# The exactness of certain randomized $C^*$-algebras

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# Part I

## Motivation

# Motivation

Many constructions of $C^*$-algebras are given by generators and relations with "perfect" or "ideal" relations .

**Examples**

$$\mathcal{O}_2 \qquad ss^* + tt^* = I \qquad s^*s = I \quad t^*t = I$$

$$C^*(\mathbb{Z}) \qquad uu^* = u^*u = I$$

Both examples have a nice property: nuclear (in particular: exact)

**Question:** What happens if we choose "arbitrary" or "less beautiful" relations?

What happens when we choose the generators by random (in $B(H)$ say)?

Properties of such constructions?

# Aim

a) Find probability measure on $B(H)$

b) Choose $x_1, \ldots, x_n$ in $B(H)$ by random and form $C^*(x_1, \ldots, x_n)$ in $B(H)$

c) Ask

$$\mathbb{P}(C^*(x_1, \ldots, x_n) \text{ has the property that... }) = ?$$

We will ask for **exactness** (suitable due to a local exactness theorem by Pisier)

# Part II

## Random $C^*$-algebras

# Borel structure on $B(H)$

Let $H$ be a separable Hilbert space

Endow $H$ and $B(H)$ with the Borel structures induced by their norms

**Lemma**

*The norm, strong operator, and weak operator topologies induce one and the same Borel structure on $B(H)$.*

# General remarks on measures on $B(H)$ (1)

$H$ separable Hilbert space

$B(H)$ is non-separable!

$\exists r > 0$ and *uncountable* many disjoint balls $B_i$ with radius $r$ in $B(H)$

Let $m$ be a measure on $B(H)$

Then

$$m(B_i) = 0$$

for all but countable many $i$

There exists no infinite dimensional Lebesgue measure

There exists no translation invariant measure on unitary group $U(B(H))$

# General remarks on measures on $B(H)$ (2)

Let $T = (T_{i,j})$ (matrix representation) a Hilbert-Schmidt operator

Let $(x_{i,j})$ i.i.d. normal distributed random variables

Then have random element $X$ in $B(H)$

$$X = \begin{pmatrix} x_{11}T_{11} & x_{12}T_{12} & x_{13}T_{13} & \dots \\ x_{21}T_{21} & x_{22}T_{22} & x_{23}T_{23} & \dots \\ x_{31}T_{31} & x_{32}T_{32} & x_{33}T_{33} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

But $X$ is Hilbert-Schmidt (a.s.)

# Random elements in a separable Hilbert space $H$

a) Wiener measure on $C([0,1]) \subseteq L^2([0,1])$

b)

$$x = \sum_{n=1}^{\infty} \frac{1}{n}(\alpha_n + i\beta_n)e_n$$

where $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ independent $N(0,1)$ normal distributed random variables

$(e_n)$ normal base in $H$

# Random elements in $U(B(H))$ (1)

Sequence $(x_n)$ of independent random elements in $H$

Apply Gram-Schmidt orthogonalization to $(x_n)$ to obtain an orthogonal sequence $(y_n)$

$$y_n = \frac{x_n - \sum_{k=1}^{n-1} < x_k, y_k > y_k}{\| - \ | \ | \ - \|}$$

Random isometry $U$

$$U(e_n) = y_n \qquad (\forall n)$$

# Random elements in $U(B(H))$ (2)

$U$ is random unitary if $(x_n)$ i.i.d.:

a) Let $B$ be a ball in $H$

We have $\mathbb{P}(x_n \in B) = \delta > 0$ for all $n$

b) Thus $\{x_1, x_2, ...\} \cap B \neq \emptyset$ almost surely

c) Hence $\{x_1, x_2, ...\}$ is dense in $H$ almost surely

d) But $\{x_1, x_2, ...\} \subseteq U(H)$

# Random elements in $B(H)$

$U, V$ unitary random elements in $U(B(H))$

$\alpha, \beta$ normal distributed random variables

Random element $X$ in $B(H)$

$$X = \alpha(U + U^*) + i\beta(V + V^*)$$

Conversely, every $X \in B(H)$ has such a from

# Random $C^*$-algebra

Let $X_1, X_2, \ldots$ random elements in $B(H)$

$$C^*(X_1, \ldots, X_n), \qquad C^*(X_1, X_2, \ldots)$$

We don't ask for measurability ...

