

SLIGHTLY LARGER THAN A GRAPH C^* -ALGEBRA

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ABSTRACT. To certain higher rank Cuntz algebras \mathcal{O}_n including the classical cases we trace a certain partial isometry U in its strong closure. Adjoining U to \mathcal{O}_n we obtain a kind of uniqueness property for this larger C^* -algebra. Its explanation is not entirely “Cuntz’s uniqueness argumentation”.

1. INTRODUCTION

In [B1] we introduced certain higher rank Cuntz algebras $\mathcal{O}_{\{n_1, n_2, \dots, n_d\}}$ for pairwise relative prime integers $n_1, \dots, n_d \geq 2$. Actually they are so-called higher rank or graph algebras [RS1, RS2, RS2, KP1, KP2, KP3], and one may also notice the tracks [P, FM] or [B2].

If the “rank” d is 1 then $\mathcal{O}_n = \mathcal{O}_{\{n_1\}}$ is just the classical Cuntz algebra [C] generated by n_1 isometries. Making this introduction readable for a preferably large audience we shall suppose that $d = 1$. In fact the following results hold also for the higher rank Cuntz algebras with minor adaption.

So let \mathcal{O}_n be the classical Cuntz algebra [C] generated by $n \geq 2$ isometries $S_0, S_1, \dots, S_{n-1} \in B(\mathcal{H})$. Then in this paper we determine a partial isometry $U \in B(\mathcal{H})$ such that

$$(1) \quad US_0 = S_1, US_1 = S_2, \dots, US_{n-2} = S_{n-1} \text{ and } US_{n-1} = S_0U.$$

Among possibly several candidates one chooses U with minimal support projection and considers the C^* -algebra $C^*(S, U)$ generated by $\mathcal{O}_n \cup \{U\}$. Let J be its smallest ideal such that $U + J$ is unitary in the quotient $C^*(S, U)/J$. Then the latter quotient C^* -algebra - denoted by $\tilde{\mathcal{O}}_n$ - contains \mathcal{O}_n canonically (“letter by letter”) and is canonically unique in the sense that it does

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not depend on the representation $S_i \in B(\mathcal{H})$, Theorem 3.3.

A second and probably more striking uniqueness variant is this. Consider \mathcal{O}_n represented on a Hilbert space \mathcal{H} and let U be a unitary satisfying (1). Now if U is unique in the sense that it is the only operator in $B(\mathcal{H})$ with property (1), then such a situation produces canonically always the same C^* -algebra $\tilde{\mathcal{O}}_n$ obtained by adjoining U to \mathcal{O}_n , Corollary 3.4.

(It remains an open question to us whether the “uniqueness of U ” here is really necessary.)

In contrast these uniqueness properties cannot be explained by the anyway very general concept in [B2], say. Indeed it is straight from the equations (1) that U^{n-1} has to be “zero-balanced”, $\text{bal}(U^{n-1}) = 0$, [B2]. But U^{n-1} does not lie in a finite dimensional C^* -algebra and property (B) [B2] fails.

This exotic example may allude - at least theoretically - to an extended uniqueness potential than seen up to now in the Cuntz-Krieger genre ([CK] is classical).

2. THE PARTIAL ISOMETRY U

In [B1] we introduced a certain higher rank Cuntz C^* -algebra $\mathcal{O}_n = \mathcal{O}_{\{n_1, \dots, n_d\}}$ which is generated by certain isometries $S_{i,j} \in B(\mathcal{H})$. In the beginning of section 2 of [B1] these isometries are explained in detail and also certain notions are introduced. We shall adopt this setting and the notions completely and would like to repeat them briefly.

Let $d \geq 1$ be an integer (corresponding to the “rank” of the algebra) and $n = (n_1, \dots, n_d) \in \mathbb{N}^d$ be a vector with mutually relative prime integer entries $n_i \geq 2$. For a Hilbert space \mathcal{H} we are given d sets of isometries $\{S_{i,0}, S_{i,1}, \dots, S_{i,n_i-1}\} \subseteq B(\mathcal{H})$ ($i = 1, \dots, d$) which satisfy the so-called Cuntz property

$$S_{i,0}S_{i,0}^* + S_{i,1}S_{i,1}^* + \dots + S_{i,n_i-1}S_{i,n_i-1}^* = I.$$

So the C^* -algebra generated by the set $\{S_{i,0}, S_{i,1}, \dots, S_{i,n_i-1}\}$ is the classical Cuntz algebra \mathcal{O}_{n_i} [C]. Assume that the generating isometries interact in the following way (sometimes referred as “permutation rules”).

$$S_{i,x}S_{j,y} = S_{j,X}S_{i,Y} \text{ whenever } x + yn_i = X + Yn_j.$$

Clearly for each tuple $(x, y) \in \{0, \dots, n_i - 1\} \times \{0, \dots, n_j - 1\}$ we find a unique corresponding tuple $(X, Y) \in \{0, \dots, n_j - 1\} \times \{0, \dots, n_i - 1\}$. One can deduce

from this setting that the following “permutation rules” must also hold.

$$S_{i,x}^* S_{j,y} = S_{j,X} S_{i,Y}^* \text{ whenever } x + X n_i = y + Y n_j.$$

Once again for each given tuple (x, y) one finds a unique tuple (X, Y) . The C^* -algebra generated by all isometries $\{S_{i,j} \in B(\mathcal{H}) \mid 1 \leq i \leq d, 0 \leq j \leq n_i - 1\}$ is denoted by $\mathcal{O}_n = \mathcal{O}_{\{n_1, \dots, n_d\}}$ and does not depend on the representation on the Hilbert space, see [B1], Theorem 2.8.

