A CLASS OF HIGHER RANK EXEL-LACA ALGEBRAS

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Abstract. A certain class of higher rank Exel-Laca algebras, satisfying a Cuntz-Krieger type uniqueness theorem, is introduced.

1. Introduction

In this paper we introduce C^*-algebras A where each A is generated by a family $(E_j)_{j \in J}$ of interacting Exel-Laca algebras E_j [EL]. The interaction between distinct Exel-Laca algebras can be different, but must satisfy a special requirement which we call *permutation rules*.

We call these algebras *A higher rank Exel-Laca algebras*. If the transition matrices of the Exel-Laca algebras are finite then we get a higher rank Exel-Laca algebra which usually allows a translation to a higher rank graph algebra [KP]. However, if the Exel-Laca algebras have infinite transition matrices then this translation fails completely. The translation fails also if one allows finitely aligned higher rank graph algebras [RSY]. Hence we are convinced that the class of higher rank Exel-Laca algebras is considerably different from finitely aligned higher rank graph algebras. See Remark 2.18 for the details.

Our main result is a typical Cuntz-Krieger type uniqueness theorem which is stated in Theorem 2.3. The approach to higher rank Exel-Laca algebras in this paper is a continuation of the work in [B1], [B2] and [B3]. We try to encapsulate the idea that a higher rank Exel-Laca algebra A is generated by a number of interacting ordinary Exel-Laca algebras E_j. The discussion of rank two Exel-Laca algebras introduced in [B2] already contains many ideas we use here. But we are considerably more general here, and the axiomatic system introduced below has the following key advantage. In the examples

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given in [B2] one had to explicitly compute the structure of a certain pre-AF-algebra $\mathbb{A}$ (for the definition of $\mathbb{A}$, which is not important to understand now, see below). In the approach of the present paper the structure of $\mathbb{A}$ need no longer be known. This progress is essentially achieved by [B4, Proposition 3.3] and Proposition 4.8.

The class of higher rank Exel-Laca algebras is described by six axioms which we introduce in section two. Afterwards we give various remarks which should help to better understand the axioms. In section three we recall some known examples. We remark that the rank two examples appearing in [B2], which are inspired by shifts of finite type in dimension two, can be generalized to rank $d \in \mathbb{N}$. But we abstain from showing this; this is rather technical since these examples are somewhat complex in higher dimensions. In the last section four we will prove Theorem 2.3.

2. The Axiomatic System

In this section we will introduce the class of higher rank Exel-Laca algebras. We will give various comments about their defining structure and compare it with the defining structure of finitely aligned higher rank graph algebras.

Definition 2.1. A *triple of generators and relations* is a triple $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ where $\mathcal{A}$ is an alphabet, $\mathbb{F}$ is the non-unital $*$-algebra over $\mathbb{C}$ which is freely generated by $\mathcal{A}$, and $\mathbb{I}$ is a two-sided self-adjoint ideal in $\mathbb{F}$.

Let $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ be a triple of generators and relations. We identify $\mathcal{A}$ with a subset of $\mathbb{F}$, and we let $\mathcal{A}^*$ be the set of adjoint letters. For any subset $\mathcal{B} \subseteq \mathcal{A}$ let $\mathcal{B}^{\circledast} = \mathcal{B} \sqcup \mathcal{B}^*$. Let V be a partition of $\mathcal{A}$, i.e. $\mathcal{A} = \bigsqcup_{v \in V} v$. For ease of notations we define two relations on $\mathcal{A}^{\circledast}$:

$a \sim b$ iff there exists some $v \in V$ such that either $a, b \in v$ or $a, b \in v^*$,

$a \parallel b$ iff there exist $v, w \in V$ such that $v \neq w$ and $a \in v^{\circledast}, b \in w^{\circledast}$.

In the sequel, if nothing else is said, we always operate in the quotient $\mathbb{F}/\mathbb{I}$ rather than in $\mathbb{F}$. Hence we simply write x rather than $x + \mathbb{I}$ for elements $x \in \mathbb{F}$; but x is always understood as an element of the quotient $\mathbb{F}/\mathbb{I}$. The set of *words* is the set

$$W = \{ x_1 \ldots x_n \in \mathbb{F}/\mathbb{I} \mid m \geq 1, \, a_i \in \mathcal{A}^{\circledast} \}.$$

We let

$$Q_a := a^* a, \quad P_a = a a^* \qquad \forall a \in \mathcal{A}.$$

We further introduce the $*$-subalgebra

$$\mathbb{A}_{00} \quad = \quad \mathrm{Alg}^*\{\, aa^* \in \mathbb{F}/\mathbb{I} \mid a \in \mathcal{A}^{\circledast} \,\}.$$

We sometimes refer to the following definition as the *axiomatic system.*

Definition 2.2 (Axiomatic System)**.** The triple $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ of generators and relations is said *to generate a higher rank Exel-Laca algebra* if there exists a partition $\mathcal{A} = \bigsqcup_{v \in V} v$ such that the following six properties hold in $\mathbb{F}/\mathbb{I}$.

(1) *Rank one Cuntz-Krieger relations.* For all $v \in V$ there exists a map $s_v : v \times v \to \{0, 1\}$, which we call a *transition matrix*, such that $aa^*a = a$, $Q_a Q_b = Q_b Q_a$, $P_a P_b = \delta_{a,b} P_a$ and $Q_a P_b = s_v(a, b) P_b$ for all $a, b \in v$.

(2) *Permutation rules.* For all $a, b \in \mathcal{A}^{\circledast}$, such that $a \parallel b$, we have $ab = 0$, or there exist $\overline{a}, \overline{b} \in \mathcal{A}^{\circledast}$ such that $\overline{a} \sim a$, $\overline{b} \sim b$,

$$ab = \overline{b}\overline{a} \quad \text{and} \quad \overline{a}b^* = \overline{b}^* a.$$

(3) *Invariance under the gauge actions.* The ideal $\mathbb{I}$ is invariant under the automorphisms

$$\phi_\lambda : \mathbb{F} \to \mathbb{F} : \phi_\lambda(a) = \lambda_v a \qquad \forall a \in v \in V$$

for all $\lambda = (\lambda_v)_{v \in V} \in \mathbb{T}^V$.

(4) *Finiteness property.* For $w \subseteq \mathcal{A}$ and $N \geq 1$ let

$$\begin{aligned} F_{w,N} \quad = \quad & \{\, a_1 \ldots a_n Q_{c_1} \ldots Q_{c_m} b_n^* \ldots b_1^* \in \mathbb{F}/\mathbb{I} \mid \\ & \qquad 0 \leq n \leq N,\ 1 \leq m,\ a_k, b_k, c_k \in w \} \cup \{0\}. \end{aligned}$$

Then we require that for all finite subsets $\{v_1, \ldots, v_m\} \subseteq V$, all finite subsets $u_i \subseteq v_i$, and all integers $N \geq 1$, there exist *finite* subsets $w_i \subseteq v_i$ such that $u_i \subseteq w_i$ and

$$F_{w_i,N} F_{w_j,N} \subseteq \mathrm{lin} F_{w_j,N} F_{w_i,N} \qquad \forall 1 \leq i, j \leq m.$$

($F_{w_j,N} F_{w_i,N}$ denotes the set of products.)

(5) *Projections property.* For all nonzero words $x = x_1 \ldots x_m$ in the letters $x_i \in \mathcal{A}$, all $v \in V$ and all sequences $(a_n)_{n \geq 1} \subseteq v$ there exists $N \geq 1$ such that $xx^* a_1 \ldots a_N a_N^* \ldots a_1^* \neq xx^*$.

(6) *Saturating* $\mathbb{A}_{00}$*-faithful representation.* There exists a Hilbert space $\mathcal{H}$ and a representation $\pi : \mathbb{F}/\mathbb{I} \to B(\mathcal{H})$ such that for all $v \in V$ the strong operator sum $P = \sum_{b \in v} \pi(P_b)$ is a unit for $\pi(\mathcal{A})$, i.e. $P\pi(a) = \pi(a)P = \pi(a)$ for all $a \in \mathcal{A}$. Such a representation π is called *saturating*. Further we require that π is faithful on $\mathbb{A}_{00}$ (we say that π is $\mathbb{A}_{00}$*-faithful*).

The aim of the latter definition is the following Cuntz-Krieger type uniqueness theorem.

Theorem 2.3. *Let* $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ *generate a higher rank Exel-Laca algebra. Let* A_1 *and* A_2 *be* C^**-algebras, and let* $\pi_1 : \mathbb{F}/\mathbb{I} \to A_1$ *and* $\pi_2 : \mathbb{F}/\mathbb{I} \to A_2$ *be* $*$*-homomorphisms with dense images. Let* π_1 *be faithful on* $\mathbb{A}_{00}$*. Then there exists a* $*$*-homomorphism* $\sigma : A_1 \to A_2$ *such that the following diagram commutes*

$$\begin{array}{ccc} \mathbb{F}/\mathbb{I} & \xrightarrow{\pi_1} & A_1 \\ & \overset{\pi_2}{\searrow} & \downarrow \sigma \\ & & A_2 \end{array}$$

By exchanging π_1 and π_2 in the last theorem we see that σ is in fact an isomorphism if π_2 is also faithful on $\mathbb{A}_{00}$. This justifies the following definition.

Definition 2.4. Consider the setting of Theorem 2.3. Then the norm closure $\overline{\pi_1(\mathbb{F}/\mathbb{I})}$ is denoted by $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$ and is called a *higher rank Exel-Laca algebra*. The cardinality of the partition V of the above definition is called the *rank* of $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$. But notice that the rank is not unique in general (think of $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2$).

Corollary 2.5. $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$ *is simple iff each nonzero representation* $\pi_2 : \mathbb{F}/\mathbb{I} \to A$ *into a* C^**-algebra* A *is faithful on* $\mathbb{A}_{00}$*.*

Proof. This is easily checked by applying Theorem 2.3 to $A_1 = \mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$. □

The next corollary is a variant of Theorem 2.3 and is analogous to (and a generalization of) the Cuntz-Krieger uniqueness theorems [B2, 4.3] and [B2, 5.7]. It is a uniqueness theorem for saturating representations, and requires only the first five properties (1)-(5) of the axiomatic system. The proof of this corollary will be done in section 4.

