NOTES ON CUNTZ–KRIEGER UNIQUENESS THEOREMS AND C^*-ALGEBRAS OF LABELLED GRAPHS

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Abstract. In this note we deal with Cuntz-Krieger uniqueness theorems and extend the class of algebras introduced in [3]. We use this analysis to show two Cuntz-Krieger uniqueness theorems for C^*-algebras of labelled graphs.

1. Introduction

One of the aims of this note is to extend the family of generalized Cuntz-Krieger algebras introduced in [3]. For convenience of the reader and to establish notation we briefly recall the basic definitions of [3] at first. We consider an alphabet $\mathcal{A}$, the free nonunital $*$-algebra $\mathbb{F}$ over $\mathbb{C}$ generated by $\mathcal{A}$, a self-adjoint two-sided ideal $\mathbb{I}$ in $\mathbb{F}$, and a compact subgroup $H \subseteq \mathbb{T}^{\mathcal{A}}$, where $\mathbb{T}$ denotes the torus. Given such a system $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ of "generators and relations" we denote by

$$W = \{ a_1 \dots a_n \in \mathbb{F}/\mathbb{I} \mid a_i \in \mathcal{A} \cup \mathcal{A}^* \}$$

the set of words in the quotient $\mathbb{F}/\mathbb{I}$. (For convenience we write $a_1 \dots a_n$ rather than $a_1 \dots a_n + \mathbb{I}$ if this is clear from the context.) Let $\mathrm{bal} : W \backslash \{0\} \to \hat{H}$ be the balance function of [3, Lemma 3.1] which is determined by the formula $\mathrm{bal}(a)(\lambda) = \lambda_a$, for all $a \in \mathcal{A}$ and $\lambda = (\lambda_b)_{b \in \mathcal{A}} \in H$, and by the identities

$$\mathrm{bal}(xy) = \mathrm{bal}(x)\mathrm{bal}(y) \quad \text{and} \quad \mathrm{bal}(z^*) = \mathrm{bal}(z)^{-1}$$

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for all $x, y, z \in W$ such that $xy \neq 0$ and $z \neq 0$. A nonzero word $x \in W$ is called *zero-balanced* if $\mathrm{bal}(x) = 1 \in \hat{H}$ (and otherwise called nonzero-balanced). Let

$$W_0 = \{ x \in W \backslash \{0\} \mid \mathrm{bal}(x) = 1 \}$$

be the set of all zero-balanced words. Let $\mathbb{A}$ be the $*$-subalgebra generated by W_0,

$$\mathbb{A} = \mathrm{Alg}^*(W_0) = \mathrm{lin}(W_0) \subseteq \mathbb{F}/\mathbb{I}, \tag{1}$$

and denote by $\mathbb{P}$ the set of *nonzero* projections of $\mathbb{A}$. Projections in $\mathbb{F}/\mathbb{I}$ are always understood to be idempotent and self-adjoint, and we have a natural order $p \leq q$ iff $pq = p$. If we want to emphasize that $\mathbb{A}$ refers to the system $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ then we write $\mathbb{A}^{(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)}$. The system $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ may satisfy the following properties.

(A) For all $\lambda = (\lambda_a)_{a \in \mathcal{A}} \in H$ the ideal $\mathbb{I}$ is invariant under the automorphism $t_\lambda : \mathbb{F} \to \mathbb{F}$ determined by $t_\lambda(a) = \lambda_a a$ for all $a \in \mathcal{A}$.

(B) $\mathbb{A}$ is the inductively ordered union of finite dimensional sub-C^*-algebras of $\mathbb{A}$. This is equivalent to saying that $\mathbb{A}$ is the algebraic direct limit of finite dimensional C^*-algebras, that is, $\mathbb{A}$ is locally matricial.

(C) For all words $x \in W \backslash W_0$ and all projections $e, e_1, e_2 \in \mathbb{P}$ there exist projections $p, p_1, p_2 \in \mathbb{P}$ such that $p \leq e, p_1 \leq e_1, p_2 \leq e_2$ and $pxp = 0$ and $p_1 x p_2 = 0$.

Theorem 1.1 ([3], Theorem 3.3)**.** *If the system $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ satisfies the properties (A), (B) and (C), and $\pi_1 : \mathbb{F}/\mathbb{I} \to A_1$ and $\pi_2 : \mathbb{F}/\mathbb{I} \to A_2$ are $*$-homomorphisms with dense images in C^*-algebras A_1, A_2, and π_1 is injective on $\mathbb{A}$, then there exists a $*$-homomorphism $\sigma : A_1 \to A_2$ such that $\sigma\pi_1 = \pi_2$. The map σ is an isomorphism if π_2 is also injective on $\mathbb{A}$.*

As explained in [3], the aim of the last theorem is to check Cuntz-Krieger uniqueness theorems for C^*-algebras induced by generators and relations, as it was done for classical Cuntz-Krieger algebras [5], graph C^*-algebras [9, 8, 10], Robertson-Steger algebras [11], or higher rank graph C^*-algebras [7]. In section 2 we will show that Theorem 1.1 still holds if we replace condition (C) by a weaker and more easily checkable condition (C'):

(C') For all words $x \in W \backslash W_0$ and all projections $e \in \mathbb{P}$ there exists a projection $p \in \mathbb{P}$ such that $p \leq e$ and $pxp = 0$.

All Exel-Laca algebras satisfying the no-terminal-loop condition satisfy the conditions (A), (B) and (C') (this is not true for condition (C), see [3]).

In section 3 we will show that we can replace the set $\mathbb{P}$ appearing in property (C') by a much smaller set $\mathbb{P}_0$ if, roughly speaking, all words are partial isometries with commuting range projections.

In the last section 4 we use the analysis of the previous sections to show two Cuntz-Krieger uniqueness theorems (Theorems 4.3 and 4.4) for C^*-algebras of (cancellable) labelled graphs [1] (including Tomforde's ultragraph, Exel-Laca, and Matsumoto algebras).

2. Condition (C')

The aim of this section is to show that we can replace property (C) by property (C') in the Cuntz-Krieger uniqueness Theorem 1.1.

