maps into $\widehat{H} \cong \mathbb{Z}^2$ and using this group isomorphism we get $\mathrm{bal}(S_a) = (1,0)$ and $\mathrm{bal}(T_b) = (0,1)$ for all $a \in \mathcal{A}^{(1)}, b \in \mathcal{A}^{(2)}$. Hence a nonzero word X is zero-balanced if and only if it contains as many letters S_a as adjoint letters S_b^* and as many letters T_c as adjoint letters T_d^*.

We shall use the following notations. $W_{\mathcal{B}}$ denotes the words in the letters $\mathcal{B} \subseteq \mathcal{A}^{(i)}$ (the empty word is allowed), and for $\alpha = (a_1, a_2, ..., a_n) \in W_{\mathcal{A}^{(i)}}$ we use $\ell(\alpha) := n$ and write

$$S_\alpha^{(i)} = S_{a_1}^{(i)} S_{a_2}^{(i)} ... S_{a_n}^{(i)},$$

$$Q_\alpha^{(i)} = Q_{a_1}^{(i)} Q_{a_2}^{(i)} ... Q_{a_n}^{(i)},$$

and $S_\emptyset^{(i)} = I$, where as in the previous section I denotes an imaginary identity which will occur only in expressions where it is not really needed (e.g. $S_\emptyset^{(i)} X := X$). Once again we should not confuse $Q_\alpha^{(i)}$ with the support projection of $S_\alpha^{(i)} \neq 0$, which is $Q_{a_n}^{(i)}$.

The set $W \subseteq \mathbb{F}/\mathbb{I}$ denotes all words, whereas $W^{(1)} \subseteq W$ denotes the words in the letters S_a, S_b^* only and $W^{(2)} \subseteq W$ the words in the letters T_c, T_d^*. The empty words are not allowed (as in $\mathbb{F}/\mathbb{I}$ too).

One can check that the relations generating $\mathbb{I}$ hold for the concrete representation on $\ell^2(J)$ we have described above. Thereby one puts $s_{a,b}^{(i)} := 0$ iff $Q_a^{(i)} P_b^{(i)} = 0$ in $B(\ell^2(J))$, what depends on the test function s. Note, however, that in our setting we permit arbitrary $s^{(i)}$.

In the sequel, if nothing else is mentioned, the calculations are done in $\mathbb{F}/\mathbb{I}$.

Lemma 5.1. (I) *For all $a \in \mathcal{A}^{(1)}, b \in \mathcal{A}^{(2)}$ we have ($Q_a := Q_a^{(1)}$)*

$$Q_a T_b = T_b Q_{\sigma(a)} \qquad Q_{\sigma(a)} T_b^* = T_b^* Q_a.$$

(II) *For all zero-balanced words $X \in W_0^{(2)}$ we have $Q_a X = X Q_a$.*

(III) *Each nonzero word $X \in W$ has a representation $X = A$ or $X = B$ or $X = AB$, where $A \in W^{(1)}$ and $B \in W^{(2)}$.*

(IV) *The words W are partial isometries. Their range and support projections,*

$$\{ S_\alpha Q_x^{(1)} S_\alpha^* \ T_\beta Q_y^{(2)} T_\beta^* \mid \alpha, x \in W_{\mathcal{A}^{(1)}}, \beta, y \in W_{\mathcal{A}^{(2)}}, \ell(x) + \ell(y) \geq 1 \} \cup \{0\},$$

generate a commutative $*$-algebra.

Proof. (I) The right equation is the adjoint one of the left one which is

$$Q_a T_b = S_a^* S_a T_b = S_a^* T_{(a_1 b_1 b_2 b_3 ...)} S_{\sigma(a)}^* = T_b S_{\sigma(a)}^* S_{\sigma(a)} = T_b Q_{\sigma(a)}.$$

(II) Let $X \in W^{(2)}$ be a zero-balanced word in the letters T. Then X contains as many letters T_b as adjoint letters T_c^*. Hence $XQ_a = Q_a X$ easily follows from the identities of Lemma 5.1.(I).

(III) This is clear as due to the permutation rules one can always move all letters S to the left and all letters T to the right in a word X. (And vice versa, beside.)

(IV) We leave it to the reader to calculate the commutation

$$S_\alpha Q_x^{(1)} S_\alpha^* \ T_\beta Q_y^{(2)} T_\beta^* = T_\beta Q_y^{(2)} T_\beta^* \ S_\alpha Q_x^{(1)} S_\alpha^*.$$

Hence with Lemma 4.1.(II) we get for a representation $X = AB$ like in Lemma 5.1.(III)

$$AB(AB)^* AB = A(BB^*)(A^*A)B = AA^*ABB^*B = AB.$$

Further one can compute that the range and support projections have the above explicit form. We suggest to prove this by induction on the length of a word X. □

