EQUIVARIANT KK-THEORY OF r-DISCRETE GROUPOIDS AND INVERSE SEMIGROUPS

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ABSTRACT. For an r-discrete Hausdorff groupoid $\mathcal{G}$ and a full inverse semigroup S of slices of $\mathcal{G}$ there is an isomorphism between $\mathcal{G}$- and S-equivariant KK-theory. We use it to define descent homomorphisms for S, and indicate a Baum–Connes map for inverse semigroups.

1. Introduction

Paterson [10] constructed a universal r-discrete groupoid $\mathcal{G}_S$ for an arbitrary inverse semigroup S such that S is a full system of compact-open slices of $\mathcal{G}_S$, that means, S generates the topology of $\mathcal{G}_S$. In [11], Quigg and Sieben found an isomorphism between the crossed product $A \rtimes \mathcal{G}$ of a C^*-algebra A by an r-discrete Hausdorff groupoid $\mathcal{G}$ and the crossed product $A \widehat{\rtimes} S$ - in Sieben's sense [12] - of A by S for an inverse semigroup S which is a full system consisting of open slices of $\mathcal{G}$. Khoshkam and Skandalis [7] introduced another full and reduced crossed products for inverse semigroups which usually significantly differ from the C^*-algebras of Sieben, but have shown the following: denoting by E the units of S, the double crossed product $(A \rtimes E) \widehat{\rtimes} S$, where the hat indicates Sieben's crossed product, is isomorphic to Khoshkam–Skandalis' crossed product $A \rtimes S$, and so they also found an isomorphism of their crossed product with $(A \rtimes E) \rtimes \mathcal{G}_S$.

In this note we extend these results to a KK-theoretical level. We use Quigg and Sieben's findings for an isomorphism ρ between $\mathcal{G}$-equivariant KK-theory and S-equivariant KK-theory when E consists of clopen sets, see Theorem 3.5. Khoshkam and Skandalis' findings will be reflected by a homomorphism $KK^S(A, B) \to KK^S(A \rtimes E, B \rtimes E)$ which we call the expansion map ϵ and which is defined like a descent homomorphism, see Theorems 4.9 and 4.10. We use ρ and ϵ to define four types of descent homomorphisms, two, $\widehat{j^S}$ and $\widehat{j_r^S}$, of groupiod- or Sieben's crossed product-type, and two, j^S and j_r^S, of Khoshkam–Skandalis' crossed product-type. j^S coincides with the descent homomorphism in [3]. Both ρ and ϵ respect functoriality, the Kasparov product, and the descent homomorphisms (where meaningful).

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Let us remark that we shall also work with a modified KK^S denoted by $\widehat{KK^S}$, which better reflects $KK^{\mathcal{G}}$. The essential difference to KK^S [4] is that in $\widehat{KK^S}(A,B)$ one requires that for any Hilbert (A,B)-bimodule $\mathcal{E}$ of a cycle, the $C_0(\mathcal{G}^{(0)})$-multiplication on $A, \mathcal{E}$ and B, as in $KK^{\mathcal{G}}$, is strictly compatible.

An application of our translation, Theorem 3.5, is that the Baum–Connes map [1] for groupoids [13] and the full crossed product readily yields a Baum–Connes map for every inverse semigroup which is a full systems of slices of an r-discrete Hausdorff groupoid with totally disconnected unit space (potentially the universal groupoid of Paterson), see Corollary 3.7. Many C^*-algebras like the Cuntz–Krieger algebras [5] are obviously generated by inverse semigroups and are potentially inverse semigroup crossed products, but are less obviously groupoid crossed products. Also, inverse semigroups and their crossed products seem simpler at first glance, so an inverse semigroup Baum–Connes theory may potentially be useful.

The overview of this note is as follows. In section 2 we shall fix the setting, introduce $\widehat{KK^S}$, and recall the bundle picture of $C_0(X)$-algebras and some constructions of crossed products for inverse semigroups. In the other two sections we shall prove the isomorphism in KK-theory (Section 3), the expansion homomorphism (Section 4), and introduce the various descent homomorphisms.

2. Preliminaries

In this note, if nothing else is said, $\mathcal{G}$ denotes an r-discrete locally compact Hausdorff groupoid, and S a full inverse semigroup consisting of slices of $\mathcal{G}$ (see [11, Section 3] and [11, Definition 5.2]). Moreover, we will assume that all slices of S which are contained in $\mathcal{G}^{(0)}$ are clopen in $\mathcal{G}^{(0)}$. Note that by Paterson's construction one may start with an arbitrary inverse semigroup S and realize it as a set of slices in Paterson's universal groupoid $\mathcal{G}_S$, provided that it happens to be Hausdorff.

The range and source maps for $\mathcal{G}$ are denoted by r and s. Recall that a slice t is an open subset of $\mathcal{G}$ such that $r|_t$ and $s|_t$ are homeomorphisms of t with $r(t)$ and $s(t)$, respectively. The set of slices $\mathcal{G}^{op}$, and in particular $S \subseteq \mathcal{G}^{op} \subseteq 2^{\mathcal{G}}$, form an inverse semigroup under set operations induced by corresponding operations in $\mathcal{G}$, see [11, Section 5]. S is called full if S generates the topology of $\mathcal{G}$. (Quigg and Sieben [11] also require S to be upward-directed, a condition which is not mentioned in Khoshkam and Skandalis' paper [7].) In general, a Hausdorff groupoid $\mathcal{G}$ is called r-discrete if $\mathcal{G}^{op}$ generates the topology of $\mathcal{G}$. We shall write

xs ($s \in S, x \in s^*s \subseteq \mathcal{G}^{(0)}$) for the unique $g \in s \subseteq \mathcal{G}$ with source x; similarly we use the product sx. We shall write E for the set of idempotent elements of S, and $1_e \in C_0(\mathcal{G}^{(0)})$ for the characteristic function of e on $\mathcal{G}^{(0)}$ ($e \in E$). By assumption, the elements of E are clopen sets and generate the topology of $\mathcal{G}^{(0)}$. So the linear span of the functions 1_e ($e \in E$) is a dense $*$-subalgebra of $C_0(\mathcal{G}^{(0)})$. We endow $\mathcal{G}$ with the Haar system of counting measures.

In this note, G denotes a semimultiplicative set [4], also called a partial semigroup [2]. That means, G is a set which is endowed with a partially defined, associative multiplication in G: $a(bc)$ is defined if and only if $(ab)c$ is defined ($a, b, c \in G$) and both expressions are equal when defined. A morphism between semimultiplicative sets is a map which respects the product if it is defined. A partial isometry U on a Hilbert module $\mathcal{E}$ is a linear map on $\mathcal{E}$ which projects on a complemented subspace and then maps it norm-isometrically to another complemented subspace, see [3]. U need not be adjoint-able, but there is a natural inverse map for U denoted by U^*, so that $U_g U_g^* U_g = U_g$ and $U_g^* U_g U_g^* = U_g^*$, and $U_g^* U_g$ and $U_g U_g^*$ are self-adjoint projections in $\mathcal{L}(\mathcal{E})$. The set of partial isometries on $\mathcal{E}$ is denoted by $\mathrm{PartIso}(\mathcal{E})$. We are going to recall the notions of G-Hilbert C^*-algebras, G-Hilbert modules and KK^G (see [4]).

