ON PARTIAL CROSSED PRODUCTS AND CUNTZ-KRIEGER TYPE ALGEBRAS

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ABSTRACT. We represent a C^*-algebra generated by partial isometries having commuting range and support projections as the quotient of a partial crossed product of an abelian C^*-algebra and a free group. Particularly we get such representations for certain Cuntz-Krieger type algebras. Sometimes the quotient can be represented directly as a partial crossed product.

1. INTRODUCTION

Consider a self-adjoint set $\mathcal{A}$ of partial isometries acting on a Hilbert space H and suppose that the support and range projections of all words $X = a_1a_2...a_n$ ($a_i \in \mathcal{A}$) commute among each other. Then each word X is a partial isometry and can be written as

$$X = a_1...a_na_n^*...a_1^*a_1...a_n = \underbrace{a_1...a_na_n^*...a_1^*}_{\in A} \ \underbrace{a_{i_1}a_{i_2}...a_{i_m}}_{\in F},$$

where $a_{i_1}...a_{i_m}$ $(1 \le i_1 \le ... \le i_m \le n)$ is the "reduced" word of $a_1...a_n$ in the sense that one cancels each occurrence a_ia_{i+1} if $a_{i+1} = a_i^*$ (see Lemma 2.3).

Formatting all words to this form we can see that the C^*-algebra $B \subseteq B(H)$ generated by $\mathcal{A}$ is a covariant representation of a partial dynamical system (A, α, F) where A is the commutative C^*-algebra generated by the range and support projections of all words,

$$A = C^*(\{\, a_1...a_na_1^*...a_n^* \mid n \ge 1,\, a_i \in \mathcal{A}\,\}),$$

F is the free group generated by $\mathcal{A}$ where we regard $a^* \cong a^{-1}$ for all $a \in \mathcal{A}$, and the partial action is given by $\alpha_{a_1...a_n}(x) = a_1...a_nxa_n^*...a_1^*$ for $x \in A$ and reduced words $a_1...a_n \in F$.

This is the basic idea of the present paper and it is inspired or mimiced the partial crossed product appearing in [EL1], also briefly described in [EL2] in section "1.Preliminaries". Moreover one may observe that the present paper shares some similar aspects with the paper [ELQ]. However, they are never really identical.

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If one considers the map $\tau : A \rtimes_\alpha F \to B$ from the full partial crossed product then B is isomorphic to the quotient $(A \rtimes_\alpha F)/\ker(\tau)$, cf. Proposition 2.5. We are particularly interested in partial crossed product representations of Cuntz-Krieger type algebras, for example as they are developed in [B2, B4]. In Corollary 2.6 we obtain the quotient representation $\mathcal{O}_{\mathbb{F},\mathbb{I},H} \cong (\overline{\mathbb{A}_0} \rtimes_\alpha F)/\overline{\sigma(\mathbb{I})}$ for all Cuntz-Krieger type algebras $\mathcal{O}_{\mathbb{F},\mathbb{I},H}$ [B4].

Under a specialized setting this quotient can be written as $\mathcal{O}_{\mathbb{F},\mathbb{I},H} \cong \overline{\mathbb{A}_0} \rtimes_\beta (F/F_0)$ for a normal subgroup $F_0 \subseteq F$, see Corollary 3.7, and the last two sections are dedicated to this improved representation. In Theorem 4.8 we combine the results.

Perhaps some day the latter representation will be useful to compute K-Theories (as in [EL2]) of some uncomplex higher rank Cuntz-Krieger algebras, by an up to now not existing generalized Pimsner-Voiculescu exact sequence for such partial crossed products with group F/F_0; for F this was done in [E, McC]. A generalized sequence exists for usual (but not partial) crossed products in [P], for example.

2. The Basic Representation

Let $\mathcal{A}$ be an (arbitrary) alphabet and let $\mathcal{A}^* := \{ a^* \mid a \in \mathcal{A} \}$ be the set of its formally involuted letter. Per definition we claim $\mathcal{A} \cap \mathcal{A}^* = \emptyset$. (Notice that we are not consistent with the introduction where $\mathcal{A}$ was supposed to be self-adjoint.) Let $\mathbb{F}$ be the free non-unital $*$-algebra generated by the alphabet $\mathcal{A}$. More precisely $\mathbb{F}$ is the complex vector space with linear basis the set of formal words of nonzero length,

$$\omega := \{ x_1 x_2 ... x_n \mid n \geq 1, x_i \in \mathcal{A} \cup \mathcal{A}^* \} \subseteq \mathbb{F}.$$

Let $\mathbb{I} \subseteq \mathbb{F}$ be a two-sided self-adjoint ideal in $\mathbb{F}$ satisfying the following definition supposed throughout.

Definition 2.1. In the quotient $\mathbb{F}/\mathbb{I}$ each letter $a + \mathbb{I}$ is a partial isometry $(aa^*a + \mathbb{I} = a + \mathbb{I})$ and the set of range and support projections of all words,

$$\Delta := \{ XX^* + \mathbb{I} \in \mathbb{F}/\mathbb{I} \mid X \in \omega \},$$

is a commuting set.

(This definition coincides with property (X) in [B4].) We denote by $\mathbb{A}_0$ the abelian $*$-algebra in $\mathbb{F}$ generated by the set Δ. It is straightforward by induction on the length that each word $X + \mathbb{I} \in \mathbb{F}/\mathbb{I}$ $(X \in \omega)$ is a partial isometry and $\mathbb{A}_0$ coincides with the linear hull of Δ (cf. [B4, Lemma 2.1]).

Let F be the free group generated by the alphabet $\mathcal{A}$. Let $X = x_1^{\epsilon_1}...x_n^{\epsilon_n}$ be a word in ω ($x_i \in \mathcal{A}, \epsilon_i \in \{1, *\}$). Then we naturally assign to X the group element $[X] := x_1^{\tau_1}...x_n^{\tau_n} \in F$, where $\tau_i := 1$ if $\epsilon_i = 1$ and $\tau_i := -1$ if $\epsilon_i = *$. Trivially we have the rules

$$[X][Y] = [XY], \quad [X]^{-1} = [X^*] \qquad \forall X, Y \in \omega.$$

Keeping on the previous notations, $X \in \omega$ is called *reduced* if $[X] = x_1^{\tau_1}...x_n^{\tau_n}$ is a reduced word in F. Clearly to each $X \in \omega$ one can assign a unique reduced word $Y \in \omega$ such that $[X] = [Y]$.

