3/05/2023

CONDENSED SETS

Seminar on Condensed Groups

- **Definition**: a **condensed set** is a contravariant functor T from Prof to Sets fulfilling $T(\emptyset) = *$ (singleton) and the following two axioms:

(A1) For every $S, S' \in$ Prof, the natural map

$$T(S \sqcup S') \longrightarrow T(S) \times T(S')$$

is a bijection, meaning that T turns finite coproducts into finite products.

(A2) For every surjection $\varphi: S \longrightarrow S'$ of profinite spaces with pullback diagram

$$\begin{array}{ccc} S \times_{S'} S & \xrightarrow{p_2} & S \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \varphi} \\ S & \xrightarrow{\varphi} & S' \end{array}$$

the following

$$\begin{array}{ccc} T(S \times_{S'} S) & \xleftarrow{p_2^*} & T(S) \\ {\scriptstyle p_1^*}\uparrow & & \uparrow{\scriptstyle \varphi^*} \\ T(S) & \xleftarrow{\varphi^*} & T(S') \end{array}$$

is an equalizer diagram. Equivalently:

$$T(S') \cong \{ x \in T(S) \mid p_1^*(x) = p_2^* \in T(S \times_{S'} S) \}$$

In other words, T transforms a coequalizer into an equalizer.

- **Definition**: a **condensed ring/group** is a contravariant functor from Prof to Rings / Groups satisfying the same axioms.

• **Definition**: if $\mathcal{C}$ is any category, the category $\mathrm{Cond}(\mathcal{C})$ of condensed objects of $\mathcal{C}$ is the category of $\mathcal{C}$-valued functors as above.

• **Remark**: since $\mathcal{P}rof$ is not small, this definition presents set-theoretic problems. To solve this, we will look at the categories $\mathcal{P}rof_{\kappa}$ whose objets are the profinite spaces of cardinality less than κ, where κ is an uncountable strong limit cardinal (i.e., κ is uncountable and $\forall\ \lambda < \kappa$, $2^{\lambda} < \kappa$). These categories are equivalent to small categories.

Example 1

As a first example we look at the Hom functor $\underline{T} = C(-,T)$ that, given a fixed Hausdorff space T, sends any profinite space S to the set of cts functions $S \to T$. A cts map of profinite spaces $\varphi: S \to S'$ gives rise to the set map $\varphi^*: \underline{T}(S') \longrightarrow \underline{T}(S)$ via precomposition ($\psi \circ \varphi : S \to S' \to T$). This makes $\underline{T}$ into a contravariant functor.

Even though everything is clear and works with the Hom functor because it preserves limits, we're going to check that both axioms are satisfied.

(A1) is clear, because any map whose domain is a disjoint union of a family of spaces can be uniquely viewed as a sequence of maps with domain each of the spaces in the family.

To check **(A2)**, note first that any surjection $\varphi: S \to S'$ of cpt Hausdorff spaces is a quotient map, hence any other map $f: S' \to T$ s.t. $f \circ \varphi$ is cts will be cts. (Namely: $U \underset{\text{open}}{\subseteq} T \overset{f\circ\varphi \text{ cts}}{\Longrightarrow} (f \circ \varphi)^{-1}(U) = \varphi^{-1} \circ f^{-1}(U) \underset{\text{open}}{\subseteq} S \overset{\varphi \text{ quotient}}{\Longrightarrow} f^{-1}(U) \underset{\text{open}}{\subseteq} S'$).

Given a surjection $\varphi : S \twoheadrightarrow S'$, we need to see:

$$\underline{T}(S') \cong \{ g : S \to T \text{ cts} \mid p_1^*(g) = p_2^*(g) \in \underline{T}(S \times_{S'} S) \} =: E.$$

Note that

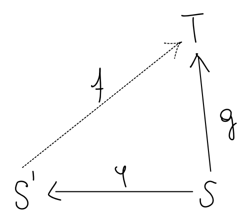
$$E = \{ g : S \to T \text{ cts} \mid g \circ p_1 = g \circ p_2 : S \times_{S'} S \to T \} =$$

$$= \{ g : S \to T \text{ cts} \mid g(a) = g(b) \ \forall (a,b) \in S \times S \text{ s.t. } \varphi(a) = \varphi(b) \}.$$

Consider now the map $\underline{T}(S') \longrightarrow E$ given by $f \mapsto f \circ \varphi$. NTS: this is a bijection.

It is well-defined, since of course $f \circ \varphi(a) = f \circ \varphi(b)$ whenever $\varphi(a) = \varphi(b)$.
Injectivity is also clear, because $f \circ \varphi = f' \circ \varphi$ implies $f = f'$ as φ is surjective.