Definition 2.1. A *G-Hilbert C^*-algebra* A is a C^*-algebra, regarded as a Hilbert module over itself under the inner product $\langle a, b\rangle = a^*b$, together with a morphism $\alpha : G \to \mathrm{End}(A) \cap \mathrm{PartIso}(A)$ satisfying

$$\langle \alpha_g(a), b\rangle = \alpha_g \langle a, \alpha_g^*(b)\rangle, \qquad \langle \alpha_g^*(a), b\rangle = \alpha_g^* \langle a, \alpha_g(b)\rangle$$

for all $a, b \in A, g \in G$. For every $g \in G$, α_g and α_g^* have to be zero graded. The action α is usually written as $\alpha_g(a) = g(a)$. A $*$-homomorphism $f : A \to B$ between S-Hilbert C^*-algebras A and B is called *G-equivariant* if $f(g(a)) = g(f(a))$ and $f(g^*(a)) = g^*(f(a))$ for all $a \in A, g \in G$. Write G-**Hil**-C^*-**Alg** for the category of G-Hilbert C^*-algebras. The morphisms are the G-equivariant $*$-homomorphisms.

Regarding $\alpha_g \alpha_g^*$ and $\alpha_g^* \alpha_g$ ($g \in G$) as elements in $\mathcal{L}(A) \cong \mathcal{M}(A)$, these are elements in the center of the multiplier algebra of A, see [4, Section 7]. Important relations are $gg^*(a)b = gg^*(ab) = agg^*(b)$ for all $a, b \in A, g \in G$, and similar identities hold for g^*g.

Definition 2.2. Let B be a G-Hilbert C^*-algebra. A *G-Hilbert B-module* $\mathcal{E}$ is a Hilbert B-module together with a morphism $U \to \mathrm{PartIso}(\mathcal{E})$ such that

$$U_g(\xi b) = U_g(\xi) g(b), \qquad U_g^*(\xi b) = U_g^*(\xi) g^*(b)$$
$$\langle U_g(\xi), \eta \rangle = g \langle \xi, U_g^*(\eta) \rangle, \qquad \langle U_g^*(\xi), \eta \rangle = g^* \langle \xi, U_g(\eta) \rangle$$

for all $\xi, \eta \in \mathcal{E}, b \in B, g \in G$. U_g and U_g^* have to be zero graded for all $g \in G$. Write G-**Hil**-B-**Mod** for the category of G-Hilbert B-modules. The morphisms are the adjoint-able operators which intertwine the G-action.

Observe that Hilbert C^*-algebras are G-Hilbert modules over themselves.

Definition 2.3. Let A and B be G-Hilbert C^*-algebras, and $\mathcal{E}$ a G-Hilbert B-module. A *G-equivariant representation* $\pi : A \to \mathcal{L}(\mathcal{E})$ is a $*$-homomorphism such that

$$[\pi(a), U_g U_g^*] = 0, \qquad [\pi(a), U_g^* U_g] = 0$$
$$\pi(g(a)) U_g U_g^* = U_g \pi(a) U_g^*, \qquad \pi(g^*(a)) U_g^* U_g = U_g^* \pi(a) U_g$$

for all $a \in A, g \in G$. This representation is called *strictly G-equivariant* if also

$$\pi(g(a)) = U_g \pi(a) U_g^*, \qquad \pi(g^*(a)) = U_g^* \pi(a) U_g$$

for all $a \in A, g \in G$.

Definition 2.4. Write G-**Hil**-(A, B)-**Bimod** for the category of pairs $(\pi, \mathcal{E})$ where $\mathcal{E}$ is a G-Hilbert B-module and $\pi : A \to \mathcal{L}(\mathcal{E})$ a G-equivariant representation. The morphisms are the G-action intertwining, adjoint-able, A-linear operators.

The class of cycles $\mathbb{E}^G(A, B)$ for G-Hilbert C^*-algebras A and B was defined in [4]. $KK^G(A, B)$ is the class of cycles in $\mathbb{E}^G(A, B)$ divided by homotopy induced by $\mathbb{E}^G(A, B[0, 1])$.

Our basic reference for $C_0(X)$-Banach spaces, -algebras and -modules for a topological space X is Paravicini's thesis [9], and also Le Gall [8] who is specific about C^*-algebras and Hilbert modules. Recall that Hilbert modules inherit their $C_0(X)$-structure from the C^*-algebra [8]. A $C_0(X)$-Banach space $\mathcal{E}$ is called locally $C_0(X)$-convex [9, Def. 4.4.1] if $\|\xi\| = \sup_{x \in X} \|\xi_x\|$ for all $\xi \in \mathcal{E}$. There is an isometric equivalence of categories between $C_0(X)$-convex $C_0(X)$-Banach spaces and upper semi-continuous fields of Banach spaces over X [9] (see Definition 4.2.8, Section 4.4 and Appendix A.2.2 in [9]). Since $C_0(X)$-C^*-algebras and Hilbert modules over $C_0(X)$-C^*-algebras automatically satisfy $C_0(X)$-convexity, they may be regarded as upper semi-continuous (u.s.c.) fields, and vice versa. In this note we will often work with the equivalent field picture, without much comment. For a $C_0(X)$-space $\mathcal{E}$ and a continuous function $f : Y \to X$ the pullback $f^*\mathcal{E}$ (see [9, Def. 2.3.1 and 3.3.1]) is a $C_0(Y)$-space consisting of functions mapping Y to fibers of $\mathcal{E}$ such that y is mapped into the fiber $\mathcal{E}_{f(y)}$. For a map $T : \mathcal{E} \to \mathcal{F}$, or in other words, a family $(T_x : \mathcal{E}_x \to \mathcal{F}_x)_{x\in X}$, $f^*T : f^*\mathcal{E} \to f^*\mathcal{F}$ maps $(\xi_y)_{y\in Y}$ to $(T_{f(y)}(\xi_y))_{y\in Y}$.

The basic reference for groupoid equivariant KK-theory is Le Gall [8]. We will next adapt KK^G-theory in such a way that we get an appropriate equivariant KK-theory $\widehat{KK^S}$ for the inverse semigroup S of slices. Note that we can extract $\mathcal{G}^{(0)}$ from S, since $E \subseteq 2^{\mathcal{G}^{(0)}}$ generates the topology of $\mathcal{G}^{(0)}$, so $\mathcal{G}^{(0)} = \bigcup E$.

