A Green–Julg isomorphism for inverse semigroups

B. Burgstaller *

Abstract

For a finite, unital inverse semigroup S we establish an isomorphism between $KK^S(\mathbb{C}, A)$ and the K-theory of $A \rtimes S$.

1 Introduction

In [2, Theorem 3.5] we gave an isomorphism between S- and $\mathcal{G}$-equivariant KK-theory for an inverse semigroup S and its associated groupoid $\mathcal{G}$ [5]. Thus, if $\mathcal{G}$ is Hausdorff, the Baum–Connes map for groupoids [8] induces a Baum–Connes map $\widehat{\mu^S} : \lim_{Y \subseteq \underline{E}\mathcal{G}} \widehat{KK^S}(C_0(Y), A) \longrightarrow K(A \widehat{\rtimes} S)$ for the inverse semigroup S and Sieben's crossed product $A \widehat{\rtimes} S$ [7] by simply exchanging $KK^{\mathcal{G}}$ with $\widehat{KK^S}$. Replacing here A by $B \rtimes E$ for the (commuting) projection set E in S, this yields a Baum–Connes map $\widehat{\mu^{S,E}} : \lim_{Y \subseteq \underline{E}\mathcal{G}} \widehat{KK^S}(C_0(Y), B \rtimes E) \longrightarrow K(B \rtimes S)$ for Khoshkam's and Skandalis' crossed product $B \rtimes S \cong (B \rtimes E) \widehat{\rtimes} S$ [4]. In this note we prove that for finite S, and $\underline{E}\mathcal{G} = X$, we can here place in front a 'compatible expansion' isomorphism $\delta^S : KK^S(\mathbb{C}, B) \longrightarrow \widehat{KK^S}(C_0(X), B \rtimes E)$, see Theorem 2.3, relying on an equivalence of categories between $C_0(X)$-incompatible Hilbert bimodules (as used in KK^S) and $C_0(X)$-compatible ones (as used in $\widehat{KK^S}$ and $KK^{\mathcal{G}}$), see Proposition 2.2. The composition $\delta^S \circ \widehat{\mu^{S,E}}$ then is a Green–Julg type isomorphism from $KK^S(\mathbb{C}, B)$ to $K(B \rtimes S)$, see Corollary 2.4, which looks compacter than $\widehat{\mu^{S,E}}$ and is reminiscent of the classical Green–Julg isomorphism [3] for compact groups S. Interestingly, this isomorphism for inverse semigroups is not a straight generalization of the classical Green–Julg isomorphism for groups, that means, it is not defined as the composition of the descent homomorphism and an averaging map.

2 The compatible expansion isomorphism

In this note, S denotes a unital, inverse semigroup which has finite idempotent set E. We let X denote the spectrum of the finite dimensional abelian

*Classification: 19K35, 46L80, 20M18 – Email: bernhardburgstaller@yahoo.de – Address: Mathematisches Institut, Einsteinstrasse 62, 48149 Münster, Germany