Lemma 2.2. *Let $\mathbb{A}_0$ be endowed with a C^*-norm and $A := \overline{\mathbb{A}_0}$ be its norm closure, so A is a commutative C^*-algebra.*

Then we have a partial C^*-dynamical system (A, F, α), where for reduced words $X \in \omega$, the isomorphisms $\alpha_{[X]} : D_{[X]^{-1}} \to D_{[X]}$ are defined by $D_{[X]} := (XX^* + \mathbb{I})A$ and $\alpha_{[X]}(a) := (X + \mathbb{I})a(X^* + \mathbb{I})$ for all $a \in D_{[X]^{-1}}$.

Proof. We have to prove the Definition of the *partial action α of G on A* given in [McC]. However, in [S, Definition 2.2] is given a shorter but equivalent definition which we in fact shall use.

In the sequel we shall use a lax syntax and omit $+\mathbb{I}$ in $X + \mathbb{I}$ for computations in $\mathbb{F}/\mathbb{I}$, where $X \in \mathbb{I}$. Let $X \in \omega$ be reduced. Notice that A is a commutative C^*-algebra and for $D_{[X]}$ we have also the representations

$$D_{[X]} = XX^*A = \overline{XX^*\mathbb{A}_0}.$$

From the definition it is immediate that $D_{[X]}$ is a two-sided closed ideal in A. As mentioned above, we have $\mathbb{A}_0 = \mathrm{lin}(\Delta)$. Thus for $a \in XX^*\mathbb{A}_0$ we have a representation $a = \sum_{i=1}^n \lambda_i Y_i Y_i^*$ for certain $\lambda_i \in \mathbb{C}, Y_i \in \omega$. Hence

$$\alpha_{[X]}(a) = XaX^* = \sum_{i=1}^n \lambda_i X Y_i Y_i^* X^* \in \mathbb{A}_0$$

and $\alpha_{[X]}(a) = XX^*(XaX^*) \in D_{[X]}$.

Note that the domain of $\alpha_{[X]}$ is $D_{[X]^{-1}} = D_{[X^*]} = X^*XA$ (X reduced $\Rightarrow X^*$ reduced). The map $\alpha_{[X]}$ is isometric by $\|XaX^*\| \leq \|a\| = \|X^*XaX^*X\| \leq \|XaX^*\|$ and can continuously be extended to X^*XA.

The last thing we have to check is the condition (ii) of Definition 2.2 in [S] which states that α_{st} extends the map $\alpha_s\alpha_t$ on its maximal possible domain $\alpha_t^{-1}(D_{s^{-1}})$.

Let $S, T \in \omega$ be reduced words and $s = [S], t = [T]$. Let $Z \in \omega$ be the reduced word such that $[Z] = st = [ST]$, so Z is the reduced word of ST. Since S, T are reduced there exists only one possibility for the appearance of reduction within ST, namely that we have representations $S = \widetilde{S}U, T = U^*\widetilde{T}$ for certain words $\widetilde{S}, \widetilde{T}, U \in \omega$, and $Z = \widetilde{S}\widetilde{T}$.

Let $x = T^*S^*SaT \in T^*S^*SAT = \alpha_t^{-1}(D_{s^{-1}})$ for $a \in A$. Then we obtain by commutativity

$$\begin{aligned} \alpha_{[S]}\alpha_{[T]}(x) &= STT^*S^*SaTT^*S^* = STT^*aS^* \\ &= \widetilde{S}UU^*\widetilde{T}\widetilde{T}^*UU^*\widetilde{S}^*\widetilde{S}UaU^*\widetilde{T}\widetilde{T}^*UU^*\widetilde{S}^* \\ &= \widetilde{S}\widetilde{T}\widetilde{T}^*UU^*\widetilde{S}^*\widetilde{S}UaU^*\widetilde{T}\widetilde{T}^*\widetilde{S}^* \\ &= \alpha_{[Z]}(x). \end{aligned}$$

□

Lemma 2.3. *Let $X, Y \in \omega$, $[Y] = [X]$ and X be reduced. Then in $\mathbb{F}/\mathbb{I}$ we have $YY^*X + \mathbb{I} = Y + \mathbb{I}$ and $YY^* + \mathbb{I} \le XX^* + \mathbb{I}$.*

Proof. In this proof we ease notations by omitting $+\mathbb{I}$ in $X + \mathbb{I}$. By induction hypothesis suppose that $YY^*X = Y$ in $\mathbb{F}/\mathbb{I}$ for all reduced words $X \in \omega$ of length n, and all $Y \in \omega$ such that $[Y] = [X]$.

Let $X_{n+1} = aX$ be a reduced word of length $n+1$, where $a \in \mathcal{A} \cup \mathcal{A}^*$ and X is a reduced word of length n. Let $Y_{n+1} \in \omega$ such that $[Y_{n+1}] = [X_{n+1}]$.

Then Y_{n+1} has shape $Y_{n+1} = AA^*aBB^*Y$ for certain $A, B, Y \in \omega$ such that $[Y] = [X]$. (More precisely A and B may also be the empty words, or in other words, we cancel their appearance. The calculation below works also in these cases.) The induction start Y_1 follows here by simply omitting Y. Then we get in $\mathbb{F}/\mathbb{I}$ by using the commutativity of Δ

$$\begin{aligned} Y_{n+1}Y_{n+1}^*X_{n+1} &= AA^* \; aBB^*YY^*BB^*a^* \; AA^* \; aX \\ &= AA^* \; a \; BB^* \; YY^*X \\ &= AA^* \; a \; BB^* \; Y \\ &= Y_{n+1}. \end{aligned}$$

The last assertion can be deduced from this by $YY^*XX^*YY^* = (YY^*X)(YY^*X)^* = YY^*$. □

Lemma 2.4. *Consider the setting and partial C^*-dynamical system (A, F, α) of Lemma 2.2. Then the $*$-homomorphism*

$$\sigma : \mathbb{F} \to A \rtimes_\alpha F : \sigma(X) = (XX^* + \mathbb{I})[X] \qquad (X \in \omega)$$

has dense image and its kern is in $\mathbb{I}$.