Definition 2.5. We denote by $C_0(\mathcal{G}^{(0)})$-S-**Hil**-C^*-**Alg** the subcategory of S-**Hil**-C^*-**Alg** consisting of S-Hilbert C^*-algebras A which are also $C_0(\mathcal{G}^{(0)})$-C^*-algebras and satisfy $1_e \cdot a = e(a)$ for all $a \in A, e \in E$.

Definition 2.6. Let B be a $C_0(\mathcal{G}^{(0)})$-S-Hilbert C^*-algebra. Denote by $C_0(\mathcal{G}^{(0)})$-S-**Hil**-B-**Mod** the subcategory of S-**Hil**-B-**Mod** consisting of S-Hilbert B-modules $\mathcal{E}$ such that $1_e \cdot \xi = U_e(\xi)$ for all $\xi \in \mathcal{E}, e \in E$.

Definition 2.7. Denote by $C_0(\mathcal{G}^{(0)})$-S-**Hil**-(A, B)-**Bimod** the subcategory of S-**Hil**-(A, B)-**Bimod** consisting of pairs $(\pi, \mathcal{E})$ where $\mathcal{E}$ is a $C_0(\mathcal{G}^{(0)})$-S-Hilbert B-module and π is a strictly S-equivariant representation.

Definition 2.8. For $C_0(\mathcal{G}^{(0)})$-S-Hilbert C^*-algebras A and B, we write $\widehat{\mathbb{E}^S}(A, B)$ for the collection of cycles $(\mathcal{E}, T)$ in $\mathbb{E}^G(A, B)$ where $\mathcal{E}$ is a $C_0(\mathcal{G}^{(0)})$-S-Hilbert (A, B)-bimodule. $\widehat{KK^S}(A, B)$ is defined to be $\widehat{\mathbb{E}^S}(A, B)$ divided by homotopy induced by $\widehat{\mathbb{E}^S}(A, B[0, 1])$.

Note that for an S-Hilbert module the S-action U satisfies $U_{s^*} = U_s^*$ for all $s \in S$, see [3, Corollary 4.6]. We shall write $\mathcal{G}$-**Hil**-B-**Mod** for the category of groupoid Hilbert modules [8] (and similarly for bimodules et cetera).

Lemma 2.9. *S-equivariant maps, and strictly S-equivariant representations are automatically $C_0(\mathcal{G}^{(0)})$-linear.*

Proof. Since $C_0(\mathcal{G}^{(0)}) = \overline{\text{span}}\{1_e | \, e \in E\}$ and $\pi(1_e a) = \pi(e(a)) = \pi(a)U_e = 1_e\pi(a)$ for a strictly S-equivariant map π. □

For crossed products of inverse semigroups we will use constructions from three sources: Sieben's full crossed product [12], Khoshkam's and Skandalis' reduced and full crossed products [7], and the author's full crossed product [3]. In any case, an S-action is an inverse semigroup homomorphism from S to some objects related to the C^*-algebra. Khoshkam and Skandalis are most general and allow morphisms from S into the isomorphisms between quotients of ideals, Sieben into partial automorphisms, and the author into endomorphisms which are also partial isometries, the smallest class. Since we will work with Hilbert C^*-algebras A, and thus with Hilbert C^*-actions α, we can use the constructions of all authors: the family of restrictions $\beta_s = (\alpha_s)|_{\alpha_{s^*s}(A)} : \alpha_{s^*s}(A) \to \alpha_{ss^*}(A)$ $(s \in S)$ are isomorphisms between ideals of A and forms an action β by partial automorphisms [12]. Conversely, since each $e \in E$ is clopen - and that is the point why we need this condition - a partial action β on a $C_0(\mathcal{G}^{(0)})$-algebra A given by isomorphisms $\beta_s : A_{s^*s} \to A_{ss^*}$, where A_e denotes the ideal $1_e \cdot A$, can be extended to endomorphisms α_s on A, by putting $\alpha_s(a) = \beta_s(1_{s^*s} \cdot a)$, and under this action α, A becomes a Hilbert C^*-algebra; one should bear this in mind when we shall use results from the other authors. The author's full crossed product coincides with the one of Khoshkam–Skandalis by [3, Corollary 8.4], and goes as follows. The crossed product $A \rtimes S$ is the enveloping C^*-algebra of the Banach $*$-algebra $\ell^1(S, A)$. The latter is the ℓ^1-norm closure of the set of formal sums $\sum_{s\in S} a_s s$ (also denoted by $\sum_{s\in S} a_s \rtimes s$) with finite support, where $a_s \in ss^*(A)$ for all $s \in S$, endowed with the natural convolution product, involution and ℓ^1-norm (see [3, Lemma 5.13]), that is, $(a_s s)^* = s^*(a_s^*)s^*$ and $(a_s s) * (b_t t) = a_s s(b_t) st$ for $a_s \in ss^*(A), b_t \in tt^*(A), s, t \in S$. The crossed product in Sieben's sense, denoted by $A \widehat{\rtimes} S$, is the universal C^*-algebra of $\ell^1(S, A)$ with respect to strictly S-equivariant representations (π, u, H) on Hilbert space H.

3. The isomorphism

In the category of $C_0(\mathcal{G}^{(0)})$-S-Hilbert modules the external tensor product $\mathcal{E} \otimes^{C_0(\mathcal{G}^{(0)})} \mathcal{F}$ is defined in the same way as in the category of $\mathcal{G}$-Hilbert modules [8, Definition 4.2]. The diagonal S-action on $\mathcal{E} \otimes \mathcal{F}$ induces also an action on the latter tensor product, since we have $U_s(1_e\xi) = U_s U_e(\xi) = U_{ses^*} U_s(\xi) = 1_{ses^*} U_s(\xi)$ $(s \in S, e \in E, \xi \in \mathcal{E})$, and so $U(1_e\xi) \otimes V(\eta) = U(\xi) \otimes V(1_e\eta)$ for the diagonal action $U \otimes V$. The internal tensor product is the usual $\mathcal{E} \otimes_B \mathcal{F}$-tensor product of Hilbert modules endowed with the diagonal S-action [4]. When here B acts strictly covariantly on $\mathcal{F}$ then $\mathcal{E} \otimes_B \mathcal{F}$ is also a $C_0(\mathcal{G}^{(0)})$-tensor product, that is, $\mathcal{E} \otimes_B \mathcal{F} \cong \mathcal{E} \otimes_B^{C_0(\mathcal{G}^{(0)})} \mathcal{F}$ (notation from [9]), see Lemma 2.9.

Definition 3.1. For a separable $C_0(\mathcal{G}^{(0)})$-S-Hilbert C^*-algebra D we will write $\widehat{\tau_D^S}$ for the map

$$\widehat{\tau_D^S} : \widehat{KK^S}(A,B) \longrightarrow \widehat{KK^S}(A \otimes^{C_0(\mathcal{G}^{(0)})} D, B \otimes^{C_0(\mathcal{G}^{(0)})} D)$$

defined by $\widehat{\tau_D^S}[\mathcal{E},T] = [\mathcal{E} \otimes^{C_0(\mathcal{G}^{(0)})} D, T \otimes 1]$ (exterior tensor product).