universal C^*-algebra $C^*(E) = \mathbb{C} \rtimes E$ generated by E, cf. [4, Section 2.6]. In general, there is a locally compact r-discrete (not necessarily Hausdorff) groupoid $\mathcal{G}$ associated to S in such a way that S can be injectively embedded as a family of slices of $\mathcal{G}$, see [5, 6, 4], and one has $X \cong \mathcal{G}^{(0)}$ homeomorphically. In our case $\mathcal{G}$ is even discrete since X is discrete. Throughout we suppose that all S-Hilbert C^*-algebras and S-Hilbert (bi)modules are full in the sense that the unit 1 of S acts identically on them. By universality of $C^*(E)$, a S-Hilbert module or a S-Hilbert C^*-algebra automatically carries the $C_0(X)$-structure induced by the action of $E \subseteq S$, cf. [2, Lemma 4.2]. The module multiplication in a S-Hilbert B-module $(\mathcal{E}, U, S)$ may be $C_0(X)$-*in*compatible (or simply 'incompatible') in the sense one only requires $U_e(\xi)b = U_e(\xi)e(b)$ (see [2, Definitions 2.1-2.4]), or $C_0(X)$-compatible in the sense that also $U_e(\xi)b = \xi e(b)$ for all $\xi \in \mathcal{E}, b \in B, e \in E$, in which case we even speak of a $C_0(X)$-S-Hilbert B-module (see [2, Definitions 2.5-2.8]). Similarly we define compatibility for a Hilbert bimodule by requiring also compatibility for the left module multiplication. Hilbert C^*-algebras have always $C_0(X)$-compatible multiplication. Our reference for KK^S- and $\widehat{KK^S}$-theory, respectively, involving incompatible and compatible, respectively, S-Hilbert bimodules, is [2]. We use, however, the unital (or full) version of KK-theory as explained in [1, Section 4] for KK^S (i.e. 1 acts as the identity map on Hilbert bimodules of cycles). We shall use the full crossed product as defined in [4] or [1]. Typical simple elements in $A \rtimes S$ will be denoted by $a \rtimes s$ with $s \in S, a \in ss^*(A) \subseteq A$. Morphisms between S-Hilbert (A, B)-bimodules are supposed to strictly respect all involving structures, i.e. the bimodule structure, the inner product and the S-action.

Let $\mathcal{E}$ be a S-Hilbert (A, B)-bimodule. Throughout, B acts on $B \rtimes E$ by left multiplication, i.e. $b(b' \rtimes e) = bb' \rtimes e$. In [2, Lemma 4.8] we have defined an expansion functor ϵ_{H} from the category of S-Hilbert (A, B)-bimodules to the category of S-Hilbert $(A \rtimes E, B \rtimes E)$-bimodules. It is given by

$$\epsilon_{\mathrm{H}}(\pi, \mathcal{E}, U, S) = (\tilde{\pi}, \mathcal{E} \otimes_B (B \rtimes E), \tilde{U}, S), \quad \epsilon_{\mathrm{H}}(T) = T \rtimes 1,$$
$$\tilde{\pi}(a \rtimes e) = (\pi(a) \rtimes 1)\tilde{U}_e, \quad \tilde{U}_s = U_s \otimes \beta_s, \quad \beta_s(b \rtimes e) = s(b) \rtimes ses^*,$$

where $T : \mathcal{E}_1 \to \mathcal{E}_2$ denotes a morphism, $T \rtimes 1$ denotes $T \otimes \mathrm{id}_{B \rtimes E}$, β denotes the S-action on $B \rtimes E$ (the usual notation is however s instead of β_s), and $a \in A, b \in B, e \in E, s \in S$. The out-coming bimodule $\mathcal{E}' = \epsilon_{\mathrm{H}}(\mathcal{E}) = \mathcal{E} \otimes_B (B \rtimes E)$ has compatible multiplication between $A \rtimes E$ and $\mathcal{E}'$, but the multiplication between $\mathcal{E}'$ and $B \rtimes E$ is compatible only if $\mathcal{E}$ and B have compatible multiplication. To obtain a functor from (incompatible) S-Hilbert modules to $C_0(X)$-S-Hilbert modules we need to modify the expansion functor ϵ_{H} in order to obtain a functor δ_{H}. We do this by replacing the last tensor product by the compatible one, so we aim to set $\delta_{\mathrm{H}}(\mathcal{E}) = \mathcal{E} \otimes_B^{C_0(X)} (B \rtimes E)$.