Let $\mathcal{A} = \{(i, j) \mid i = 1, \dots, d \text{ and } j = 0, \dots, n_i - 1\}$ be our alphabet which corresponds to the given generators S_a , $a \in \mathcal{A}$. Then $V = \cup_{m \geq 0} \mathcal{A}^m$ denotes all finite words in this alphabet. But very often we will use the extraordinary subset of “ordered” words

$$W = \{((i_1, j_1), \dots, (i_m, j_m)) \in V \mid m \geq 0 \text{ and } i_1 \leq i_2 \leq \dots \leq i_m\}.$$

Let $\alpha = (a_1, \dots, a_m) \in V$. Then we write $S_\alpha = S_{a_1} S_{a_2} \dots S_{a_m}$. Further we associate an integer vector $|\alpha| := (N_1, \dots, N_d) \in \mathbb{Z}^d$ to α , where

$$N_i = \text{card}\{k \in \{1, \dots, m\} \mid a_k = (i, j) \text{ for some } j\}.$$

For $\alpha = ((i_1, j_1), \dots, (i_m, j_m)) \in V$ ($(i_k, j_k) \in \mathcal{A}$) we introduce the integer

$$Z(\alpha) = \sum_{k=1}^m j_k n_{i_1} \dots n_{i_{k-1}}.$$

This is the number which is associated to the digit representation (j_1, \dots, j_m) with respect to the basis $(n_{i_1}, \dots, n_{i_m})$, where j_1 is the least significant digit. Given $k \in \mathbb{Z}_+^d$ we abbreviate the product $n_1^{k_1} n_2^{k_2} \dots n_d^{k_d}$ by n^k . One should then observe that for all $\alpha \in V$ we have

$$0 \leq Z(\alpha) < n_{i_1} n_{i_2} \dots n_{i_m} = n^{|\alpha|}.$$

So far the “dimension” or “rank” $d \in \mathbb{N}$ is finite. However we shall permit also $d = \infty$ in the sequel. That means that in this case \mathcal{O}_n is generated by infinitely many isometries $S_{i,j}$, $i \in \mathbb{N}$, $0 \leq j < n_i$, where $n = (n_1, n_2, \dots)$ is a sequence.

It is no problem to adopt the above notions to the infinite rank case. For example the set of words V remains the set of words, may the alphabet be finite or infinite. But one point deserves explanation, namely we put, when $d = \infty$,

$$\mathbb{Z}_+^\infty := \{(x_1, x_2, \dots) \in (\mathbb{Z}_+)^{\mathbb{N}} \mid x_i = 0 \text{ for all but finitely many } i \geq 1\}.$$

As also noticed in [B1], an ordered word $\alpha \in W$ is uniquely determined by $|\alpha|$ and $Z(\alpha)$. More precisely the following map t is a bijection.

$$t : W \rightarrow \{ (k, j) \mid k \in \mathbb{Z}_+^d, 0 \leq j < n^k \} : t(\alpha) = (|\alpha|, Z(\alpha)).$$

(In principle such a correlation appears in [BJ], section 2, too.)

Using this bijection and given $k \in \mathbb{Z}_+^d$ we put $\min(k) := \alpha$ for the unique $\alpha \in W$ such that $|\alpha| = k$ and $Z(\alpha) = 0$. Analogously we use $\max(k) := \alpha$ for the unique $\alpha \in W$ such that $|\alpha| = k$ and $Z(\alpha) = n^k - 1$.

Hence $S_{\min(k)}$ consists of isometries $S_{i,0}$ only, whereas $S_{\max(k)}$ consists of isometries of the form S_{i,n_i-1} only.

For given $\gamma \in W$ we write $\gamma + 1 := \alpha$ where $\alpha \in W$ is uniquely determined by $|\alpha| = |\gamma|$ and $Z(\alpha) = Z(\gamma) + 1$ modulo $n^{|\alpha|}$.

Recall that the least significant digit in a word $\alpha \in W$ is the most left one, so for example $((1, 3), a_2, a_3, \dots) + 1 = ((1, 4), a_2, a_3, \dots)$. If we had here an overflow in the first digit then we would get $((1, 3), (5, 1), a_3, \dots) + 1 = ((1, 0), (5, 2), a_3, \dots)$.