# Part III

## Gilles Pisier's local theory of exactness

# Exactness

## Definition

$C^*$-algebra $E$ is *exact*, if for all $C^*$-algebras $A$ and $I$, and each exact sequence

$$0 \to I \to A \to A/I \to 0,$$

the following sequence is exact:

$$0 \to E \otimes_{\min} I \to E \otimes_{\min} A \to E \otimes_{\min} (A/I) \to 0$$

# Operator spaces

An *operator space* $E$ is a (closed) linear subspace $E$ of $B(H)$

Let $E, F \subseteq B(H)$ be operator spaces
A linear map $u : E \to F$ is *completely bounded* if the map

$$\mathrm{id} \otimes u : \mathbb{K} \otimes_{\min} E \to \mathbb{K} \otimes_{\min} F$$

is bounded as a linear map between normed spaces
$(\mathbb{K} \otimes_{\min} E \subseteq \mathbb{K} \otimes_{\min} B(H))$

$$\|u\|_{cb} = \|\mathrm{id} \otimes u\| \qquad \text{(completely bounded norm)}$$

Subspaces of $B(H)$ and completely bounded maps form category of operator spaces

# Exactness-Criterion by G. Pisier (1)

Let $E, F$ be finite dimensional operator spaces with same linear dimension
Completely bounded Banach-Mazur distance is

$$d_{cb}(E,F) = \{\, \|u\|_{cb}\ \|u^{-1}\|_{cb} \mid u : E \to F \text{ linear isomorphism} \,\}$$

Distance to compacts $\mathbb{K} = \mathbb{K}(B(H))$:

$$d_{S\mathbb{K}}(E) = \inf\{\, d_{cb}(E,F) \mid F \subseteq \mathbb{K},\ E \cong F \text{ linearly isomorphic} \,\}$$

Any infinite dimensional operator space $E$:

$$d_{S\mathbb{K}}(E) = \sup d_{S\mathbb{K}}(E_0)$$

sup over all finite dimensional subspaces $E_0 \subseteq E$

# Exactness-Criterion by G. Pisier (2)

**Theorem (Pisier)**

*A $C^*$-algebra $A$ is exact iff*

$$d_{S\mathbb{K}}(A) = 1.$$

**Theorem (Pisier)**

*Let $V \subseteq U(B(H))$ be a set of unitaries containing the $1$.*

*Let $A$ be the $C^*$-algebra generated by $\mathbb{V}$ in $B(H)$.*

*Then $A$ is exact iff*

$$d_{S\mathbb{K}}(\overline{\mathrm{lin}}(V)) = 1.$$

# Exactness-Criterion by G. Pisier (3)

Let $\mathbb{F}_n$ be the free group with $n$ free generators

Let $U_1, ..., U_n$ be the canonical unitary generators of $C^*(\mathbb{F}_n)$

Let $E_U^n = \mathrm{lin}(U_1, ..., U_n)$

## Theorem (Pisier)

$$d_{S\mathbb{K}}(E_U^n) \geq \frac{n}{2\sqrt{n-1}}.$$

*Hence* $d_{S\mathbb{K}}(E_U^n) > 1$ *for* $n \geq 3$.

## Corollary

$C^*(\mathbb{F}_n)$ *is not exact* $(n \geq 3)$ *since*

$$d_{S\mathbb{K}}(C^*(\mathbb{F}_n)) \geq d_{S\mathbb{K}}(E_U^n) > 1$$

# The non-separability of $OS_n$ (1)

In this context the following is interesting:

$OS_n$ = $n$-dim operator spaces endowed with the metric

$$\log d_{cb}(E, F) \qquad (E, F \ n\text{-dim o.s.})$$

**Theorem (M. Junge and G. Pisier)**

*$OS_n$ is not separabel.*

# The non-separability of $OS_n$ (2)

Take $x_1, ..., x_n$ in $B(H)$ by random.