To define such a tensor product, consider two S-Hilbert modules $(\mathcal{E}, U, S)$

and $(\mathcal{F}, V, S)$. We may express the spectrum X by means of common refinements of projections $e \in E$ in $C_0(X)$. That is, every $x \in X$ can be expressed as a *formal* expression

$$P = u_{e_1} \dots u_{e_n}(1 - u_{f_1}) \dots (1 - u_{f_m}) \tag{1}$$

where $E = \{e_1 \dots e_n, f_1, \dots, f_m\}$, P is nonzero, and $\delta_x = P(1_{C_0(X)})$. Here, the formal projection P is naturally evaluated by replacing each formal action u_e with the corresponding E-action e on $C_0(X)$. We shall use such an evaluation of P also for other S-Hilbert modules. It is not difficult to see that any $P \in X$ can be expressed in standard form $P = u_e \prod_{f \in E, f < e}(1 - u_f)$ with $e \in E$, and vice versa, every expression in standard form is an element of X. For all $s \in S$ there is an order preserving isomorphism $\gamma_s : (s^*s)E \to (ss^*)E$ by $\gamma_s((s^*s)e) = s(s^*s)es^*$ for $e \in E$.

We may linearly identify the vector space quotient of $\mathcal{E} \otimes_B \mathcal{F}$ divided by the linear span of the elements $U_e(\xi) \otimes \eta - \xi \otimes V_e(\eta)$ $(e \in E, \xi \in \mathcal{E}, \eta \in \mathcal{F})$ with the image of the 'diagonal projection'

$$\mathbb{D} = \sum_{P \in X} P \otimes P$$

acting on $\mathcal{E} \otimes_B \mathcal{F}$. In this way we can define the S-Hilbert module structure of the $C_0(X)$-balanced tensor product by considering it as a submodule:

Definition 2.1 *Define $\mathcal{E} \otimes_B^{C_0(X)} \mathcal{F}$ as the sub-S-Hilbert module $\mathbb{D}(\mathcal{E} \otimes_B \mathcal{F})$.*

Note that for a (formal) S-action u one has $u_s P = P_s u_s$ for $s \in S$, where

$$P_s = u_{se_1s^*} \dots u_{se_ns^*}(1 - u_{sf_1s^*}) \dots (1 - u_{sf_ms^*}).$$

Using the standard form of P, one checks with the isomorphism γ_s that $P \mapsto P_s$ restricts to an isomorphism $\{P \in X | \, P \le u_{s^*s}\} \to \{Q \in X | \, Q \le u_{ss^*}\}$. In this way we see that $(u_s \otimes u_s)\mathbb{D} = \mathbb{D}(u_s \otimes u_s)$. In particular, the diagonal action $U \otimes V$ is invariant under the submodule $\mathbb{D}(\mathcal{E} \otimes_B \mathcal{F})$, and so Definition 2.1 makes sense. If the module multiplication between $\mathcal{E}$ and B is $C_0(X)$-compatible then $P(\xi)b = \xi P(b) = P(\xi b) = P(\xi)P(b)$ $(\xi \in \mathcal{E}, b \in B)$ by induction. If also the module multiplication between B and $\mathcal{F}$ is $C_0(X)$-compatible then $\mathbb{D} = 1$, i.e. $\mathcal{E} \otimes_B^{C_0(X)} \mathcal{F} = \mathcal{E} \otimes_B \mathcal{F}$.

Proposition 2.2 *Let A and B be (full) S-Hilbert C^*-algebras. There is an equivalence of additive categories, δ_{H}, between S-Hilbert (A, B)-bimodules and $C_0(X)$-S-Hilbert (A, B)-bimodules. δ_{H} is given by restricting the evaluated expansion functor ϵ_H to the $C_0(X)$-balanced tensor product, that means precisely,*

$$\begin{aligned} \delta_H(\pi, \mathcal{E}, U, S) &= \epsilon_{\mathrm{H}}(\pi, \mathcal{E}, U, S)|_{\mathcal{E} \otimes_B^{C_0(X)} (B \rtimes E)} \\ &= (\tilde{\pi}\mathbb{D}, \mathbb{D}(\mathcal{E} \otimes_B (B \rtimes E)), \tilde{U}\mathbb{D}, S) \end{aligned} \tag{2}$$

and $\delta_{\rm H}(T) = \epsilon_{\rm H}(T)\mathbb{D} = (T \rtimes 1)\mathbb{D}$ *for a morphism* $T : \mathcal{E}_1 \to \mathcal{E}_2$*.*