In the sequel we shall often abbreviate $\sum_{\gamma \in W, |\gamma|=k} X(\gamma)$ by $\sum_{|\gamma|=k} X(\gamma)$, where $k \in \mathbb{Z}_+^d$, so in such sums the index γ is always understood to be taken from W .

Let $\mathcal{O}_n \subseteq B(\mathcal{H})$ be the C^* -algebra that is generated by the isometries $S_{i,j} \in B(\mathcal{H})$. In what follows we shall trace a partial isometry U in the strong closure of \mathcal{O}_n which obeys the following rules for all $1 \leq i \leq d$.

$$\begin{aligned} \text{(A)} \quad & \begin{cases} US_{i,j} = S_{i,j+1} & \text{for } 0 \leq j < n_i - 1 \\ US_{i,n_i-1} = S_{i,0}U \end{cases} \\ \text{(B)} \quad & \begin{cases} U^*S_{i,j} = S_{i,j-1} & \text{for } 0 < j \leq n_i - 1 \\ U^*S_{i,0} = S_{i,n_i-1}U^* \end{cases} \\ \text{(C)} \quad & I = U^*U + P_{\max(\infty)} \\ \text{(D)} \quad & I = UU^* + P_{\min(\infty)} \end{aligned}$$

Here $P_{\max(\infty)}$ and $P_{\min(\infty)}$ denote the limits of the nets of projections $S_{\max(N)}S_{\max(N)}^*$ and $S_{\min(N)}S_{\min(N)}^*$ respectively ($N \in \mathbb{Z}_+^d$, N tends to ‘‘infinity’’).

In the next lemma we deduce some useful algebraic relations from properties (A)-(B). One should notice here that $Z(\alpha + 1) = 0$ simply means $\alpha = \max(k)$ for $k = |\alpha|$.

Lemma 2.1. *From (A)-(B) follows ($\alpha \in W$)*

$$\begin{aligned} US_\alpha &= S_{\alpha+1}, & U^*S_{\alpha+1} &= S_\alpha & \text{if } Z(\alpha + 1) &\neq 0 \\ US_\alpha &= S_{\alpha+1}U, & U^*S_{\alpha+1} &= S_\alpha U^* & \text{if } Z(\alpha + 1) &= 0 \end{aligned}$$

and $S_\alpha = U^{Z(\alpha)}S^{|\alpha|}$ for all $\alpha \in V$.

Proof. We demonstrate here the first asserted column. Indeed, by induction hypothesis let $US_\alpha = S_{\alpha+1}$ be valid for all $\alpha \in W$ with fixed length $|\alpha| = k$ and $Z(\alpha + 1) \neq 0$, and $US_\alpha = S_{\alpha+1}U$ if $Z(\alpha + 1) = 0$ respectively.

Let $\beta = \alpha \circ (i, j)$ be the word α added by the letter (i, j) . One may notice here that

$$Z(\beta) = Z(\alpha) + jn^{|\alpha|}.$$

Then $US_\beta = US_\alpha S_{i,j}$ is either $S_{\alpha+1}S_{i,j}$ if $Z(\alpha + 1) \neq 0$ and we get $US_\beta = US_{\beta+1}$. Or we get, if $Z(\alpha + 1) = 0$,

$$US_\beta = S_{\alpha+1}US_{i,j} = S_{\alpha+1}S_{i,j+1}$$

by property (A) and once again, since $Z(\alpha) = n^{|\alpha|} - 1$, we have $US_\beta = US_{\beta+1}$. \square

Lemma 2.2. *Define $P_\alpha = S_\alpha S_\alpha^*$ for $\alpha \in W$. Particularly we have ($k \in \mathbb{Z}_+^d$)*

$$(2) \quad P_{\min(k)} = S_{\min(k)} S_{\min(k)}^*, \quad P_{\max(k)} = S_{\max(k)} S_{\max(k)}^*.$$

Both nets (2) are decreasing and converge to $P_{\min(\infty)}$ and $P_{\max(\infty)}$ respectively.

Proof. Notice that per definition of the $S_{i,j}$ [B1] we have $S_{i,0}S_{j,0} = S_{j,0}S_{i,0}$ and $S_{i,n_i-1}S_{j,n_j-1} = S_{j,n_j-1}S_{i,n_i-1}$ for all $1 \leq i, j \leq d$. Thus both sequences (2) are decreasing for increasing $k \in \mathbb{Z}_+^d$. \square

Lemma 2.3. *The properties (A) and (C) determine U uniquely.*

All U satisfying (A) yield the same operator $U(I - P_{\max(\infty)})$.

Proof. We shall use the left column of Lemma 2.1. As demonstrated above it can be proved by just using (A) and so for what follows we use (A) only.