The corresponding well-known maps for KK^S and $KK^{\mathcal{G}}$ are denoted by τ_D^S and $\tau_D^{\mathcal{G}}$, [4] and [8].

We define a Kasparov product for $\widehat{KK^S}$ in the same way as in [4]. In the last Section 7 of [4] one has to use $\widehat{\tau^S}$ rather than τ^S and the one element $1 = [C_0(\mathcal{G}^{(0)}), 0] \in KK^S(C_0(\mathcal{G}^{(0)}), C_0(\mathcal{G}^{(0)}))$ instead of $[\mathbb{C}, 0] \in KK^S(\mathbb{C},\mathbb{C})$ to prove associativity of the cup-cap product. Alternatively, and simpler, one may use the isomorphism of Theorem 3.5 to define the Kasparov product for $\widehat{KK^S}$ or to see its basic properties.

Theorem 3.2 (Quigg and Sieben [11])**.** *There is an equivalence of categories*

$$\rho_{C^*} : \ \mathcal{G}\text{-}C^*\text{-}\mathbf{Alg} \longrightarrow \ C_0(\mathcal{G}^{(0)})\text{-}S\text{-}\mathbf{Hil}\text{-}C^*\text{-}\mathbf{Alg}$$

More precisely, for $\rho_{C^*}(A,\alpha,\mathcal{G}) = (A,\beta,S)$ the $\mathcal{G}$-action α and S-action β determine each other by

$$\begin{aligned} (\beta_s(a))_x &= 1_{\{x \in ss^*\}} \alpha_{xs}(a(s^*xs)) \\ \alpha_g(a_{s(g)}) &= (\beta_s(a))_{r(g)} \end{aligned}$$

for $a \in A, x \in \mathcal{G}^{(0)}, g \in s \in S$.

Moreover, there is an isomorphism $\psi : A \widehat{\rtimes}_\beta S \to A \rtimes_\alpha \mathcal{G}$ determined by

$$(\psi(a \widehat{\rtimes} s))_g = 1_{\{g \in s\}} a_{r(g)}$$

for $a \in ss^*(A) = 1_{ss^*} A, s \in S, g \in \mathcal{G}$.

Proof. The maps and the isomorphism can be found in Theorems 5.3, 6.2 and 7.1 in [11]. ☐

Proposition 3.3. *There is an equivalence of categories*

$$\rho_{\mathrm{H}} : \ \mathcal{G}\text{-}\mathbf{Hil}\text{-}(A,B)\text{-}\mathbf{Bimod} \longrightarrow \ C_0(\mathcal{G}^{(0)})\text{-}S\text{-}\mathbf{Hil}\text{-}(\rho_{C^*}(A),\rho_{C^*}(B))\text{-}\mathbf{Bimod}$$

More precisely, for $\rho_{\mathrm{H}}(\pi;\mathcal{E},V,\mathcal{G};B) = (\pi;\mathcal{E},U,S;\rho_{C^*}(B))$ the $\mathcal{G}$-action V and S-action U determine each other by

$$\begin{aligned} (U_s(\xi))_x &= 1_{\{x \in ss^*\}} V_{xs}(\xi(s^*xs)) \\ V_g(\xi_{s(g)}) &= (U_s(\xi))_{r(g)} \end{aligned}$$

for $\xi \in \mathcal{E}, x \in \mathcal{G}^{(0)}, g \in s \in S$. *Actually,* $\rho_{\mathrm{Hil}} \circ \rho_{\mathrm{Hil}}^{-1}$ *and* $\rho_{\mathrm{Hil}}^{-1} \circ \rho_{\mathrm{Hil}}$ *are the identic functors on the respective category. This equivalence respects the internal and external tensor products.*

Proof. Let us be given a $\mathcal{G}$-action $V : s^*\mathcal{E} \to r^*\mathcal{E}$, and define U as in the theorem. Let $s \in S$ and $\xi \in \mathcal{E}$. We need to show that $U_s(\xi)$ is upper semi-continuous. Introduce the homeomorphism, and transformation, $g : ss^* \to s$, $g(x) = xs$. Then $x = r(g)$, and $(U_s(\xi))_{r(g)} = 1_{\{r(g) \in ss^*\}} V_g(\xi(s(g)))$ for $g \in s$, and so the identity

$$U_s(\xi) \circ r = V(\xi \circ s)$$

holds on the set s. Since $\xi \circ s \in s^*\mathcal{E}$, $V(\xi \circ s) \in r^*\mathcal{E}$ by [9, Def. 3.3.1]. Thus $U_s(\xi) \circ r$ is upper semi-continuous on the set s. It is now easy to see that $U_s(\xi)$ is upper semi-continuous, as $r|_s$ is a homeomorphism and ss^* is clopen.

Let us be given an S-action U, and define the family $V = (V_g)_{g \in \mathcal{G}}$ as in the theorem. Recall that the pullback $s^*\mathcal{E}$ is the "closure" in the sense of [9, Prop. 3.1.26] of the set of simple functions $\mathcal{E} \circ s = \{\xi \circ s | \, \xi \in \mathcal{E}\}$ by [9, Def. 3.3.1]. Note that V is (globally) contractive: given $\varepsilon > 0$ and $g \in \mathcal{G}$, the elements $\xi_{s(g)}$ satisfying $\|\xi\| \le \|\xi_{s(g)}\| + \varepsilon$ for some $\xi \in \mathcal{E}$ are dense in $\mathcal{E}_{s(g)}$; then $\|V_g(\xi_{s(g)})\| = \|U_s(\xi)_{r(g)}\| \le \|U_s\| \|\xi\| \le \|\xi_{s(g)}\| + \varepsilon$. We need to show that the family V maps elements of $s^*\mathcal{E}$ to $r^*\mathcal{E}$. Since V is locally bounded and by definition carries a simple function $\xi \circ s$ to the simple function $U_s(\xi) \circ r$, V is a continuous $\mathcal{G}$-action on $\mathcal{E}$ by [9, Prop. 3.1.30].

For the inner product on $\mathcal{E}$ note that $(\langle \xi, \eta \rangle_{\mathcal{E}})_x = \langle \xi_x, \eta_x \rangle_{\mathcal{E}_x}$ for all $\xi, \eta \in \mathcal{E}, x \in \mathcal{G}^{(0)}$. It is now straightforward computation on fibers to verify the formulas that U is an S-action if V is a $\mathcal{G}$-action (and vice versa), thereby bearing in mind that the corresponding actions on B are the one of Theorem 3.2. Also the remaining straightforward assertions are left to the reader. □

We will now often omit notating ρ_{C^*}; it is understood that we have both the $\mathcal{G}$- and the S-action present as soon as one of these actions is defined.