Lemma 5.2. (I) *Let $i = 1,2$ and $N \geq 1$, $S_{\mathcal{B}}^{(i)} := \{ S_b^{(i)} \mid b \in \mathcal{B} \}$ and*

$$O_N^{(i)} := \{ S_\alpha^{(i)} Q_\gamma^{(i)} S_\beta^{(i)*} \mid \alpha, \beta, \gamma \in W_{\mathcal{A}^{(i)}}, 0 \leq \ell(\alpha) = \ell(\beta) \leq N, 1 \leq \ell(\gamma) \}.$$

Then for a finite set $\mathcal{B} \subseteq \mathcal{A}^{(1)}$ the finite set $\mathcal{C} := \sigma^{-N}\sigma^N(\mathcal{B}) \supseteq \mathcal{B}$ satisfies

$$S_{\mathcal{C}}^{(1)} O_N^{(2)} \backslash \{0\} = O_N^{(2)} S_{\mathcal{C}}^{(1)} \backslash \{0\}. \tag{11}$$

(II) *$\mathbb{A}$ is the inductively ordered union of finite dimensional $*$-algebras.*

Proof. (I) Consider the finite set

$$\mathcal{C} := \sigma^{-N}\sigma^N(\mathcal{B}) = \tag{12}$$

$$\{ (a_1 a_2 a_3 ... a_n b_{n+1} b_{n+2} ...) \in \mathcal{A}^{(1)} \mid a_i \in \Omega \text{ and } (b_1 b_2 b_3 ...) \in \mathcal{B} \}.$$

Let $a \in \mathcal{C}$, $\ell(\alpha) = \ell(\beta) \leq N$. Then by applying the permutation rules and Lemma 5.1.(I), we find some $a' \in \mathcal{C}$ and α', β', γ' with $\ell(\alpha) = \ell(\alpha') = \ell(\beta')$ such that, if nonzero,

$$S_a \ T_\alpha Q_\gamma T_\beta^* = T_{\alpha'} Q_{\gamma'} T_{\beta'}^* \ S_{a'}.$$

Vice versa to any $a' \in \mathcal{C}$ we find $a \in \mathcal{C}$ and we have the result.

(II) According to Lemma 5.1.(III) and Lemma 4.1.(III) finitely many words in $\mathbb{A}$ are always contained in $\mathcal{F}_{\mathcal{B}_1,N}^{(1)} + \mathcal{F}_{\mathcal{B}_2,N}^{(2)} + \mathcal{F}_{\mathcal{B}_1,N}^{(1)} \mathcal{F}_{\mathcal{B}_2,N}^{(2)}$ for certain finite sets $\mathcal{B}_i \subseteq \mathcal{A}^{(i)}$ $(i = 1,2)$ and integer $N \geq 1$.

Like in (I), we choose $\mathcal{C}_i := \sigma^{-N}\sigma^N(\mathcal{B}_i)$ $(i = 1,2)$, see (12). Then by a similar computation as was done in (I), for words $A_i \in \mathcal{F}_{\mathcal{C}_i,N}^{(i)}$ $(i = 1,2)$ there exist words $B_i \in \mathcal{F}_{\mathcal{C}_i,N}^{(i)}$ such that, if nonzero, $A_1 A_2 = B_2 B_1$.

It is thus evident that $\mathcal{F}_{\mathcal{C}_1,N}^{(1)} + \mathcal{F}_{\mathcal{C}_2,N}^{(2)} + \mathcal{F}_{\mathcal{C}_1,N}^{(1)} \mathcal{F}_{\mathcal{C}_2,N}^{(2)}$ is a finite-dimensional $*$-algebra. □

Lemma 5.3. *$\mathbb{A}$ is the inductively ordered union of finite-dimensional C^*-algebras $\mathcal{F}_{\mathcal{C}^{(1)},\mathcal{C}^{(2)},N}$ for integers $N \geq 1$ and certain suitable finite subsets $\mathcal{C}^{(i)} \subseteq \mathcal{A}^{(i)}$ $(i = 1, 2)$.*

Each $\mathcal{F}_{\mathcal{C}^{(1)},\mathcal{C}^{(2)},N}$ is the linear span of the entirety of the following three types of elements (the lines (13)- (15))

$$S_{\alpha_1} V^{(1)} (I - P^{(2)})(I - \overline{Q}^{(2)}) S^*_{\beta_1}, \tag{13}$$

$$(I - P^{(1)})(I - \overline{Q}^{(1)}) T_{\alpha_2} V^{(2)} T^*_{\beta_2}, \tag{14}$$

$$S_{\alpha_1} V^{(1)} T_{\alpha_2} V^{(2)} T^*_{\beta_2} S^*_{\beta_1}, \tag{15}$$

where

$$\alpha_i, \beta_i \in W_{\mathcal{C}^{(i)}}, \quad 0 \leq \ell(\alpha_i) = \ell(\beta_i) \leq N, \qquad L^{(i)} \in \mathbb{Q}^{(i)}(\mathcal{C}^{(i)}),$$

$$V^{(i)} = \begin{cases} (I - P^{(i)}) L^{(i)} & \text{if } \ell(\alpha_i) < N \\ L^{(i)} & \text{if } \ell(\alpha_i) = N, \end{cases}$$

$$\mathbb{Q}^{(i)}(\mathcal{C}^{(i)}) = \Big\{ \prod_{a \in \mathcal{C}^{(i)}} X_a \mid X_a = Q^{(i)}_a \text{ or } X_a = I - Q^{(i)}_a, \exists c : X_c = Q^{(i)}_c \Big\},$$

$$P^{(i)} = \sum_{a \in \mathcal{C}^{(i)}} S^{(i)}_a S^{(i)*}_a, \qquad \overline{Q}^{(i)} = \bigvee_{a \in \mathcal{C}^{(i)}} Q^{(i)}_a.$$