Proof. Everything is clear or straightforward. It only remains to show that the kern lies in $\mathbb{I}$. To this end consider an arbitrary $Z \in \mathbb{F}$. It admits a representation

$$Z = \sum_{i=1}^{N} \sum_{j=1}^{n_i} \lambda_{ij} X_{ij}, \tag{1}$$

where $\lambda_{ij} \in \mathbb{C}, X_{ij} \in \omega$ and such that $[X_{i_1 j_1}] = [X_{i_2 j_2}]$ if and only if $i_1 = i_2$, for all i_1, j_1, i_2, j_2. Let $X_i \in \omega$ be the reduced version of X_{ij}, that is $[X_i] = [X_{ij}]$. Then we have

$$\sigma(Z) = \sum_{i=1}^{N} \sum_{j=1}^{n_i} \lambda_{ij} (X_{ij} X_{ij}^* + \mathbb{I})[X_{ij}] =: \sum_{i=1}^{N} Z_i [X_i],$$

$Z_i \in \mathbb{F}/\mathbb{I}$ let be obviously defined. Now suppose $\sigma(Z) = 0$. Then $Z_i = 0$ for all $1 \leq i \leq N$ and by Lemma 2.3 we have

$$0 = Z_i(X_i + \mathbb{I}) = \sum_{j=1}^{n_i} \lambda_{ij} X_{ij} X_{ij}^* X_i + \mathbb{I} = \sum_{j=1}^{n_i} \lambda_{ij} X_{ij} + \mathbb{I}.$$

Recalling the representation (1) this yields $Z \in \mathbb{I}$. □

We can build the map $\widetilde{\sigma} : \mathbb{F}/\mathbb{I} \to (A \rtimes F)/\overline{\sigma(\mathbb{I})}$. However, we could not decide whether this map is injective on $\mathbb{A}_0$. This is useful since otherwise the quotient would loose too much information on $\mathbb{F}/\mathbb{I}$, particularly to get a representation of Cuntz-Krieger type algebras (cf. Corollary 2.6). But we have the following factorization, which can ensure injectivity on $\mathbb{A}_0$.

Proposition 2.5. *Suppose Definition 2.1, let $\mathcal{O}$ be a C^*-algebra and $\pi : \mathbb{F}/\mathbb{I} \to \mathcal{O}$ be a $*$-homomorphism which is injective on $\mathbb{A}_0$. Endow $\mathbb{A}_0$ with the norm inherited from $\mathcal{O}$, let σ as in Lemma 2.4 and $\widetilde{\sigma}$ its canonical map. Then π can be factorized as sketched in the following commutative diagram. Particularly $\widetilde{\sigma}$ has dense image and is injective on $\mathbb{A}_0$.*

$$\begin{array}{ccccc} \mathbb{F} & \longrightarrow & \mathbb{F}/\mathbb{I} & \xrightarrow{\pi} & \mathcal{O} \\ \downarrow \sigma & & \downarrow \widetilde{\sigma} & \nearrow & \\ A \rtimes_\alpha F & \longrightarrow & (A \rtimes_\alpha F)/\overline{\sigma(\mathbb{I})} & & \end{array}$$

Proof. We have the diagram

$$\begin{array}{ccccc} \mathbb{F} & \longrightarrow & \mathbb{F}/\mathbb{I} & \xrightarrow{\pi} & \mathcal{O} \\ \downarrow \sigma & & & & \\ A \rtimes G & & & & \end{array}$$

By Lemma 2.4 we have $\ker(\sigma) \subseteq \mathbb{I}$. Thus in the above diagram we obtain a $*$-homomorphism $\tau_0 : \sigma(\mathbb{F}) \to \mathcal{O}$. Let $X = \sum_i \sum_j \lambda_{ij} X_{ij} \in \mathbb{F}$ where $\lambda_{ij} \in \mathbb{C}$, $X_{ij} \in \omega$ and the reduced word of X_{ij} is X_i. Then by Lemma 2.3 we get

$$\Big\|\pi(X+\mathbb{I})\Big\| = \Big\|\pi(\sum_i \sum_j \lambda_{ij} X_{ij} X_{ij}^* X_i + \mathbb{I})\Big\|$$

$$\leq \sum_i \Big\|\pi(\sum_j \lambda_{ij} X_{ij} X_{ij}^* + \mathbb{I})\Big\| = \|\sigma(X)\|.$$

We can thus continuously extend τ_0 to $\tau_1 : A \rtimes F \to \mathcal{O}$. From τ_1 we can derive the quotient map $A \rtimes G/\overline{\sigma(\mathbb{I})} \to \mathcal{O}$ and we easily can complete the diagram. □

If we combine the last proposition with [B4, Theorem 2.10] then we get

Corollary 2.6. *Let $\mathcal{O}_{\mathbb{F},\mathbb{I},H}$ be a Cuntz-Krieger type algebra as introduced in* [B4]*. Then one has an isomorphism $\mathcal{O}_{\mathbb{F},\mathbb{I},H} \cong (\overline{\mathbb{A}_0} \rtimes_\alpha F)/\overline{\sigma(\mathbb{I})}$.*

We remark that by the axiomatic system [B4], $\mathbb{A}_0$ is endowed with a C^*-norm and it is unique since $\mathbb{A}_0$ is the inductively ordered union of finite dimensional C^*-algebras.

3. An Equivalence Relation on ω

We have seen in Proposition 2.5 that C^*-representations $\pi : \mathbb{F}/\mathbb{I} \to \mathcal{O}$ being faithful on $\mathbb{A}_0$ factors through $(A \rtimes F)/\overline{\sigma(\mathbb{I})}$. Our aim of this section is the following. We suppose that $\mathbb{I}$ is plain enough such that we obtain an isomorphism

$$(A \rtimes F)/\overline{\sigma(\mathbb{I})} \cong A \rtimes (F/F_0)$$

for some normal subgroup F_0 of F. Beside the necessary assumptions that the words $\omega + \mathbb{I}$ are partial isometries with commuting range and source projections, we suppose that $\mathbb{I}$ is generated by an equivalence relation R on ω. This R should carry over to F, induce F_0 and give the above isomorphism (under unfortunately non-mild conditions as we believe, see Definition 3.2 below). Throughout this section we assume

Definition 3.1. Let R be an equivalence relation on ω which respects multiplication and involution. Moreover suppose that $X \equiv_R Y \Rightarrow X - Y \in \mathbb{I}$ for all $X, Y \in \omega$.

The set of equivalence classes ω_R of ω with respect to R forms an involutive semigroup. We introduce an equivalence $\equiv_F$ on F which is inherited from $\equiv_R$ on ω. More precisely we put $[X] \equiv_F [Y]$ if $X \equiv_R Y$ $(X, Y \in \omega)$. Let $F_0 \subseteq F$ be the smallest normal subgroup generated by the equivalence $\equiv_F$ (i.e. generated by $\{\, [X][Y]^{-1} \mid X \equiv_R Y,\, X, Y \in \omega \,\}$).