Recall ([9, 5.1.2]) that for a continuous field of bilinear maps, $\mu : \mathcal{E} \times \mathcal{F} \to \mathcal{N}$, of $\mathcal{G}$-Banach spaces $\mathcal{E}, \mathcal{F}, \mathcal{N}$, there is a convolution product $* : C_c(r^*\mathcal{E}) \times C_c(r^*\mathcal{F}) \longrightarrow C_c(r^*\mathcal{N})$ defined by

$$(\xi * \eta)_g = \int_{\mathcal{G}^{r(g)}} \mu(\xi_h, h(\eta(h^{-1}g))) d\lambda^{r(g)}(h)$$

for $g \in \mathcal{G}$.

Definition 3.4. The groupoid descent homomorphism $j^{\mathcal{G}} : KK^{\mathcal{G}}(A,B) \longrightarrow KK(A \rtimes \mathcal{G}, B \rtimes \mathcal{G})$ is defined by $j^{\mathcal{G}}[\pi,\mathcal{E},T] = [\tilde{\pi}, \overline{C_c(r^*\mathcal{E})}, r^*T]$, where the bar denotes the closure with respect to the $B \rtimes \mathcal{G}$-valued inner product defined by convolution. Also $\tilde{\pi}$ acts by convolution.

We may express $\overline{C_c(r^*\mathcal{E})}$ as

$$\overline{C_c(r^*\mathcal{E})} = \overline{C_c(\mathcal{E} \otimes_B r^*B)} = \mathcal{E} \otimes_B (B \rtimes \mathcal{G}), \tag{1}$$

where B acts on $B \rtimes \mathcal{G}$ by $(b_x)_{x\in\mathcal{G}^{(0)}} \cdot (c_g)_{g\in\mathcal{G}} = (b_{r(g)}c_g)_{g\in\mathcal{G}}$ for $b \in B, c \in r^*B$.

Theorem 3.5. *(1) Let A, B be $\mathcal{G}$-C^*-algebras. Then there exists a group isomorphism*

$$\rho : KK^{\mathcal{G}}(A,B) \longrightarrow \widehat{KK^S}(A,B)$$

defined by $\rho[\mathcal{E},T] = [\rho_{\mathrm{H}}(\mathcal{E}),T]$ for $(\mathcal{E},T) \in \mathbb{E}^G(A,B)$. The inverse map is given by $\rho^{-1}[\mathcal{E},T] = [\rho_{\mathrm{H}}^{-1}(\mathcal{E}),T]$.

(2) ρ respects the Kasparov product, that is,

$$\rho(x \otimes_B y) = \rho(x) \otimes_B \rho(y)$$

if the product $x \otimes_B y$ is defined ($x \in KK^{\mathcal{G}}(A,B), y \in KK^{\mathcal{G}}(B,C)$).

(3) ρ respects functoriality in A and B. That is, equivariant homomorphisms $f : A' \to A$ and $g : B \to B'$ enjoy $f^*\rho = \rho f^*$ and $g_*\rho = \rho g_*$.

(4) ρ respects tensoring, that is, $\rho \circ \tau_D^{\mathcal{G}} = \widehat{\tau_D^S} \circ \rho$ for separable D.

(5) ρ respects the Kasparov cup-cap product.

(6) Define a descent homomorphism $\widehat{j^S} = \nu \circ j^{\mathcal{G}} \circ \rho^{-1}$ by the commuting diagram

$$\begin{array}{ccc} KK^{\mathcal{G}}(A,B) & \xrightarrow{\rho} & \widehat{KK^S}(A,B) \\ j^{\mathcal{G}} \downarrow & & \downarrow \widehat{j^S} \\ KK(A \rtimes \mathcal{G}, B \rtimes \mathcal{G}) & \xrightarrow{\nu} & KK(A \widehat{\rtimes} S, B \widehat{\rtimes} S) \end{array}$$

The isomorphism ν is induced by the isomorphism ψ of Theorem 3.2. One has

$$\begin{aligned} \widehat{j^S}[\pi,\mathcal{E},T] &= [\tilde{\pi}, \mathcal{E} \otimes_B (B \widehat{\rtimes} S), T \otimes 1] \\ \tilde{\pi}(a \widehat{\rtimes} s)(\xi \otimes b \widehat{\rtimes} t) &= \pi(a)U_s(\xi) \otimes s(b) \widehat{\rtimes} st \end{aligned}$$

for $s,t \in S, \xi \in \mathcal{E}, a \in ss^*(A), b \in tt^*(B)$.

(7) Corresponding invariance properties as in points (2)-(6) hold also for ρ^{-1}.

Proof. (1) Let $(\pi, \mathcal{E}, T)$ be a cycle in $\widehat{\mathbb{E}^S}(A,B)$. Define the $\mathcal{G}$-action V as in Proposition 3.3. To show that $(\pi, \mathcal{E}, T) \in \mathbb{E}^{\mathcal{G}}(A,B)$, we need to check that for all $a \in r^*A$, $Q = ((r^*\pi)(a))(V(s^*T)V^* - r^*T) \in \mathcal{K}(r^*\mathcal{E})$ [8, Def. 5.2]. Recall that $\mathcal{K}(r^*\mathcal{E}) = r^*\mathcal{K}(\mathcal{E})$. Let $\xi \in \mathcal{E}$, $s \in S$ an open neighborhood in $\mathcal{G}$, and $a \in A$. Then for all $g \in s$ we have

$$\begin{aligned} & \big(((r^*\pi)(a \circ r))(V(s^*T)V^* - (r^*T)VV^*)(\xi \circ r)\big)_g \\ = \; & (\pi_{r(g)}(a_{r(g)}))(V_g T_{s(g)} V_{g^{-1}} - T_{r(g)} V_g V_{g^{-1}})\xi_{r(g)} \\ = \; & (\pi(a)(U_s T U_{s^*} - T U_s U_{s^*})\xi)_{r(g)} \\ = \; & (k(\xi))_{r(g)} = \big((k \circ r)(\xi \circ r)\big)_g \end{aligned}$$

with $k = \pi(a)(U_s T U_{s^*} - T U_s U_{s^*}) \in \mathcal{K}(\mathcal{E})$ by the definition of a cycle in $\mathbb{E}^S$ [4]. So we have proved that on the local neighborhood s, $(Q - (k \circ r))(\xi \circ r) = 0$. Write $D = Q - (k \circ r)$. Let $\varepsilon > 0$. For any η in $r^*\mathcal{E}$ and $g \in \mathcal{G}$ we may choose a $\xi \in \mathcal{E}$ such that $\|\eta_h - \xi_{r(h)}\| \le \varepsilon$ for all $h \in u_h$ in a neighborhood u_h of g by [9, Def. 3.3.1 and Prop. 3.1.26]. Thus $\|D_g(\eta_g)\| \le \|D\| \|\eta_g\| \varepsilon$ for all $g \in s, \eta_g \in (r^*\mathcal{E})_g$. Consequently, $\|D_g\| = 0$. Hence, on the neighborhood s we have $\|D|_s\| = \sup_{g \in s} \|D_g\| = 0$. So Q looks locally like $k \circ r \in r^*\mathcal{K}(\mathcal{E})$, and so $Q \in r^*\mathcal{K}(\mathcal{E})$.