If we fix one type of the above elements (13)*,* (14)*,* (15)*, fix $V^{(1)}$ and $V^{(2)}$, and fix the lengths $\ell(\alpha_i)$ and just vary $\alpha_i, \beta_i \in W_{\mathcal{C}^{(i)}}$, then they form a simple matrix algebra $\mathcal{M}$. $\mathcal{F}_{\mathcal{C}^{(1)},\mathcal{C}^{(2)},N}$ is then the finite direct sum of all these simple matrix algebras $\mathcal{M}$.*

Moreover, $(I - P^{(1)})(I - \overline{Q}^{(1)})$ commutes with $T_{\alpha_2} V^{(2)} T^*_{\beta_2}$ in (14)*, and $V^{(1)}$ with $T_{\alpha_2} V^{(2)} T^*_{\beta_2}$ in* (15)*.*

Proof. Step 1: Consider finitely many zero-balanced words $X_1, ..., X_n \in W_0$. We shall show that they are in such a finite dimensional C^*-algebra A as described. For $N \geq 1$ and $\mathcal{B}^{(i)} \subseteq \mathcal{A}^{(i)}$ $(i = 1, 2)$ we set

$$O^{(i)}_N(\mathcal{B}^{(i)}) := \{ S^{(i)}_\alpha Q^{(i)}_\gamma S^{(i)*}_\beta \mid \alpha, \beta, \gamma \in W_{\mathcal{B}^{(i)}}, \ell(\alpha) = \ell(\beta) \leq N \}.$$

One clearly finds finite sets of letters $\mathcal{B}^{(i)} \subseteq \mathcal{A}^{(i)}$ and $N \geq 1$ such that the words X_k are contained in the finite set of words $O^{(1)}_N(\mathcal{B}^{(1)}) \cup O^{(2)}_N(\mathcal{B}^{(2)}) \cup O^{(1)}_N(\mathcal{B}^{(1)}) O^{(2)}_N(\mathcal{B}^{(2)})$, see Lemma 4.1.(I) and Lemma 5.1.(III). We then choose the finite set $\mathcal{C}^{(1)} := \sigma^{-N} \sigma^N(\mathcal{B}^{(1)}) \supseteq \mathcal{B}^{(1)}$ such that (11) holds.

Step 2a: Put $P^{(1)} = \sum_{a \in \mathcal{C}^{(1)}} S_a S^*_a$. Applying Lemma 4.1, every $Y^{(1)} \in O^{(1)}_N(\mathcal{C}^{(1)})$ has a representation

$$Y^{(1)} = S_\alpha (I - P^{(1)}) L^{(1)} S^*_\beta \quad \text{if} \quad \ell(\alpha) < N$$
$$\text{and} \quad Y^{(1)} = S_\alpha L^{(1)} S^*_\beta \quad \text{if} \quad \ell(\alpha) = N$$

for some $L^{(1)} \in \mathbb{Q}^{(1)}(\mathcal{C}^{(1)})$. Let $V^{(1)}$ denote either the expression $(I - P^{(1)}) L^{(1)}$ or $L^{(1)}$, such that we can write without definition by cases $Y^{(1)} = S_\alpha V^{(1)} S^*_\beta$.

Step 2b: Hence we can bring the nonzero words $Y \in O_N^{(1)}(\mathcal{C}^{(1)})O_N^{(2)}(\mathcal{B}^{(2)})$ to the form

$$Y = S_\alpha V^{(1)} Y^{(2)} S_\beta^* \tag{16}$$

for certain $Y^{(2)} \in O_N^{(2)}$ (see Lemma 5.2.(I)) and $\alpha, \beta \in W_{\mathcal{C}^{(1)}}$, $\ell(\alpha) = \ell(\beta) \le N$.

Step 2c: Next we focus on the $Y^{(2)} \in O_N^{(2)}(\mathcal{B}^{(2)})$. Let $\overline{Q}^{(1)}$ be the projection $\bigvee_{a \in \mathcal{C}^{(1)}} Q_a^{(1)}$, which is obviously an algebraic expression in the support projections $Q_a^{(1)}$ (for example $Q_a \vee Q_b = Q_a + Q_b - Q_a Q_b$). Then we split

$$Y^{(2)} = (I - P^{(1)})(I - \overline{Q}^{(1)})Y^{(2)} + (P^{(1)} - P^{(1)}\overline{Q}^{(1)} + \overline{Q}^{(1)})Y^{(2)}. \tag{17}$$

The left summand has its final form (14) and the words occurring in the right summand are handled as in step 2b to get the form (16) for them.