Theorem 2.1. *Theorem 1.1 still holds if one replaces the required condition (C) by the weaker condition (C').*

Proof. Assume that the system $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ satisfies the conditions (A), (B) and (C'). Introduce the formal alphabet

$$\mathcal{P} := \{p_{(1)}\} \cup \{\, p_{(1,c_2,\ldots,c_n)} \in \{1,2\}^{\{1,\ldots,n\}} \mid n \geq 2, c_i \in \{1,2\}\,\},$$

and define $\mathbb{F}'$ to be the free nonunital $*$-algebra generated by the alphabet $\mathcal{P}$. Let $\mathbb{I}'$ be the two-sided self-adjoint ideal in $\mathbb{F}'$ which is generated by $pp - p$, $p^* - p$, $pq - qp$, $p_{(1)}q - q$ and

$$p_{(1,c_2,\ldots,c_n)} - p_{(1,c_2,\ldots,c_n,1)} - p_{(1,c_2,\ldots,c_n,2)} \tag{2}$$

for all $p, q \in \mathcal{P}$, $n \geq 1$ and $c_1, \ldots, c_n \in \{1,2\}$. In this proof we will regard $\mathcal{P}$ as a subset of $\mathbb{F}'/\mathbb{I}'$. Note that $\mathcal{P}$ is a commuting set of projections which forms a kind of binary tree: p_1 is a unit, $p_1 = p_{11} + p_{12}$, $p_{11} = p_{111} + p_{112}$, $p_{12} = p_{121} + p_{122}$, and so on. Trivially, the system $(\mathcal{P}, \mathbb{F}', \mathbb{I}', \{e\})$ (where e denotes the neutral element of the group $\mathbb{T}^{\mathcal{P}}$) satisfies the properties (A), (B) and (C'), as $\mathrm{bal}(p) = 1$ for all $p \in \mathcal{P}$, in other words, all words of $\mathbb{F}'/\mathbb{I}'$ are zero-balanced, whence $\mathbb{F}'/\mathbb{I}' =$

$\mathbb{A}^{(\mathcal{P},\mathbb{F}',\mathbb{I}',\{e\})}$. One also has $\mathbb{F}'/\mathbb{I}' = \mathrm{Alg}^*(\mathcal{P}) = \mathrm{lin}(\mathcal{P})$ since the elements in $\mathcal{P}$ are either orthogonal or comparable. Write $C = \overline{\mathbb{F}'/\mathbb{I}'}$ for the unique C^*-norm closure of $\mathbb{F}'/\mathbb{I}'$.

Write $S = (\mathcal{A} \times \mathcal{P}, \mathbb{G}, \mathbb{J}, H \times \{e\})$ for the "tensor product system" of $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ and $(\mathcal{P}, \mathbb{F}', \mathbb{I}', \{e\})$, that is, we define $\mathbb{G}$ to be the free nonuntal $*$-algebra generated by $\mathcal{A} \times \mathcal{P}$, and $\mathbb{J}$ to be the kernel of the $*$-epimorphism $\varphi : \mathbb{G} \to \mathbb{F}/\mathbb{I} \otimes \mathbb{F}'/\mathbb{I}'$ determined by $\varphi((a,p)) = a \otimes p$ for all $a \in \mathcal{A}, p \in \mathcal{P}$. We will identify $\mathbb{G}/\mathbb{J}$ with the image of φ. Note that every element $x \in \mathbb{G}/\mathbb{J}$ allows a representation

$$x = y_1 \otimes q_1 + \ldots y_n \otimes q_n, \tag{3}$$

for some elements $y_i \in \mathbb{F}/\mathbb{I}$, and mutually orthogonal projections $q_i \in \mathcal{P}$ (indeed, think of choosing the projections q_i on a sufficiently large kth level on the tree of $\mathcal{P}$, that is, $q_1 = p_{1,1,1,\ldots,1}, q_2 = p_{1,1,1,\ldots,2}, \ldots, q_n = p_{1,2,2,\ldots,2}$). That the system S satisfies property (A) may be realized by the fact that when $\tilde{t}_\lambda$ is the induced gauge action on $\mathbb{F}/\mathbb{I}$ for $\lambda \in H$, and $\tilde{\mathrm{id}}$ is the identity gauge action on $\mathbb{F}'/\mathbb{I}'$, then $\tilde{t}_\lambda \otimes \tilde{\mathrm{id}}$ defines a gauge action on $\mathbb{G}/\mathbb{J}$, showing that $\mathbb{J}$ is invariant under $t_\lambda \times \mathrm{id}$. The system S satisfies property (B), as $\mathbb{A}^{(S)} \cong \mathbb{A}^{(\mathcal{A},\mathbb{F},\mathbb{I},H)} \otimes \mathbb{F}'/\mathbb{I}'$.

Now assume that we are given two $*$- homomorphisms $\pi_k : \mathbb{F}/\mathbb{I} \to A_k$ mapping into C^*-algebras A_k, and the π_k's are faithful on $\mathbb{A}^{(\mathcal{A},\mathbb{F},\mathbb{I},H)}$ and have dense images ($k = 1, 2$). Define homomorphisms $\pi_k \otimes \mathrm{id} : \mathbb{G}/\mathbb{J} \to A_k \otimes C$ by $(\pi_k \otimes \mathrm{id})(a \otimes p) = \pi_k(a) \otimes p$ ($k = 1, 2$).