The inverse functor is given by $\delta_{\rm H}^{-1}(\mathcal{F}, U, S) = (\mathcal{E}, U, S)$*, where* $\mathcal{E}$ *is defined to be* $\mathcal{F}$ *as a graded vector space, the* S*-actions on* $\mathcal{E}$ *and* $\mathcal{F}$ *are the same, and the Hilbert* (A, B)*-bimodule structure on* $\mathcal{E}$ *is determined by*

$$\begin{aligned} \xi \cdot b &= \xi(b \rtimes 1) \\ \langle P(\xi), P(\eta)\rangle_{\mathcal{E}} P(1_B \rtimes 1) &= \langle P(\xi), P(\eta)\rangle_{\mathcal{F}} \\ a \cdot \xi &= (a \rtimes 1)\xi \end{aligned} \tag{3}$$

for all $a \in A, b \in B, \xi, \eta \in \mathcal{F}$ *and* $P \in X$*. Thereby,* $\langle P(\xi), P(\eta)\rangle_{\mathcal{E}}$ *has to be uniquely chosen in the image of the projection* $\rho_P = e_1 \dots e_n$ *acting on* B*, for* P *having a representation as in (1). For a morphism* $T : \mathcal{F}_1 \to \mathcal{F}_2$ *one defines* $\delta_{\rm H}^{-1}(T) = T$*.*

Proof We begin with $\delta_{\rm H}$. We have already seen that $\tilde{U}\mathbb{D} = \mathbb{D}\tilde{U}$. Also, as $\pi(a) \otimes 1$ $(a \in A)$ commutes with $\tilde{U}_e$ $(e \in E)$, $\tilde{\pi}\mathbb{D} = \mathbb{D}\tilde{\pi}$, and so $\tilde{\pi}$ is invariant under $\mathbb{D}$. The $(B \rtimes E)$-module multiplication is now compatible in $\delta_{\rm H}(\mathcal{E})$, as in $\mathcal{E} \otimes_B (B \rtimes E)$ one has

$$\sum_{P \in X} P(\xi) \otimes P(x)e(y) = \sum_{P(e) \neq 0} P(\xi) \otimes eP(xy) = \sum_{P \in X} U_e P(\xi) \otimes eP(xy)$$

for $\xi \in \mathcal{E}, x, y \in B \rtimes E, e \in E$.

We are going to discuss the inverse functor $\delta_{\rm H}^{-1}(\mathcal{F}) = \mathcal{E}$. Suppose for simplicity that B has a unit 1_B; otherwise we replace 1_B by an approximate unit and take the limit on the left hand side in (3). Choose for P a representation as in (1). If $b \rtimes e \in B \rtimes E$ then a small computation shows that

$$(b \rtimes e)P(1_B \rtimes 1) = \rho_P(b)P(1_B \rtimes 1) \tag{4}$$

Thus, for any $x \in B \rtimes E$ there is a $b \in \rho_P(B)$ such that $xP(1_B \rtimes 1) = bP(1_B \rtimes 1)$. This b is unique, since it is the leading coefficient of the expansion

$$bP(1_B \rtimes 1) = b \rtimes e_1 \dots e_n - f_1(b) \rtimes e_1 \dots e_n f_1$$

(for simplicity we assumed $m = 1$). This shows that $\langle P(\xi), P(\eta)\rangle_{\mathcal{E}}$ in (3) is the unique $b \in \rho_P(B)$ such that

$$bP(1_B \rtimes 1) = \langle P(\xi), P(\eta)\rangle_{\mathcal{F}} P(1_B \rtimes 1) = \langle P(\xi), P(\eta)\rangle_{\mathcal{F}}.$$

Of course, the inner product on $\mathcal{E}$ is now determined by $\langle \xi, \eta\rangle_{\mathcal{E}} = \langle \sum_{P \in X} P(\xi), \sum_{Q \in X} Q(\eta)\rangle_{\mathcal{E}} := \sum_{P \in X} \langle P(\xi), P(\eta)\rangle_{\mathcal{E}}$. One checks that $\langle P\xi, \eta\rangle_{\mathcal{E}} = \langle \xi, P\eta\rangle_{\mathcal{E}}$.