Step 3: So far we note that the finitely many manipulations in steps 2b-c have produced finitely many new words $Y^{(2)} \in O_N^{(2)}$ in (16). We now choose a finite set $\mathcal{C}^{(2)} \supseteq \mathcal{B}^{(2)}$ which contains all letters that may occur in these $Y^{(2)}$.

We put $\overline{Q}^{(2)} = \bigvee_{a \in \mathcal{C}^{(2)}} Q_a^{(2)}$ and represent the words $Y^{(1)} = S_\alpha V^{(1)} S_\beta^* \in O_N^{(1)}(\mathcal{C}^{(1)})$ as

$$Y^{(1)} = S_\alpha V^{(1)} (I - P^{(2)})(I - \overline{Q}^{(2)}) S_\beta^* + S_\alpha V^{(1)} (P^{(2)} - P^{(2)}\overline{Q}^{(2)} + \overline{Q}^{(2)}) S_\beta^*.$$

The first summand has its final form (13) here and the second summand obviously consists of words of the form represented in (16).

Like in step 2a we now apply Lemma 4.1 to $Y^{(2)}$ in (16) and the first summand of (17) to get the final representations (15) and (14).

Step 4: We have seen that the words X_k are spanned by the elements (13), (14), (15). It is now straight forward to prove that they are the direct sum of matrix algebras $\mathcal{M}$ as described. We just should note that

$$(I - P^{(1)})(I - \overline{Q}^{(1)}) \;\; T_{\alpha_2} V^{(2)} T_{\beta_2}^* = T_{\alpha_2} V^{(2)} T_{\beta_2}^* \;\; (I - P^{(1)})(I - \overline{Q}^{(1)})$$

in (14) and

$$V^{(1)} \;\; T_{\alpha_2} V^{(2)} T_{\beta_2}^* = T_{\alpha_2} V^{(2)} T_{\beta_2}^* \;\; V^{(1)}$$

in (15). Indeed, we will demonstrate the last equation and abbreviate $X = T_{\alpha_2} V^{(2)} T_{\beta_2}^*$. We have $L^{(1)} X = X L^{(1)}$ for all $L^{(1)} \in \mathbb{Q}^{(1)}(\mathcal{C}^{(1)})$, see Lemma 5.1.(II), and hence also $(I - P^{(1)}) L^{(1)} X = X (I - P^{(1)}) L^{(1)}$, since $(I - P^{(1)}) X P^{(1)} = 0$ due to our special choice of $\mathcal{C}^{(1)}$. □

We now assume any representation $\pi : \mathbb{F}/\mathbb{I} \to B(\mathcal{H})$ such that $P^{(i)} = \sum_{b \in \mathcal{A}^{(i)}} \pi(P_b^{(i)})$ are units for $\pi(\mathbb{F}/\mathbb{I})$, where $i = 1, 2$. Using the same argument from the previous section we have representations $\pi(Q_a^{(i)}) = \sum s_{a,b}^{(i)} P_b^{(i)}$ for $i = 1, 2$ and $a \in \mathcal{A}^{(i)}$, confer (9) and the discussion there.

Thereby notice that such a representation π was constructed at the beginning of this section for certain but not arbitrary $s^{(i)}$, provided that $\mathcal{A}^{(i)}$ is chosen large enough, $\mathcal{A}^{(i)} := \Omega^{\mathbb{N}}$ say, such that $P^{(i)}$ is a unit.

As in the previous section, we are now passing from the ideal $\mathbb{I}$ to the larger ideal $\mathbb{J} \subseteq \mathbb{F}$ generated by $\ker(\pi|\mathbb{A})$. More precisely we define $\mathbb{J}$ as in Lemma 3.8 such that $\tilde{\pi}$, derived from π, becomes injective on $\mathbb{A}$. (All notations, $\mathbb{A}$ etc., are now related to the ideal $\mathbb{J}$.)

The representations in the last lemmas then remain valid and simple matrix algebras $\mathcal{M}$ can only either vanish completely or survive unaltered the transition $\mathbb{F}/\mathbb{I} \to \mathbb{F}/\mathbb{J}$. Therefore property (B) also holds for the $*$-algebra $\mathbb{F}/\mathbb{J}$ by Lemma 5.3. Recall also the analogous discussion in the previous section.