Since $US_\alpha S_\alpha^* = S_{\alpha+1} S_\alpha^*$ by Lemma 2.1, U is uniquely determined on the range projections $P_\alpha = S_\alpha S_\alpha^*$ whenever $Z(\alpha + 1) \neq 0$. Recall that

$$(3) \quad \sum_{|\alpha|=k} P_\alpha = I$$

for arbitrary $k \in \mathbb{Z}_+^d$ (see [B1]). Hence the operator $U(I - P_{\max(k)})$ is unique. Hence also its strong limit $U(I - P_{\max(\infty)})$. Since we assume (C) we have $UP_{\max(\infty)} = 0$ and U is unique. \square

Proposition 2.4. *There exists a partial isometry U in the strong closure of \mathcal{O}_n satisfying (A)-(D).*

Proof. Step 1. We can already see from the previous proof how U should be defined.

We define $UP_\alpha = S_{\alpha+1}^* S_\alpha$ as long as $Z(\alpha + 1) \neq 0$. For a concatenated word $\alpha \circ \beta$ we have, if $|\alpha| = |\gamma|$,

$$\begin{aligned} P_{\alpha \circ \beta} &\leq P_\gamma && \text{if } \gamma = \alpha \\ P_\gamma P_{\alpha \circ \beta} &= 0 && \text{if } \gamma \neq \alpha. \end{aligned}$$

So we should prove whether U is well defined since we have per definition two possibilities for $P_{\alpha \circ \beta} \leq P_\alpha$. But we are consistent since for a concatenated word $\alpha \circ \beta$ such that $Z(\alpha + 1) \neq 0$ and $Z(\alpha \circ \beta + 1) \neq 0$ we have

$$\begin{aligned} US_{\alpha \circ \beta} &= S_{\alpha \circ \beta + 1} = S_{(\alpha+1) \circ \beta}, \text{ or} \\ US_{\alpha \circ \beta} &= US_\alpha S_\beta = S_{\alpha+1} S_\beta = S_{(\alpha+1) \circ \beta}. \end{aligned}$$

Hence the operator $U(I - P_{\max(k)})$ is defined for all $k \in \mathbb{Z}_+^d$, recall (3). In other words U is defined on the subspace

$$\mathcal{H}_0 = \{ \xi \in \mathcal{H} \mid \exists k \in \mathbb{Z}_+^d : P_{\max(k)} \xi = 0 \}$$

and we extend U isometrically to $\overline{\mathcal{H}_0}$ and let U be zero on the complement \mathcal{H}_0^\perp to get the final U .

Step 2. For all $k \in \mathbb{Z}_+^d$ we put

$$U_k := U(I - P_{\max(k)}) = \sum_{|\alpha|=k, Z(\alpha+1) \neq 0} S_{\alpha+1} S_\alpha^* \in \mathcal{O}_n$$

and it is clear from the left equivalence that U_k converges strongly to U .

Step 3. We shall prove property (A). First we have $US_{i,j} = S_{i,j+1}$ per definition of U for a letter (i, j) and as long as $j \neq n_i - 1$.

For the next computation we notice the following. Let $k \in \mathbb{Z}_+^d$, $1 \leq i \leq d$ and identify the delta function for the i -th coordinate by $\delta_i \in \mathbb{Z}_+^d$. Then for each $\alpha \in W$ with $|\alpha| = k + \delta_i$ we find unique $0 \leq j < n_i$ and unique $\beta \in W$

with $|\beta| = k$ such that $\alpha \equiv (i, j) \circ \beta$ (see [B1] Lemma 2.1), and vice versa, uniquely determined by the formula

$$Z(\alpha) = j + Z(\beta)n_i,$$

and consequently $S_\alpha = S_{(i,j) \circ \beta} = S_{(i,j)}S_\beta$ by [B1] Lemma 2.1.

Watching this fact we can compute

$$\begin{aligned} U_{k+\delta_i}S_{i,n_i-1} &= \sum_{|\alpha|=k+\delta_i, Z(\alpha+1) \neq 0} S_{\alpha+1}S_\alpha^*S_{i,n_i-1} \\ &= \sum_{|\beta|=k, Z(\beta+1) \neq 0} S_{i,0}S_{\beta+1}S_\beta^*S_{i,n_i-1}^*S_{i,n_i-1} \\ &= S_{i,0}U_k. \end{aligned}$$

Now we let k tend to infinity to obtain $US_{i,n_i-1} = S_{i,0}U$ by Step 2.

Step 4. Property (C) is valid since per definition the support projection U^*U is those one which projects onto $\overline{\mathcal{H}_0}$. But $P_{\max(\infty)}$ projects onto \mathcal{H}_0^\perp .

Step 5. Property (B) follows from (A) and (C) by

$$U^*S_{i,j} = U^*US_{i,j-1} = (I - P_{\max(\infty)})S_{i,j-1} = S_{i,j-1}$$

if $j > 1$. But if $j = 0$ then we compute very similarly as in Step 3 ($k \in \mathbb{Z}_+^d$)

$$\begin{aligned} U_{k+\delta_i}^*S_{i,0} &= \sum_{|\alpha|=k+\delta_i, Z(\alpha+1) \neq 0} S_\alpha S_{\alpha+1}^*S_{i,0} \\ &= \sum_{|\beta|=k, Z(\beta+1) \neq 0} S_{i,n_i-1}S_\beta S_{\beta+1}^*S_{i,0}^*S_{i,0} \\ &= S_{i,n_i-1}U_k^* \end{aligned}$$

and let k tend to infinity.