The inclusion $\mathbb{E}^{\mathcal{G}}(A,B) \subseteq \widehat{\mathbb{E}^S}(A,B)$ is proved similarly. Clearly, we also have thus proved homotopy invariance of ρ, as $\mathbb{E}^{\mathcal{G}}(A, B[0,1]) = \widehat{\mathbb{E}^S}(A, B[0,1])$.

(2) The Kasparov products are essentially defined in the same way: see [8, Def. 6.1] and [4, Def. 19]. Also, the points (3)-(5) are easy, where (5) follows directly from (2) and (4).

(6) Replacing $B \rtimes \mathcal{G}$ by $B \widehat{\rtimes} S$ in Definition 3.4 and (1), we are almost there where we want to be. Let us take the proposed formula for $\tilde{\pi}$ and transform it by the formulas in Theorem 3.2 and Proposition 3.3:

$$\begin{aligned} \tilde{\pi}(a \widehat{\rtimes} s)(\xi \otimes b \widehat{\rtimes} t) \quad &= \quad \pi(a) U_s (\xi b) \widehat{\rtimes} st, \\ \tilde{\pi}\big((1_{\{g \in s\}} a_{r(g)})_{g \in \mathcal{G}}\big)\big((1_{\{g \in t\}} \xi_{r(g)} b_{r(g)})_{g \in \mathcal{G}}\big) & \qquad (2) \\ = \; \pi\big((a_x)_{x \in \mathcal{G}^{(0)}}\big) \Big(1_{\{x \in ss^*\}} V_{xs}((\xi b)(s^* x s))\Big)_{x \in \mathcal{G}^{(0)}} \widehat{\rtimes} st & \\ = \; \Big(1_{\{g \in st\}} 1_{\{r(g) \in ss^*\}} \pi(a_{r(g)}) V_{r(g)s}((\xi b)(s^* r(g) s))\Big)_{g \in \mathcal{G}} & \qquad (3) \end{aligned}$$

(For brevity, we have used the suggestive notation $\xi b \rtimes s$ instead of $\xi \otimes b \rtimes s$.) On the other hand, using convolution for $\tilde{\pi}$ (Def. 3.4) in line (2), we get

$$\Big(\int_{\mathcal{G}^{r(g)}} \pi(1_{\{h \in s\}} a_{r(h)}) V_h \big(1_{\{h^{-1} g \in t\}} (\xi b)_{r(h^{-1} g)}\big) d\lambda^{r(g)}(h) \Big)_{g \in \mathcal{G}} \qquad (4)$$

Since $h \in s$ and $h^{-1}g \in t$, there is only one solution for h, namely $h = r(g)s$. Recalling that $\lambda^{r(g)}$ is chosen to be the counting measure, (4) is exactly (3). (Note that if $g \in st$ then $g = (r(g)s)(ts(g))$, and this is the only possible decomposition in s and t.) □

Definition 3.6. Somewhat improperly, we may define a descent homomorphism $\widehat{j_r^S}$ for the reduced crossed product by $\widehat{j_r^S} = j_r^{\mathcal{G}} \circ \rho^{-1}$.

Corollary 3.7. *If we apply the isomorphism ρ to Tu's Baum–Connes map*

$$\mu^{\mathcal{G}} : \lim_{X \subseteq \underline{E}\mathcal{G}} KK^{\mathcal{G}}(C_0(X), A) \longrightarrow K(A \rtimes \mathcal{G})$$

for groupoids, and combine it with Quigg and Sieben's and Khoshkam and Skandalis', respectively, isomorphisms of crossed products, then we obtain Baum–Connes maps

$$\widehat{\mu^S} : \lim_{X \subseteq \underline{E}\mathcal{G}} \widehat{KK^S}(C_0(X), A) \longrightarrow K(A \widehat{\rtimes} S)$$

$$\mu^S : \lim_{X \subseteq \underline{E}\mathcal{G}} \widehat{KK^S}(C_0(X) \rtimes E, A \rtimes E) \longrightarrow K(A \rtimes S)$$

for inverse semigroups and Sieben's crossed product and Khoshkam-Skandalis' crossed product, respectively.

In the last Baum–Connes map μ^S we have used the isomorphism of Theorem 4.2 below from Khoskam–Skandalis' paper [7]. For Tu's Baum–Connes map $\mu^{\mathcal{G}}$ see [13].

4. The expansion homomorphism

In this section we assume that $\mathcal{G}$ is Paterson's groupoid $\mathcal{G}_S$ associated to S, see Paterson [10], or [6, p. 64]. In particular, $\mathcal{G}^{(0)}$ is homeomorphic to the totally disconnected spectrum X of $C^*(E) \cong C_0(X)$. Since one can construct $\mathcal{G}_S$ for any inverse semigroup S, the only restriction for S in this section is that $\mathcal{G}_S$ is Hausdorff (an exception being Theorem 4.10).

By the universality of $C^*(E)$, the S-action of a S-Hilbert C^*-algebra A extends to a homomorphism from $C^*(E) \cong C_0(X)$ to $Z(\mathcal{L}(A)) \cong Z(\mathcal{M}(A))$. Then A is a $C_0(X)$-algebra if and only if $C_0(X)A = A$.

Proposition 4.1 (Khoshkam and Skandalis [7])**.** *There is a covariant 'expansion' functor*

$$\epsilon_{C^*} : \ S\text{-}\mathbf{Hil}\text{-}C^*\text{-}\mathbf{Alg} \longrightarrow \ C_0(X)\text{-}S\text{-}\mathbf{Hil}\text{-}C^*\text{-}\mathbf{Alg}$$

given by $\epsilon_{C^*}(A, \alpha, S) = (A \rtimes_\alpha E, \beta, S)$, where the S-action β is given by

$$\beta_s(a \rtimes e) = \alpha_s(a) \rtimes ses^*$$

for $a \in e(A), e \in E, s \in S$*. For a morphism* $\pi : A \to B$ *one defines* $\epsilon_{C^*}(\pi) = \pi \rtimes 1$*, where* $(\pi \rtimes 1)(a \rtimes e) := \pi(a) \rtimes e$ *for* $a \in e(A), e \in E$*.*

Proof. The action β was defined in Khoshkam–Skandalis [7, Lemma 6.3]. For the $C_0(X)$-structure see [7, Proposition 5.13]. □