Notice, however, that it is already clear from Lemma 5.2.(II) that $\mathbb{A} \subseteq \mathbb{F}/\mathbb{J}$ is the inductively ordered union of finite-dimensional C^*-algebras since we get a C^*-norm from the image of $\tilde{\pi}$. Nevertheless, we have computed the structure of $\mathbb{A} \subseteq \mathbb{F}/\mathbb{I}$ explicitly in Lemma 5.3, and the main reason is to obtain the following Corollary 5.4. (Informally, we think this is a somewhat unsatisfying situation, since, anticipating the corollary, it is so easy to see that $\tilde{\pi}(\mathbb{P}_2) \precsim \tilde{\pi}(\mathbb{P})$ in $B(\mathcal{H})$, but we need $\precsim$ in $\mathbb{A} \cong \tilde{\pi}(\mathbb{A})$.) However, it can also make things easier if one knows the structure of the finite dimensional C^*-algebras explicitly. We shall use it in Proposition 5.7, and one could also use it to get an analogous result as in Lemma 4.4.

Corollary 5.4. *In $\mathbb{F}/\mathbb{J}$ for all $Q \in \mathbb{P}$ there exists $P \in \mathbb{P}_2$ such that $P \precsim Q$ in $\mathbb{A}$, where*

$$\mathbb{P}_2 := \{\, S_\alpha S_\alpha^* T_\beta T_\beta^* \mid \alpha \in W_{\mathcal{A}^{(1)}}, \beta \in W_{\mathcal{A}^{(2)}}, \ell(\alpha) + \ell(\beta) \geq 1 \,\} \backslash \{0\}.$$

Proof. In the following proof we think that we argue in the algebra $\tilde{\pi}(\mathbb{A})$, where we can use the strong operator topology, rather than in $\mathbb{A}$. This evokes no problems since the two $*$-algebras $\tilde{\pi}(\mathbb{A})$ and $\mathbb{A}$ are isomorphic by the injectivity of $\tilde{\pi}$ on $\mathbb{A}$.

Let $Q \in \mathbb{P}$. Then there exists a smaller projection $Q' \precsim Q$ which sits on the diagonal of a matrix algebra computed in Lemma 5.3. Hence Q' takes the form of (13), (14) or (15) (where $\alpha_1 = \beta_1, \alpha_2 = \beta_2$). We can replace the projections $V^{(i)}$ and $(I - P^{(j)})(I - \overline{Q}^{(j)})$ there where they occur by infinite sums of orthogonal range projections $P_a^{(i)}$ and $P_b^{(j)}$ respectively. Since not all summands vanish, there are letters $a \in \mathcal{A}^{(1)}, b \in \mathcal{A}^{(2)}$ such that, depending on the case (13), (14) or (15) in this order, the following projection P is nonzero,

$$P := S_{\alpha_1} P_a^{(1)} P_b^{(2)} S_{\alpha_1}^*, \text{ or}$$
$$P := P_a^{(1)} T_{\alpha_2} P_b^{(2)} T_{\alpha_2}^*, \text{ or}$$
$$P := S_{\alpha_1} P_a^{(1)} T_{\alpha_2} P_b^{(2)} T_{\alpha_2}^* S_{\alpha_1}^*.$$

The corollary is thus proved by checking $P \leq Q' \precsim Q$ and $P \in \mathbb{P}_2$. □

For the criterion below we introduce the following notations. For $A, B \in W_{\mathcal{A}^{(i)}}$ let $AB \in W_{\mathcal{A}^{(i)}}$ be the concatenated word. Further we denote $P_A^{(i)} := S_A^{(i)} S_A^{(i)*}$ and $\sigma(A) := (\sigma(a_1), \sigma(a_2), ..., \sigma(a_N)) \in W_{\mathcal{A}^{(i)}}$ for all $i = 1, 2$ and $A = (a_1, ..., a_N)$ with $a_k \in \mathcal{A}^{(i)}$.

(Rank2) For all $i, j \in \{1,2\}$, $i \neq j$, all integers $n, m \geq 0$, all $A \in W_{\mathcal{A}^{(i)}}$ and all $P^{(i)}_{A_k} P^{(j)}_{B_k} \neq 0$ for $k = 1,2,3$ we have the following. There exist $C_1, C_2, C_3 \in W_{\mathcal{A}^{(i)}}$ such that $P^{(i)}_{A_k C_k} P^{(j)}_{B_k} \neq 0$ for $k = 1,2,3$ and

$$\big(S^{(i)}_{\sigma^m(C_1)}\big)^* S^{(i)}_A S^{(i)}_{\sigma^n(C_1)} = 0, \qquad \big(S^{(i)}_{\sigma^m(C_2)}\big)^* S^{(i)}_{\sigma^n(C_3)} = 0.$$

Lemma 5.5. *Property* (C) *holds in* $\mathbb{F}/\mathbb{J}$ *if criterion* (Rank2) *is satisfied in* $\mathbb{F}/\mathbb{J}$.

Proof. We shall prove criterion (C*) of Lemma 2.4. Thereby we choose for $\mathbb{P}_2$ the definition of Corollary 5.4.