Step 6. Property (D) can be seen as follows. Let $P_\alpha = S_\alpha S_\alpha^*$ for $\alpha \in W$. By (C) we get $\lim_{N \in \mathbb{Z}_+^d} UP_{\max(N)} = UU^*UP_{\max(\infty)} = 0$. By Lemma 2.1 we have $UP_\alpha U^* = P_{\alpha+1}$ as long as $Z(\alpha+1) \neq 0$. So with (3)

$$\begin{aligned} UU^* &= \lim_{N \in \mathbb{Z}_+^d} \sum_{|\alpha|=N} UP_\alpha U^* = \lim_{N \in \mathbb{Z}_+^d} \sum_{|\alpha|=N, Z(\alpha+1) \neq 0} P_{\alpha+1} \\ &= \lim_{N \in \mathbb{Z}_+^d} (I - P_{\min(N)}) = I - P_{\min(\infty)}. \end{aligned}$$

□

3. THE ALGEBRAS $\tilde{\mathcal{O}}_n$

In this section, if nothing else is mentioned, U denotes the unique operator of Proposition 2.4 satisfying properties (A)-(D). In the sequel we shall denote by $C^*(S) \cong \mathcal{O}_n \subseteq B(\mathcal{H})$ the C^* -algebra which is generated by the isometries $S_{i,j} \in B(\mathcal{H})$, and by $C^*(S, U)$ the C^* -algebra generated by the $S_{i,j} \in B(\mathcal{H})$ together with $U \in B(\mathcal{H})$.

The analogous (smaller) $*$ -algebras generated by this generators are denoted by $\text{Alg}^*(S)$ and $\text{Alg}^*(S, U)$ respectively.

In the next lemma we shall divide out an ideal J in $C^*(S, U)$ such that the resulting quotient converts the partial isometry U to a unitary.

Lemma 3.1. *Let $J \subseteq C^*(S, U)$ be the closed ideal generated by $I - U^*U$ and $I - UU^*$. Then the canonical map $\pi : C^*(S) \rightarrow C^*(S, U)/J$ is injective.*

Proof. Step 1. Assume π were not injective. Then $J \cap C^*(S) \neq \{0\}$ is a nonzero ideal in the simple C^* -algebra $C^*(S)$, see [B1] Theorem 3.3, and we have $I \in J$.

Consider the algebraic ideal J_0 generated by $P = I - U^*U$ and $Q = I - UU^*$ in the $*$ -algebra $\text{Alg}^*(S, U)$. Then we have $\overline{J_0} = J$ for the norm closure in $C^*(S, U)$.

We shall show in the next ‘‘Step 2’’ that for all $z \in J_0$ there exists a word $\alpha \in W$ such that $zS_\alpha = 0$. Therefore, since $\|I - z\| < 1$ for some $z \in J_0$,

$$\|S_\alpha\| = \|(I - z)S_\alpha\| < 1,$$

a contradiction.

Step 2. Let $Y \in J_0$ be a word in the letters $S_{i,j}, U, P = I - U^*U, Q = I - UU^*$ and their involuted letters. So Y contains at least either P or Q . We shall find $\alpha \in W$ such that $YS_\alpha = 0$. (For arbitrary $Y \in J_0$ we achieve the same by successively cancelling each summand.)

Assume by induction hypothesis that we have already found $\beta \in V$ such that $YS_\beta = Y_2T$ where T (or sometimes \tilde{T}) denotes a product in the letters $S_{i,j}$ only, and Y_2 is a shorter word than Y . Let y be the most right letter of Y_2 , i.e. $Y_2 = \tilde{Y}y$.

If $y = S_{i,j}$ then we simply will put it to T and obtain $Y_2T = \tilde{Y}\tilde{T}$. If $y = S_{i,j}^*$ then by multiplying Y_2T from the right with the letter $S_{i,0}$ and

“permuting” (using the Definitions of the $S_{i,j}$ [B1]) this letter to y we obtain $Y_2TS_{i,0} = \tilde{Y}\tilde{T}$ or zero (using the fact that $S_{i,j}^*S_{i,k}$ is either 0 or I).

If $y = U$, then successively applying the rules stated in Lemma 2.1 we obtain $Y_2T = \tilde{Y}\tilde{T}$ or $\tilde{Y}\tilde{T}U$. In the latter case we multiply from the right with $S_{1,0}$ and get $\tilde{Y}\tilde{T}S_{1,1}$. Similarly we act in the case $y = U^*$.

Now if $y = P$ then we multiply from the right with $S_{1,0}$. Then we certainly have $Y_2TS_{1,0} = 0$ and at the latest we are done here. This is because we can move P to the right by $PS_{i,n_i-1} = S_{i,n_i-1}P$ (use Lemma 2.1) or we have zero by $PS_{i,j} = 0$ whenever $0 \leq j < n_i - 1$. Similarly if $y = Q$ then we multiply with S_{1,n_1-1} . \square

The following proposition approximates the norm of elements in $C^*(S, U)/J$ by those of \mathcal{O}_n (which has the uniqueness property).