We obtain a well defined map $\omega_R \to F/F_0$ which maps a representative $X \in \omega$ to the representative $[X]F_0 \in F/F_0$. We shall denote this map by [.] too. Let $F_R := F/F_0$.

$$\begin{array}{ccccc} & & \omega & \xrightarrow{[\,]} & F \\ & \swarrow & \downarrow & & \downarrow \\ \mathbb{F}/\mathbb{I} & \longleftarrow & \omega_R & \underset{\xleftarrow{r}}{\xrightarrow{[\,]}} & F_R \end{array}$$

Since we suppose $X \equiv_R Y \Rightarrow X - Y \in \mathbb{I}$, we get a well defined map $\omega_R \to \mathbb{F}/\mathbb{I}$ which assigns a representative of $X \in \omega$ to $X + \mathbb{I}$. So we can extend the notion $X + \mathbb{I}$ to $X \in \omega_R$.

In the last section we could assign to each group element $g \in F$ a reduced word $X \in \omega$ such that $g = [X]$. An analogous assignment r we need here between F_R and ω_R. Let $\omega^+ := \omega \cup \{\emptyset\}$ be ω adjoint by the empty word and identity $\emptyset$. Similarly let $\omega_R^+ := \omega_R \cup \{\emptyset\}$.

Definition 3.2. There exists a "reduction" map $r : F_R \backslash \{e\} \to \omega_R$ satisfying:

(a) $[r(g)] = g$ for all $g \in F_R \backslash \{e\}$.

(b) $r(g^{-1}) = r(g)^*$ for all $g \in F_R \backslash \{e\}$.

(c) For all $g, h \in F_R \backslash \{e\}$, $gh \neq e$, there exist $A, y, B \in \omega_R^+$ such that $r(g) = Ay, r(h) = y^* B$ and $r(gh) = AB$.

(d) $r([a]) = a$ for all letters $a \in \mathcal{A}$. If $[a] = e$ then a is an identity in $\mathbb{F}/\mathbb{I}$.

Such a lifting r seems to choose that representative $r_g \in \omega_R$ such that $r_g r_g^* + \mathbb{I}$ is maximal. In Example 4.11 we give an example where such a maximum does not seem to exist.

Lemma 3.3. *Suppose Definitions 3.2, let $\mathbb{A}_0$ be a pre-C^*-algebra and $A := \overline{\mathbb{A}_0}$ be its C^*-closure, so A is a commutative C^*-algebra.*

Then we have a partial C^*-dynamical system $(A, F/F_0, \beta)$, where the isomorphisms $\beta_g : D_{g^{-1}} \to D_g$ ($g \in F/F_0$) are given by

$$D_g = D_{[r(g)]} := (r_g r_g^* + \mathbb{I})A,$$
$$\beta_g(a) := (r_g + \mathbb{I})a(r_g^* + \mathbb{I}) \qquad (a \in D_g^{-1}).$$

Proof. The proof is quite the same as the one of Lemma 2.2, with easy adaption. One just replaces the reduced word $X \in \omega$ for $g = [X] \in G$ in the proof of Lemma 2.2 by the reduced word $r_g \in \omega_R$ associated to $g = [r_g] \in F_R$ here. □

From the last lemma we immediately obtain the following natural representation.

Lemma 3.4. *Suppose Definition 3.2. Then one has a $*$-homomorphism σ with dense image,*

$$\sigma : \mathbb{F} \to A \rtimes_\beta (F/F_0), \quad \sigma(X) = (XX^* + \mathbb{I})[X], \quad \forall X \in \omega_R.$$

Now we shall consider the case that $\mathbb{I}$ is generated by the set

$$\begin{aligned} \mathbb{I}_0 &:= \{ XX^*X - X \in \mathbb{F} \mid X \in \omega \} \\ &\cup \ \mathbb{I} \cap \mathrm{Alg}^*\{ XX^* \in \mathbb{F} \mid X \in \omega \} \\ &\cup \ \{ X - Y \in \mathbb{F} \mid X, Y \in \omega, X \equiv_R Y \} \end{aligned}$$

in $\mathbb{F}$. It is immediate from the definition of σ that $\mathbb{I}_0$ lies in the kern of σ, and the following proposition is evident.

Proposition 3.5. *Suppose Definitions 2.1, 3.1 and 3.2, let $\mathbb{A}_0$ be a pre-C^*-algebra and let $\mathbb{I}$ be generated by $\mathbb{I}_0$. Then one obtains a $*$-homomorphism $\widehat{\sigma} : \mathbb{F}/\mathbb{I} \to A \rtimes_\beta (F/F_0)$ deduced from σ of Lemma 3.4, which is injective on $\mathbb{A}_0$ and has dense image.*

Corollary 3.6. *Consider the setting of Proposition 3.5. Then τ is a $*$-isomorphism in the following commutative diagram.*

$$\begin{array}{ccc} \mathbb{F}/\mathbb{I} & \xrightarrow{\widehat{\sigma}} & A \rtimes_\beta (F/F_0) \\ \downarrow \widetilde{\sigma} & \nearrow_\tau & \\ (A \rtimes_\alpha F)/\overline{\sigma(\mathbb{I})} & & \end{array}$$

Proof. Applying Proposition 2.5 to $\mathcal{O} := A \rtimes_\beta (F/F_0)$ of Proposition 3.5 we immediately get the diagram. The only thing we need to show is that τ is injective. To this end think that $Q := (A \rtimes_\alpha F)/\overline{\sigma(\mathbb{I})}$ is represented on a Hilbert space H. Then it is straight forward to show that (π, u, H) is a covariant representation (see [McC]) of $(A, F/F_0, \beta)$, where

$$\begin{aligned} \pi(a) &= \widetilde{\sigma}(a) \in B(H) \qquad (a \in \mathbb{A}_0), \\ u_g &= \widetilde{\sigma}(r_g + \mathbb{I}) \in B(H) \qquad (g \in F/F_0 \backslash \{e\}). \end{aligned}$$

Thus we get a $*$-homomorphism $\pi \times u : A \rtimes (F/F_0) \to Q$ which is the inverse of τ (since $(\pi \times u)\tau(\widetilde{\sigma}(a + \mathbb{I})) = \widetilde{\sigma}(a + \mathbb{I})$ for all letters $a \in \mathcal{A}$ by Definition 3.2.(d)). □

Combining Proposition 2.5 with [B4, Theorem 2.10] yields

Corollary 3.7. *Let $\mathcal{O}_{\mathbb{F},\mathbb{I},H}$ be a Cuntz-Krieger type algebra of* [B4], *suppose Definition 3.2 and let $\mathbb{I}$ be generated by $\mathbb{I}_0$. Then one has an isomorphism $\mathcal{O}_{\mathbb{F},\mathbb{I},H} \cong \overline{\mathbb{A}_0} \rtimes_\beta (F/F_0)$.*

We remark that the above Proposition 3.5 ensures the existence of a C^*-representation $\widehat{\sigma} : \mathbb{F}/\mathbb{I} \to \mathcal{O}$ being faithful on $\mathbb{A}_0$ if the axiomatic system of [B4] is fulfilled, and if one is lacking a concrete representation; at least under the very special setting of this section.