Let $X = AB \in W$ be a nonzero-balanced word where $A \in W^{(1)}$ and $B \in W^{(2)}$ using the representation of Lemma 5.1.(III). Without loss of generality we assume that $\mathrm{bal}(A_1) \neq 0$. Using the permutation rules we can choose a representation

$$X = T_{\alpha_2} S_{\alpha_1} Q^{(1)}_{\gamma_1} S^*_{\beta_1} Q^{(2)}_{\gamma_2} T^*_{\beta_2},$$

where $\ell(\alpha_1) \neq \ell(\beta_1)$. Even more we may suppose that $\ell(\alpha_1) > \ell(\beta_1)$, otherwise we consider X^*.

Let $E = P^{(2)}_Y P^{(1)}_B \in \mathbb{P}_2$ where we assume w.l.o.g. that $\ell(B) \geq \ell(\alpha_1) > \ell(\beta_1)$, otherwise we replace B by a longer word B' and replace E by $E' = P^{(2)}_Y P^{(1)}_{B'} \leq E$. A nonzero choice of E' is possible by property (Rank2).

Let $F = P^{(2)}_Y P^{(1)}_{BC} \leq E$ be nonzero for some $C \in W_{\mathcal{A}^{(1)}}$ by property (Rank2). We shall specify C more precisely below. If $FXF = 0$ then we have finished the task, otherwise we compute ($m := \ell(\alpha_2)$, $n := \ell(\beta_2)$)

$$\begin{aligned} FXF &= P^{(2)}_Y T_{\alpha_2} P^{(1)}_{\sigma^m(BC)} \; S_{\alpha_1} Q^{(1)}_{\gamma_1} S^*_{\beta_1} \; P^{(1)}_{\sigma^n(BC)} Q^{(2)}_{\gamma_2} T^*_{\beta_2} P^{(2)}_Y \\ &= P^{(2)}_Y T_{\alpha_2} S_{\sigma^m(BC)} S^*_{\sigma^m(C)} \\ &\quad S^*_{\sigma^m(B)} S_{\alpha_1} Q^{(1)}_{\gamma_1} S^*_{\beta_1} S_{\sigma^n(B)} \qquad (18) \\ &\quad S_{\sigma^n(C)} S^*_{\sigma^n(BC)} Q^{(2)}_{\gamma_2} T^*_{\beta_2} P^{(2)}_Y. \end{aligned}$$

Now (18) is S_A for some $A \in W_{\mathcal{A}^{(1)}}$ and we can choose C such that $S^*_{\sigma^m(C)} S_A S_{\sigma^n(C)} = 0$ by property (Rank2). Hence $FXF = 0$ with $0 \neq F \leq E$.

Similarly we handle the second yet open case where we have given two projections $E_1 = P^{(2)}_Y P^{(1)}_{B_1}, E_2 = P^{(2)}_Y P^{(1)}_{B_2}$ in $\mathbb{P}_2$. The only difference is that we choose the lengths $\ell(B_1), \ell(B_2)$ such that the corresponding expression (18) is $Q^{(1)}_a$ for some letter $a \in \mathcal{A}^{(1)}$. □

Lemma 5.6. *Criterion* (Rank2) *is satisfied in* $\mathbb{F}/\mathbb{J}$ *if* $\mathbb{J}$ *is induced by the representation* π *on the Hilbert space* $\ell^2(J)$ *which we considered at the beginning of this section.*

Proof. To show criterion (Rank2) it is enough to prove it in the image under the map $\tilde{\pi}$ since the criterion is a criterion on the words W, but $\tilde{\pi}$ is faithful on the set of words W.

Indeed if $X \in W$ is a nonzero word and therefore a partial isometry by Lemma 5.1.(IV), then we get $X = XX^*X \neq 0$ and thus $\tilde{\pi}(X^*X) \neq 0$ by the faithfulness of $\tilde{\pi}$ on $\mathbb{A}$.

Now we think that we argue in the image of $\tilde{\pi}$. Let $P_A^{(i)} P_B^{(j)} \neq 0$. Notice that this orthogonal projection projects onto the Hilbert subspace $\ell^2(K) \subseteq \ell^2(J)$, where

$$K = \{\, x \in J \mid \forall 1 \leq k \leq \ell(A) : x_{k,(.)} = A_k \text{ and } \forall 1 \leq k \leq \ell(B) : x_{(.),k} = B_k \,\}$$

for $A = (A_1, A_2, ...) \in W_{\mathcal{A}^{(i)}}$ and $B = (B_1, B_2 ...) \in W_{\mathcal{A}^{(j)}}$.