Proposition 3.2. *If $X \in \text{Alg}^*(S, U)$ then we find a net of projections $Q_N \rightarrow Q_\infty$, $Q_N \in \text{Alg}^*(S)$, $N \in \mathbb{Z}_+^d$, such that $X(I - Q_N) \in \text{Alg}^*(S)$ and*

$$\|X + J\|_{C^*(S, U)/J} = \|X(I - Q_\infty)\|_{C^*(S, U)} = \lim_{N \in \mathbb{Z}_+^d} \|X(I - Q_N)\|_{C^*(S)}.$$

Neither the choice of the net $(Q_N) \subseteq \mathcal{O}_n$ nor $X(I - Q_N) \in \mathcal{O}_n$ does depend on the representation on the Hilbert space \mathcal{H} .

Proof. Step 1. Let X be a word in the letters $S_{i,j}, U$ and their involuted letters. Then by using the rules of properties (A) and (B) each occurrence of both U and U^* in X can be moved to the right or is absorbed by a right-handed letter $S_{i,j}$. Not so obvious is this for expressions $US_{i,j}^*$. But here we argue by using (B)

$$\begin{aligned} US_{i,j}^* &= (S_{i,j}U^*)^* = ((U^*)^{n_i-1-j}S_{i,n_i-1}U^*)^* \\ &= ((U^*)^{n_i-1-j}U^*S_{i,0})^* = S_{i,0}^*U^{n_i-j}. \end{aligned}$$

Similarly U^* can “skip” the involuted letter $S_{i,j}^*$.

Step 2. So it is clear that each element $X \in \text{Alg}^*(S, U)$ can be written as sum $X = \sum_{i=1}^m \lambda_i X_i U_i$ where $\lambda_i \in \mathbb{C}$, $X_i \in \text{Alg}^*(S)$ and U_i is a word in the two letters $\{U, U^*\}$.

Let $M \in \mathbb{N}$ be the maximal length of all words U_i . Let $P_\alpha := S_\alpha S_\alpha^*$ for $\alpha \in W$.

Then for $N \in \mathbb{Z}_+^d$ and $\alpha \in W$ with $|\alpha| = N$ and $M < Z(\alpha) < n^N - M$ we find a certain $\beta_i \in W$ with $|\beta_i| = N$ (see the rules of Lemma 2.1) such that

$$(4) \quad XP_\alpha = \sum_{i=1}^m \lambda_i X_i U_i S_\alpha S_\alpha^* = \sum_{i=1}^m \lambda_i X_i S_{\beta_i} S_\alpha^* \in C^*(S).$$

Step 3. Since $\lim_{N \in \mathbb{Z}_+^d} P_{\max(N)} = I - U^*U \in J$ by property (C), we get by Lemma 2.1 for all integers $k \geq 0$

$$U^{*k}(I - U^*U)U^k = \lim_{N \in \mathbb{Z}_+^d} U^{*k} P_{\max(N)} U^k = \lim_{N \in \mathbb{Z}_+^d} P_{\max(N)-k} \in J.$$

Similarly we get $\lim_{N \in \mathbb{Z}_+^d} P_{\min(N)+k} \in J$ by considering $I - UU^* \in J$. Let

$$(5) \quad Q_N := \sum_{k=-M}^M P_{\min(N)+k}$$

and $Q_\infty = \lim_{N \in \mathbb{Z}_+^d} Q_N$ strongly. (Note that $P_{\min(N)-1} = P_{\max(N)}$, $P_{\min(N)-2} = P_{\max(N)-1}$ etc., per definition, say.) Then by the above computations we have $Q_\infty \in J$.

Step 4. Abbreviate $A := C^*(S, U)$. For arbitrary $\varepsilon > 0$ we find some normalized $\xi \in \mathcal{H}$ and $N_0 \in \mathbb{Z}_+^d$ such that for all $N \geq N_0$

$$(6) \quad \begin{aligned} \|X + J\|_{A/J} &\leq \|X - XQ_\infty\|_A \\ &\leq \|X(I - Q_\infty)\xi\|_{\mathcal{H}} + \varepsilon \\ &\leq \|X(I - Q_N)\xi\|_{\mathcal{H}} + \|X(Q_N - Q_\infty)\xi\|_{\mathcal{H}} + \varepsilon \\ &\leq \|X(I - Q_N)\|_A + 2\varepsilon \\ &= \|X(I - Q_N) + J\|_{A/J} + 2\varepsilon \\ &\leq \|X + J\|_{A/J} + 2\varepsilon \end{aligned}$$

The equality (6) is due to Lemma 3.1 because by (3), (4) and (5) we have

$$X(I - Q_N) = X \sum_{|\alpha|=N, M < Z(\alpha) < n^N - M} P_\alpha \in C^*(S).$$

Considering the above proof we notice that we have only used the properties (A)-(B) of U to determine Q_N and $X(I - Q_N)$, and the formal shape of $X(I - Q_N)$ in the letters $S_{i,j}$ only does not depend on the representation on the Hilbert space \mathcal{H} . (Also notice [B1], the uniqueness Theorem 2.8.)