Theorem 4.2 (Khoshkam and Skandalis [7])**.** *Let* $\mathcal{G} = \mathcal{G}_S$*. For an* S*-Hilbert* C^**-algebra* A *define* β *as in Proposition 4.1. Then there are isomorphisms*

$$A \rtimes_\alpha S \overset{\gamma}{\longleftarrow} (A \rtimes_\alpha E) \widehat{\rtimes}_\beta S \overset{\psi}{\longrightarrow} (A \rtimes_\alpha E) \rtimes \mathcal{G}$$

$$A \rtimes_{\alpha,\mathrm{r}} S \overset{\mu}{\longrightarrow} (A \rtimes_\alpha E) \rtimes_{\mathrm{r}} \mathcal{G}$$

where $\gamma((a \rtimes e)\widehat{\rtimes} s) = a \rtimes es$ *for* $s \in S, e \in E, a \in e(A)$ *and* $e \le ss^*$*. (*ψ *is the isomorphism of Quigg–Sieben, Theorem 3.2.)*

Proof. For these results see [7, Theorems 6.2 and 6.5]. □

Definition 4.3. For an S-Hilbert B-module $\mathcal{E}$, the tensor product $\mathcal{E} \otimes_B (B \rtimes E)$ endowed with the diagonal S-action $\tilde{U} = U \otimes \beta$, that is,

$$\tilde{U}_s(\xi \otimes b \rtimes e) = U_s(\xi) \otimes s(b) \rtimes ses^*$$

for $\xi \in \mathcal{E}, b \in e(B), e \in E$, is an S-Hilbert $B \rtimes E$-module denoted by $\mathcal{E} \rtimes E$.

B acts here on $B \rtimes E$ by left multiplication. Note that β_s commutes with this multiplication, whence $\tilde{U}_s$ is well defined. We denote $T \rtimes 1 := T \otimes \mathrm{id}_{B \rtimes E}$ for $T \in \mathcal{L}(\mathcal{E})$. If $\mathcal{E}$ is a $C_0(X)$-S-Hilbert module, so is $\mathcal{E} \rtimes E$.

Lemma 4.4. *There is a covariant 'expansion' functor*

$$\epsilon_{\mathrm{H}} : S\text{-}\mathbf{Hil}\text{-}(A, B)\text{-}\mathbf{Bimod} \longrightarrow S\text{-}\mathbf{Hil}\text{-}(\epsilon_{C^*}(A), \epsilon_{C^*}(B))\text{-}\mathbf{Bimod}$$

given by $\epsilon_{\mathrm{H}}(\pi; \mathcal{E}, U; S) = (\tilde{\pi}; \mathcal{E} \rtimes E, \tilde{U}; S)$ *(Definition 4.3), where* $\tilde{\pi} : A \rtimes E \longrightarrow \mathcal{L}(\mathcal{E} \rtimes E)$ *is the strictly* S*-equivariant map*

$$\tilde{\pi}(a \rtimes e) = (\pi(a) \rtimes 1)\tilde{U}_e$$

for $a \in A_e, e \in E$*. For a morphism* $T : \mathcal{E} \to \mathcal{F}$ *one defines* $\epsilon_{\mathrm{H}}(T) = T \rtimes 1$*.* ϵ_{H} *leaves the subcategory of* $C_0(X)$*-*S*-Hilbert bimodules invariant.*

Proof. Straightforward; note that $\tilde{\pi}(a \rtimes e)(\xi \otimes b \rtimes f) = \pi(a)U_e(\xi) \otimes e(b) \rtimes ef$. □

Lemma 4.5. *Let $\mathcal{E}$ and $\mathcal{F}$ be $C_0(X)$-S-Hilbert bimodules. Then there is a canonical isomorphism of $C_0(X)$-S-Hilbert bimodules*

$$(\mathcal{E} \rtimes E) \otimes_{B \rtimes E} (\mathcal{F} \rtimes E) \;\cong\; (\mathcal{E} \otimes_B \mathcal{F}) \rtimes E$$

Proof. We leave the detailed verification to the reader. Let $\mathcal{E}$ and $\mathcal{F}$ modules over B and C, respectively, with G-actions denoted by U and V, respectively. $B \rtimes E$ acts on $\mathcal{F} \rtimes E$ according to Lemma 4.4. The transformation is given by

$$(\xi \otimes b \rtimes e) \otimes (\eta \otimes c \rtimes f) \longmapsto (\xi b \otimes V_e(\eta)) \otimes e(c) \rtimes ef$$

for $e, f \in E, \xi \in \mathcal{E}, \eta \in \mathcal{F}, b \in e(B)$ and $c \in f(C)$. □

Definition 4.6. In [3], a descent homomorphism

$$j^G : KK^G(A, B) \longrightarrow KK(A \rtimes G, B \rtimes G)$$

is defined by mapping a cycle $(\pi, \mathcal{E}, T)$ to the cycle $(\tilde{\pi}, \mathcal{E} \otimes_B (B \rtimes G), T \otimes 1)$, where $\tilde{\pi}(a \rtimes g) = (\pi(a) \otimes 1)(U_g \otimes V_g)$, and $V_g(b \rtimes h) = g(b) \rtimes gh$ for $g, h \in G, a \in gg^*(A), b \in hh^*(B)$.

Definition 4.7. We define a descent homomorphism, also denoted by j^S, by the commuting diagram

$$\begin{array}{ccc} \widehat{KK^S}(A, B) & \stackrel{\omega}{\longrightarrow} & KK^S(A, B) \\ & j^S \searrow & \downarrow j^S \\ & & KK(A \rtimes S, B \rtimes S) \end{array}$$

Here, ω is defined to be the induced map of the identity map on cycles.

Lemma 4.8. *ω respects the Kasparov product and functoriality in A and B.*

Theorem 4.9. *Let A and B be $C_0(X)$-S-Hilbert C^*-algebras ($X = \mathcal{G}_S^{(0)}$).*

(1) There exists an 'expansion' group homomorphism

$$\epsilon : \widehat{KK^S}(A, B) \longrightarrow \widehat{KK^S}(A \rtimes E, B \rtimes E)$$

defined by $\epsilon[\pi, \mathcal{E}, T] = [\epsilon_{\mathrm{H}}(\pi, \mathcal{E}), T \rtimes 1] = [\tilde{\pi}, \mathcal{E} \rtimes E, T \rtimes 1]$ for $(\pi, \mathcal{E}, T) \in \mathbb{E}^S(A, B)$.

(2) If S has a unit, then the expansion homomorphism respects the intersection product, that is,

$$\epsilon(x \otimes_B y) = \epsilon(x) \otimes_{B \rtimes E} \epsilon(y)$$

if the product $x \otimes_B y$ is defined ($x \in KK^S(A, B), y \in KK^S(B, C)$).

(3) ϵ respects functoriality in A and B, i.e. equivariant homomorphisms $f : A' \to A$ and $g : B \to B'$ enjoy $(f \rtimes 1)^* \epsilon = \epsilon f^*$ and $(g \rtimes 1)_* \epsilon = \epsilon g_*$.