The set $M$ of all $n$-dim operator subspaces of $\mathbb{K}$ is separabel in $OS_n$

Can cover $M$ in $OS_n$ by countable many balls $B_n$ with fixed radius $r$ and centers in a dense subset of $M$

However, uncountable many disjoint balls with radius $r$ lie in $OS_n$

Is it then really likely that

$$F = \mathrm{lin}(x_1, ..., x_n) \in \bigcup_{n \in \mathbb{N}} B_n \quad ?$$

We ask this, because, if this is not the case, then we have

$$d_{S\mathbb{K}}(F) > 1$$

and hence

$$C^*(x_1, ..., x_n)$$

is not exact (theorem of Pisier)

# The non-separability of $OS_n$ (3)

However this question does not seem to be easy, because:

Fix $n \geq 1$

Consider distance

$$d_n(E, F) = \inf\{ \|u_n\| \, \|u_n^{-1}\| \mid u : E \longrightarrow F \text{ linear isomorphism} \}$$

$u_n : M_n(E) \longrightarrow M_n(F)$ map on the $n$-th matrix level

Let $E$ be any $n$-dim operator space

Then for all $\varepsilon > 0$ there exists a $n$-dim subspace $F$ of $\mathbb{K}$ such that

$$d_n(E, F) < 1 + \varepsilon$$

# Part IV

## The YES/NO-answer of the question "$A$ is exact?" is measurable

# Exactness of randomized $C^*$-algebras (1)

Let $x = (x_1, ..., x_n)$ where $x_i$ random elements in $B(H)$

Let $A = C^*(x_1, ..., x_n)$ random $C^*$-algebra

Then

$$d_{S\mathbb{K}}(A) = \sup_{E \subseteq A} \inf_{F \subseteq \mathbb{K}} d_{cb}(E, F)$$

is measurable:

$$d_{S\mathbb{K}}(A) = \underbrace{\sup_{n \geq 1} \sup_{f_1, \ldots f_n \in \Gamma} \inf_{y_1, \ldots, y_n \in D} d_{cb}\Big(\mathrm{lin}\big(f_1(x), ..., f_n(x)\big), \mathrm{lin}\big(y_1, ..., y_n\big)\Big)}_{\text{measurable in } x \in B(H)^n}$$

$\Gamma$ is the countable set of formal $*$-algebraic expressions with scalars only in $\mathbb{Q} + i\mathbb{Q} \subseteq \mathbb{C}$

$D$ is a countable dense subset of $\mathbb{K}$

# Exactness of randomized $C^*$-algebras (2)

Can ask

$$\mathbb{P}\Big(A \text{ is exact}\Big) = \mathbb{P}\Big(d_{S\mathbb{K}}(A) = 1\Big) = ?$$

# Part V

## Widely spread isometries

# Widely spread isometries (1)

**Definition**

A tuple $(S_1, ..., S_n)$ of isometries $S_i \in B(H)$ is *widely spread*, if $\forall\ \varepsilon > 0$, $\forall\ k \geq 1$, $\forall$ isometries $T_1, ..., T_n$, there exists an isometry $V$ such that

$$| < S_a V e_i, S_b V e_j > - < T_a e_i, T_b e_j > | \leq \varepsilon$$

for all $1 \leq a, b \leq n$, all $1 \leq i, j \leq k$.

**Lemma**

*Let $(S_1, ..., S_n)$ be widely spread isometries, $F = \mathrm{lin}(S_1, ..., S_n)$. Then*

$$F \cong E_U^n \qquad \textit{completely isometric}$$

# Widely spread isometries (2)

**Theorem**

*Let $(S_1, ..., S_n)$ be widely spread isometries, and $n \geq 3$.*

*Let $A = C^*(S_1, ..., S_n)$ be generated in $B(H)$.*

*Then $A$ is not exact.*

**Proof.**

$F = \mathrm{lin}(S_1, ..., S_n)$

$$d_{S\mathbb{K}}(A) \geq d_{S\mathbb{K}}(F) = d_{S\mathbb{K}}(E_U^n) > 1$$

Pisier: $A$ exact $\Leftrightarrow d_{S\mathbb{K}}(A) = 1$ □

# Widely spread isometries (3)

**Corollary**

*Let $n \geq 3$.*

*There exits a countable strong operator topology dense subset $D$ in $U(B(H))^n$ such that*

$$C^*(u_1, ..., u_n)$$

*is not exact for all $(u_1, ..., u_n) \in D$.*

**Proof.**

Choose suitable widely spread tuples $(u_1, ..., u_n)$ of unitaries $u_i$ (However, it would be enough to deal with the generators $U_1, ..., U_n$ of $C^*(\mathbb{F}_n)$) □

Part VI

# Main result: probability measure resulting in non-exactness

# Some remarks (1)

**Definition**

A probability measure $\mathbb{P}$ on $B(H)$ is *strong operator topology-dense* if $\mathbb{P}(X) > 0$ for all open strong operator topology neighborhoods $X$