4. Existence of a Reduction Map

In the last section we had a representation $\mathbb{F}/\mathbb{I} \to A \rtimes F_R$ under certain conditions. In particular we supposed a "reduction map" $r : F_R \backslash \{e\} \to \omega_R$ in Defintion 3.2, which is analogous to the assignment to reduced words of free group elements. This section is dedicated to give a condition on R such that a reduction map r exists. The equivalence relation R on ω is supposed to be generated by a focused smaller set $R_0 \subseteq \omega^2$. This is motivated by higher rank Cuntz-Krieger algebras [B1, B2, B3, B5], where a table R_0 of "permutation rules" determines the interaction between generator sets of several rank 1 Cuntz-Krieger algebras. However, the necessary claims on R_0 turn out to be very restrictive here, see Definition 4.1; much more restrictive than what is necessary to ensure "uniqueness" of a Cuntz-Krieger algebra, see Example 4.11 below.

We assume that the alphabet $\mathcal{A}$ is endowed with a partition, i.e. $\mathcal{A} = \bigcup_{v \in V} v$. For any subset $w \subseteq \mathcal{A}$ we use the notation $w^{\circledast} := w \cup w^* \subseteq \mathcal{A} \cup \mathcal{A}^*$. To simplify notations we write

$$
\begin{aligned}
a \sim b \quad &:\Leftrightarrow \quad \exists v \in V : a, b \in v \text{ or } a, b \in v^*, \\
a \parallel b \quad &:\Leftrightarrow \quad \exists v, w \in V : v \neq w, a \in v^{\circledast}, b \in w^{\circledast}
\end{aligned}
$$

for all $a, b \in \mathcal{A}^{\circledast}$.

Definition 4.1. Suppose that the equivalence relation R on ω as in Definition 3.1 is generated by a set of so-called *permutation rules*

$$
R_0 \subseteq \{ (ab, BA) \in \omega \times \omega \mid a, b, A, B \in \mathcal{A}^{\circledast}, a \sim A, b \sim B, a \parallel b \}
$$

with the following properties. For all letters $a, b, A, B, ...$ in $\mathcal{A}^{\circledast}$ we have

$$
\begin{aligned}
(ab, BA) \in R_0 \quad &\Rightarrow \quad (BA, ab), ((ab)^*, (BA)^*), (Ab^*, B^*a) \in R_0, \\
(ab, BA), (ab, \widetilde{B}\widetilde{A}) \in R_0 \quad &\Rightarrow \quad BA = \widetilde{B}\widetilde{A}, \\
(ab, BA), (xa, \widetilde{a}x_2), (x_2 b, \widetilde{b}\widetilde{x}) \in R_0 \quad &\Rightarrow \quad \exists \widetilde{A}, \widetilde{B}, x_3 : (\widetilde{a}\widetilde{b}, \widetilde{B}\widetilde{A}), (xB, \widetilde{B}x_3), (x_3 A, \widetilde{A}\widetilde{x}) \in R_0.
\end{aligned}
$$

Notice that the last rule means that if $ab = BA$ and $xab = \widetilde{a}\widetilde{b}\widetilde{x}$ then $\widetilde{a}\widetilde{b} = \widetilde{B}\widetilde{A}$ and $xBA = \widetilde{B}\widetilde{A}\widetilde{x}$. In other words, it does not matter whether we permute a and b to the right or to the left of x.

Moreover we shall use the observation that $x, y \in \omega$ are equivalent modulo R (in the sequel denoted by $x \equiv y$) if and only if there exists a sequence $x_1, ..., x_n \in \omega$ such that for all $1 \leq i \leq n-1$, x_i differs from x_{i+1} by a single application of a permutation rule R_0 to two neighboring letters of x_i.

Further we emphasize that the above definition of R_0 does not say that two letters a, b are even permutable; R_0 could also be the empty set.

Lemma 4.2. *ω_R has cancellation. Even more if $Ax \equiv By \mod R$ for $A, B \in \omega$ and $x, y \in v^{⊛}$, $v \in V$, then $A \equiv B$ and $x = y$.*

Proof. We have a sequence $x_1, ..., x_n \in \omega$ with $x_1 = Ax$ and $x_n = By$ and such that x_i and x_{i+1} differ by a single R_0-permutation rule for all $1 \leq i \leq n-1$. Fix the rightmost letter a_m in the word $x_i = a_1...a_M$ ($a_k \in \mathcal{A}^{⊛}$) which is in $v^{⊛}$. By induction hypothesis on i suppose that you can move a_m straight entirely to the right by applying the permutation rules R_0, and suppose you obtain $x_i \equiv b_1...b_{M-1}x$ with this procedure. Further suppose that you have $b_1...b_{M-1} \equiv A$.

If $i = n$ then the lemma is proved, because $m = M$ and you necessarily have $B = b_1...b_{M-1} \equiv A$ and $y = x$ by hypothesis. Otherwise you have a single manipulation which "R_0-permutes" two neighboring letters $a_k a_{k+1}$, what yields x_{i+1}. If this permutation is to the left of a_m (i.e. $k+1 < m$), then afterwards you clearly can move a_m entirely to the right and obtain, say, $x_{i+1} \equiv c_1...c_{M-1}x$ and $c_1...c_{M-1} \equiv A$ by hypothesis.

On the other hand, if the R_0-permutation transiting x_i to x_{i+1} involves the letter a_m itself (i.e. $k = m$ or $k+1 = m$), then reversing this step and moving a_m straight entirely to the right comes to the same thing as doing not reverse this step and moving the said letter entirely to the right. So you obtain the induction hypothesis for x_{i+1} once again.

Finally suppose that $m < k$. In this case due to our assumption on R_0 (the last point of Definition 4.1) moving a_m entirely to the right in the word x_{i+1} comes to the same as firstly moving a_m entirely to the right in the word x_i and afterwards doing the R_0-permutation on the adequate position. But this procedure yields $x_{i+1} \equiv b_1...b_{M-1}x$ with $b_1...b_{M-1} \equiv A$. □

In the last proof we have in fact also shown the following result.