Corollary 2.6. *Let* $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ *satisfy the first four properties (1)-(4) of the axiomatic system. Let* $\pi_i : \mathbb{F}/\mathbb{I} \to B(\mathcal{H}_i)$ *be saturating representations for* $1 \le i \le 2$*. Let* A_i *be the norm closures of* $\pi_i(\mathbb{F}/\mathbb{I})$*. Further assume that* *the projections property, property (5) of the axiomatic system, holds in the image $\pi_1(\mathbb{F}/\mathbb{I})$ rather than in $\mathbb{F}/\mathbb{I}$ (that is, replace each letter $a \in \mathcal{A}$ by the letter $\pi_1(a)$). Then there exists a $*$-homomorphism $\sigma : A_1 \to A_2$ such that $\sigma\pi_1 = \pi_2$ if the following condition holds.*

$$\pi_1(P_{a_1} \dots P_{a_n}) = 0 \quad \Rightarrow \quad \pi_2(P_{a_1} \dots P_{a_n}) = 0 \quad \forall n \geq 1 \, \forall a_i \in \mathcal{A}.$$

σ is an isomorphism if also the reverse implication holds.

In the sequel let $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ generate a higher rank Exel-Laca algebra (if nothing else is said).

Remark 2.7. If the rank one Cuntz-Krieger relations hold and $\pi : \mathbb{F}/\mathbb{I} \to B(\mathcal{H})$ is a saturating representation then for all $v \in V$ and all $a \in v$ we have

$$\pi(Q_a) = \sum_{b \in v} s_v(a, b)\pi(P_b),$$

where the sum is w.r.t. the strong operator topology.

Remark 2.8. We comment on the permutation rules.

(a) Let $a, b \in \mathcal{A}^{\circledast}$ so that $a \parallel b$ and $ab \neq 0$. Then by the permutation rules there exists a pair $(\overline{b}, \overline{a})$ such that $ab = \overline{b}\overline{a}$. We remark that the pair $(\overline{b}, \overline{a})$ need not be unique. In Example 3.4 there exist letters $a, b \in \mathcal{A}$ such that $ab^* = \overline{b}^*\overline{a}$ for different pairs $(\overline{b}^*, \overline{a})$. However, $(\overline{b}, \overline{a})$ is unique if $a, b \in \mathcal{A}$.

(b) Observe that we permit a, b in $\mathcal{A}^{\circledast}$ rather than in $\mathcal{A}$. That means that we can 'permute' adjoint letters a^* with non-adjoint letters b (as long as $a \parallel b$).

(c) If $ab \neq 0$ then the relations $ab = \overline{b}\overline{a}$ and $\overline{a}b^* = \overline{b}^*a$ must really hold *simultaneously*. The deeper reason for this requirement is that then the range projections xx^* of all words $x \in W$ commute among each other (see Lemma 4.4). However, the even stronger and important fact stated in Lemma 4.1 holds.

Remark 2.9. In the usual examples of higher rank Exel-Laca algebras the relations, which are encoded in $\mathbb{I}$, are defined by rank one Cuntz-Krieger relations, some permutation rules, and some relations which can be expressed in $\text{lin}\{ xx^* \in \mathbb{F} \mid x \in W \}$. All these relations are invariant under the gauge actions. So in a usual example axiom (3) is automatically satisfied.

Remark 2.10. If each set $v \in V$ is finite and the first two properties (1)-(2) of the axiomatic system hold, then the finiteness property is automatically satisfied. To prove this one puts $w_i := v_i$ in the finiteness property and applies the permutation rules. Much of the strength of Theorem 2.3 (and many of the complexities of the proof) is due to the fact that the equivalence classes $v \in V$ can be *infinite* sets. That means that each v is the generator set of an Exel-Laca algebra which is a subalgebra of the higher rank Exel-Laca algebra $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$. That is also the reason why we speak about higher rank *Exel-Laca* algebras rather than of Cuntz-Krieger algebras. The finiteness property ensures that the fixed point algebra $\mathbb{A}$ is an AF-algebra. A comparable property in the world of higher rank graph C^*-algebras is the requirement of finite alignment.

Remark 2.11. The projections property can be regarded as the counterpart to the property (I) appearing in [CK], or to the condition in [EL] that a graph has no terminal circuit, or to certain aperiodicity conditions known for graphs in the context of higher rank graph C^*-algebras.

Remark 2.12. Theorem 2.3 still holds if one replaces the last property of the axiomatic system, i.e. the existence of a saturating $\mathbb{A}_{00}$-faithful representation π, by a weaker property which is a purely algebraic characterization of $\mathbb{F}/\mathbb{I}$ (see Remark 4.16).

Remark 2.13. The idea of the axiomatic system is that one starts with a saturating, not necessarily $\mathbb{A}_{00}$-faithful (see the remark below, why), representation π. It is not solved yet how to construct saturating representations if one starts just with rank one Cuntz-Krieger relations and permutation rules. [1]

Remark 2.14. If one is provided with a saturating representation π which is not necessarily faithful on $\mathbb{A}_{00}$, then one can do the following. Let J be the two-sided self-adjoint ideal in $\mathbb{F}/\mathbb{I}$ which is generated by $\ker(\pi|\mathbb{A}_{00})$, and let $\mathbb{F}/\mathbb{J} \cong (\mathbb{F}/\mathbb{I})/J$ canonically. Then consider the triple $(\mathcal{A}, \mathbb{F}, \mathbb{J})$. The canonical map $\tilde{\pi} : \mathbb{F}/\mathbb{J} \to B(\mathcal{H})$ associated to π is then faithful on $\mathbb{A}_{00}$.

We remark that $\mathbb{A}_{00}$ is an abelian algebra, and by a standard argument for abelian algebras generated by projections we can indicate quite concrete relations which induce $\ker(\pi|\mathbb{A}_{00})$, see Lemma 2.17. (At least the relations are of the same structure or 'complexity' as the 'extra relations' (see the next paragraphs for what we mean by that) appearing in [EL] and [RSY].)

[1]On the other hand, in the theory of graph C^*-algebras one starts with a k-graph satisfying certain technical requirements like finite alignment. Notice that it seems also be a quite non-trivial task to construct or find such k-graphs.

This 'trick' (of making π faithful on $\mathbb{A}_{00}$) yields the 'extra relations' introduced by Exel and Laca [EL] (equations (1.3) on page 121), but not used by Cuntz and Krieger [CK], very naturally: the extra relations are encoded in $\ker(\pi|\mathbb{A}_{00})$. This was already proved in [B2]. On the other hand the relations follow quite directly from Lemma 2.17; we will demonstrate this for convenience of the reader in Example 3.3.

In this respect property (6) of the axiomatic system is also comparable to some 'extra relations' (CK) induced by a set $\mathbb{E} \subseteq \mathrm{FE}(\Lambda)$ one requires in the theory of *relative finitely aligned higher rank graph algebras* [RSY], [S] and [BE2]. This comparison suggests itself if one compares the relations (CK) with the relations stated in Lemma 2.17.

For more general Cuntz-Krieger type algebras [B2] and [B4], 'missing relations' would be encoded in $\ker(\pi|\mathbb{A})$ (since there the uniqueness theorems ask for $\mathbb{A}$-faithful representations). However, it is Proposition 4.8 which shows us that in the case of higher rank Exel-Laca algebras the extra relations $\ker(\pi|\mathbb{A})$ are redundant and are already generated by the extra relations $\ker(\pi|\mathbb{A}_{00})$, which is a much smaller set than $\ker(\pi|\mathbb{A})$. [2]

Notice that in the approaches of [EL] and [RSY, S, BE2] all necessary relations are part of the axioms there. We finally point out that the application of Corollary 2.6 does not necessitate to deal with 'extra relations' at all.

The following lemma is often useful with respect to the last remark.

Lemma 2.15. *Let* $\mathbb{J}$ *be induced by* $\ker(\pi|\mathbb{A}_{00})$ *(see Remark 2.14). Then if a property of the first four properties (1)-(4) of the axiomatic system holds for the triple* $(\mathcal{A}, \mathbb{F}, \mathbb{I})$*, then this property is also valid for the triple* $(\mathcal{A}, \mathbb{F}, \mathbb{J})$*.*

[2] We think the idea that missing extra relations are encoded in $\ker(\pi|\mathbb{A}_{00})$, or encoded in $\ker(\pi|\mathbb{A})$ in the more general settings [B2] and [B4], may provide an automatic - or at least systematic - way to find such relations (when starting with an insufficient system of relations). Whereas it may be difficult to find necessary 'extra relations', a search which presumably includes guessing, if starting with insufficiently many relations, we think it is a promising approach to read off the extra relations from a representation: consider a representation π, then the missing relations are $\ker(\pi|\mathbb{A})$; the final task will be to describe $\ker(\pi|\mathbb{A})$ by a possibly small set $X \subseteq \mathbb{F}/\mathbb{I}$ such that X and $\ker(\pi|\mathbb{A})$ generate the same ideal in $\mathbb{F}/\mathbb{I}$ (it will be very helpful for this task if one knows the structure of $\mathbb{A}$ as explicitly as possible). Perhaps this approach could also (at least theoretically) reveal all possible essentially different 'extra relations' which 'close' an insufficient system of relations (if one makes constraints like 'it is only allowed to search in $\mathbb{A}$') in some cases.

Proof. We only prove the invariance of $\mathbb{J}$ under the gauge actions, point (3) of the axiomatic system, since the other points are trivial. Assume that the ideal $\mathbb{I}$ is invariant under the gauge actions. Let $X \in \mathbb{J}$ and $\lambda \in \mathbb{T}^V$. Then X permits a representation $X = \sum_i \alpha_i a_i Y_i b_i$ for scalars $\alpha_i \in \mathbb{C}$ and elements $a_i, Y_i, b_i \in \mathbb{F}$ such that $Y_i + \mathbb{I} \in \ker(\pi|\mathbb{A}_{00})$, and where a_i, b_i might be identities (or omitted in the sum).

We have $\phi_\lambda(aa^*) = aa^*$ for all $a \in \mathcal{A}^{\circledast}$. Therefore we obtain $\phi_\lambda(Y) = Y \in \mathbb{J}$ for all $Y + \mathbb{I} \in \ker(\pi|\mathbb{A}_{00})$. Consequently we have $\phi_\lambda(X) \in \mathbb{J}$ by the above sum representation of X. This shows that $\mathbb{J}$ is invariant under the gauge actions. □

We introduce the notion of an imaginary identity I.

Definition 2.16. Though we do not assume an identity I in $\mathbb{F}/\mathbb{I}$, we sometimes act as if we had one in order to ease notations. We use I only in expressions where I is not really needed, for example $PI := P$ or $(I - P)Q := Q - PQ$. We call I an *imaginary identity* for $\mathbb{F}/\mathbb{I}$.

The following lemma was announced in Remark 2.14.