By representation (3), for any nonzero projection $x \in \mathbb{G}/\mathbb{J}$ there are nonzero projections $y \in \mathbb{F}/\mathbb{I}$ and $q \in \mathcal{P}$ such that $y \otimes q \le x$. We are going to show that the system S satisfies condition (C). To this end, suppose we are given three nonzero projections $e_1, e_2, e_3 \in \mathbb{P}$ and a nonzero-balanced word $x \in W \backslash W_e$. We have to show that there are smaller projections $p_i \le e_i$ in $\mathbb{P}$ such that $p_1 x p_1 = p_2 x p_3 = 0$. At first we choose nonzero smaller projections $f_i \otimes q_i \le e_i$ for some $f_i \in \mathbb{P}^{(\mathcal{A},\mathbb{F},\mathbb{I},H)}$ and $q_i \in \mathcal{P}$ ($1 \le i \le 3$). We may write $x = y \otimes q$ for some word y in the letters of $\mathcal{A}$, and some projection $q \in \mathcal{P}$. By construction of $\mathcal{P}$, there exist smaller projections $a_i \le q_i$ in $\mathcal{P}$ ($2 \le i \le 3$) such that $a_2 a_3 = 0$, and consequently $p_2 x p_3 = f_2 x f_3 \otimes a_2 q a_3 = 0$ for $p_i = f_i \otimes a_i \le e_i$. Assuming condition (C') for the system $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$, we may choose a projection $g_1 \in \mathbb{P}^{(\mathcal{A},\mathbb{F},\mathbb{I},H)}$ such that $g_1 \leq f_1$ and $g_1 y g_1 \otimes q_1 = 0$, and thus we may set $p_1 = g_1 \otimes q_1 \leq e_1$.

By Theorem 1.1 there is a homomorphism $\tilde{\sigma} : A_1 \otimes C \to A_2 \otimes C$ such that $\tilde{\sigma}(\pi_1 \otimes \mathrm{id}) = \pi_2 \otimes \mathrm{id}$. The restriction $\sigma = \tilde{\sigma}|_{A_1 \otimes \mathbb{C}}$ is the desired map $\sigma : A_1 \to A_2$ satisfying $\sigma\pi_1 = \pi_2$. □

Definition 2.2. If a system $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ satisfies the conditions (A), (B) and (C'), and there exists an $\mathbb{A}$-faithful representation $\pi : \mathbb{F}/\mathbb{I} \to A$ into a C^*-algebra A, then we call $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I},H} := \overline{\pi(\mathbb{F}/\mathbb{I})}$ (norm closure) the *Cuntz-Krieger type algebra* associated with $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$. (By the last uniqueness theorem, $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I},H}$ is uniquely determined up to $*$-isomorphism.)

Given projections $p, q \in \mathbb{A}$, we write "$p \precsim q$ in $\mathbb{A}$" if there exists a partial isometry $s \in \mathbb{A}$ (i.e. $ss^*s = s$) such that $p = ss^*$ and $s^*s \leq q$.

We introduce the following condition:

(C*) There exist subsets $W' \subseteq \mathbb{F}/\mathbb{I}$ and $\mathbb{P}_2 \subseteq \mathbb{P}$ such that
- (i) $W \backslash W_0 \subseteq \mathrm{lin}(W')$,
- (ii) for all $p \in \mathbb{P}$ there exists $p_2 \in \mathbb{P}_2$ such that $p_2 \precsim p$ in $\mathbb{A}$, and
- (iii) for all $x \in W'$ and $e \in \mathbb{P}_2$ there exists $p \in \mathbb{P}_2$ such that $p \leq e$ and $pxp = 0$.

Notice that $W' = W \backslash W_0$ is a valid candidate satisfying (i), and $\mathbb{P}_2 = \mathbb{P}$ is a valid candidate satisfying (ii). Hence (C') trivially implies (C*). But also the inverse implication holds true:

Lemma 2.3. *Assume that (A) holds. Then (C*) and (C') are equivalent.*

Proof. Fix $x \in W \backslash W_0$ and $e \in \mathbb{P}_2$. By (C*)(i) we may write x as $x = \sum_{i=1}^n \lambda_i x_i$ for some $\lambda_i \in \mathbb{C}, x_i \in W'$. By condition (C*)(iii) we may choose a decreasing sequence of projections $e \geq p_1 \geq \ldots \geq p_n$, $p_i \in \mathbb{P}_2$, such that $p_1 x_1 p_1 = p_2 x_2 p_2 = \ldots = p_n x_n p_n = 0$. Hence $p_n x p_n = 0$ as required in condition (C'). For the general case, when $e \in \mathbb{P}$, we refer the reader to "Step 2" of the proof of Lemma 2.4 in [3]. □

3. Condition (D)

In most cases of which we are aware a generalized Cuntz-Krieger algebra is generated by partial isometries with commuting range and source projections. The aim of this section is to assume such a situation and to analyze the structure of $\mathbb{A}$ to obtain a condition (D) which implies (C') and which may be easier to check sometimes. As proved in [4], the higher rank Exel-Laca algebras [4] satisfy (A), (B) and (D). Moreover, we will use condition (D) in the next section for C^*-algebras of labelled graphs. Let $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ be a system satisfying (A). By [3, Lemma 3.1] we have a balance function as described in the introduction. Define

$$\Delta = \{ xx^* \in \mathbb{F}/\mathbb{I} \mid x \in W \},$$

$\mathbb{A}_0 = \mathrm{Alg}^*(\Delta)$, and denote by $\mathbb{P}_0$ the set of nonzero projections of $\mathbb{A}_0$. Observe that $\mathbb{A}_0 \subseteq \mathbb{A}$ and $\mathbb{P}_0 \subseteq \mathbb{P}$. Put $\mathrm{Rg}(s) = ss^*$ and $\mathrm{Supp}(s) = s^*s$ for a partial isometry s (i.e. $ss^*s = s$). We introduce the following properties.

(D)(a) The alphabet $\mathcal{A}$ consists of partial isometries (i.e. $aa^*a = a$ in $\mathbb{F}/\mathbb{I}$) and Δ is a commuting set.

(D)(b) If $x \in W_0$, $p \in \mathbb{P}_0$ and $\mathrm{Supp}(pxp) = \mathrm{Rg}(pxp)$ then pxp is a projection.

(D)(c) If $x \in W \backslash W_0$ and $e \in \mathbb{P}_0$ then there exists $p \in \mathbb{P}_0$ such that $p \le e$ and $pxp = 0$.

All three properties (D)(a)-(c) together are denoted by (D).