Lemma 4.3. *Let $x, a_i \in \mathcal{A}^{⊛}$, $B \in \omega$ and $a_1...a_M \equiv Bx$. Then the rightmost letter $a_m \sim x$ in the word $a_1...a_M$ can moved straight entirely to the right by "skipping" $a_{m+1}, ..., a_M$ via R_0-permutation.*

Lemma 4.4. *For all $a, b, ...\mathcal{A}^{⊛}$, if $(ab, BA), (\widetilde{a}b, \widetilde{B}A) \in R_0$ then $a = \widetilde{a}, B = \widetilde{B}$.*

Proof. By Definition 3.1 we get $(Ab^*, B^*a), (Ab^*, \widetilde{B}^*\widetilde{a}) \in R_0$ and hence $B^*a = \widetilde{B}^*\widetilde{a}$. □

Lemma 4.5. *Let $(ab, BA) \in R_0$. Then there exist $X_1, X_2 \in \omega$ such that $X_1 b \equiv X_2 A \mod R$ iff there exists $Y \in \omega^+$ such that $X_1 \equiv Ya \mod R$.*

Proof. Since the if part is trivial, we consider the only if part. By Lemma 4.3, the rightmost letter which is $\sim A$ in the word $X_1 b$, can moved as far to the right as we like. Thus we get $X_1 b \equiv Ycb$ for some $c \sim A$ and $Y \in \omega$. Similarly we argue that $X_2 A \equiv UdA$ for some $d \sim b$ and $U \in \omega$. By moving d entirely to the right we obtain

$$Ycb \equiv UdA \equiv UA'd'$$

for some $A' \sim A$ and $d' \sim d$. By Lemma 4.2 we get $cb = A'd' \equiv dA$. Due to Lemma 4.4 we have $c = a$ and $d = B$, and we finally cancel b in $X_1 b \equiv Yab$ by Lemma 4.2. □

The following lemma is straight forward by induction on the length of the word B.

Lemma 4.6. *Let $c \in \mathcal{A}^{\circledast}$ and $B \in \omega$. If $c^* B \equiv B' c'^* \mod R$ $(c \sim c')$ by skipping via R_0-permutation, then $cB' \equiv Bc$ via R_0-permutation.*

We shall now consider the following picture of the group F_R. Recall that $\omega^+ := \omega \cup \{\emptyset\}$. If we consider the surjective map

$$\psi : \omega^+ \to F/F_0 : \psi(X) = [X]F_0,\ \psi(\emptyset) = e$$

and its equivalence relation S induced on ω^+, then we can write

$$\omega^+/S \cong F/F_0, \tag{2}$$

purely as bijection between sets for the moment. But it turns out that the natural concatenation and involution on ω^+ coincides with the multiplication and inversion on ω^+ inherited from F/F_0, and (2) is a group isomorphism.

Moreover S is the smallest equivalence relation generated by $R_0 \cup \{\, (aa^*, \emptyset) \in \omega^+ \times \omega^+ \mid a \in \mathcal{A}^{\circledast} \,\}$ which respects multiplication and involution. Further we notice that two words $X, Y \in \omega$ are equivalent modulo S if and only if there exists a sequence $x_1, ..., x_n \in \omega$ such that $X = x_1, Y = x_n$ and for all $1 \le i \le n-1$, x_i differs from x_{i+1} either by a single R_0-permutation, or by adding or cancelling the expression aa^* $(a \in \mathcal{A}^{\circledast})$ once.

Proposition 4.7. *Suppose Definition 4.1. Then there exists a reduction map $r : F_R \backslash \{e\} \to \omega_R$ satisfying Definition 3.2.*

Proof. Step 1. Let $a_1 ... a_n \in \omega$ be a representative of $g \in \omega^+/S \cong F_R$. Then we define $r(g)$ inductively by $r(a_1) := a_1 \in \omega_R$ if $n = 1$. If $n \ge 1$ then we put

$$r(a_1 ... a_{n+1}) := \begin{cases} y & \text{if } \exists y \in \omega^+ : r(a_1 ... a_n) \equiv y a_{n+1}^* \mod R \\ r(a_1 ... a_n) a_{n+1} & \text{otherwise.} \end{cases} \tag{3}$$

Notice that due to Lemma 4.2, y is unique modulo R. Nevertheless we have to show that r is well defined. In the sequel we exclusively use the equivalence character $\equiv$ for equivalence modulo R.

Step 2. Firstly we show that $z :\equiv r(a_1...a_n)$ is a reduced word in the sense that it allows no cancellation in F_R, i.e. $z \not\equiv Axx^*B \mod R$ for any $A, B \in \omega^+$, $x \in \mathcal{A}^{\circledast}$.

Assume to the contrary that $r(a_1...a_{n+1})$ was not reduced, that is $r(a_1...a_{n+1}) \equiv Axx^*B$. By induction hypothesis let $r(a_1...a_n)$ be reduced. In the first case in (3) we get a contradiction by

$$r(a_1...a_n) \equiv ya^*_{n+1} \equiv Axx^*Ba^*_{n+1}.$$

In the second case in (3) we have

$$r(a_1...a_{n+1}) = Axx^*B \equiv r(a_1...a_n)a_{n+1},$$

and by Lemma 4.3 we can move the appearance of the rightmost letter $\sim a_{n+1}$ in the word Axx^*B entirely to the right, and we get $Axx^*B \equiv A'x'x'^*B'a_{n+1} \equiv r(a_1...a_n)a_{n+1}$ and thus $r(a_1...a_n) \equiv A'x'x'^*B'$ by Lemma 4.2, contradicting that $r(a_1...a_n)$ is reduced.

However, if x^* is the utmost right letter $\sim a_{n+1}$ in Axx^*B then we get $Axx^*B \equiv ABa^*_{n+1}a_{n+1}$ by Lemma 4.6, thus $r(a_1...a_n) \equiv ABa^*_{n+1}$ by cancellation, contradicting that we handle the second case in (3).

Step 3. Let $a = a_1...a_n, b = b_1...b_m \in \omega$ two representatives of F_R which are equivalent modulo S. We are going to show that $r(a) \equiv r(b)$, that is r is well defined. It is enough to restrict us to the case that a and b differ only by a single elementary manipulation, i.e. by a manipulation like

$$b_1...b_m = a_1...a_{l-1}b_lb^*_la_l...a_n$$

$(m = n + 2)$, or by a single permutation relation R_0 like

$$b_1...b_m = a_1...a_{l-1}A_lA_{l+1}a_{l+2}...a_m,$$

where $(a_la_{l+1}, A_lA_{l+1}) \in R_0$ and $n = m$.

The first type of manipulation easily yields $r(a) = r(b)$ by the definition of r and by the result of step 2. So we consider the second type. Let

$$z :\equiv r(a_1...a_{l-1}) \qquad (z \in \omega^+).$$

We are going to consider four cases, depending on four possible situations. One has to show that $r(a_1...a_{l-1}a_la_{l+1}) = r(a_1...a_{l-1}A_lA_{l+1})$.