Lemma 2.17. *Assume that the first two properties (1)-(2) of the axiomatic system hold, and let*

$$\mathbb{W}_{00} := \{ y_1 \dots y_n \in \mathbb{A}_{00} \,|\, n \geq 1, \exists a_i \in \mathcal{A}^{\circledast} : y_i = a_i a_i^* \text{ or } y_i = I - a_i a_i^*, \ \exists 1 \leq k \leq n : y_k = a_k a_k^* \}.$$

(a) Then $\mathbb{A}_{00}$ *is an abelian* $*$*-algebra, and we have* $\mathbb{A}_{00} = \mathrm{lin}(\mathbb{W}_{00})$.

(b) Let $\pi : \mathbb{F}/\mathbb{I} \to B(\mathcal{H})$ *be any representation. Then* π *is injective on* $\mathbb{A}_{00}$ *if and only if* π *is injective on* $\mathbb{W}_{00}$*. Furthermore,* $\ker(\pi|\mathbb{A}_{00}) = \mathrm{lin}(\ker(\pi|\mathbb{W}_{00}))$*. In particular, the 'extra relations'* $\ker(\pi|\mathbb{A}_{00})$ *are already induced by the extra relations* $\ker(\pi|\mathbb{W}_{00})$*, i.e.* $\ker(\pi|\mathbb{A}_{00})$ *and* $\ker(\pi|\mathbb{W}_{00})$ *generate the same ideal in* $\mathbb{F}/\mathbb{I}$.

Proof. It is not difficult to see by the first two properties of the axiomatic system that $\mathbb{A}_{00}$ is an abelian algebra. Alternatively see Lemma 4.4; this causes no problem since the current lemma will not be used for the proof of Lemma 4.4. Let $x \in \mathbb{A}_{00}$. Then $x = \sum_i \alpha_i x_i$ ($\alpha_i \in \mathbb{C}$) is a finite linear combination of words x_i in the letters aa^* where $a \in \mathcal{A}^{\circledast}$. By inductively rewriting the words x_i as $x_i = x_i aa^* + x_i(I - aa^*)$, we can achieve that x is the linear combination $x = \sum_i \beta_i z_i$ ($\beta_i \in \mathbb{C}$, $\beta_i \neq 0$) for elements $z_i \in \mathbb{W}_{00}$ such that $z_i z_j = 0$ for $i \neq j$. Now $\pi(x) = 0$ if and only if $\sum_i \beta_i \pi(z_i) = 0$ if and only if $\pi(z_i) = 0$ for all i. This proves the claim (b). □

Remark 2.18. Let $\mathcal{A}$ be a finite set. In particular the partition $V = \{v_1, \ldots, v_d\}$ of $\mathcal{A}$ is finite. Let π be the saturating $\mathbb{A}_{00}$-faithful representation of the axiomatic system. There exists a higher rank graph Λ such that $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$ is $*$-isomorphic to the higher rank graph algebra $C^*(\Lambda)$ defined in [KP] (see [BE2] for a proof). The object set of Λ is defined by

$$\Lambda^0 := \{\, (a_1, a_2, \ldots, a_d) \in v_1 \times v_2 \times \ldots \times v_d \mid \pi(P_{a_1} P_{a_2} \ldots P_{a_d}) \neq 0 \,\}.$$

We introduce exactly one morphism $\theta^i_{a,b}$ of degree $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with range $a = (a_1, \ldots, a_d) \in \Lambda^0$ and source $b = (b_1, \ldots, b_d) \in \Lambda^0$ in case that

$$0 \neq \pi(P_{a_1} P_{a_2} \ldots P_{a_d} a_i P_{b_1} P_{b_2} \ldots P_{b_d}).$$

If $\pi(P_{a_1} P_{a_2} \ldots P_{a_d} a_i P_{b_1} P_{b_2} \ldots P_{b_d})$ vanishes then there is no morphism of degree e_i with range a and source b. Let Λ be the so associated higher rank graph. Then

$$\begin{aligned} s_a &= \pi(P_{a_1} P_{a_2} \ldots P_{a_d}) \quad \text{if } a = (a_1, \ldots, a_d) \in \Lambda^0, \\ s_{\theta^i_{a,b}} &= \pi(a_i P_{b_1} P_{b_2} \ldots P_{b_d}) \quad \text{for } a, b \in \Lambda^0,\ 1 \leq i \leq d \end{aligned}$$

is a Cuntz-Krieger family in $\pi(X)$ in the sense of [KP], and this family generates the C^*-algebra $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$.

Notice that this approach does not seem to be successful if some partition $v_i \in V$ is infinite. The reason is that $\pi(a) \in \mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$, but in general $\pi(a) \notin C^*(\Lambda)$; $\pi(a)$ could only be expressed as an *infinite* strong operator sum

$$\pi(a) = \sum_{b_1 \in v_1, \cdots, b_d \in v_d} \pi(a P_{b_1} \ldots P_{b_d}) = \sum_{b \in \Lambda^0} s_{\theta^i_{a,b}}.$$

Remark 2.19. For a special subclass of higher rank Exel-Laca algebras we compute the K-theory in [B5]. In this case we have $K_1(\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}) = 0$ and $K_0(\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}) \cong K_0(B)$, where B denotes the abelian C^*-algebra which is the norm closure of $\mathrm{Alg}^*\{\, Q_a \in \mathbb{A} \mid a \in \mathcal{A} \,\}$.

It is useful to get the following picture of the axiomatic system. One usually starts with the rank one Cuntz-Krieger relations, some permutation rules, and some further relations which are induced by $\ker(\pi|\mathbb{A}_{00})$. If one has even further relations, then they must not be too bizarre such that the invariance under the gauge actions is satisfied. Hence, the properties (1), (2), (3) and (6) are the *defining* properties (more precisely: defining the ideal $\mathbb{I}$, or defining the relations), and the properties (4) and (5) are purely *technical* properties.

It might be interesting to weaken the permutation rules in a way that, say, one had $ab = \bar{b}\bar{a}$ for certain $\bar{a} \sim a$ and $\bar{b} \sim b$, however, on the other hand one had a sum of words for the product a^*b, i.e.

$$a^*b = \sum_i \overline{w}_i \overline{v}_i^*$$

for certain words $\overline{w}_i$ and $\overline{v}_i$. This would be more general, and also closer to the graph algebras picture.

3. Examples

Since the present paper is a continuation of the work in [B1] and [B2] there exist already some examples of higher rank Exel-Laca algebras.

Example 3.1 (Higher dimensional Cuntz algebras)**.** The C^*-algebras introduced in [B1] are higher rank Exel-Laca algebras in the sense of this paper. Under a suitable translation these algebras turn out to be special cases of the algebras introduced by Robertson and Steger in [RS1] and [RS2].

Example 3.2 (Robertson-Steger algebras and graph algebras)**.** By the translation of certain higher rank Exel-Laca algebras to graph algebras in Remark 2.18 we can conclude the reverse that some Robertson-Steger algebras [RS1, RS2] and higher rank graph algebras [KP] are higher rank Exel-Laca algebras. But notice that by far not all Robertson-Steger and graph algebras are higher rank Exel-Laca algebras because of an incompatibility with the permutation rules (see the last paragraph of section 2).

Example 3.3 (Rank one Exel-Laca algebras)**.** If the set $V = \{v\}$ consists of a single point v then we get an Exel-Laca algebra [EL]. For a broader discussion on this see also [B2]. We announced in Remark 2.14 that we will show that the 'extra relations' introduced in [EL] are encoded in $\ker(\pi|\mathbb{A}_{00})$ for any saturating representation π satisfying $\pi(P_a) \neq 0$ for all $a \in \mathcal{A}$. Indeed, by the rank one Cuntz-Krieger relations it is easy to see that the set $\mathbb{W}_{00}$ of Lemma 2.17 consists of the elements 0, P_a for $a \in \mathcal{A}$, and

$$\begin{aligned} X \;\; = \;\; \{\, & y_1 \ldots y_n(I-P) \in \mathbb{A}_{00} \,|\, m,n \geq 0, \exists a_i \in \mathcal{A} : y_i = Q_a \text{ or} \\ & y_i = I - Q_a,\, P = \sum_{k=1}^m P_{b_k} \text{ for different } b_k \in \mathcal{A}, \\ & m = 0 \Rightarrow \exists k : y_k = Q_{a_k} \}. \end{aligned}$$

By replacing $\pi(Q_a)$ by the strong operator sum $\sum_b A(a,b)\pi(P_b)$, it is easy to give conditions when $\pi(x) = 0$ for $x \in X$ by means of the transition matrix A; then it turns out that $\ker(\pi|X)$ coincides with the extra relations introduced by Exel and Laca, see also the discussion in [B2, p. 224].

Example 3.4 (Rank two Exel-Laca algebras inspired by shift spaces)**.** In [B2] we introduced special rank two Exel-Laca algebras which are inspired by shifts of finite type in dimension two. It is relatively easy to show that these algebras are rank two Exel-Laca algebras in the sense of the present paper. (The proof that these algebras satisfy a uniqueness theorem is considerably longer in [B2].) We will briefly recall the definition of the alphabet and the permutation rules of these algebras. Let Ω be an arbitrary set, and consider two alphabets v_1 and v_2 which are copies of $\Omega^{\mathbb{N}}$. Let $\mathcal{A} = v_1 \sqcup v_2$ and $V = \{v_1, v_2\}$. We introduce the permutation rules

$$\begin{array}{rcll} (a_1a_2a_3\ldots)(b_1b_2b_3\ldots) & = & (a_1b_1b_2\ldots)(a_2a_3a_4\ldots), & \\ (a_1a_2a_3\ldots)^*(b_1b_2b_3\ldots) & = & (b_2b_3b_4\ldots)(a_2a_3a_4\ldots)^* & \text{if } a_1 = b_1, \\ (a_1a_2a_3\ldots)^*(b_1b_2b_3\ldots) & = & 0 & \text{if } a_1 \neq b_1 \end{array}$$

for all $(a_1a_2\ldots) \in v_1$ and $(b_1b_2\ldots) \in v_2$, and $a_k, b_k \in \Omega$.