We complete the proof by varying $\varepsilon > 0$. \square

Theorem 3.3. Uniqueness variant 1. *Consider the higher rank Cuntz algebra $\mathcal{O}_n = \mathcal{O}_{\{n_1, n_2, \dots, n_d\}}$ [B1] ($d \leq \infty$) generated by the isometries $S_{i,j} \in B(\mathcal{H})$. (If $d = 1$ then this is the classical Cuntz algebra [C].) Then we can find a partial isometry $U \in B(\mathcal{H})$ satisfying (A)-(D). We adjoin U to \mathcal{O}_n and obtain a C^* -algebra $C^*(S, U) \subseteq B(\mathcal{H})$. Next we divide out the ideal $J \subseteq C^*(S, U)$ generated by $I - U^*U$ and $I - UU^*$ to obtain the quotient $\tilde{\mathcal{O}}_n := C^*(S, U)/J$. Then we have $\mathcal{O}_n \subseteq \tilde{\mathcal{O}}_n$ canonically.*

Now if we have given another representation of \mathcal{O}_n generated by the isometries $\hat{S}_{i,j} \subseteq B(\hat{\mathcal{H}})$ and consider the procedure there, then the canonical map

$$\sigma : C^*(S, U)/J \rightarrow C^*(\hat{S}, \hat{U})/\hat{J} : \sigma(S_{i,j} + J) = \hat{S}_{i,j} + \hat{J}, \sigma(U + J) = \hat{U} + \hat{J}$$

is a well defined $$ -isomorphism.*

Proof. The theorem is a corollary of Proposition 3.2. If $X \in \text{Alg}^*(S, U)$ and \hat{X} is the formally translated word, then by Proposition 3.2 we get

$$\begin{aligned} \|X + J\|_{C^*(S, U)/J} &= \|X(I - Q_N)\|_{C^*(S)} \\ \|\hat{X} + \hat{J}\|_{C^*(\hat{S}, \hat{U})/\hat{J}} &= \|\hat{X}(\hat{I} - \hat{Q}_N)\|_{C^*(\hat{S})}. \end{aligned}$$

But by the uniqueness property for \mathcal{O}_n [B1] and Proposition 3.2 the right side of both equations are identic. Hence σ is an isometry and in particular well defined. \square

Corollary 3.4. Uniqueness variant 2. *Consider the higher rank Cuntz algebra $\mathcal{O}_n = \mathcal{O}_{\{n_1, n_2, \dots, n_d\}}$ [B1] ($d \leq \infty$) represented on \mathcal{H} and suppose that there exists only one operator $U \in B(\mathcal{H})$ satisfying (A) and suppose U is unitary. (This setting is the case if and only if $P_{\min(\infty)} = P_{\max(\infty)} = 0$.)*

The same fact suppose for a further representation on $\hat{\mathcal{H}}$. Then the canonical map $\sigma : \tilde{\mathcal{O}}_n \cong C^(S, U) \rightarrow C^*(\hat{S}, \hat{U})$ is a well defined $*$ -isomorphism.*

Proof. Since U is the only operator satisfying (A) it must be the operator of Proposition 2.4 with properties (A)-(D). But U is unitary and properties (C)-(D) yield $P_{\min(\infty)} = P_{\max(\infty)} = 0$. The canonical map σ follows then by Theorem 3.3 since both ideals J, \hat{J} are zero.

(On the other hand if $P_{\min(\infty)} = P_{\max(\infty)} = 0$ then U is unique by Lemma 2.3, hence satisfies (A)-(D) by Proposition 2.4 and is therefore unitary by (C)-(D).) \square

Example 3.5. Consider the higher rank Cuntz algebra $\mathcal{O}_n = \mathcal{O}_{\{n_1, n_2, \dots, n_d\}}$ represented on $\ell^2(\mathbb{Z})$ by the isometries $S_{i,j}(\delta_k) = \delta_{n_i k + j}$ where $1 \leq i \leq d, 0 \leq$

$j < n_i$. Then one can compute that $P_{\max(N)}$, $N \in \mathbb{Z}_+^d$, projects onto the coordinates $\delta_{\mathbb{Z}nN-1}$ and consequently $P_{\max(\infty)}$ projects onto δ_{-1} .

(This representation is permutative [BJ], section 2, and if we denote $\sigma_{i,j}(k) = n_i k + j$ then for each i the set $\{\sigma_{i,0}, \dots, \sigma_{i,n_i-1}\}$ forms a multiplicity free branching function system [BJ], Definition 2.1.)

Then the partial isometry $U(\delta_{-1}) = 0$ and $U(\delta_k) = \delta_{k+1}$, $k \in \mathbb{Z} \setminus \{-1\}$, satisfies the rules (A) and (C) and consequently (A)-(D) by 2.3 and 2.4.