Let $s \in S$. We have $Pu_s = u_s P_{s^*}$ and $s(P(1_B \rtimes 1)) = P_s s(1_B \rtimes 1) = P_s(1_B \rtimes 1)$. Then

$$\begin{aligned} &\langle PU_s\xi, P\eta\rangle_{\mathcal{E}} P(1_B \rtimes 1) = \langle U_s P_{s^*}\xi, P\eta\rangle_{\mathcal{F}} = s(\langle P_{s^*}, P_{s^*}U_s^*\eta\rangle_{\mathcal{F}}) \\ = \;& s(\langle P_{s^*}, P_{s^*}U_s^*\eta\rangle_{\mathcal{E}} P_{s^*}(1_B \rtimes 1)) = s(\langle P_{s^*}, P_{s^*}U_s^*\eta\rangle_{\mathcal{E}}) P_{ss^*}(1_B \rtimes 1) \end{aligned}$$

Thus, $\langle PU_s\xi, P\eta\rangle_{\mathcal{E}} = s\langle P_{s^*}\xi, P_{s^*}U_s^*\eta\rangle_{\mathcal{E}}$, because this is obviously true if $P_{s^*} = 0$, and otherwise this follows from the last computation and the facts that $P_{ss^*} = P$, and $\rho_{P_{s^*}}(b) = b$ implies $\rho_P(s(b)) = s(b)$ for all $b \in B$. Then

$$\langle U_s\xi, \eta\rangle_{\mathcal{E}} = \sum_{P\in X} \langle PU_s\xi, P\eta\rangle_{\mathcal{E}} = \sum_{P\in X} s\langle P_{s^*}\xi, P_{s^*}U_s^*\eta\rangle_{\mathcal{E}} = s\langle \xi, U_s^*\eta\rangle_{\mathcal{E}}$$

Let $T \in \mathcal{L}(\mathcal{F})$ and denote its adjoint in $\mathcal{L}(\mathcal{F})$ by $T^\times$ and in $\mathcal{L}(\mathcal{E})$ by T^*. Since the module multiplication $\mathcal{F}\times B \to \mathcal{F}$ is $C_0(X)$-compatible, $TP = PT$ for all $P \in X$. Then the computation

$$\langle T\xi, P\eta\rangle_{\mathcal{E}} P(1_B \rtimes 1) = \langle T\xi, P\eta\rangle_{\mathcal{F}} = \langle P\xi, T^\times\eta\rangle_{\mathcal{F}} = \langle P\xi, T^\times\eta\rangle_{\mathcal{E}} P(1_B \rtimes 1)$$

shows $\langle T\xi, P\eta\rangle_{\mathcal{E}} = \langle P\xi, T^\times\eta\rangle_{\mathcal{E}}$, and summing up over all $P \in X$ this yields $T^* = T^\times$. This shows that A acts via adjoint-able operators on $\mathcal{E}$. We leave the complete verification that $\mathcal{E}$ is a S-Hilbert (A, B)-bimodule to the reader.

We are going to show the categorial equivalence. The natural isomorphisms u und v, given by isomorphisms $u_{\mathcal{E}} : \mathcal{E} \to \delta_{\mathrm{H}}^{-1}\delta_{\mathrm{H}}(\mathcal{E})$ and $v_{\mathcal{F}} : \mathcal{F} \to \delta_{\mathrm{H}}\delta_{\mathrm{H}}^{-1}(\mathcal{F})$ for all S-Hilbert (A, B)-bimodules $\mathcal{E}$, and all $C_0(X)$-Hilbert $(A \rtimes E, B \rtimes E)$-bimodules $\mathcal{F}$, are defined by the linear maps