Let $p_n(a) = a_n$ be the n-th entry of the letter $(a_1a_2a_3\ldots) \in \Omega^{\mathbb{N}}$. In order to check the finiteness property fix an integer $N \geq 1$ and finite subsets $u_1 \subseteq v_1$ and $u_2 \subseteq v_2$. Then the permutation rules show that the finite sets

$$\begin{aligned} w_i \;=\; \{\, & (x_1 \ldots x_N a_{N+1} a_{N+2} \ldots) \in v_i \,|\, (a_1a_2a_3\ldots) \in u_1 \cup u_2, \\ & x_1, \ldots, x_N \in \bigcup_{n=1}^{N} p_n(u_1 \cup u_2) \,\} \end{aligned}$$

satisfy the requirement of the finiteness property that

$$F_{w_1,N} F_{w_2,N} \subseteq F_{w_2,N} F_{w_1,N}.$$

Note that we also strengthen the results of [B2] here in that the alphabet Ω may be infinite.

Other examples of higher rank Exel-Laca algebras could be constructed by considering not only two alphabets v_1, v_2 but a whole family (v_i) of alphabets v_i which are copies of $\Omega^{\mathbb{N}}$ and using the above permutation rules. But we know nothing about $\mathbb{A}_{00}$-faithful saturating representations.

Example 3.5. In the last example we briefly recalled rank two Exel-Laca algebras which where introduced in [B2], and which are inspired by shifts of finite type in dimension two. The whole construction would also work in higher dimensions. That is, we obtain rank $d \geq 2$ Exel-Laca algebras which are associated to shifts of finite type in dimension d. (However, unfortunately the 'association' is not natural in the sense that topologically conjugated shifts of finite type yield isomorphic higher rank Exel-Laca algebras. There is some basic 'flaw' in the construction in that the topology of the associated higher rank Exel-Laca algebra does not reflect the topology of the underlying shift of finite type. The same phenomenon appears if one tries to associate Exel-Laca algebras to one-dimensional shifts of finite type with an infinite alphabet.) The whole example is somewhat technical in dimensions $d > 2$, so we abstain from a detailed discussion.

Example 3.6. The class of higher rank Exel-Laca algebras is closed under tensor products. By the uniqueness theorem 2.3 we have no choice for the C^*-tensor product. In fact, higher rank Exel-Laca algebras are always nuclear ([BE1]).

4. Proof of Theorem 2.3

In this section we aim to prove Theorem 2.3. We will show that a triple $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ generating a higher rank Exel-Laca algebra satisfies the properties (A), (B) and (D) defined in [B4]. Then Theorem 2.3 follows from [B4, Corollary 3.4].

Throughout let $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ be a triple of generators and relations generating a higher rank Exel-Laca algebra. That means that we have a partition $\mathcal{A} = \bigsqcup_{v \in V} v$, a family of transition matrices $(s_v)_{v \in V}$, and a saturating $\mathbb{A}_{00}$-faithful representation $\pi : \mathbb{F}/\mathbb{I} \to B(\mathcal{H})$ as in Definition 2.2. We fix these data throughout of this section. Let

$$H = \{ (\mu_a)_{a \in \mathcal{A}} \in \mathbb{T}^{\mathcal{A}} \mid \mu_a = \mu_b \text{ whenever } a \sim b \}.$$

Let $(\mu_a)_{a \in \mathcal{A}} \in H$. Let $\gamma_\mu : \mathbb{F} \to \mathbb{F}$ be the automorphism given by $\gamma_\mu(a) = \mu_a a$ for $a \in \mathcal{A}$. By the gauge actions property (3) of the axiomatic system it is clear that $\mathbb{I}$ is also invariant under the actions γ_μ for all $\mu \in H$. Hence property (A) of [B4] holds for the group H. Hence there exists a balance function bal as defined in [B4]. Identifying the multiplicative group $(\widehat{H}, \cdot, 1)$ with the additive group $(\mathbb{Z}^V, +, 0)$ (notice that $H \cong \mathbb{T}^V$), the balance function is the involutive semigroup homomorphism

$$\mathrm{bal} : W \backslash \{0\} \to \mathbb{Z}^V : \mathrm{bal}(a) = 1_v \qquad \forall a \in v \in V,$$

satisfying $\mathrm{bal}(xy) = \mathrm{bal}(x) + \mathrm{bal}(y)$, and $\mathrm{bal}(x^*) = -\mathrm{bal}(x)$ for all words $x, y \in W$ such that $xy \neq 0$. ($1_v(v) = 1$ and $1_v(w) = 0$ for $w \neq v$, $w \in V$.)

We introduce the following distinguished sets.

$$\begin{aligned} W_0 &:= \{w \in W\backslash\{0\} \mid \mathrm{bal}(w) = 0\}, \\ \mathbb{A} &:= \mathrm{lin}(W_0) = \mathrm{Alg}^*(W_0) \subseteq \mathbb{F}/\mathbb{I}, \\ \mathbb{A}_0 &:= \mathrm{Alg}^*\{\, xx^* \in \mathbb{F}/\mathbb{I} \mid x \in W\,\}, \\ \mathbb{P} &:= \{p \in \mathbb{A} \mid p = p^* = p^2 \neq 0\}, \\ \mathbb{P}_0 &:= \{p \in \mathbb{A}_0 \mid p = p^* = p^2 \neq 0\}. \end{aligned}$$

The elements of W_0 are called *zero-balanced words.*

We recall that if nothing else is said then all computations are done in the quotient $\mathbb{F}/\mathbb{I}$. In the proofs of 4.1-4.8 we use only the first three properties (1)-(3) of the axiomatic system.

Lemma 4.1. *Let $a, b_1, \ldots, b_n \in \mathcal{A} \sqcup \mathcal{A}^*$ with $a \parallel b_i$ for all $i = 1, \ldots, n$. Then there exist $\overline{b}_i \sim b_i$ such that, if $ab_1 \ldots b_n \neq 0$,*

$$ab_1 \ldots b_n b_n^* \ldots b_1^* = \overline{b}_1 \ldots \overline{b}_n \overline{b}_n^* \ldots \overline{b}_1^* a.$$

Proof. Let $n = 1$. By the permutation rules there exist $\overline{a} \sim a$ and $\overline{b}_1 \sim b_1$ such that $ab_1 = \overline{b}_1\overline{a}$ and $\overline{a}b_1^* = \overline{b}_1^* a$. Hence we get

$$ab_1b_1^* = \overline{b}_1\overline{a}b_1^* = \overline{b}_1\overline{b}_1^* a.$$

By induction hypothesis we assume that the claim holds for $n-1$. Then by the permutation rules there exist $\overline{a} \sim a$ and $\overline{b}_1 \sim b_1$ such that $ab_1 = \overline{b}_1\overline{a}$ and $\overline{a}b_1^* = \overline{b}_1^* a$, and by the induction hypothesis there exist certain $\overline{b}_i \sim b_i$ for $i \geq 2$ such that we have

$$\begin{aligned} &ab_1(b_2 \ldots b_n b_n^* \ldots b_2^*)b_1^* = \overline{b}_1\overline{a}(b_2 \ldots b_n b_n^* \ldots b_2^*)b_1^* \\ &= \overline{b}_1(\overline{b}_2 \ldots \overline{b}_n \overline{b}_n^* \ldots \overline{b}_2^*)\overline{a}b_1^* = \overline{b}_1(\overline{b}_2 \ldots \overline{b}_n \overline{b}_n^* \ldots \overline{b}_2^*)\overline{b}_1^* a. \end{aligned}$$

□

Definition 4.2. We say a word $x = a_1 \ldots a_n$ ($a_i \in \mathcal{A}^\circledast$) is represented as an *ordered word*, if the letters a_i are sorted by the partition V. More precisely we require that for all $1 \leq i_1 \leq i_2 \leq i_3 \leq n$ we have the following: if $a_{i_1}, a_{i_3} \in v^\circledast$ for some $v \in V$ then also $a_{i_2} \in v^\circledast$.

Lemma 4.3. *Let $x \in W\backslash\{0\}$ be a nonzero word. Then*

(a) x is representable as an ordered word;

(b) x admits a representation

$$x = a_1 \ldots a_n Q_{c_1} \ldots Q_{c_k} b_m^* \ldots b_1^* \tag{1}$$

for some $a_i, b_i, c_i \in \mathcal{A}$, $0 \leq n, m$ *and* $1 \leq k$, *such* $a_1 \ldots a_n$ *and* $b_m^* \ldots b_1^*$ *are ordered words;*

(c) if x *is zero-balanced then* $m = n$ *in (b), and we can choose* a_i, b_i *in such a way that* $a_i \sim b_i$ *for all* $1 \leq i \leq n$.

Proof. (a) It is easy to realize that by an successive application of the permutation rules we can permute neighboring letters x_i and x_{i+1} in a nonzero word $x = x_1 \ldots x_n$ ($x_i \in \mathcal{A}^{\circledast}$) until the letters $x_1, \ldots, x_n$ are sorted by the partition V, and so x is represented as an ordered word.

(b) By (a) x admits a representation $x = A_1 \ldots A_d$ where A_i is a word in the letters of $v_i^{\circledast}$ and $v_1, \ldots, v_d \in V$. At first suppose that $x = A_1$. Then x is a word in the rank one Cuntz-Krieger algebra generated by the alphabet v_1, and (b) follows from [B2, Lemma 4.1].

Next suppose that $x = A_1 \ldots A_d$. By what we have proved so far we can choose representations

$$A_i = a_{i,1} \ldots a_{i,n_i} Q_{c_{i,1}} \ldots Q_{c_{i,k_i}} b_{i,m_i}^* \ldots b_{i,1}^*$$

for all $1 \leq i \leq d$ and some $a_{i,j}, b_{i,j}, c_{i,j} \in v_i$. Then, by using the permutation rules and Lemma 4.1 we can move all letters $a_{i,j}$ to the left and all adjoint letters $b_{i,j}^*$ to the right in the product $A_1 \ldots A_N$. Let us demonstrate this by a small example where $d = 2$:

$$\begin{aligned} A_1 A_2 &= a_{1,1} Q_{c_{1,1}} b_{1,1}^* a_{2,1} Q_{c_{2,1}} b_{2,1}^* \\ &= a_{1,1} Q_{c_{1,1}} \overline{a}_{2,1} \overline{b}_{1,1}^* Q_{c_{2,1}} b_{2,1}^* \\ &= a_{1,1} \overline{a}_{2,1} Q_{\overline{c}_{1,1}} Q_{\overline{c}_{2,1}} \overline{b}_{1,1}^* b_{2,1}^*. \end{aligned}$$

In this way we obtain the representation (1) for x. Finally, if necessary, by an application of the permutation rules we can sort the letters of the words $a_1 \ldots a_n$ and $b_1 \ldots b_m$ by the partition V such that these words represent ordered words.