Contrary, the unitary $\tilde{U}(\delta_k) = \delta_{k+1}$ ($k \in \mathbb{Z}$) satisfies (A) but not (C).

Example 3.6. This example refers to Corollary 3.4. Consider the classical Cuntz algebra \mathcal{O}_2 represented on $\ell^2(\mathbb{N}_0)$ by the isometries S_0, S_1 sketched in the following diagram.

δ_k	0	1	2	3	4	5	6	...
$S_0(\delta_k)$	1	3	4	6	8	10	12	...
$S_1(\delta_k)$	2	0	5	7	9	11	13	...
$U(\delta_k)$	4	2	8	0	5	16	7	...

Then we have $P_{\max(\infty)} = P_{\min(\infty)} = 0$ and only one operator U satisfies (A) and U is unitary, see Corollary 3.4.

4. AN OPEN QUESTION

We have seen by Corollary 3.4 that a unitary operator U satisfying (A), as long as it is the only operator fulfilling (A), generates $\tilde{\mathcal{O}}_n$ canonically together with the isometries $S_{i,j}$.

If we could abandon the uniqueness of U then we would have a familiar uniqueness. The uniqueness determined by generators and relations. However we neither can prove nor falsify whether a unitary U together with the isometries $S_{i,j}$ always yields $\tilde{\mathcal{O}}_n$; see for example Example 3.5 where a unitary \tilde{U} is not the only operator with property (A).

Towards solving this problem we finally would like to present the following minor results. The first result Lemma 4.1 tells us - in combination with Lemma 4.2 - that if a unitary U satisfies (A) then we can map $C^*(S, U)$ canonically to $\tilde{\mathcal{O}}_n$.

So solving the above question comes to showing that this map is *isometric*.

Lemma 4.1. *Consider $\mathcal{O}_n = \mathcal{O}_{\{n_1, n_2, \dots, n_d\}}$ ($d \leq \infty$) represented on \mathcal{H} and suppose $U \in B(\mathcal{H})$ satisfies (A)-(B). Then the canonical map $\sigma : C^*(S, U) \rightarrow \tilde{\mathcal{O}}_n$ is well defined.*

Proof. Let $X \in \text{Alg}^*(S, U)$. Although the setting of Proposition 3.2 does not seem to be satisfied completely we nevertheless can apply it to the point that we can choose $Q_N \in \text{Alg}^*(S)$ such that $X(I - Q_N) \in \text{Alg}^*(S)$. The reason is that in the proof of Proposition 3.2 this fact follows already from the properties (A)-(B) which we assume.

The same fact is also true in the image σ and it yields the same canonically translated corresponding expression $\sigma(X(I - Q_N))$ in the letters of $S_{i,j}$ since we have the rules (A)-(B) also there.

Note that $\sigma|_{\mathcal{O}_n}$ is an isometry by [B1], Theorem 2.8, wherefore the boundedness of σ follows from Proposition 3.2 by

$$\begin{aligned} \|\sigma(X)\| &= \lim_{N \in \mathbb{Z}_+^d} \|\sigma(X(I - Q_N))\|_{\mathcal{O}_n} \\ &= \lim_{N \in \mathbb{Z}_+^d} \|X(I - Q_N)\|_{C^*(S)} \leq \|X\|_{C^*(S, U)}. \end{aligned}$$

□

Lemma 4.2. *Let U be a unitary satisfying (A). Then U fulfills also (B).*

Proof. Let U_0 be the unique operator with properties (A)-(D) according to Proposition 2.4. By property (C) and Lemma 2.3 we have

$$U = U_0(I - P_{\max(\infty)}) = U(I - P_{\max(\infty)})$$

wherefore $U = U_0 + D$ for the partial isometry $D = UP_{\max(\infty)}$. The support and range projections of D are $P_{\max(\infty)}$ and $P_{\min(\infty)}$ respectively by properties (C)-(D).

Now for $j \neq 0$ we have $U^*S_{i,j} = U^*US_{i,j-1} = S_{i,j-1}$ which shows (B) in that case.

It is easily computed that

$$\begin{aligned} DD^*S_{i,0} &= P_{\min(\infty)}S_{i,0} = S_{i,0}P_{\min(\infty)} = S_{i,0}DD^*, \\ D^*DS_{i,n_i-1} &= P_{\max(\infty)}S_{i,n_i-1} = S_{i,n_i-1}P_{\max(\infty)} = S_{i,n_i-1}D^*D. \end{aligned}$$

By property (A)

$$DS_{i,n_i-1} = (U - U_0)S_{i,n_i-1} = S_{i,0}(U - U_0) = S_{i,0}D.$$

If we use these formulas then we obtain

$$D^*S_{i,0} = D^*DD^*S_{i,0} = D^*S_{i,0}DD^* = D^*DS_{i,n_i-1}D^* = S_{i,n_i-1}D^*.$$

So the still open equivalence $U^*S_{i,0} = S_{i,n_i-1}U^*$ is now immediate. □

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