Lemma 3.1. *The following points hold when (D)(a) is satisfied.*

(1) *Each word $x \in W$ is a partial isometry, $\Delta\Delta \subseteq \Delta$, and $\mathbb{A}_0 = \mathrm{Alg}^*(\Delta) = \mathrm{lin}(\Delta)$ is a commutative $*$-algebra.*

(2) *If $x \in W$ and $p, q \in \mathbb{P}_0$ then px, xq and pxq are partial isometries with range and source projections in $\mathbb{A}_0$.*

(3) *The following two conditions are equivalent for $x \in W$ and $p \in \mathbb{P}_0$. First, $px = xp$ and $px^*x = pxx^* = p$. Second, $\mathrm{Supp}(pxp) = \mathrm{Rg}(pxp) = p$.*

(4) *Suppose that each $x \in W_0$ satisfying $xx \neq 0$ is a projection. Then property (D)(b) holds.*

(5) *Assume that there does not exist $e \in \mathbb{P}_0$ and $x \in W \backslash W_0$ such that $e \le (x^n x^{*n})(x^{*n} x^n)$ for all $n \ge 1$. Then property (D)(c) holds.*

(6) *Under (D), the set of words W forms an inverse semigroup under multiplication.*

Proof. Points (1) and (2) are standard. Point (3) is "playing around with partial isometries and projections" (we skip the tedious proof). (4): We have to check (D)(b). Let $x \in W_0$ and $p \in \mathbb{P}_0$. Assume that $0 \neq \mathrm{Supp}(pxp) = \mathrm{Rg}(pxp)$. Then $\mathrm{Supp}(pxp)\mathrm{Rg}(pxp) \neq 0$. Then $xpx \neq 0$, and so $x^2 \neq 0$. By assumption x is a projection, and hence pxp is a projection. (5): We have to check (D)(c). Let $e \in \mathbb{P}_0$ and $x \in W \backslash W_0$. Let $y = exe$. If $\mathrm{Rg}(y) < e$, and notice that necessarily $\mathrm{Rg}(y) \leq e$, then we put $p = e - \mathrm{Rg}(y)$, and we get $pxp = 0$ as required in condition (D)(c). A similar argument works when $\mathrm{Supp}(y) < e$. So assume the last possible case that $\mathrm{Rg}(y) = \mathrm{Supp}(y) = e$. Then $ex = xe$ and $ex^*x = exx^* = e$ by Lemma 3.1.(3). So we get $e = exx^* = xex^*$. Hence we obtain $e = xex^* = xxex^*x^* = \ldots = x^n ex^{*n} \leq x^n x^{*n}$ and similarly $e \leq x^{*n}x^n$ for all $n \geq 1$. Consequently, $e \leq x^n x^{*n} \wedge x^{*n}x^n$, which contradicts the assumption. (6): By Lemma 3.1.(1) we have inverses. By a well known characterization for inverse semigroups, it is enough to show that idempotent elements in W commute. Let $x \in W$ and assume that $xx = x$. Then either $x = 0$, or $\mathrm{bal}(x) = 1$. Applying Lemma 3.1.(3) to $x = x$ and $p = xx^*$ we get $x = xx^* \in \Delta$. □

Lemma 3.2. *Suppose (A) and (B). Then $\mathbb{A}$ is the inductively ordered union of a family $(M)_{M\in\Omega}$ of finite dimensional C^*-subalgebras M of $\mathbb{A}$, where each M is the linear span of finitely many words in W.*

Proof. Given words $x_1, \ldots, x_n \in W_0$ define M to be the finite dimensional C^*-algebra generated by $x_1, \ldots, x_n$. Then $M = \mathrm{lin}(Y)$, where Y denotes the set of all finite products generated by $\{x_i\} \cup \{x_j^*\}$. By choosing a finite linear basis in $\mathrm{lin}(Y)$, we obtain finitely many words $Y_0 \subseteq Y$ such that $\mathrm{lin}(Y_0) = \mathrm{lin}(Y) = M$. Then we set $\Omega = \{M = M_{x_1,\ldots,x_n} | \, n \geq 1, \, x_i \in W_0\}$. □

Proposition 3.3. *Assume that (A), (B), (D)(a) and (D)(b) hold.*

(i) *Then $\mathbb{A}$ is the inductively ordered union of a family $(\mathcal{M})_{\mathcal{M}\in\Omega}$ of finite dimensional C^*-subalgebras $\mathcal{M} \subseteq \mathbb{A}$ such that $\mathcal{M} \cap \mathbb{A}_0$ is a maximal abelian subalgebra of $\mathcal{M}$.*

(ii) *In particular, each $\mathcal{M} \in \Omega$ allows a representation $\mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_N$ of simple factors $\mathcal{M}_t \subseteq \mathcal{M}$ (denote the generating matrix units of $\mathcal{M}_t$ by $\{e^t_{i,j}\}_{1\leq i,j\leq n_t}$) such that*

$$\mathcal{M} \cap \mathbb{A}_0 = \mathrm{lin}\{ \, e^t_{ii} \mid 1 \leq t \leq N, \, 1 \leq i \leq n_t \, \}.$$

(iii) *In particular, for all $p \in \mathbb{P}$ there exists $p_0 \in \mathbb{P}_0$ such that $p_0 \precsim p$ in $\mathbb{A}$.*

Proof. Step 1. Consider the family $(\mathcal{M})_{\mathcal{M} \in \Omega}$ of Lemma 3.2. Fix a finite dimensional C^*-algebra $\mathcal{M} \in \Omega$, and chose words $x_1, \ldots, x_n \in W$ such that $\mathcal{M} = \mathrm{lin}\{x_1, \ldots, x_n\}$. Let $C = \mathcal{M} \cap \mathbb{A}_0$ and notice that C is an abelian self-adjoint subalgebra of $\mathcal{M}$. Enlarge C to a maximal abelian self-adjoint subalgebra $\widehat{C} \subseteq \mathcal{M}$. We want to show that $C = \widehat{C}$. It is sufficient if we can show that each minimal projection z of $\widehat{C}$ is contained in C.