(c) For a nonzero zero-balanced word x as represented in (1) we have

$$\begin{aligned} 0 = \mathrm{bal}(X) &= \mathrm{bal}(a_1 \ldots a_n) + \mathrm{bal}(Q_{c_1}) + \ldots + \mathrm{bal}(Q_{c_k}) + \mathrm{bal}(b_m^* \ldots b_1^*) \\ &= \mathrm{bal}(a_1 \ldots a_n) - \mathrm{bal}(b_1 \ldots b_m). \end{aligned}$$

Hence we have $n = m$, and for each $v_i \in V$ we have

$$|\{a_1, \ldots, a_n\} \cap v_i| = |\{b_1, \ldots, b_n\} \cap v_i|.$$

So we can apply the permutation rules to the words $a_1 \ldots a_n$ and $b_1 \ldots b_n$ to ensure that $a_i \sim b_i$ for all $i = 1, \ldots, n$. □

Lemma 4.4. *All words $x \in W$ are partial isometries, i.e. $xx^*x = x$, and their range and source projections commute among each other. In particular, $\mathbb{A}_{00}$ and $\mathbb{A}_0$ are abelian $*$-algebras, and $\mathbb{A}_0 = \text{lin}\{ xx^* \in \mathbb{F}/\mathbb{I} \mid x \in W \}$.*

Proof. If we have a rank one Exel-Laca algebra, i.e. if V consists of a single partition $V = \{v_1\}$, then the claim is already proved in [B2, Lemma 4.1].

We will proof the claim where V consists only of two parts $V = \{v, w\}$. The proof of the general case is similar. By induction hypothesis on $(n, m) \in \{1, \ldots, N\}^2$ (lexicographical order) let xx^* and yy^* commute for all words $x = a_1 \ldots a_n$ and $y = b_1 \ldots b_m$ where $a_i \in v^{\circledast}$ and $b \in w^{\circledast}$. (If $xx^*yy^* = 0$, then also $yy^*xx^* = 0$ by taking the adjoint.) Let $a \in v^{\circledast}$. Then by Lemma 4.1 we find some $\overline{y} = \overline{b}_1 \ldots \overline{b}_m$ such that $a^*yy^* = \overline{y}\overline{y}^*a^*$, and thus

$$axx^*a^*yy^* = axx^*\overline{y}\overline{y}^*a^* = a\overline{y}\overline{y}^*xx^*a^* = yy^*axx^*a^*.$$

This completes the induction. The base case $(n, m) = (1, 1)$ follows analogoulsy by ommiting xx^* in the above equations.

Now assume that x_1, x_2 are words in the alphabet $\mathcal{A}^{\circledast}$. By Lemma 4.3 we can choose an ordered representation $x_i = A_iB_i$ such that A_i is a word in the letters of $v^{\circledast}$ and B_i is a word in the letters of $w^{\circledast}$ $(i = 1, 2)$. Then by Lemma 4.1 we have $x_ix_i^* = A_iB_iB_i^*A_i^* = A_iA_i^*\overline{B}_i\overline{B}_i^*$ for some word $\overline{B}_i$ in the letters of $w^{\circledast}$. Hence, by what we have proved before we get

$$x_1x_1^*x_2x_2^* = A_1A_1^*\overline{B}_1\overline{B}_1^*A_2A_2^*\overline{B}_2\overline{B}_2^* = A_2A_2^*\overline{B}_2\overline{B}_2^*A_1A_1^*\overline{B}_1\overline{B}_1^* = x_2x_2^*x_1x_1^*.$$

By a similar computation we see that x is a partial isometry. Finally, $\mathbb{A}_0$ is the linear span of $\{ xx^* \mid x \in W \}$ since the latter set is closed under multiplication: for words xx^* and yy^* we have $xx^*yy^* = (yy^*x)(yy^*x)^*$. □

Lemma 4.5. *If $a_i, b_i \in \mathcal{A}$ such that $a_i \sim b_i$ for all $i = 1, \ldots, n$, then there exist certain $c_j \in \mathcal{A}$ such that*

$$a_n^* \ldots a_1^*b_1 \ldots b_n = \delta_{a_1,b_1} \ldots \delta_{a_n,b_n} Q_{c_1} \ldots Q_{c_m} \qquad (\textit{if } b_1 \ldots b_n \neq 0).$$

Proof. If $n = 1$ then $a_1^*a_1 = Q_{a_1}$ and $a_1^*b_1 = a_1^*P_{a_1}P_{b_1}b_1 = 0$ for $a_1 \neq b_1$. By induction hypothesis assume that the claim holds for fixed $n \geq 1$. Then we have

$$\begin{aligned} a^*a_n^* \ldots a_1^*b_1 \ldots b_nb &= a^*\delta_{a_1,b_1} \ldots \delta_{a_n,b_n}Q_{c_1} \ldots Q_{c_m}b \\ &= \delta_{a_1,b_1} \ldots \delta_{a_n,b_n} \ \delta_{a,b}a^*b \ Q_{c'_1} \ldots Q_{c'_M} \end{aligned}$$

for certain $c'_1, \ldots, c'_M \in \mathcal{A}$ by successively 'moving' the letter b to the left. More precisely, when b successively 'skips' Q_{c_i}, then the following situations can occur. First, if $c_i \parallel b$ and $Q_{c_i} b \neq 0$, then b can 'skip' Q_{c_i} by Lemma 4.1. Second, if $c_i \sim b$, then $Q_{c_i} b = Q_{c_i} P_b b = b$, if $Q_{c_i} b \neq 0$, by the the rank one Cuntz-Krieger relations. However, if $Q_{c_i} b = 0$ then this is no problem for the case that $a_j \neq b_j$ for some j or $a \neq b$, because in this case we should anyway get zero. If, however, $a_j = b_j$ for all j and $a = b$, then $Q_{c_i} b$ must always be nonzero since we assume $0 \neq b_1 \dots b_n b$, and thus $b^* b_n^* \dots b_1^* b_1 \dots b_n b$ cannot vanish. □

Corollary 4.6. *A zero-balanced word $x \in W_0$ satisfying $xx \neq 0$ is a projection.*

Proof. Consider the representation (1) of Lemma 4.3.(c) for a zero-balanced word x. If $xx \neq 0$ then we have $a_i = b_i$ for all i by Lemma 4.5. Hence $x = yy^*$ for the partial isometry $y = a_1 \dots a_n Q_{c_1} \dots Q_{c_k}$ (Lemma 4.4). □

Lemma 4.7. *Let $v \in V$, let $\mathcal{B} \subseteq v$ be a finite subset and $a \in (\mathcal{A} \backslash v)^{\circledast}$. Then there exists a (possibly empty) finite subset $\mathcal{C} \subseteq v$ such that $\sum_{b \in \mathcal{B}} P_b a = a \sum_{c \in \mathcal{C}} P_c$.*

Proof. By the permutation rules and Lemma 4.1 $P_b a$ either vanishes or $P_b a = a P_c$ for some letter $c \in v$. So we get $\sum_{b \in \mathcal{B}} P_b a = a \sum_{i=1}^n P_{c_i}$ for some $n \geq 0$ and $c_i \in v$. We only have to show that the c_i are distinct. Assume that we had $0 \neq P_{b_1} a = a P_c = P_{b_2} a$ for $b_1 \neq b_2$. Then $P_{b_1} a = P_{b_1} P_{b_1} a = P_{b_1} P_{b_2} a = 0$, a contradiction. □

Proposition 4.8. *Suppose that the first three properties (1)-(3) of the axiomatic system hold (these are the rank one Cuntz-Krieger relations, the permutation rules and the invariance under the gauge actions). Let $\Phi : \mathbb{F}/\mathbb{I} \to A$ be a homomorphism into a ring A. If then Φ is injective on $\mathbb{A}_{00}$ then Φ is injective on $\mathbb{A}$.*

Proof. Let $X \in \mathbb{A}$ and suppose that $\Phi(X) = 0$. We must show by induction that $X = 0$. We will give a brief overview of the lengthy inductive proof.

- At first we rewrite X in a nice form using Lemma 4.3.(c); in particular, only finitely many different 'shapes' of words appear in this expression.
- Next we define a total order on the 'shapes' of words appearing in this expression for X; since there were only finitely many shapes involved, there exists $(N, \dots, N)$ which dominates them all.
- Next (this is the bulk of the proof) we show that if we suppose that n is the smallest shape occurring in our expression for X, and

$n \neq (N, \ldots, N)$, then the sum of those terms of shape n which cannot themselves be rewritten as sums of terms of shapes m where $n < m \leq (N, \ldots, N)$ is equal to 0.

- The induction is completed when we show that n is the successor of $(N, \ldots, N)$, which proves that X is the empty sum, or in other words, $X = 0$.

First proof part: base case of the induction. By Lemma 4.3 there exist an integer $N \geq 0$ and a finite set $\widehat{V} := \{v_1, \ldots, v_M\} \subseteq V$ such that X is a linear combination of words of the set

$$\begin{aligned} F \;\; = \;\; \{\, & a_1 \ldots a_s Q_{c_1} \ldots Q_{c_k} b_s^* \ldots b_1^* \in W \mid 0 \leq s \leq N,\, 1 \leq k, \\ & a_i, b_i, c_i \in \widehat{\mathcal{A}},\, \mathrm{bal}(a_1 \ldots a_s) = \mathrm{bal}(b_1 \ldots b_s) \,\}, \end{aligned}$$

where $\widehat{\mathcal{A}} := v_1 \sqcup \ldots \sqcup v_M$. Let $\widehat{W} := \{\, a_1 \ldots a_m \in W \mid m \geq 1, a_i \in (\widehat{\mathcal{A}})^{\circledast} \,\}$ be the set of words in the letters of $(\widehat{\mathcal{A}})^{\circledast}$. Restricting the balance functions to the words $\widehat{W} \backslash \{0\}$ we obtain, or define,

$$\widehat{\mathrm{bal}} : \widehat{W} \backslash \{0\} \to \mathbb{Z}^M : \widehat{\mathrm{bal}} = p \circ \mathrm{bal},$$

where $p : \bigoplus_{v \in V} \mathbb{Z} \to \bigoplus_{v \in \widehat{V}} \mathbb{Z} = \mathbb{Z}^M$ is the projection which maps the coordinate v_i to the i-th coordinate of $\mathbb{Z}^M$.