Case 1. Firstly suppose that $z \equiv y a_{l+1}^* a_l^* \mod R$ for some $y \in \omega^+$. Then we clearly get $r(a) = y = r(b)$, since also $z \equiv y A_{l+1}^* A_l^*$.

Case 2. Our next case let be $z \equiv y a_l^*$ for some $y \in \omega^+$, but $y \not\equiv u a_{l+1}^*$ for any $u \in \omega^+$. So

$$r(a_1 ... a_l a_{l+1}) \equiv y a_{l+1}.$$

Assume that $y a_l^* \equiv z \equiv u A_l^*$ for some $u \in \omega^+$. Then by Lemma 4.5 we find some $Y \in \omega^+$ such that $y \equiv Y a_{l+1}^*$, contradicting the assumption. Hence $z \not\equiv u A_l^*$ and $r(a_1 ... a_{l-1} A_l) = z A_l$. From $(a_l a_{l+1}, A_l A_{l+1}) \in R_0$ we get

$$(A_{l+1} a_{l+1}^*, A_l^* a_l) \in R_0$$

and thus $z A_l \equiv y a_l^* A_l \equiv y a_{l+1} A_{l+1}^*$. Therefore we get consistence by $r(a_1 ... A_l A_{l+1}) \equiv y a_{l+1}$.

Case 3. The third case is $z \not\equiv u a_l^*$ for all $u \in \omega^+$ and $z a_l \equiv y a_{l+1}^*$ for some $y \in \omega^+$. This yields

$$r(a_1 ... a_{l+1}) \equiv y.$$

From $(a_l a_{l+1}, A_l A_{l+1}) \in R_0$ we get $(A_{l+1} a_{l+1}^*, A_l^* a_l) \in R_0$ and thus by Lemma 4.5, $z \equiv Y A_l^*$ for some $Y \in \omega^+$. The assumption $Y \equiv v A_{l+1}^*$ would contradict $z \equiv Y A_l^* \equiv v A_{l+1}^* A_l^* \equiv v a_{l+1}^* a_l^*$. So we get $r(a_1 ... A_l A_{l+1}) \equiv Y A_{l+1}^*$. Since $y a_{l+1}^* \equiv z a_l \equiv Y A_l^* a_l \equiv Y A_{l+1} a_{l+1}^*$ we get $y \equiv Y A_{l+1}$ by cancellation, and $r(a) = r(b)$ once again.

Case 4. The fourth case $r(a_1 ... a_n) \equiv z a_l a_{l+1}$ can be proved similar.

STEP 4. Next we are going to check Definition 3.2.

We start with (d). In $F/F_0 \cong \omega^+/S$ we have no chance to get $a \equiv \emptyset \mod S$ for a letter $a \in \mathcal{A}$ since such a transition can be written as a finite sequence of manipulations in ω^+ by exchanging letters, adding xx^* or cancelling xx^*. In all that cases the number of letters in a word remain odd if we start with a, whereas $\emptyset$ has an even number.

We check (c). By induction hypothesis on n let $r(a_1 ... a_m) \equiv Ay$, $r(b_1 ... b_{n-1}) \equiv y^* B$ and $r(a_1 ... a_m b_1 ... b_{n-1}) \equiv AB$ for some $A, y, B \in \omega^+$. We may signal such a formula by writing

$$r(a_1 ... a_m b_1 ... b_{n-1}) \equiv r(a_1 ... a_m) \odot r(b_1 ... b_{n-1}).$$

Per definition $Z :\equiv r(a_1 ... a_m b_1 ... b_{n-1} b_n)$ is either equal $Z \equiv AB b_n$, or equal $Z \equiv z$ if we have a factorization $AB \equiv z b_n^*$.

Case 1. Consider the first case $Z \equiv ABb_n$. If $y^*B \not\equiv ub_n^*$ for any $u \in \omega^+$ then $r(b_1...b_n) \equiv y^*Bb_n$ and so $r(a_1...a_m) \odot r(b_1...b_n) \equiv ABb_n$ as desired.

If $y^*B \equiv ub_n^*$, then by Lemma 4.3, ub_n^* can be seen as the result by moving the most right occurrence $\sim b_n^*$ to the right in the word y^*B. Since $B \equiv u'b_n^*$ would contradict $Z \equiv ABb_n$, the source of b_n^* must lie in y^*, i.e. we must have a pattern like

$$y^*B \equiv y'^*c^*B \equiv y'^*B'b_n^* \tag{4}$$

for $c \sim b_n$. Thus $r(a_1...a_m) \equiv Ay \equiv Acy'$ and $r(b_1...b_n) \equiv y'^*B'$, and we get

$$r(a_1...a_m) \odot r(b_1...b_n) \equiv AcB' \equiv ABb_n \equiv r(a_1...a_mb_1...b_n)$$

by Lemma 4.6.

Case 2. Next consider the case that $AB \equiv zb_n^*$. If according to Lemma 4.3 the letter b_n^* has its source in B, i.e. we have $B \equiv B''b_n^*$, then $r(b_1...b_n) \equiv y^*B''$, and $zb_n^* \equiv AB \equiv AB''b_n^*$. So

$$r(a_1...a_mb_1...b_n) \equiv z \equiv AB'' \equiv r(a_1...a_m) \odot r(b_1...b_n).$$

If the source of b_n^* is in A then we have a pattern (by Lemma 4.3) like

$$AB \equiv A_1d^*B \equiv A_1B_1b_n^*$$

for $d \sim b_n$. We consequently have $z \equiv A_1B_1$ by cancellation and $r(a_1...a_m) \equiv A_1d^*y$. If $r(b_1...b_n) \equiv y^*Bb_n \equiv y^*dB_1$ (we have $dB_1 \equiv Bb_n$ by Lemma 4.6) then we obtain

$$r(a_1...a_m) \odot r(b_1...b_n) \equiv A_1B_1 \equiv z \equiv r(a_1...a_mb_1...b_n).$$

If $r(b_1...b_{n-1}) \equiv ub_n^*$ for some $u \in \omega^+$, then we necessarily have the pattern (4), since B cannot contain a letter similarly to b_n^*. Consequently we get

$$r(a_1...a_m) \equiv Ay \equiv Acy' \equiv A_1d^*cy'.$$

Summing up we have $c^*B \equiv B'b_n^*$ and $d^*B \equiv B_1b_n^*$ via straight R_0-permutations. Thus $cB' \equiv Bb_n$ and $dB_1 \equiv Bb_n$ via R_0-permutation by Lemma 4.6. Since the permutation rules are unique by Definition 4.1 we get $c \equiv d$. However, this contradicts "Step 2" above.