(4) ϵ respects the descent homomorphism, that is, there is a commuting diagram

$$\begin{array}{ccc} \widehat{KK^S}(A,B) & \stackrel{\epsilon}{\longrightarrow} & \widehat{KK^S}(A \rtimes E, B \rtimes E) \\ j^S \downarrow & & \downarrow \widehat{j^S} \\ KK(A \rtimes S, B \rtimes S) & \longrightarrow & KK((A \rtimes E)\widehat{\rtimes} S, (B \rtimes E)\widehat{\rtimes} S) \end{array}$$

The bottom isomorphism is induced by γ of Theorem 4.2.

Proof. (1) At first, we have a descent homomorphism $j^E : KK^S(A,B) \to KK(A \rtimes E, B \rtimes E)$ by Definition 4.6. Now note that j^E is defined on cycles exactly as ϵ. In view of Lemma 4.4 it remains to show that $\tilde{\pi}(a)[\tilde{U}_e, T \rtimes 1]$ and $\tilde{\pi}(a)(\tilde{U}_s(T \rtimes 1)\tilde{U}_s^* - (T \rtimes 1)\tilde{U}_{ss^*})$ are compact operators ($a \in A, e \in E, s \in S$); this follows easily from $\tilde{U}_e(\mathcal{K}(\mathcal{E}) \rtimes 1) \subseteq \mathcal{K}(\mathcal{E} \rtimes E)$ which appears in the proof of [3, Theorem 13.4].

(2) Since on cycles ϵ is defined like j^S, we can use [3, Theorem 13.4] (which requires a unit for S). It is proved there that if x_1, x_2, x_{12} are Kasparov cycles in $KK^S(A,B), KK^S(B,C), KK(A,C)$, then $j^S(x_{12})$ is a Kasparov product [4, Def. 19] for $j^S(x_1)$ and $j^S(x_2)$. So the claim follows immediately if we take here cycles x_1, x_2, x_{12} in $\widehat{KK^S}(A,B), \widehat{KK^S}(B,C)$ and $\widehat{KK^S}(A,C)$, respectively.

(3) The verification is left to the reader, one uses Lemma 4.5.

(4) We leave the straightforward details to the reader, but only outline how the Hilbert module $\mathcal{E}$ of a cycle transforms. Denoting $\mathcal{E}\widehat{\rtimes} S = \mathcal{E} \otimes_B (B\widehat{\rtimes} S)$, and using γ, one exploits an obvious isomorphism

$$(\mathcal{E} \rtimes E)\widehat{\rtimes} S = \mathcal{E} \rtimes_B (B \rtimes E) \otimes_{B \rtimes E} ((B \rtimes E)\widehat{\rtimes} S) \cong \mathcal{E} \otimes_B (B \rtimes S) \cong \mathcal{E} \rtimes S$$

which also intertwines the action of $A \rtimes S$. ☐

Theorem 4.10. *Let S be an arbitrary inverse semigroup (i.e. $\mathcal{G}_S$ not necessarily Hausdorff). Let A and B be S-Hilbert C^*-algebras. Similarly as in Theorem 4.9 there exists an expansion homomorphism*

$$\epsilon : KK^S(A,B) \longrightarrow KK^S(A \rtimes E, B \rtimes E)$$

The points (2)-(3) of Theorem 4.9 are also valid for this expansion homomorphism.

Proof. One proves this along the lines of Theorem 4.9. ☐

Definition 4.11. Let $\mathcal{G} = \mathcal{G}_S$. We define a descent homomorphism j_r^S by the commuting diagram

$$\begin{array}{ccc} \widehat{KK^S}(A,B) & \stackrel{\epsilon}{\longrightarrow} & \widehat{KK^S}(A \rtimes E, B \rtimes E) \\ & & \downarrow \rho^{-1} \\ \downarrow j_{\rm r}^S & & KK^{\mathcal{G}}(A \rtimes E, B \rtimes E) \\ & & \downarrow j_{\rm r}^{\mathcal{G}} \\ KK(A \rtimes_{\rm r} S, B \rtimes_{\rm r} S) & \longleftarrow & KK((A \rtimes E) \rtimes_{\rm r} \mathcal{G}, (B \rtimes E) \rtimes_{\rm r} \mathcal{G}) \end{array}$$

The bottom isomorphism is the canonical one induced by μ of Theorem 4.2.

If we replace in the last diagram the reduced crossed products and $j_r^S, j_r^{\mathcal{G}}$ with the full crossed products and $j^S, j^{\mathcal{G}}$ then we obtain the diagram of Theorem 4.9.(4).

References

[1] P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and K-theory of group C^*-algebras. Doran, Robert S. (ed.), C^*-Algebras: 1943-1993. Providence, RI: American Mathematical Society. Contemp. Math. 167, 241-291 (1994).

[2] V. Bergelson, A. Blass, and N. Hindman. Partition theorems for spaces of variable words. *Proc. Lond. Math. Soc., III. Ser.*, 68(3):449–476, 1994.

[3] B. Burgstaller. A descent homomorphism for semimultiplicative sets. preprint.

[4] B. Burgstaller. Equivariant KK-theory for semimultiplicative sets. *New York J. Math.*, 15:505–531, 2009.

[5] J. Cuntz and W. Krieger. A class of C^*-algebras and topological Markov chains. *Invent. Math.*, 56:251–268, 1980.

[6] M. Khoshkam and G. Skandalis. Regular representation of groupoid C^*-algebras and applications to inverse semigroups. *J. Reine Angew. Math.*, 546:47–72, 2002.

[7] M. Khoshkam and G. Skandalis. Crossed products of C^*-algebras by groupoids and inverse semigroups. *J. Oper. Theory*, 51(2):255–279, 2004.

[8] P.-Y. Le Gall. Equivariant Kasparov theory and groupoids. I. (Théorie de Kasparov équivariante et groupoïdes. I.). *K-Theory*, 16(4):361–390, 1999.

[9] W. Paravicini. *KK-theory for Banach algebras and proper groupoids.* PhD thesis, University of Münster, 2006.

[10] A.L.T. Paterson. *Groupoids, inverse semigroups, and their operator algebras.* Progress in Mathematics (Boston, Mass.). 170. Boston, MA: Birkhäuser. xvi, 274 p. , 1999.

[11] J. Quigg and N. Sieben. C^*-actions of r-discrete groupoids and inverse semigroups. *J. Aust. Math. Soc., Ser. A*, 66(2):143–167, 1999.

[12] N. Sieben. C^*-crossed products by partial actions and actions of inverse semigroups. *J. Aust. Math. Soc., Ser. A*, 63(1):32–46, 1997.

[13] J. L. Tu. The Novikov conjecture for hyperbolic foliations. (La conjecture de Novikov pour les feuilletages hyperboliques.). *K-Theory*, 16(2):129–184, 1999.

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