We endow the set $\gamma := \{0, \ldots, N\}^M \subseteq \mathbb{Z}^M$ with its lexicographical order $\precsim$ where the first coordinate (corresponding to v_1) is the most significant one and the last coordinate (corresponding to v_M) is the least significant one. Notice that $\precsim$ defines a total order on γ and we can enumerate the elements of γ. For $n = (n_1, \ldots, n_M) \in \gamma$ let

$$\phi_n := \{ \prod_{i=1}^{M} \prod_{j=1}^{n_i} a_{i,j} \in \mathbb{F}/\mathbb{I} \mid a_{i,j} \in v_i \}.$$

Thereby we put $\phi_0 := \{I\}$ where I is an imaginary identity (Definition 2.16). Notice that $\widehat{\mathrm{bal}}(x) = n$ for $x \in \phi_n$. For $n \in \gamma$ let L_n be the linear hull

$$L_n := \mathrm{lin}\{\, a Q_{c_1} \ldots Q_{c_k} b^* \in \mathbb{F}/\mathbb{I} \mid a, b \in \phi_n, k \geq 1, c_i \in \widehat{\mathcal{A}} \}.$$

Notice that by Lemma 4.3 we can order the words $a_1 \ldots a_s$ and $b_1 \ldots b_s$ appearing in the definition of F, if necessary, and so we have

$$F \subseteq \bigcup_{m \in \gamma} L_m.$$

Hence we have a representation

$$X = \sum_{m \in \gamma} X_m$$

for certain $X_m \in L_m$. This shows the base case of the induction, i.e. the case $n = 0 \in \gamma$, because $m \gtrsim 0$ in the above sum.

Second proof part: inductive step. Step 1. Let $n \in \gamma$. Assume by induction hypothesis on $n \in \gamma$, with respect to the total order $\precsim$ on γ, that there exist elements $X_m \in L_m$ for all $m \in \gamma$ and $m \gtrsim n$, such that X allows a representation

$$X = \sum_{n \precsim m \in \gamma} X_m. \tag{2}$$

Step 2. In this step we will show that X allows a representation

$$X = \sum_{n \precsim m \in \gamma} Y_m \tag{3}$$

for certain $Y_m \in L_m$ such that

$$Y_n = \sum_{a,b \in \Gamma} a U_{a,b} b^*$$

for certain $U_{a,b} \in q$ with the property that, provided that $n \neq (N, \ldots, N)$, we have (for the definitions of Γ, q and p_a see below)

$$aU_{a,b}b^* \neq 0 \quad \Rightarrow \quad \nexists Y \in p_a \text{ which is a unit for } a^*ab^*bU_{a,b}. \tag{4}$$

Indeed, by the definition of L_n, X_n can be written as a finite sum

$$X_n = \sum_{a,b \in \Gamma} a Z_{a,b} b^*, \tag{5}$$

where $\Gamma \subseteq \phi_n$ is some finite subset of ϕ_n, and each $Z_{a,b}$ is some element of the algebra

$$q := \mathrm{Alg}\{\, Q_c \in \mathbb{F}/\mathbb{I} \mid c \in \widehat{\mathcal{A}}\,\}.$$

Next fix $a, b \in \Gamma$. For all $1 \leq i \leq M$ let

$$\varepsilon_i := \begin{cases} 1 & \text{if } \widehat{\mathrm{bal}}(a)_i < N \\ 0 & \text{if } \widehat{\mathrm{bal}}(a)_i = N, \end{cases} \tag{6}$$

where $\widehat{\mathrm{bal}}(a)_i$ denotes the i-th coordinate of $\widehat{\mathrm{bal}}(a)$. Let

$$p_a := \mathrm{Alg}\{\, P_c \in \mathbb{F}/\mathbb{I} \mid c \in v_1^{\varepsilon_1} \sqcup \ldots \sqcup v_M^{\varepsilon_M}\,\},$$

where per definition $v_i^0 := \emptyset$ and $v_i^1 := v_i$. Assume that $n \neq (N, \ldots, N)$. Then we want to prove the following property:

$$(\exists Y \in p_a \text{ which is a unit for } a^*ab^*bZ_{a,b}) \quad \Rightarrow \quad aZ_{a,b}b^* \in \sum_{n \prec m \in \gamma} L_m. \tag{7}$$

(Here $\prec$ means $\precsim$ but not equal.) Assume that $Y \in p_a$ and Y is a unit for $a^*ab^*bZ_{a,b}$. By Lemma 4.4 we have

$$aZ_{a,b}b^* = aa^*aZ_{a,b}b^*bb^* = aa^*aZ_{a,b}Yb^*bb^* = aZ_{a,b}Yb^*. \tag{8}$$

$Z_{a,b}$ is a linear combination of words of the form $Q_{c_1} \dots Q_{c_m}$ for $c_k \in \widehat{\mathcal{A}}$, and Y is a linear combination of words of the form $P_{d_1} \dots P_{d_s}$ where $s \geq 1$, $1 \leq i_1 < i_2 < \dots < i_s \leq M$, $d_k \in v_{i_k}$ and $\varepsilon_{i_k} = 1$ for all $1 \leq k \leq s$. Then, by using the permutation rules (or Lemma 4.1) and the rank one Cuntz-Krieger relations we get (if the following expression is nonzero)

$$\begin{aligned} aQ_{c_1} \dots Q_{c_m} P_{d_1} \dots P_{d_s} b^* &= aQ_{c_1} \dots Q_{c_m} d_1 d_1^* d_2 d_2^* \dots d_s d_s^* b^* \\ &= a\tilde{d}_s \dots \tilde{d}_1 Q_{u_1} \dots Q_{u_l} \tilde{d}_1^* \dots \tilde{d}_s^* b^* \\ &= \tilde{a} Q_{u_1} \dots Q_{u_l} \tilde{b}^* \in L_t \end{aligned}$$

for certain $\tilde{d}_k \in v_{i_k}$, $u_k \in \widehat{\mathcal{A}}$ and $\tilde{a} := a\tilde{d}_s \dots \tilde{d}_1 \in \phi_t, \tilde{b} := b\tilde{d}_s \dots \tilde{d}_1 \in \phi_t$, where

$$t := \widehat{\mathrm{bal}}(\tilde{a}) = \widehat{\mathrm{bal}}(a) + \widehat{\mathrm{bal}}(\tilde{d}_s) + \dots + \widehat{\mathrm{bal}}(\tilde{d}_1) \in \gamma$$

by the choice of ε_j, and where $n \prec t$ since $\widehat{\mathrm{bal}}(a) = n$. Together with (8) this proves that $aZ_{a,b}b^* \in \sum_{n \prec m \in \gamma} L_m$. This proves the property (7).

If $n \neq (N, \dots, N)$, then by property (7), for all $a, b \in \Gamma$ we can write $aZ_{a,b}b^*$ as an element in $\sum_{n \prec m \in \gamma} L_m$, *if* there exists a unit $Y \in p_a$ for $a^*ab^*bZ_{a,b}$. Consequently we get the result we claimed in the beginning of Step 2.

Step 3. In this final step we want to show that Y_n is zero. Hence $X = \sum_{n \prec m \in \gamma} Y_m$ and the inductive step for the induction hypothesis (see Step 1) is completed. Notice that the proof of the proposition is then complete because the induction hypothesis for $n = (N, \dots, N)$ states that X is the empty sum, that means that $X = 0$. We will divide Step 3 into two smaller steps.

Step 3.a. Since each Y_m is a linear combination of words, we can choose suitable finite subsets $\mathcal{B}_i \subseteq v_i$ $(1 \leq i \leq M)$, such that in the above representation (3) of X only letters of the set $\mathcal{B}_1 \sqcup \dots \sqcup \mathcal{B}_M$ occur. Let

$$P_i := \sum_{c \in \mathcal{B}_i} P_c.$$

Fix $a, b \in \Gamma$. Then b has a representation $b = b_1 \dots b_M$ where each b_i is a word in the letters of $\mathcal{B}_i$. For all $1 \leq i \leq M$ let ε_i be defined as in (6). Let I be an imaginary identity (Definition 2.16) and let

$$D_b := b_1(I - P_1)^{\varepsilon_1} b_2 (I - P_2)^{\varepsilon_2} \dots b_M (I - P_M)^{\varepsilon_M},$$

where $(I-P_i)^0 := I$ per definition. Then our next aim, which is the aim of Step 3.a, is to show that

$$a^* X D_b = a^* a U_{a,b} b^* D_b. \tag{9}$$

Indeed, at first we consider a summand $AU_{A,B}B^*$ in Y_n, where $A, B \in \Gamma$ and $(A,B) \neq (a,b)$. By Lemma 4.7 we can write

$$D_b = b_1 b_2 \ldots b_M (I - \widetilde{P}_1)^{\varepsilon_1} (I - \widetilde{P}_2)^{\varepsilon_2} \ldots (I - \widetilde{P}_M)^{\varepsilon_M} \tag{10}$$

for certain finite subsets $\widetilde{\mathcal{B}}_i \subseteq v_i$ and where $\widetilde{P}_i := \sum_{c \in \widetilde{\mathcal{B}}_i} P_c$. Therefore, since $B^* b$ or $a^* A$ vanishes (notice Lemma 4.5 and that $\widehat{\mathrm{bal}}(a) = \widehat{\mathrm{bal}}(A) = n$), we get

$$a^* A U_{A,B} B^* D_b = 0.$$

This already proves one part of the identity (9).

Next we consider a word AQB^* which appears in the linear combination $\sum_{n \prec m \in \gamma} Y_m$, where $Q \in q$ and $A, B \in \phi_m$ for some $n \prec m \in \gamma$. We have, by the choice of the $\mathcal{B}_i$, a representation $B = B_1 \ldots B_M$ such that B_i is a word in the letters of $\mathcal{B}_i$. Since $\widehat{\mathrm{bal}}(b) = n \prec m = \widehat{\mathrm{bal}}(B)$, there exists $0 \le k < M$ such that $\widehat{\mathrm{bal}}(b_1 \ldots b_k) = \widehat{\mathrm{bal}}(B_1 \ldots B_k)$ but $\widehat{\mathrm{bal}}(b_{k+1}) < \widehat{\mathrm{bal}}(B_{k+1})$, by the lexicographical order $\precsim$. In particular $\varepsilon_{k+1} = 1$ since $\widehat{\mathrm{bal}}(a)_{k+1} = \widehat{\mathrm{bal}}(b)_{k+1} = \widehat{\mathrm{bal}}(b_{k+1}) < N$. By Lemma 4.7 we can write

$$\begin{aligned} D_b \;=\; & b_1 \ldots b_k b_{k+1} (I - \widetilde{P}_1)^{\varepsilon_1} \ldots (I - \widetilde{P}_k)^{\varepsilon_k} \; (I - P_{k+1}) \\ & \cdot b_{k+2} (I - P_{k+2})^{\varepsilon_{k+2}} \ldots b_M (I - P_M)^{\varepsilon_M} \end{aligned} \tag{11}$$

for certain finite subsets $\widetilde{\mathcal{B}}_i \subseteq v_i$ and where $\widetilde{P}_i := \sum_{c \in \widetilde{\mathcal{B}}_i} P_c$.