Then we find a continuation $c \in \mathcal{A}^{(i)}$ such that $P_{Ac}^{(i)} P_B^{(j)} \neq 0$. By modifying $c = (c_1 c_2 c_3 ...)$ on finitely many positions $c_i \in \Omega$ we, in fact, obtain infinitely many letters c such that $P_{Ac}^{(i)} P_B^{(j)} \neq 0$ (recall that we allow finitely many errors in J). With that freedom of choice it is not too difficult to choose $C_1, C_2, C_3 \in W_{\mathcal{A}^{(i)}}$ as required in criterion (Rank2). □

Proposition 5.7. Uniqueness of rank two Cuntz-Krieger algebras

Let $\pi_k : \mathbb{F}/\mathbb{I} \to B(\mathcal{H}_k)$ be two representations ($k = 1, 2$) such that the strong operator sums $\sum_{a \in \mathcal{A}^{(i)}} \pi_k(P_a^{(i)})$ are units for $\pi_k(\mathbb{F}/\mathbb{I})$ for all $i = 1, 2$ and $k = 1, 2$. Suppose that condition (Rank2) *holds in the image $\pi_1(\mathbb{F}/\mathbb{I})$ (or in $\mathbb{F}/\mathbb{J}$) and assume the implication $\pi_1(P_a^{(1)} P_b^{(2)}) = 0 \Rightarrow \pi_2(P_a^{(1)} P_b^{(2)}) = 0$ for all $a \in \mathcal{A}^{(1)}$ and $b \in \mathcal{A}^{(2)}$.*

Then there exists a canonical $*$-homomorphism $\sigma : \overline{\pi_1(\mathbb{F}/\mathbb{I})} \to \overline{\pi_2(\mathbb{F}/\mathbb{I})}$ such that $\sigma\pi_1 = \pi_2$, where the line denotes the norm closure.

Proof. This can be proved in the same way as Proposition 4.3 was proved by applying Theorem 3.3.

Firstly we pass from the ideal $\mathbb{I} \subseteq \mathbb{F}$ to the larger ideal $\mathbb{J} \subseteq \mathbb{F}$ as was done below the proof of Lemma 5.3. The only difference is that we define $\mathbb{J}$ with respect to π_1 rather than π. The canonical map $\tilde{\pi}_1 : \mathbb{F}/\mathbb{J} \to B(\mathcal{H}_1)$, which is injective on $\mathbb{A}$, is associated to π_1.

By Lemma 3.8 property (A) holds for the ideal $\mathbb{J}$. By the representations in Lemma 5.3 (also recall the discussion before Corollary 5.4) we are sure that property (B) holds in the $*$-algebra $\mathbb{F}/\mathbb{J}$.

As discussed in the first part of the proof of Lemma 5.6, the condition (Rank2) holds in $\mathbb{F}/\mathbb{J}$ too if it holds in the image of $\tilde{\pi}_1$ (which is the image of π_1), since this map is faithful on the set of words. More precisely we have $\pi_1(X + \mathbb{I}) = 0 \Leftrightarrow \tilde{\pi}_1(X + \mathbb{J}) = 0 \Leftrightarrow X + \mathbb{J} = 0$ for all words $X \in \mathbb{F}$. Therefore we get property (C) in $\mathbb{F}/\mathbb{J}$ by Lemma 5.5.

The last thing we have to show is that one obtains a quotient map $\tilde{\pi}_2 : \mathbb{F}/\mathbb{J} \to B(\mathcal{H}_2)$ deduced from π_2. For this we must show that $\ker(\pi_1|\mathbb{A}) \subseteq \ker(\pi_2)$ (recall that $\mathbb{J}$ is generated by $\ker(\pi_1|\mathbb{A})$).

It is enough to consider the matrix units we have computed in Lemma 5.3. Consider, for example, the type (15) and let $e := S_{\alpha_1} V^{(1)} T_{\alpha_2} V^{(2)} T_{\beta_2}^* S_{\beta_1}^* \neq 0$. Suppose that $\pi_1(e) = 0$.

Then

$$\begin{aligned} 0 &= \pi_1(T^*_{\alpha_2})\pi_1(S^*_{\alpha_1})\pi_1(e)\pi_1(S_{\beta_1})\pi_1(T_{\beta_2}) \\ &= \pi_1(T^*_{\alpha_2})\pi_1(V^{(1)}T_{\alpha_2}V^{(2)}T^*_{\beta_2})\pi_1(T_{\beta_2}) \\ &= \pi_1(\widetilde{V}^{(1)}V^{(2)}) \\ &= \sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{A}}\lambda_{a,b}\pi_1(P^{(1)}_a P^{(2)}_b) \end{aligned}$$

for certain $\lambda_{a,b} \in \{0,1\}$, only depending on the transition matrices $s^{(1)}$ and $s^{(2)}$, by expanding each of the range projections $Q^{(i)}_a$ as the strong operator sum of the range projections $P^{(i)}_b$ like in (9).

Further we have applied the permutation rules such that $T^*_{\alpha_2}V^{(1)} = \widetilde{V}^{(1)}T^*_{\alpha_2}$ by Lemma 5.1.(I).

Thus $\lambda_{a,b}\pi_1(P^{(1)}_a P^{(2)}_b) = 0$ for all $a, b \in \mathcal{A}$ and by the assumption of the proposition we get

$$0 = \sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{A}}\lambda_{a,b}\pi_2(P^{(1)}_a P^{(2)}_b) = \pi_2(\widetilde{V}^{(1)}V^{(2)}).$$

Consequently we obtain $\pi_2(e) = \pi_2(S_{\alpha_1}T_{\alpha_2})\pi_2(\widetilde{V}^{(1)}V^{(2)})\pi_2(T^*_{\beta_2}S^*_{\beta_1}) = 0$ as desired.