**Example**

$\mathbb{P}_D$ atomic on a dense subset $D$ of the compacts $\mathbb{K}$, and vanishing outside of $D$

*Not so satisfying...*

If we ask for exactness:

Choose Independent $\mathbb{P}_D$-distributed $x_1, ..., x_n \in B(H)$

Then

$$\mathbb{P}\Big(C^*(x_1, ..., x_n) \text{ is exact}\Big) = 1$$

# Some remarks (2)

## Example

Fix any $m \in \mathbb{N}$

There exists a probability measure $P$ on $B(H)$ such that for independent $P$-distributed $x_1, ..., x_n$ we have:

$$\mathbb{P}\Big(C^*(x_1, ..., x_n) \text{ is exact}\Big) = \frac{1}{m^{n-1}}$$

# Some remarks (2)

## Example

Fix any $m \in \mathbb{N}$

There exists a probability measure $P$ on $B(H)$ such that for independent $P$-distributed $x_1, ..., x_n$ we have:

$$\mathbb{P}\Big(C^*(x_1, ..., x_n) \text{ is exact}\Big) = \frac{1}{m^{n-1}}$$

**Solution:**

Let $U_1, ..., U_m$ be the canonical generators of $C^*(\mathbb{F}_m)$

Take the Dirac-measure on $\{U_1, ..., U_m\}$ and let $P$ vanish outside of this set:

$$P(U_1) = P(U_2) = ... = P(U_m) = \frac{1}{m}$$

# Probability measure resulting in non-exactness

**Theorem**

*There exists a non-atomic, strong operator topology-dense probability measure $\mathbb{P}_U$ on $U(B(H))$, such that independent $\mathbb{P}_U$-distributed random elements $U_1, \ldots, U_n$ are almost surely widely spread for $n \geq 1$.*
*Hence, $C^*(U_1, \ldots, U_n)$ is almost surely non-exact for $n \geq 3$.*

(Recall: $U_1, \ldots, U_n$ widely spread, $n \geq 3 \quad \Rightarrow \quad C^*(U_1, \ldots, U_n)$ not exact)

# Construction of $\mathbb{P}_U$ (1)

Let $H = \ell^2(\mathbb{Z})$, and $\sigma : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the shift operator

Let $x_1, x_2, x_3, \ldots \in H$ and $w_1, w_2, w_3, \ldots \in H$ be independent, norm-dense and non-degenerated random elements in $H$, $(x_i)$, $(w_i)$ i.i.d.

Let $k_1, k_2, k_3, \ldots$ be a sequence of integers (increasing very fast)

$$
\begin{aligned}
y_1 &= \sigma^{k_1}(x_1) && \text{"line 1"} \\
y_2 &= w_1 \\
(y_3, y_4) &= (\sigma^{k_2}(x_3), \sigma^{k_2}(x_4)) && \text{"line 2"} \\
y_5 &= w_2 \\
(y_6, y_7, y_8) &= (\sigma^{k_3}(x_6), \sigma^{k_3}(x_7), \sigma^{k_3}(x_8)) && \text{"line 3"} \\
y_9 &= w_3 \\
\ldots &= \ldots
\end{aligned}
$$

# Construction of $\mathbb{P}_U$ (2)

Apply Gram-Schmidt to $(y_i)$ to obtain an orthogonal sequence $(z_i)$
Let $U \in U(B(H))$ be the random element

$$U(e_i) = z_i \qquad \forall i \geq 1$$

$U$ is random unitary because $(w_1, w_2, ...)$ is i.i.d. and hence $\{w_1, w_2, ...\}$ dense in $H$; but $\{w_1, w_2, ...\} \subseteq U(H)$

# Proof of the theorem (1)

To show that $(U_1, ..., U_n)$ is widely spread:

Let $\varepsilon > 0$

Let $k \geq 1$

Let $T_1, ..., T_n$ be isometries

We want isometry $V$ such that

$$| < U_a V e_i, U_b V e_j > - < T_a e_i, T_b e_j > | \leq \varepsilon$$

$$\forall a, b = 1, ..., n \qquad \forall i, j = 1, ..., k$$

almost surely

# Proof of the theorem (2)

$$\begin{aligned} \dots &= \dots \\ y_{m-1} &= w_{v-1} \\ (y_m, y_{m+1}, \dots, y_{m+v}) &= (\sigma^{k_v}(x_m), \dots, \sigma^{k_v}(x_{m+v})) \qquad \text{"line v"} \\ y_{m+v+1} &= w_v \\ \dots &= \dots \end{aligned}$$