Finally Definition 3.2 (b) can be easily proved by induction and using Definition 3.2 (c) and "Step 2" above. □

Summarizing our results we may formulate the following theorem.

Theorem 4.8. *Let $\mathbb{I}$ be an ideal in $\mathbb{F}$ which is generated by $aa^*a - a$ ($a \in \mathcal{A}$), $AA^*BB^* - BB^*AA^*$ ($A, B \in \omega$), $\mathbb{I} \cap \mathrm{Alg}^*\{ AA^* \in \mathbb{F} \mid A \in \omega \}$ and $\{ A - B \mid (A,B) \in R_0 \}$ for an equivalence relation R_0 on ω satisfying Definition 4.1. Then any $*$-homomorphism $\pi : \mathbb{F}/\mathbb{I} \to \mathcal{O}$ into a C^*-algebra $\mathcal{O}$ being faithful on $\mathbb{A}_0$ can be factorized by the following commutative diagram.*

$$\begin{array}{ccc} & \mathbb{F}/\mathbb{I} & \stackrel{\pi}{\longrightarrow} \quad \mathcal{O} \\ & \downarrow \widehat{\sigma} & \nearrow \\ (\overline{\mathbb{A}_0} \rtimes_\alpha F)/\overline{\sigma(\mathbb{I})} & \cong & \overline{\mathbb{A}_0} \rtimes_\beta (F/F_0) \end{array}$$

Proof. We combine Proposition 2.5, Proposition 3.5, Corollary 3.6 and Proposition 4.7. □

Example 4.9. Consider the higher rank Cuntz algebras $\mathcal{O}_{\{n_1,\ldots,n_N\}}$ [B1] generated by isometries $\mathcal{A} = \bigcup_{i=1}^N \{S_{i,0}, S_{i,1}, \ldots S_{i,n_i-1}\}$ (with this partition of $\mathcal{A}$) for relative prime numbers $n_1, \ldots, n_N$, with the Cuntz properties

$$\sum_{k=1}^{n_i} S_{i,k} S_{i,k}^* - I = 0$$

for all i and with interactions

$$R_{00} := \{ (S_{i,x}S_{j,y}, S_{j,X}S_{i,Y}) \in \omega^2 \mid x + n_i y = X + n_j Y,\ 0 \le x, Y < n_i, 0 \le y, X < n_j \}.$$

Under this rules the following interactions R_{01} are satisfied automatically (cf. [B1]),

$$R_{01} = \{ (S_{i,Y}S_{j,y}^*, S_{j,X}^* S_{i,x}) \in \omega^2 \mid x + n_i y = X + n_j Y,\ 0 \le x, Y < n_i, 0 \le y, X < n_j \}.$$

We put $R_0 := R_{00} \cup R_{01} \cup R_{00}^* \cup R_{01}^*$ and the properties of Definition 4.1 are satisfied. Thereby it is essential that the numbers n_i are relative prime. The last property of Definition 4.1 follows from [B1] where we have shown (Lemma 2.1) that the order of applying permutation rules does not matter.

Remember the ideal $\mathbb{I}_0$ of section 3. Then $\mathbb{I}_0$ contains both the Cuntz properties and the permutation rules, and we get by Proposition 4.7 and Corollary 3.7 that

$$\mathcal{O}_{\{n_1,\ldots,n_N\}} \cong \overline{\mathbb{A}_0} \rtimes F_R.$$

Example 4.10. Consider a partition $\mathcal{A} = \bigcup_{v \in V} v$ such that each v is a finite generator set of a classical rank 1 Cuntz-Krieger algebra [CK] in the usual manner, i.e.

$$\sum_{b \in v} A_v(a,b) bb^* = a^* a$$

for all $v \in V$ and certain so-called transition matrices $A_v : v \times v \to \{0,1\}$ (also cf. [B5]).

Now assume that we have given bijections $\sigma_b : \mathcal{A} \to \mathcal{A}$ for all $b \in \mathcal{A}$, such that $\sigma_b(v) = v$, $\sigma_b \sigma_c = \sigma_c \sigma_b$ and $\sigma_c = \sigma_d$ for all $v \in V$ and all $b, c, d \in \mathcal{A}$ with $c \sim d$.

We claim the permutation rules

$$R_{00} = \{ (a^* b, \sigma_a(b)\sigma_b(a)^*) \in \omega^2 \mid a, b \in \mathcal{A}, a \parallel b \},$$
$$R_{01} = \{ (a\sigma_a(b), b\sigma_b(a) \in \omega^2 \mid a, b \in \mathcal{A}, a \parallel b \},$$

between the rank 1 Cuntz-Krieger algebras, and put $R_0 := R_{00} \cup R_{01} \cup R_{00}^* \cup R_{01}^*$. Then it is straight forward to check the validity of Definition 4.1. Since both the Cuntz-Krieger relations and the R_0-rules are contained in the ideal $\mathbb{I}_0$, Proposition 4.7 and Corollary 3.7 yield a representation $\mathbb{F}/\mathbb{I} \to \overline{\mathbb{A}_0} \rtimes F_R$ which is injective on $\mathbb{A}_0$. The feature is that we are supported with such a representation. In many cases we would expect that $\overline{\mathbb{A}}_0 \rtimes F_R$ is a higher rank Cuntz-Krieger algebra satisfying a canonical uniqueness theorem, confer the class developed in [B5].

Example 4.11. The properties of a reduction map, Definition 3.2, are very restrictive. Consider the rank 2 Cuntz-Krieger algebras of [B2] (a concise description appears in [B5, 4.3]). Then their permutation rules fail the Definition 4.1. One can also see quite directly by the following example that a reduction map does not seem to exist. Consider the word

$$(a_1 b_1 b_2 ...)(a_2 a_3 ...)(a_2 a_3 ...)^* (A_1 b_1 b_2 ...)^* \equiv_R (a_1 a_2 a_3 ...)(b_1 b_2 ...)(b_1 b_2 ...)^* (A_1 a_2 a_3 ...)^*,$$

where $a_i, b_i, A_1 \in \Omega$ and $(a_1 a_2 ...) \parallel (b_1 b_2 ...)$ in $\mathcal{A}$.

Then this word could be reduced (modulo S) to the different words $(a_1 b_1 b_2 ...)(A_1 b_1 b_2 ...)^*$ or $(a_1 a_2 a_3 ...)(A_1 a_2 a_3 ...)^*$, and this is not what we expect.

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