$$u_{\mathcal{E}}(\xi) = \mathbb{D}(\xi \otimes (1_B \rtimes 1)), \qquad v_{\mathcal{F}}(\eta) = \mathbb{D}(\eta \otimes (1_B \rtimes 1)) \tag{5}$$

for all $\xi \in \mathcal{E}, \eta \in \mathcal{F}$, the images being here in the vector space $\mathcal{E} \otimes_B^{C_0(X)} (B \rtimes E)$. It is essentially straight forward to check the categorial equivalence; let us demonstrate some identities for convenience of the reader:

$$\begin{aligned} (v_{\mathcal{F}}(\eta))(b \rtimes e) &= \mathbb{D}(\eta \otimes (1_B \rtimes 1))(b \rtimes e) = \sum_{P(e)\neq 0} P(\eta) \otimes P(b \rtimes 1) \\ &= \sum_{P(e)\neq 0} P(\eta(b \rtimes 1)) \otimes P(1_B \rtimes 1) = \mathbb{D}(\eta(b \rtimes e) \otimes (1_B \rtimes 1)) \\ &= v_{\mathcal{F}}(\eta(b \rtimes e)) \end{aligned}$$

for all $\eta \in \mathcal{F}, b \rtimes e \in B \rtimes E$. The S-equivariance of $u_{\mathcal{E}}$ follows from

$$\begin{aligned} u_{\mathcal{E}}(U_s(\xi)) &= \mathbb{D}(U_s(\xi) \otimes (1_B \rtimes 1)) = \mathbb{D}(U_{ss^*}U_s(\xi)s(1_B) \otimes (1_B \rtimes 1)) \\ &= \mathbb{D}(U_s(\xi) \otimes ss^*(s(1_B)(1_B \rtimes 1))) = \mathbb{D}(U_s(\xi) \otimes s(1_B \rtimes 1)) \\ &= s(u_{\mathcal{E}}(\xi)) \end{aligned}$$

QED

Theorem 2.3 *Let E be finite. Let A and B be S-Hilbert C^*-algebras. There is a 'compatible expansion'-isomorphism*

$$\delta^S : KK^S(A, B) \longrightarrow \widehat{KK^S}(A \rtimes E, B \rtimes E)$$

(or δ for brevity) induced by the compatible expansion functor, that is, $\delta[\mathcal{E},T]=[\delta_{\mathrm{H}}(\mathcal{E}),\mathbb{D}(T\rtimes 1)\mathbb{D}]$ and $\delta^{-1}[\mathcal{E},T]=[\delta_{\mathrm{H}}^{-1}(\mathcal{E}),T]$. Both δ and δ^{-1} respect functoriality in A and B, i.e. $(f\rtimes 1)^*\delta=\delta f^*$ and $(g\rtimes 1)_*\delta=\delta g_*$ for equivariant homomorphisms $f:A'\to A$ and $g:B\to B'$.

Proof For a cycle $(\mathcal{E},T)\in\mathbb{E}^S(A,B)$, δ is given by cutting down the cycle $\epsilon(\mathcal{E},T)=(\epsilon_{\mathrm{H}}(\mathcal{E}),T\rtimes 1)\in\mathbb{E}^G(A\rtimes E,B\rtimes E)$ produced by the expansion homomorphism ϵ, see [2, Theorem 4.14], by the projection $\mathbb{D}$ as described in (2). Since $\mathbb{D}$ commutes with $\tilde{\pi}$ and $\tilde{U}$, and $\mathbb{D}\mathcal{K}(\mathcal{E}')\mathbb{D}=\mathcal{K}(\mathbb{D}(\mathcal{E}'))$ for $\mathcal{E}'=\epsilon_{\mathrm{H}}(\mathcal{E})$, $\delta(\mathcal{E},T)$ is a cycle again. It remains to show that δ respects homotopy. Let $(\mathcal{E},T)\in\mathbb{E}^S(A,B[0,1])$. Identify $(B\rtimes E)[0,1]$ with $B[0,1]\rtimes E$. The evaluation map $B[0,1]\rtimes E\to B\rtimes E$ at time t is $C_0(X)$-compatible, and so

$$\begin{aligned}\mathbb{D}(\mathcal{E}\otimes_{B[0,1]}(B[0,1]\rtimes E))\otimes_{B[0,1]\rtimes E}(B\rtimes E) &= \mathbb{D}(\mathcal{E}\otimes_{B[0,1]}(B\rtimes E))\\ &= \mathbb{D}(\mathcal{E}_t\otimes_B(B\rtimes E))\end{aligned}$$