Step 2. Let $z \in \widehat{C}$ be a minimal projection of $\widehat{C}$. For each projection $p \in C$ we either have $zp = 0$ or $zp = z$. We choose a representation

$$z = \sum_{i=1}^{n} \alpha_i p_i x_i q_i \tag{4}$$

where $\alpha_i \in \mathbb{C}$ and $p_i, q_i \in C$ (for example put $p_i = x_i x_i^*$ and $q_i = x_i^* x_i$). By Lemma 3.1.(2) each $p_i x_i q_i$ is a partial isometry with range projection $r_i \in \mathbb{A}_0$ and source projection $s_i \in \mathbb{A}_0$. Hence $r_i, s_i \in C$. Let $I \in C$ be the identity of C, and let $p^\perp = I - p \in C$ for $p \in C$. By representation (4) we have $zI = z$. Hence

$$\begin{aligned} z = IzI &= (r_1 + r_1^\perp) \ldots (r_n + r_n^\perp) z (s_1 + s_1^\perp) \ldots (s_n + s_n^\perp) \\ &= r_1 r_2 \ldots r_n z s_1 s_2 \ldots s_n + \ldots\ldots + r_1^\perp r_2^\perp \ldots r_n^\perp z s_1^\perp s_2^\perp \ldots s_n^\perp . \end{aligned} \tag{5}$$

Since z is minimal in $\widehat{C}$, each summand of (5) is either 0 or z, and consequently only one summand of (5) does not vanish. Hence for certain $\epsilon_i, \nu_i \in \{1, \perp\}$ we have (by commutativity)

$$z = r_1^{\epsilon_1} \ldots r_n^{\epsilon_n} z s_1^{\nu_1} \ldots s_n^{\nu_n} = r_1^{\epsilon_1} \ldots r_n^{\epsilon_n} s_1^{\nu_1} \ldots s_n^{\nu_n} z r_1^{\epsilon_1} \ldots r_n^{\epsilon_n} s_1^{\nu_1} \ldots s_n^{\nu_n},$$

$$z = \sum_{i=1}^{n} \alpha_i p_i' x_i q_i', \tag{6}$$

where $p_i' := r_1^{\epsilon_1} \ldots r_n^{\epsilon_n} s_1^{\nu_1} \ldots s_n^{\nu_n} p_i \in C$ and $q_i' := q_i r_1^{\epsilon_1} \ldots r_n^{\epsilon_n} s_1^{\nu_1} \ldots s_n^{\nu_n} \in C$. Notice that $p_i' \leq p_i$ and $q_i' \leq q_i$.

Now we replace representation (4) by representation (6), in other words we replace p_i by p_i' and q_i by q_i' in (4), and repeat the last procedure with this new representation (6) of z. We will then obtain new $p_i'', q_i'' \in C$ such that $z = \sum_i p_i'' x_i q_i''$. Then we apply the procedure again, and so on. The procedure is repeated until it no longer provides a new representation of z. Clearly this loop will halt after some finite steps since the projections $p_i \geq p_i' \geq p_i'' \geq \ldots$ and $q_i \geq q_i' \geq q_i'' \geq \ldots$ form a decreasing sequence in the finite dimensional C^*-algebra C.

Step 3. We assume the loop has stopped with a representation as in (4), i.e. $z = \sum_i p_i x_i q_i$ for some $p_i, q_i \in C$. We apply the above procedure once more and obtain new $r_i, s_i, \epsilon_i, \nu_i$ and clearly $p_i' = p_i$ and $q_i' = q_i$. Then we have (since $p_i = p_i'$ and $q_i = q_i'$)

$$\begin{aligned} p_i &= r_1^{\epsilon_1} \ldots r_n^{\epsilon_n} s_1^{\nu_1} \ldots s_n^{\nu_n} p_i \\ q_i &= r_1^{\epsilon_1} \ldots r_n^{\epsilon_n} s_1^{\nu_1} \ldots s_n^{\nu_n} q_i. \end{aligned}$$

Thus $r_i = \mathrm{Rg}(p_i x_i q_i) \leq p_i \leq r_i^{\epsilon_i}$. If $\epsilon_i = \perp$ then such an inequality can only hold if $r_i = 0$. Hence we have $\epsilon_i = 1$ and $r_i = p_i$ for all $i \in \Gamma$ where

$$\Gamma = \{\, j \in \{1, ..., n\} \mid \alpha_j p_j x_j q_j \neq 0 \,\}.$$

By a similar argument we get $\nu_i = 1$ and $s_i = q_i$ for all $i \in \Gamma$. Hence $r_i = p_i \leq s_j^{\nu_j} = s_j = q_j \leq r_i^{\epsilon_i} = r_i$ for all $i, j \in \Gamma$. Hence $p := p_i = q_i = r_i = s_i$ for all $i \in \Gamma$. This yields

$$p = s_i = \mathrm{Supp}(p_i x_i q_i) = \mathrm{Supp}(p x_i p) = \mathrm{Rg}(p x_i p)$$

for all $i \in \Gamma$. By condition (D)(b) we get $p x_i p = p$ and consequently

$$z = \sum_{i=1}^{n} \alpha_i p_i x_i q_i = \sum_{i \in \Gamma} \alpha_i p x_i p = \sum_{i \in \Gamma} \alpha_i p = \Big(\sum_{i \in \Gamma} \alpha_i \Big) p.$$

Hence $z = p \in C$. This shows that $C = \widehat{C}$.

Step 4. We have proved that $C = \mathcal{M} \cap \mathbb{A}_0 = \widehat{C}$ is a maximal abelian subalgebra of $\mathcal{M}$. This proves the claim (i). Thus C contains the center $C(\mathcal{M})$ of $\mathcal{M}$, and for each minimal projection $p \in C(\mathcal{M})$, $pC \subseteq C$ is a maximal abelian subalgebra of the simple factor $p\mathcal{M}$. It is well-known, see for example [12, 11.2], that one can construct a matrix representation $M = \mathrm{lin}\{e_{ij}\}$ of a simple finite dimensional C^*-algebra M (in our case $M = p\mathcal{M}$) such that $\mathrm{lin}\{e_{ii}\} = S$ for a beforehand given maximal subalgebra S (in our case $S = pC$) of M. This proves the point (ii).