We have chosen $k_v$ so large, such that $(y_m, y_{m+1}, \dots, y_{m+v})$ is with high probability almost orthogonal to $(y_1, y_2, \dots, y_{m-1})$

Hence $(y_1, y_2, \dots, y_{m-1})$ does not much affect $(z_m, z_{m+1} \dots, z_{m+v})$

$$(z_m, z_{m+1}, \dots, z_{m+v}) \approx \text{Gram-Schmidt}(y_m, y_{m+1}, \dots, y_{m+v})$$

# Proof of the theorem (3)

The probability that

$$\begin{aligned}(x_m, x_{m+1}, \dots, x_{m+k}) &\approx (T_1 e_1, T_1 e_2, \dots, T_1 e_k)\\ (y_m, y_{m+1}, \dots, y_{m+k}) &\approx (\sigma^{k_v} T_1 e_1, \sigma^{k_v} T_1 e_2, \dots, \sigma^{k_v} T_1 e_k)\end{aligned}$$

is $\delta$, independent from $v$

Let $V$ be the shift operator $Ve_i = Ve_{i+m-1}$

$$\begin{aligned}(U_1 V e_1, U_1 V e_2, \dots, U_1 V e_k) &= (U_1 e_m, U_1 e_{m+1}, \dots, U_1 e_{m+k})\\ &= (z_m, z_{m+1}, \dots, z_{m+k})\\ &\approx \text{Gram-Schmidt}(y_m, y_{m+1}, \dots, y_{m+k})\\ &\approx (\sigma^{k_v} T_1 e_1, \sigma^{k_v} T_1 e_2, \dots, \sigma^{k_v} T_1 e_k)\end{aligned}$$

Infinitely many chances with $\mathbb{P} = \delta$ for this; hence

$$\mathbb{P}\Big(\exists v, V : (U_1 V e_1, \dots, U_1 V e_k) \approx (\sigma^{k_v} T_1 e_1, \dots, \sigma^{k_v} T_1 e_k)\Big) = 1$$

# Proof of the theorem (4)

Similarly

$$\mathbb{P}\Big(\exists v, V : \forall a = 1, ..., n : (U_a V e_1, ..., U_a V e_k) \approx (\sigma^{k_v} T_a e_1, ..., \sigma^{k_v} T_a e_k)\Big) = 1$$

Hence almost surely there exists $v, V$ such that

$$| < U_a V e_i, U_b V e_j > - < \sigma^{k_v} T_a e_i, \sigma^{k_v} T_b e_j > | \leq \varepsilon$$

$$\forall a, b = 1, ..., n \qquad \forall i, j = 1, ..., k$$

Show this for countably many $k \geq 1$, $\varepsilon = 1/m$, $T_1, ..., T_k$

$(U_1, ..., U_n)$ widely spread a.s.

Q.E.D.

# Further Remarks (1)

The construction of the measure $\mathbb{P}_U$ could be modified as follows

$$\begin{aligned} y_1 &= \sigma^{k_1}(x_1) \\ (y_2, y_3) &= (\sigma^{k_2}(x_2), \sigma^{k_2}(x_3)) \\ (y_4, y_5, y_6) &= (\sigma^{k_3}(x_4), \sigma^{k_3}(x_5), \sigma^{k_3}(x_6)) \\ \ldots &= \ldots \end{aligned}$$

Obtain random *isometry U*

However, all other claims of the theorem persist valid

# Further Remarks (2)

Perhaps more natural measure

Let $(x_i)$ be i.i.d.

$$\begin{aligned} y_1 &= \sigma^1(x_1) \\ y_2 &= \sigma^2(x_2) \\ y_3 &= \sigma^3(x_3) \\ \ldots &= \ldots \end{aligned}$$

Unclear if then $(U_1, \ldots, U_n)$ is also widely spread

# Further Remarks (3)

But there exists a rearrangement

$$f : \mathbb{N} \rightarrow \mathbb{N}$$

such that $(U_1, ..., U_n)$ is widely spread when

$$\begin{aligned} y_1 &= \sigma^{f(1)}(x_1) \\ y_2 &= \sigma^{f(2)}(x_2) \\ y_3 &= \sigma^{f(3)}(x_3) \\ ... &= ... \end{aligned}$$