Let us now discuss δ_{H}^{-1}. Let $(\mathcal{F},T)\in\mathbb{E}^S(A\rtimes E,B\rtimes E)$ and $\delta^{-1}(\mathcal{F},T)=(\mathcal{E},T)$. The norms in $\mathcal{F}$ and $\mathcal{E}$ are equivalent, as

$$\begin{aligned}\|\langle\xi,\xi\rangle_{\mathcal{F}}\| &= \Big\|\sum_{P\in X}\langle P\xi,P\xi\rangle_{\mathcal{E}}P(1_B\rtimes 1)\Big\|\le\sup_{P\in X}\|\langle P\xi,P\xi\rangle_{\mathcal{E}}\|\\ \le\ \|\langle\xi,\xi\rangle_{\mathcal{E}}\| &= \Big\|\sum_{P\in X}\langle P\xi,P\xi\rangle_{\mathcal{E}}\Big\|\le\sum_{P\in X}\Big\|\langle P\xi,P\xi\rangle_{\mathcal{F}}\Big\|\le n\|\langle\xi,\xi\rangle_{\mathcal{F}}\|\end{aligned}$$

for $n=|X|$. An elementary compact operator $\theta_{\xi,\eta}\in\mathcal{K}(\mathcal{F})$, regarded as an operator on $\mathcal{E}$, is in $\mathcal{K}(\mathcal{E})$ by the computation

$$\xi\langle\eta,x\rangle_{\mathcal{F}}=\sum_P P(\xi)\langle P\eta,x\rangle_{\mathcal{F}}=\sum_P P(\xi)(\langle P\eta,x\rangle_{\mathcal{E}}\rtimes 1)=\sum_P P(\xi)\cdot\langle P\eta,x\rangle_{\mathcal{E}},$$

where $\cdot$ denotes the module multiplication in $\mathcal{E}$. We have proved that the identity map $\mathcal{L}(\mathcal{F})\to\mathcal{L}(\mathcal{E})$ maps $\mathcal{K}(\mathcal{F})$ into $\mathcal{K}(\mathcal{E})$. It is then easy to check that $(\mathcal{E},T)$ is a cycle.

It remains to show that δ^{-1} respects homotopy. Let $(\mathcal{F},T)\in\mathbb{E}^S(A\rtimes E,(B\rtimes E)[0,1])$ and $\delta^{-1}(\mathcal{F},T)=(\mathcal{E},T)\in\mathbb{E}^S(A,B[0,1])$. Let $\varphi_t:(B\rtimes E)[0,1]\to B\rtimes E$ and $\psi_t:B[0,1]\to B$ be the evaluation maps at time t, and set $\mathcal{F}_t=\mathcal{F}\otimes_{\varphi_t}(B\rtimes E)$ and $\mathcal{E}_t=\mathcal{E}\otimes_{\psi_t}B$. Let $\delta^{-1}(\mathcal{F}_t,T_t)=(\mathcal{G}_t,T_t)$. We want to show that $\omega:\mathcal{E}_t\to\mathcal{G}_t$ ($\cong\mathcal{F}_t$ linearly), $\omega(\xi\otimes b)=\xi\otimes(b\rtimes 1)$, is an isomorphism of S-Hilbert bimodules. The inner product of $\mathcal{G}_t$ is computed from the one of $\mathcal{F}_t$ by