Therefore $a^* AQB^* D_b$ vanishes if the product $B_{k+1}^* B_k^* \ldots B_1^* b_1 \ldots b_k b_{k+1}$ is zero. This proves another part of the identity (9).

If the latter product is nonzero, then we have (see Lemma 4.5) $B_1 \ldots B_k = b_1 \ldots b_k$ and $B_{k+1}^* = C^* b_{k+1}^*$, since $\widehat{\mathrm{bal}}(B_{k+1}) > \widehat{\mathrm{bal}}(b_{k+1})$, where C is a word in the letters of $\mathcal{B}_{k+1}$. Thus we obtain, by using (11) and Lemma 4.4,

$$\begin{aligned} & a^* AQB^* D_b \\ = \;& a^* AQB_M^* \ldots B_{k+2}^* B_{k+1}^* \ldots B_1^* \; b_1 \ldots b_{k+1} (I - P_{k+1})(I - \widetilde{P}_1)^{\varepsilon_1} \ldots\ldots \\ = \;& a^* AQB_M^* \ldots B_{k+2}^* C^* \; b_{k+1}^* \ldots b_1^* b_1 \ldots b_{k+1} \; (I - P_{k+1})(I - \widetilde{P}_1)^{\varepsilon_1} \ldots\ldots \\ = \;& a^* AQB_M^* \ldots B_{k+2}^* C^* (I - P_{k+1}) b_{k+1}^* \ldots b_1^* b_1 \ldots b_{k+1} (I - \widetilde{P}_1)^{\varepsilon_1} \ldots\ldots \\ = \;& 0, \end{aligned}$$

since $C(I - P_{k+1}) = 0$ by the choice of $\mathcal{B}_{k+1}$ and P_{k+1}. These considerations prove the identity (9).

Step 3.b. Fix $a, b \in \Gamma$ and assume that $aU_{a,b}b^* \neq 0$. Then the identities (9) and (10) yield

$$a^* X D_b = a^* a U_{a,b} b^* b (I - \widetilde{P}_1)^{\varepsilon_1} (I - \widetilde{P}_2)^{\varepsilon_2} \dots (I - \widetilde{P}_M)^{\varepsilon_M} =: R_{a,b}.$$

Recall that we suppose that $\Phi(X) = 0$. In particular we have $\Phi(R_{a,b}) = 0$. Since $R_{a,b} \in \mathbb{A}_{00}$ (recall Lemma 4.5) and Φ is injective on $\mathbb{A}_{00}$ we obtain $R_{a,b} = 0$. Hence, if $n \neq (N, \dots, N)$ then

$$I - (I - \widetilde{P}_1)^{\varepsilon_1} (I - \widetilde{P}_2)^{\varepsilon_2} \dots (I - \widetilde{P}_M)^{\varepsilon_M} \in p_a$$

is an identity for $a^* a U_{a,b} b^* b$. However, this contradicts the claim in (4). Consequently $aU_{a,b}b^*$ must vanish.

If $n = (N, \dots, N)$ then $0 = R_{a,b} = a^* a U_{a,b} b^* b I$ since all $\varepsilon_i = 0$. So in this case we also obtain $aU_{a,b}b^* = 0$. Therefore we have proven that $Y_n = 0$, as announced in the beginning of Step 3. □

Definition 4.9. A $*$-algebra A is called the *inductively ordered union of finite dimensional $*$-algebras (C^*-algebras)* if for all $n \geq 1$ and all $x_1, \dots, x_n \in A$ there exists a finite dimensional $*$-algebra (C^*-algebra) $B \subseteq A$ such that $x_1, \dots, x_n \in B$.

In the proof of the following lemma we will need the finiteness property of the axiomatic system. It will be used only here and nowhere else.

Lemma 4.10. *$\mathbb{A}$ is the inductively ordered union of finite dimensional $*$-algebras.*

Proof. For $x \in W_0$ we choose an ordered representation $x = A_1 \dots A_n$ (Lemma 4.3) where A_i is a word in the letters of $v_i^{\circledast}$ for different $v_i \in V$. Since $\mathrm{bal}(x) = 0$ we have $\mathrm{bal}(A_i) = 0$. Thus by Lemma 4.3.(c), $A_i \in F_{w_i,N}$ for some finite subsets $w_i \in v_i$, where $F_{w_i,N}$ is defined in the finiteness property. Thus $x \in F_{w_1,N} \dots F_{w_n,N}$. Hence, if $x_1, \dots, x_M$ are zero-balanced words, then there exist a finite subset $\{v_1, \dots, v_m\} \subseteq V$, finite subsets $u_i \subseteq v_i$, and an integer N, such that all $x_1, \dots, x_M$ are contained in the finite set

$$\gamma := \bigcup_{n=1}^{m} \bigcup_{1 \leq i_1 < i_2 < \dots < i_n \leq m} F_{u_{i_1},N} \dots F_{u_{i_n},N}. \tag{12}$$

We next enlarge the finite subsets u_i to finite subsets $w_i \subseteq v_i$ as claimed in the finiteness property. Define γ' like γ in (12) with the difference that one replaces each occurrence of u_i with w_i. A straight computation shows that we have $F_{w,N} F_{w,N} \subseteq F_{w,N}$ for arbitrary $w \in v \in V$. Noticing this, and successively applying $F_{w_i,N} F_{w_j,N} \subseteq \mathrm{lin} F_{w_j,N} F_{w_i,N}$ (see the finiteness property), we see that $\mathrm{lin}(\gamma')$ is a finite dimensional $*$-algebra containing $\mathrm{lin}(\gamma)$. This proves the claim, since $\mathbb{A}$ is just the linear span of all zero-balanced words. ☐

Recall that π is a $\mathbb{A}_{00}$-faithful saturating representation. However, we remark that the $\mathbb{A}_{00}$-faithfulness is actually not needed for the lemmas 4.12-4.13.

Corollary 4.11. *$\mathbb{A}$ is the inductively ordered union of finite dimensional C^*-algebras.*

Proof. Since $\pi(\mathbb{A}) \cong \mathbb{A}$ by Proposition 4.8, $\mathbb{A}$ is equipped with a C^*-norm. By Lemma 4.10 $\mathbb{A}$ is the inductively ordered union of finite dimensional $*$-algebras, but these finite dimensional $*$-algebras must actually be C^*-algebras. ☐

Lemma 4.12. *If $X \in \mathbb{A}$ then there exist an integer $N \geq 1$, a finite subset $\{v_1, \ldots, v_m\} \subseteq V$, and scalars $\lambda_{a,b} \in \mathbb{C}$ such that $\pi(X)$ is representable as the strong operator sum*

$$\pi(X) = \sum_{a=(a_1,\ldots,a_{Nm}),b=(b_1,\ldots,b_{Nm})\in v_1^N\times\ldots\times v_m^N} \lambda_{a,b}\pi(a_1\ldots a_{Nm}b_{Nm}^*\ldots b_1^*).$$

Proof. At first recall that for $a \in v \in V$ we have

$$\pi(Q_a) = \pi(Q_a)\sum_{b\in v}\pi(P_b) = \sum_{b\in v} s_v(a,b)\pi(P_b).$$

Let $v \in V$ and Z be a word in the letters of $v^{\circledast}$. By Lemma 4.3 we get

$$\begin{aligned}
\pi(Z) &= \pi(a_1\ldots a_n)\pi(Q_{c_1})\ldots\pi(Q_{c_m})\pi(b_n^*\ldots b_1^*) \\
&= \sum_{b\in v}\lambda_b\pi(a_1\ldots a_n)\pi(bb^*)\pi(b_n^*\ldots b_1^*) \\
&= \sum_{b\in v}\lambda_b\pi(a_1\ldots a_n b)\Big(\sum_{c_1\in v}\pi(c_1c_1^*)\Big)\pi(b^*b_n^*\ldots b_1^*) \\
&= \sum_{b\in v}\sum_{c_1\in v}\lambda_b\pi(a_1\ldots a_n bc_1c_1^*b^*b_n^*\ldots b_1^*) \\
&= \sum_{b\in v}\sum_{c_1,\ldots,c_{N-1-n}\in v}\lambda_b\pi(a_1\ldots a_n bc_1\ldots c_{N-1-n}c_{N-1-n}^*\ldots c_1^*b^*b_n^*\ldots b_1^*) \\
&= \sum_{d=(d_1,\ldots,d_N),e=(e_1,\ldots,e_N)\in v^N}\lambda_{d,e}\pi(d_1\ldots d_Ne_N^*\ldots e_1^*)
\end{aligned}$$

for the scalars $\lambda_b = s_v(c_1, b) \dots s_v(c_m, b) \in \{0, 1\}$ and certain scalars $\lambda_{d,e} \in \mathbb{C}$. (The series can be rearranged due to similar arguments we will show below.)

If we have given any word $Z \in W_0$ then we choose an ordered version $Z = A_1 \dots A_n$, where $A_i \in W_0$ is a zero-balanced word in the letters of $v_i \in V$ ($v_1, \dots, v_n$ mutually different). We suppose $n = 2$, the general case is analogous. By the above computation and by successive application of the permutation rules we get a representation

$$\begin{aligned} \pi(A_1 A_2) &= \sum_{a,b \in v_1^N} \lambda_{a,b} \pi(a_1 \dots a_N b_1^* \dots b_N^*) \sum_{c,d \in v_2^N} \mu_{c,d} \pi(c_1 \dots c_N d_1^* \dots d_N^*) \\ &= \sum_{a,b \in v_1^N} \sum_{c,d \in v_2^N} \lambda_{a,b} \mu_{c,d} \gamma_{a,b,c,d} \pi(f(a,b,c,d) g(a,b,c,d)^*), \qquad (13) \end{aligned}$$

where $\gamma_{a,b,c,d} \in \{0, 1\}$, and $f(a, b, c, d)$ and $g(a, b, c, d)$ are words of the form

$$x_1 \dots x_N y_1 \dots y_N,$$

where $x_i \in v_1, y_i \in v_2$. Next we will rearrange the series.