We are now ready to apply Theorem 3.3 to $\mathbb{F}, \mathbb{J}$ and H and we get the proposition. □

We shall close this section with some comments. Like in the previous section, where we discussed the rank one Cuntz-Krieger algebras, it is possible to formulate the above proposition without using strong operator topology. In Lemma 4.4 and Proposition 4.5 we did this for the rank one algebras and we suggest the interested reader to proceed similarly for the rank two algebras.

In [CK], Proposition 2.17, it was shown that two classical Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic if both one-sided shifts of finite type X_A and X_B, associated with the transition matrices A and B, respectively, are topologically conjugated. However, we tend to believe that this may no longer be the case for our rank two algebras. The problem we have in mind turns up by the two contained rank one Cuntz-Krieger algebras with infinitely many generators. We could imagine namely that projections of the form P_a $(a \in \mathcal{A})$ of the Cuntz-Krieger algebra associated with a shift S are only representable as *infinite* sums of projections $\widetilde{P}_a$ $(a \in \widetilde{\mathcal{A}})$ in the Cuntz-Krieger algebra associated with a topologically conjugated shift $\widetilde{S}$. (Consider for example a sliding block code of a full shift.) But such sums are in the strong closure rather than in the norm closure.

Finally let us state an open problem which we find very interesting in the context of higher rank Cuntz-Krieger algebras. Given two uniquely generated C^*-algebras $\mathcal{O}_{\mathbb{F}_i,\mathbb{I}_i,H_i}$ $(i = 1, 2)$, which ideals $\mathbb{I} \subseteq \mathbb{F}_1 \star \mathbb{F}_2$, such that $\mathbb{I}_1 \cup \mathbb{I}_2 \subseteq \mathbb{I}$, permit a uniquely generated C^*-algebra $\mathcal{O}_{\mathbb{F}_1\star\mathbb{F}_2,\mathbb{I},H_1\times H_2}$? Or, in other words, which interactions between two given Cuntz-Krieger type algebras must or can we claim such that their fusion forms a Cuntz-Krieger type algebra, too.

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References

[B] Burgstaller, B., *Some multidimensional Cuntz algebras.* submitted.

[CE] Choi, M.-D., Effros, E., *Nuclear C^*-algebras and the approximation property.* Amer. J. Math. **100**, 61-79 (1978).

[CD] Ceccherini, T., Dopplicher, S., Pinzari, C., Roberts, J.E., *A generalization of the Cuntz algebras and model actions.* J. Funct. Anal. **125**, 416-437 (1994).

[C] Cuntz, J., *Simple C^*-algebras generated by isometries.* Commun. Math. Phys. **57**, 173-185 (1977).

[CK] Cuntz, J., Krieger, W., *A class of C^*-algebras and topological markov chains.* Invent. Math. **56**, 251-268 (1980).

[D] Deaconu, V., *Generalized Cuntz-Krieger algebras.* Proc. Amer. Math. Soc. **124**, 3427-3435 (1996).

[EL] Exel, R., Laca, M., *Cuntz-Krieger algebras for infinite matrices.* J. reine angew. Math. **512**, 119-172 (1999).

[KP1] Kumjian, A., Pask, D., *Higher rank graph C^*-algebras.* New York J. Math. **6**, 1-20 (2000).

[KP2] Kumjian, A., Pask, D., Raeburn, I., *Cuntz-Krieger algebras of directed graphs.* Pacific J. Math. **184**, 161-174 (1998).

[KP3] Kumjian, A., Pask, D., Raeburn, I., Renault, J., *Graphs, groupoids, and Cuntz-Krieger algebras.* J. Funct. Anal. **144**, 505-541 (1997).

[Pim] Pimsner, M., *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by* $\mathbb{Z}$. (Voiculescu D., ed.), Fields Inst. Commun. 12, 189-212 (1997).

[Put] Putnam, I.F., *C^*-algebras from smale spaces.* Canad. J. Math. **48** (1), 175-195 (1996).

[R] Renault, J., *A groupoid approach to C^*-algebras.* Lecture Notes in Math. **793**, Springer, 1980.

[RS1] Robertson, G., Steger, T., *C^*-algebras arising from group actions on the boundary of a triangle building.* Proc. London Math. Soc. **72**, 613-637 (1996).

[RS2] Robertson, G., Steger, T., *Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras.* J. reine angew. Math. **513**, 115-144 (1999).

[RøS] Rørdam, M., Størmer, E., *Classification of Nuclear C^*-Algebras. Entropy in Operator Algebras.* Operator Algebras and Non-Commutative Geometry, Encyclopaedia of Mathematical Sciences, Vol. **126**, Springer, 2002.

[Ru] Ruelle, D., *Noncommutative algebras for hyperbolic diffeomorphisms.* Invent. Math. **93**, 1-13 (1988).

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