$$\langle P(\xi\otimes(b\rtimes 1)),P(\eta\otimes(c\rtimes 1))\rangle_{\mathcal{G}_t}P(1_B\rtimes 1)=P(b\rtimes 1)^*\varphi_t(\langle P\xi,P\eta\rangle_{\mathcal{F}})P(c\rtimes 1))$$

for $\xi,\eta\in\mathcal{F},b,c\in B$. We get

$$\psi_t(\langle P\xi,P\eta\rangle_{\mathcal{E}})P(1_B\rtimes 1)=\varphi_t(\langle P\xi,P\eta\rangle_{\mathcal{F}})$$

from the connection between the inner products of $\mathcal{E}$ and $\mathcal{F}$. Multiplying from the left and right with $P(b^* \rtimes 1)$ and $P(c \rtimes 1)$, respectively, gives

$$(b^* \psi_t(\langle P\xi, P\eta\rangle_{\mathcal{E}})c)P(1_B \rtimes 1) = P(b^* \rtimes 1)\varphi_t(\langle P\xi, P\eta\rangle_{\mathcal{F}})P(c \rtimes 1)$$

A compare of the last identities shows that ω respects the inner product.

It is now evident with Proposition 2.2 that δ and δ^{-1} are inverses of each others. Finally, using $(f \rtimes 1)^*\epsilon = \epsilon f^*$ ([2, Theorem 4.13.(3)]), one gets $(f \rtimes 1)^*\delta = (f \rtimes 1)^*\mathbb{D}\epsilon = \mathbb{D}(f \rtimes 1)^*\epsilon = \delta f^*$.

QED

For finite S, and hence finite $\mathcal{G}$, we may choose the universal space $\underline{E}\mathcal{G}$ to be X. Then Theorem 2.3, the Baum–Connes isomorphism $\widehat{\mu^S} \equiv \mu^{\mathcal{G}}$, and the isomorphism $\gamma : A \rtimes S \longrightarrow (A \rtimes E)\widehat{\rtimes}S$ [4, Theorems 6.2 and 6.5] yield

Corollary 2.4 *Let S be a finite, unital inverse semigroup. There exists a Green–Julg type isomorphism μ^S determined by the commutative diagram*

$$\begin{array}{ccccc} KK^S(\mathbb{C}, A) & \xrightarrow{\delta^S} & \widehat{KK^S}(C_0(X), A \rtimes E) & \xrightarrow{\widehat{\mu^S}} & K((A \rtimes E)\widehat{\rtimes}S) \\ & \searrow^{\mu^S} & & & \downarrow \gamma_* \\ & & & & K(A \rtimes S) \end{array}$$

References

[1] B. Burgstaller. A KK-theoretical descent homomorphism for semimultiplicative sets. preprint. A revised version will be sent to Münster J. Math.

[2] B. Burgstaller. Equivariant KK-theory of r-discrete groupoids and inverse semigroups. preprint.

[3] P. Julg. K-théorie équivariante et produits croises. *C. R. Acad. Sci., Paris, Sér. I*, 292:629–632, 1981.

[4] M. Khoshkam and G. Skandalis. Crossed products of C^*-algebras by groupoids and inverse semigroups. *J. Oper. Theory*, 51(2):255–279, 2004.

[5] A.L.T. Paterson. *Groupoids, inverse semigroups, and their operator algebras.* Progress in Mathematics (Boston, Mass.). 170. Boston, MA: Birkhäuser., 1999.

[6] J. Quigg and N. Sieben. C^*-actions of r-discrete groupoids and inverse semigroups. *J. Aust. Math. Soc., Ser. A*, 66(2):143–167, 1999.

[7] N. Sieben. C^*-crossed products by partial actions and actions of inverse semigroups. *J. Aust. Math. Soc., Ser. A*, 63(1):32–46, 1997.

[8] Jean-Louis Tu. The Baum-Connes conjecture for groupoids. Cuntz, Joachim (ed.) et al., C*-algebras. Proceedings of the SFB-workshop, Münster, Germany, March 8-12, 1999. Berlin: Springer. 227-242, 2000.