Hence for each projection $p \in \mathcal{M}$ there exists a projection $q \in C$ such that $q \precsim p$ in the algebra $\mathcal{M}$. This proves the point (iii). □

Corollary 3.4. *Assume that the system* $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ *satisfies (A), (B) and (D). Then it also satisfies (C'). Moreover, a representation* $\pi : \mathbb{F}/\mathbb{I} \to A$ *is faithful on* $\mathbb{A}$ *if and only if it is faithful on* $\mathbb{P}_0$*.*

Proof. By Proposition 3.3.(iii), condition (D)(c) and Lemma 2.3 (one sets $W' := W \backslash W_0$ and $\mathbb{P}_2 := \mathbb{P}_0$ in condition (C*)) we get condition (C'). Using Proposition 3.3, it is easy to check that π is faithful on $\mathbb{A}$ if and only if π is faithful on all matrix diagonal entries $\{e_{ii}^t\} \subseteq \mathbb{P}_0$ for all $\mathcal{M} \in \Omega$. □

4. C^*-algebras of labelled graphs

Our next aim is to prove two Cuntz-Krieger uniqueness theorems for C^*-algebras associated to weakly left-resolving labelled spaces $(E, \mathcal{L}, \mathcal{B})$ introduced by Bates and Pask in [1]. We will adopt the notations introduced in [1] and briefly recall it. The system $E = (E^0, E^1, r, s)$ denotes a directed graph with range and source maps $r, s : E^1 \to E^0$. A labelled space consists of a system $(E, \mathcal{L}, \mathcal{B})$ where E is a directed graph, $\mathcal{L} : E^1 \to \mathcal{A}$ is an (arbitrary) labelling map into an alphabet $\mathcal{A}$, and $\mathcal{B} \subseteq 2^{E^0}$ is an accommodation for $(E, \mathcal{L})$ (see [1]). Write $E^* = \bigcup_{n \geq 1} E^n$ for the set of finite paths in E. We will assume that $\mathcal{A} = \mathcal{L}(E^1)$, and write $\mathcal{L}^*(E) = \bigcup_{n \geq 1} \mathcal{L}(E^1)^n$ for the set of all finite labelled paths $\mathcal{L}(x_1) \ldots \mathcal{L}(x_n) \in \mathcal{L}(E^1)^n$ $(x_i \in E^1, x_1 x_2 \ldots x_n \in E^n)$. Labelled paths can be concatenated (written as a product) whenever the concatenation is also in $\mathcal{L}^*(E)$. We introduce source and range maps $s, r : \mathcal{L}^*(E) \to 2^{E^0}$ by $s(\alpha) = \{s(x) |\, \mathcal{L}(x) = \alpha, x \in E^*\}$ and $r(\alpha) = \{r(x) |\, \mathcal{L}(x) = \alpha, x \in E^*\}$ for all $\alpha \in \mathcal{L}^*(E)$. The *relative range of* $\alpha \in \mathcal{L}^*(E)$ *with respect to* $A \in 2^{E^0}$ is defined to be

$$r(A, \alpha) = \{r(x) \in E^0 |\, s(x) \in A, \mathcal{L}(x) = \alpha, x \in E^*\}.$$

The labelled space $(E, \mathcal{L}, \mathcal{B})$ is called *weakly left-resolving* if for every $A, B \in \mathcal{B}$ and every $\alpha \in \mathcal{L}^*(E)$ we have $r(A, \alpha) \cap r(B, \alpha) = r(A \cap B, \alpha)$. For $A \in \mathcal{B}$ one sets

$$L_A^1 = \{b \in \mathcal{L}(E^1) |\, s(b) \cap A \neq \emptyset\},$$

and $(E, \mathcal{L}, \mathcal{B})$ is called *set-finite* if L_A^1 is finite for all $A \in \mathcal{B}$. For the definition of a representation $(s_a, p_A)_{a \in \mathcal{L}(E^1), A \in \mathcal{B}}$ of $(E, \mathcal{L}, \mathcal{B})$ we refer to Definition 4.1 of [1]. We define $\mathcal{L}^*(E)' = \mathcal{L}^*(E) \cup \{\emptyset\}$ (not used in

[1]), with the convention that $\alpha\emptyset = \emptyset\alpha = \alpha$ for all $\alpha \in \mathcal{L}^*(E)'$ and $r(\emptyset) = 2^{E^0}$.

By Theorem 4.5 of [1], there exists a representation $(S_a, P_A)_{a\in\mathcal{L}(E^1), A\in\mathcal{B}}$ of $(E, \mathcal{L}, \mathcal{B})$ which does not annihilate any of the generators S_α, P_A. Hence the generators p_A, s_a of the universal representation $C^*(E, \mathcal{L}, \mathcal{B})$ (also denoted by $C^*(s_a, p_A)$) of $(E, \mathcal{L}, \mathcal{B})$ are all nonzero.

In the rest of this section *we will assume that the graph E has no sink* (that is, every vertex emits an edge).

Lemma 4.1. *If $A, B \in \mathcal{B}$ and $B \subsetneqq A$ then $p_A - p_B > 0$ and $P_A - P_B > 0$.*

Proof. By the proof of Theorem 4.5 of [1] there is a representation S_a, P_A of $(E, \mathcal{L}, \mathcal{B})$ such that P_A is the orthogonal projection onto

$$\mathcal{H}_A = \bigoplus_{b\in L^1_A} \bigoplus_{v\in s(b)\cap A} \bigoplus_{\{e\in\mathcal{L}^{-1}(b) \mid s(e)=v\}} \mathcal{H}_{(b,e)},$$

where each $\mathcal{H}_{(b,e)}$ is a copy of an infinite dimensional Hilbert space. By assumption there is a $v \in A\backslash B$, which (by assumption) also emits an edge $e \in E^1$ with label $b = \mathcal{L}(e)$. Clearly, the vertex v distinguishes $\mathcal{H}_A$ from $\mathcal{H}_B$. □

We give here a combinatorial (though rather technical) condition for a labelled space.