Let $x_i, X_i \in v_1$ and $y_i, Y_i \in v_2$. We abbreviate $x := x_1 \dots x_N$. Then we consider

$$\begin{aligned} &\pi(xyy^*x^*)\pi(A_1A_2)\pi(XYY^*X^*) \\ = &\sum_{\substack{a,b \in v_1^N, c,d \in v_2^N, \\ f(a,b,c,d) = xy, g(a,b,c,d) = XY}} \lambda_{a,b} \mu_{c,d} \gamma_{a,b,c,d} \pi(xyY^*X^*). \end{aligned}$$

Hence, if xyY^*X^* is nonzero, then the number series

$$\eta(x, y, Y, X) := \sum_{\substack{a,b \in v_1^N, c,d \in v_2^N, \\ f(a,b,c,d) = xy, g(a,b,c,d) = XY}} \lambda_{a,b} \mu_{c,d} \gamma_{a,b,c,d}$$

converges. Otherwise we put $\eta(x, y, Y, X) = 0$. By representation (13) we see that $\sum_{x \in v_1} \sum_{y \in v_2} \pi(xyy^*x^*)$ is a unit for $\pi(A_1 A_2)$. Therefore we get, as claimed,

$$\begin{aligned} \pi(A_1 A_2) &= \sum_{x \in v_1} \sum_{y \in v_2} \pi(xyy^*x^*) \; \pi(A_1 A_2) \sum_{X \in v_1} \sum_{Y \in v_2} \pi(XYY^*X^*) \\ &= \sum_{x \in v_1} \sum_{y \in v_2} \sum_{X \in v_1} \sum_{Y \in v_2} \eta(x, y, Y, X) \pi(xyY^*X^*). \end{aligned}$$

So far we have proved the lemma for words in $\mathbb{A}$. If we have given several words (like in a linear combination) then we do the above procedure for each word. To this end we remark that if we have given a word Z which is a word in the letters of $v^{\circledast}$, say, then, since π is saturating, we can write $\pi(Z)$ as the (usually infinite) sum $\pi(Z) = \sum_{b \in w} \pi(Z)\pi(P_b)$ of words in the letters of $v^{\circledast}$ *and* $w^{\circledast}$ $(v, w \in V)$. (By similar arguments as above it is always possible to rearrange all series to the desired format at the end.) □

Lemma 4.13. *If* $X \in \mathbb{A}_0$ *then we find an integer* $N \geq 1$*, a finite subset* $\{v_1, \ldots, v_m\} \subseteq V$*, and scalars* $\lambda_a \in \mathbb{C}$ *such that* $\pi(X)$ *is representable as the strong operator sum*

$$\pi(X) = \sum_{a=(a_1,\ldots,a_{Nm}) \in v_1^N \times \ldots \times v_m^N} \lambda_a \pi(a_1 \ldots a_{Nm} a_{Nm}^* \ldots a_1^*).$$

Proof. Choose for $\pi(X)$ a representation as stated in Lemma 4.12. Let $a = (a_1, \ldots, a_{Nm}), b = (b_1, \ldots, b_{Nm})$ distinct elements of $v_1^N \times \ldots \times v_m^N$, and abbreviate $a' := a_1 \ldots a_{Nm}$ and $b' := b_1 \ldots b_{Nm}$. Since $\mathbb{A}_0$ is abelian (Lemma 4.4) and $a'^* b' = 0$ (Lemma 4.5) we have

$$0 = \pi(a' a'^*)\pi(X)\pi(b' b'^*) = \lambda_{a,b}\pi(a' b'^*).$$

□

Corollary 4.14. *For every nonzero projection* $P \in \mathbb{A}_0$ *we find a nonzero word* x *in the letters of* $\mathcal{A}$ *such that* $xx^* \leq P$.

Proof. By Proposition 4.8 π is faithful on $\mathbb{A}$ and we obtain $\pi(\mathbb{A}) \cong \mathbb{A}$, and in particular we have $\pi(\mathbb{A}_0) \cong \mathbb{A}_0$. By Lemma 4.13 we find a nonzero word x in the letters of $\mathcal{A}$ such that $\pi(xx^*) \leq \pi(P)$. Hence $xx^* \leq P$. □

For the proof of the next lemma we will need the projections property, and actually it will be needed only here.

Lemma 4.15. *There does not exist an* $E \in \mathbb{P}_0$ *and a nonzero-balanced word* $X \in W$ *such that* $E \leq X^n X^{*n} X^{*n} X^n$ *for all* $n \geq 1$.

Proof. Assume that $E \leq X^n X^{*n} X^{*n} X^n$ for all $n \geq 1$ for some $E \in \mathbb{P}_0$ and $X \in W \backslash W_0$. By Corollary 4.14 there exists a word x in the letters of $\mathcal{A}$ such that $xx^* \leq E$.

Since X is nonzero-balanced, we have a $v \in V$ such that $\mathrm{bal}(X)_v \neq 0$. W.l.o.g. we assume that $\mathrm{bal}(X)_v > 0$ (otherwise we would deal with X^* rather than X). Thus by Lemma 4.3, X^n has a representation $X^n = A_n C_n Q_n B_n^*$ where A_n is a word in the letters v, and C_n and B_n are words in the letters of $\mathcal{A}$, and Q_n is a product of source projections of letters. Notice that

$$X^n X^{n*} \leq A_n A_n^*.$$

Let $A_n = a_1 a_2 \dots a_{k_n}$ for $a_i \in v$, and notice that the sequence k_n is increasing, since we have

$$\mathrm{bal}(X^n)_v = \mathrm{bal}(A_n)_v - \mathrm{bal}(B_n)_v,$$

the sequence $\mathrm{bal}(X^n)_v$ is increasing, and the sequence $\mathrm{bal}(B^n)_v$ is constant, which can be seen for example when $n = 2$ by

$$X^2 = A_1 C_1 Q_1 B_1^* \; A_1 C_1 Q_1 B_1^* = A_n \overline{A_1} \; \overline{C_1 Q_1 B_1}^* \; C_1 Q_1 B_1^*$$

for some word $\overline{A}_1$ in the letters of v and some word $\overline{B}_1$ which does not contain a letter of v by an application of the permutation rules and rank one Cuntz-Krieger relations.

We observe that the letters a_i do not depend on n, more precisely the first k_n letters a_i of A_n and A_{n+1} coincide, since the assumption of the opposite would force the product $X^{n+1*} X^n$ to zero, which cannot be true since $X^{n+1} X^{n+1*} \leq X^n X^{n*}$.

By the projections property there exists some $N \geq 1$ such that

$$Exx^* = xx^* > xx^* a_1 \dots a_N a_N^* \dots a_1^* \geq xx^* X^m X^{*m}$$

for some $m \geq 1$. However, this contradicts the assumption that $E \leq X^m X^{*m} X^{*m} X^m$ (recall Lemma 4.4). □

Proof of Theorem 2.3. Let $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ be a triple of generators and relations generating a higher rank Exel-Laca algebra, and let π be a saturating $\mathbb{A}_{00}$-faithful representation. We will check the axioms (A), (B) and (D) of [B4]. (Notice that the definitions of $\mathcal{A}, \mathbb{F}, \mathbb{I}, H, \mathrm{bal}, W, W_0, \mathbb{A}, \mathbb{A}_0$ used here, and used in [B4], are consistent.) We already mentioned in the beginning of this section that property (A) holds. Property (B) is nothing else than Corollary 4.11. Notice that π and π_1 are faithful on $\mathbb{A}$ by Proposition 4.8. Property (D) is evident by Lemma 4.4, Corollary 4.6, Lemma 4.15 and [B4, Lemma 3.1 (4)-(5)]. Hence [B4, Corollary 3.4] proves Theorem 2.3. □

Proof of Corollary 2.6. As described in Remark 2.14 let $\mathbb{J}$ be the ideal in $\mathbb{F}$ such that $\mathbb{F}/\mathbb{J} \cong (\mathbb{F}/\mathbb{I})/\ker(\pi_1|\mathbb{A}_{00})$. Let $\tilde{\pi}_1 : \mathbb{F}/\mathbb{J} \to B(\mathcal{H}_1)$ be the quotient map induced by π_1. Then $\tilde{\pi}_1$ is a $\mathbb{A}_{00}$-faithful saturating representation for the triple $(\mathcal{A}, \mathbb{F}, \mathbb{J})$; that is, we have property (6) of the axiomatic system for this triple. By Lemma 2.15 and by the assumptions of Corollary 2.6 the triple $(\mathcal{A}, \mathbb{F}, \mathbb{J})$ satisfies the properties (1)-(5) of the axiomatic system (notice that $\tilde{\pi}_1$ is injective on $\mathbb{A}$ by Proposition 4.8). We have to show that the quotient map $\tilde{\pi}_2 : \mathbb{F}/\mathbb{J} \to B(\mathcal{H}_2)$ associated to π_2 is well defined; that is, by Lemma 2.17 we have to show that

$$\pi_1(x) = 0 \quad \Rightarrow \quad \pi_2(x) = 0 \quad \forall x \in \mathbb{W}_{00}. \tag{14}$$

Expanding in $\pi_i(x)$ each occurrence of $\pi_i(Q_a)$ by a strong operator sum $\sum_b \pi_i(P_b)$ it is easy to realize that the condition given in Corollary 2.17 ensures the claim (14). The existence of σ now follows from Theorem 2.3 applied to the triple $(\mathcal{A}, \mathbb{F}, \mathbb{J})$ and $\tilde{\pi}_1$ and $\tilde{\pi}_2$.

Furthermore, observe that if the reverse implication of the condition given in Corollary 2.6 also holds then we have $\ker(\pi_1|\mathbb{A}_{00}) = \ker(\pi_2|\mathbb{A}_{00})$ by Lemma 2.17, and Theorem 2.3 also yields an inverse σ^{-1} if we exchange $\tilde{\pi}_1$ and $\tilde{\pi}_2$. □

Remark 4.16. If we replace the last property in the axiomatic system, i.e. the existence of a saturating $\mathbb{A}_{00}$-faithful representation π, by the property that

$$\forall p \in \mathbb{P}_0 \; \exists x \in \{ a_1 \cdots a_m \mid m \geq 1, \, a_i \in \mathcal{A} \} \quad xx^* \leq p,$$

then Theorem 2.3 persists valid.

Indeed, in the proof of Theorem 2.3 a saturating representation π is only needed to get the claim of Corollary 4.14. (We also use π in Corollary 4.11, but there we can use π_1 just as well.) The claim of Corollary 4.14 is used in Lemma 4.15, which is used in the proof of Theorem 2.3. So if we suppose the above stated property, which is the claim of Corollary 4.14, then Theorem 2.3 persists valid.

Notice, however, that a saturating representation π gives us a hint which relations, namely $\ker(\pi|\mathbb{A}_{00})$ (cf. 2.14), we might assume such that the claim of Corollary 4.14 holds.

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