Definition 4.2. A labelled space $(E, \mathcal{L}, \mathcal{B})$ is called *cancellable* if for all $\alpha, \mu, \nu \in \mathcal{L}^*(E)'$ with $|\mu| \neq |\nu|$, and all $A, B, X \in \mathcal{B}$ with $B \subseteq A$ and

$$r(\alpha) \cap A \neq r(\alpha) \cap B,$$

there exist $\beta \in \mathcal{L}^*(E)'$ with $\alpha\beta \in \mathcal{L}^*(E)'$ and $\hat{A}, \hat{B} \in \mathcal{B}$ with $B \subseteq \hat{B} \subseteq \hat{A} \subseteq A$ such that

$$U := r(\alpha\beta) \cap r(\hat{A}, \beta) \neq V := r(\alpha\beta) \cap r(\hat{B}, \beta),$$

and

$$(p_U - p_V) s^*_{\alpha\beta} s_\mu p_X s^*_\nu s_{\alpha\beta} (p_U - p_V) = 0.$$

It is not difficult to check that the usual aperiodicity condition (that is, condition (L) in [8]) implies cancelability in case that E is an ordinary directed graph (meaning that $\mathcal{L} : E^1 \to E^1$ is the identity map): Indeed, if $r(\alpha)\cap A \neq r(\alpha)\cap B$ then $r(\alpha) \in A\backslash B$ and so $s(\beta) = r(\alpha) \notin B$, which shows that $U \neq V$ if we choose $\hat{A} = A$ and $\hat{B} = B$. Further, we choose an "aperiodic" continuation $\alpha\beta$ of α such that $|\alpha\beta| \geq |\mu| \vee |\nu|$ and $s^*_{\alpha\beta} s_\mu s^*_\nu s_{\alpha\beta} = 0$. This proves the claim.

Theorem 4.3. *The C^*-algebra of a cancellable set-finite weakly left-resolving labelled space $(E, \mathcal{L}, \mathcal{B})$ satisfies the Cuntz-Krieger uniqueness theorem: a C^*-algebra $C^*(s'_a, p'_A)$ generated by a representation (s'_a, p'_A) of $(E, \mathcal{L}, \mathcal{B})$ is canonically isomorphic to the universal C^*-algebra $C^*(s_a, p_A)$ if $p'_A - p'_B \neq 0$ for all $A, B \in \mathcal{B}$ with $B \subsetneq A$.*

Proof. Consider the alphabet $\mathcal{A} = \{ p_A, s_a \mid A \in \mathcal{B}, a \in \mathcal{L}(E^1) \}$. Let $\sigma : \mathbb{F} \to C^*(E, \mathcal{L}, \mathcal{B})$ be the canonical map into the universal C^*-algebra. Write $\mathbb{I}$ for the kernel of σ. It is clear that the canonical map $\pi_1 : \mathbb{F}/\mathbb{I} \to C^*(E, \mathcal{L}, \mathcal{B})$ derived from σ is injective, all identities introduced in [1, Definition 4.1] hold in $\mathbb{F}/\mathbb{I}$, and there is a canonical representation $\pi_2 : \mathbb{F}/\mathbb{I} \to C^*(s'_a, p'_A)$. Recall that each word $x \in W \subseteq \mathbb{F}/\mathbb{I}$ allows a representation $x = s_\alpha p_A s^*_\beta$ $(\alpha, \beta \in \mathcal{L}^*(E)', A \in \mathcal{B})$ (with the convention that $s_\emptyset = 1$). Then the system $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ satisfies property (A) for

$$H = \{(\lambda_a) \in \mathbb{T}^{\mathcal{L}(E^1) \cup \mathcal{B}} | \, \lambda_a = \lambda_b \text{ if } a, b \in \mathcal{L}(E^1), \, \lambda_c = 1 \text{ if } c \in \mathcal{B}\},$$

as by [1], after Lemma 4.8, there is a gauge action on $C^*(E, \mathcal{L}, \mathcal{B})$. One computes that $H \cong \mathbb{T}$, and $\mathrm{bal}(s_\alpha p_A s^*_\beta) \cong |\alpha| - |\beta|$ under the identification $\hat{H} \cong \mathbb{Z}$. Moreover, the system $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ satisfies property (B) (see the proof of [1, Theorem 5.3]), and the conditions (D)(a)-(D)(b) (follows immediately from Lemma 3.1.(4)).

We are going to check condition (C*). Given a $p \in \mathbb{P}$, there is some $v = \sum_{i=1}^k \gamma_k x_i x_i^*$ $(\gamma_i \in \mathbb{C}, x_i \in W)$ in $\mathbb{P}_0$ such that $v \precsim p$ by Proposition 3.3.(iii). By choosing a common refinement of the commuting $x_i x_i^*$'s and $(1 - x_i x_i^*)$'s $(1 \leq i \leq k)$, we may choose a $u \leq v$, $u \in \mathbb{P}_0$, of the form

$$u = s_{a_0} p_{A_0} s^*_{a_0} (1 - s_{a_1} p_{A_1} s^*_{a_1}) \ldots (1 - s_{a_n} p_{A_n} s^*_{a_n}) \tag{7}$$

for some $a_i \in \mathcal{L}^*(E)', A_i \in \mathcal{B}$. Using set-finiteness and Definition 4.1.(iv) of [1], we get

$$p_A = p_A \sum_{a \in \mathcal{L}(E^1)} s_a s^*_a, \qquad \forall A \in \mathcal{B}, \tag{8}$$

and the appearing sum becomes actually finite.

Define $N = |a_0| \vee \ldots \vee |a_n| \vee 1$, and rewrite each $s_{a_i} p_{A_i} s_{a_i}^*$ $(1 \le i \le n)$ as a (finite) sum

$$s_{a_i} p_{A_i} s_{a_i}^* = \sum_{\alpha \in \mathcal{L}^*(E^N)} s_\alpha p_{A_{i,\alpha}} s_\alpha^* \tag{9}$$

for some $A_{i,\alpha} \in \mathcal{B}$ by successive application of identity (8), for instance,

$$s_{a_i} p_{A_i} s_{a_i}^* = s_{a_i} p_{A_i} \Big(\sum_{a \in \mathcal{L}(E^1)} s_a s_a^* \Big) s_{a_i}^* = \sum_{a \in \mathcal{L}(E^1)} s_{a_i} s_a p_{r(A_i,a)} s_a^* s_{a_i}^*. \tag{10}$$

Entering the formulas (9) into (7) and then expanding (7) yields

$$u = \sum_{\alpha \in \mathcal{L}^*(E^N)} s_\alpha (p_{A_{0,\alpha}} - p_{A_{0,\alpha} \cap A_\alpha}) s_\alpha^* \tag{11}$$

for some $A_\alpha \in \mathcal{B}$ by the lattice rules for the map $A \mapsto p_A$.

Since there is at least one non-vanishing summand q in (11), we can pick a $q \in \mathbb{P}_2$ such that $q \le u \le v \precsim p$, where

$$\mathbb{P}_2 = \{ s_\alpha (p_A - p_B) s_\alpha^* \mid \alpha \in \mathcal{L}^*(E),\, A, B \in \mathcal{B},\, B \subseteq A \} \backslash \{0\}.$$

So we checked condition (C*)(ii). Note that $s_\alpha (p_A - p_B) s_\alpha^* = 0$ if and only if $p_{r(\alpha)} (p_A - p_B) p_{r(\alpha)} = 0$ if and only if $r(\alpha) \cap A = r(\alpha) \cap B$ by Lemma 4.1. Condition (C*)(ii) shows that the C^*-representations π_1 and π_2, which are faithful on $\{ p_A - p_B \mid A, B \in \mathcal{B}, B \subseteq A \}$ (and hence faithful on $\mathbb{P}_2$ by Lemma 4.1), are automatically faithful on $\mathbb{A}$. Set $W' = W \backslash W_0$ in condition (C*)(i).

Given $x = s_\mu p_X s_\nu^* \in W'$ (where $|\mu| \ne |\nu|$) and $e = s_\alpha (p_A - p_B) s_\alpha^* \in \mathbb{P}_2$, by the cancelability condition we may choose a $\beta \in \mathcal{L}^*(E)'$ and $\hat{A}, \hat{B} \in \mathcal{B}$ such that

$$e \ge p := s_{\alpha\beta} (p_{r(\hat{A},\beta)} - p_{r(\hat{B},\beta)}) s_{\alpha\beta}^* \in \mathbb{P}_2,$$

$$r(\alpha\beta) \cap r(\hat{A}, \beta) \ne r(\alpha\beta) \cap r(\hat{B}, \beta))$$

(this inequality implies $p \ne 0$ by Lemma 4.1), and $pxp = 0$. This verifies condition (C*)(iii). The claim then follows by Lemma 2.3 and Theorem 2.1. ☐

Lemma 4.4. *$C^*(S_a, P_A)$ is represented on a Hilbert space such that, with respect to strong operator topology,*

$$P_A \le \sum_{a \in \mathcal{L}(E^1)} S_a S_a^* \qquad \forall A \in \mathcal{B}. \tag{12}$$

Proof. One checks that the representation constructed in Theorem 4.5 of [1] satisfies the claim. □

Without assuming set-finiteness we can say the following:

Theorem 4.5. *Let $(E, \mathcal{L}, \mathcal{B})$ be a cancellable weakly left-resolving labelled space. Let (S'_a, P'_A) be a representation of $(E, \mathcal{L}, \mathcal{B})$ acting on Hilbert space and satisfying $P'_A \le \sum_{a \in \mathcal{L}(E^1)} S'_a {S'_a}^*$ in the strong operator topology for all $A \in \mathcal{B}$, and assume that $P'_A - P'_B > 0$ for all $B \subsetneq A$. Then $C^*(S_a, P_A)$ and $C^*(S'_a, P'_A)$ are canonically isomorphic.*

Proof. (Sketch.) The proof is quite similar to the proof of Theorem 4.3 with the following adaption. Assume (S_a, P_A) and (S'_a, P'_A) are represented on Hilbert spaces H and H', respectively. Redefine s_a and p_A, where $a \in \mathcal{L}(E^1)$ and $A \in \mathcal{B}$, by $s_a := \bigoplus_{\lambda \in \mathbb{T}} \lambda S_a \oplus \lambda S'_a$ and $p_A := \bigoplus_{\lambda \in \mathbb{T}} P_A \oplus P'_A$, that is, they are represented on the Hilbert space $\bigoplus_{\lambda \in \mathbb{T}} H \oplus H'$. On $C^*(s_a, p_A)$ we can define a gauge action $\Gamma_\mu(s_a) = \mu s_a$, $\Gamma_\mu(p_A) = p_A$ ($\mu \in \mathbb{T}$), as Γ_μ acts only as a shifting operator along the direct sum $\bigoplus_{\lambda \in \mathbb{T}}$. Moreover, one has $p_A \le \sum_{a \in \mathcal{L}(E^1)} s_a s_a^*$ (strong operator topology sum) for all $A \in \mathcal{B}$. We copy then the proof of Theorem 4.3 two times, once putting $(s'_a, p'_A) := (S_a, P_A)$ and the second time putting $(s'_a, p'_A) := (S'_a, P'_A)$. We also replace in the original proof the universal C^*-algebra $C^*(E, \mathcal{L}, \mathcal{B})$ by our redefined C^*-algebra $C^*(s_a, p_A)$. This shows that $C^*(s_a, p_A) \cong C^*(S_a, P_A) \cong C^*(S'_a, P'_A)$. What we still have not said is, how we circumvent the problem that $(E, \mathcal{L}, \mathcal{B})$ is not set-finite. We do this by simply replacing in the identities (8), (9), (10) and (11) the appearing finite sums by (possibly infinite) strong operator topology sums. □

We remark that Bates and Pask also proved a Cuntz–Krieger uniqueness theorem for labelled graph C^*-algebras in [2] completely independently from the author (and vice versa) at the same time (a previous version of this paper appeared as a preprint in November 2007).

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