To see surjectivity we need to solve the lifting problem



for every $g \in E$. This is easy again by the surjectivity of φ: given $x \in S'$, $\exists y \in S$ with $\varphi(y) = x$. Put then $f(x) := g(a)$ and note that this construction is well-defined, as $g(a) = g(b)$ whenever $\varphi(a) = \varphi(b)$. Finally, by the fact discussed at the beginning, we know that this map f is cts.

• **Proposition 1.7:** the functor $X \longrightarrow \underline{X}$ from $\mathcal{T}op$ to $Cond_K$ has a left adjoint.

Proof:

Given a condensed set T, consider the following top. space:

> $T(*)$ (image of T of a singleton).
> Consider $S \in Prof$ and follow the construction:
>
> • $\forall\, s \in S$, get a map $s : * \longrightarrow S$ $\rightsquigarrow$ $s^* : T(S) \longrightarrow T(*)$
>
> • Each $f \in T(S)$ has an underlying map $\underline{f} : S \longrightarrow T(*)$, $s \mapsto s^*(f)$
>
> • Define topology on $T(*)$ as the strongest topology for which $\underline{f}$ is cts. $\forall\, f \in T(S)$, $\forall\, S \in Prof$.

This is the left adjoint:

$$L : Cond_K \rightleftarrows \mathcal{T}op : R$$

$T \longmapsto T(*)$ with above top.

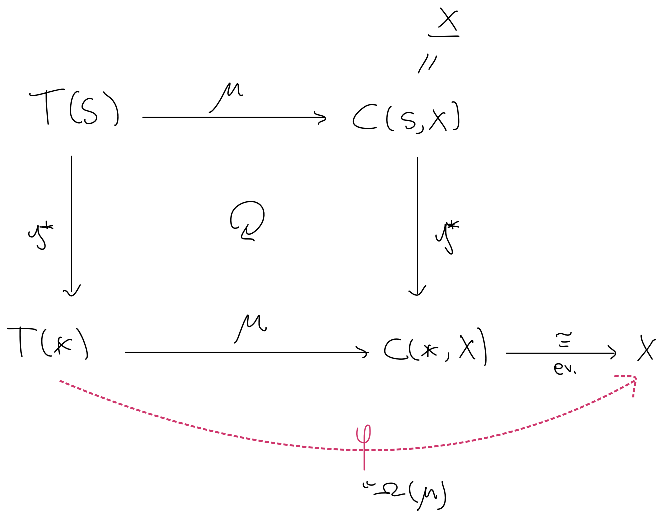
$\underline{X} \longleftarrow X$

So we need a natural bijection

$$\Omega : mor_{Cond_K}(T, \underline{X}) \longrightarrow C(T(*), X)$$

Note that:
- the morphisms between two condensed sets are natural transf.
- $\underline{*}(S) = C(*, S) \underset{ev.}{\cong} S$
- in the case of $T = \underline{X}$, the above map s^* is given by $s^* : C(S, X) \longrightarrow C(*, X)$, $f \longmapsto f(s)$

Define $\Omega(\mu) := ev \circ \mu$ and observe the following diagram, which must commute since μ is a nat. transf. between T and $\underline{X}$:



Take $a \in T(S)$, so that $\mu(a) \in C(S,X)$. By naturality and definition of y^* above, we get:

$$ev \circ \mu(a)(y) = (ev \circ y^* \circ \mu)(a) = (ev \circ \mu \circ y^*)(a) =: \varphi(y^*(a)) =: \Omega(\mu)(y^*(a))$$

↑ by [green] above ↑ by [blue] above

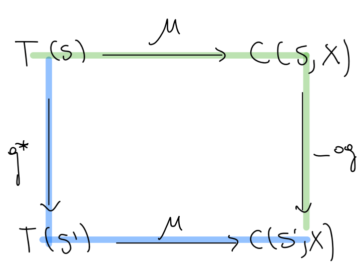
It is then clear that μ determines φ completely, and so Ω is injective.

Besides, φ is cts because of the choice of topology on $T(*)$.

For surjectivity, take $\varphi : T(*) \longrightarrow X$ cts and put

$$\mu : T(S) \longrightarrow C(S,X), \quad a \longmapsto \left(\mu_a : S \longrightarrow X,\ y \longmapsto \varphi(y^*(a)) \right)$$

It is clear that $\Omega(\mu)$ will be φ, so we only need to check its naturality. Therefore, take $g : S' \longrightarrow S$ in Prof. and get the diagram



The green path is as follows:

$$T(S) \xrightarrow{\ \mu\ } C(S,X) \xrightarrow{\ -\circ g\ } C(S',X)$$

$$a \longmapsto \left(\mu_a : S \longrightarrow X,\ y \longmapsto \varphi(y^*(a)) \right) \longmapsto \left(\mu_a \circ g : S' \longrightarrow X,\ y' \longmapsto \varphi(\underbrace{(g \circ y')^*}_{= y'^* \circ g^*}(a)) \right)$$

by contrafunctoriality

Similarly, the blue path gives us:

$$\begin{array}{ccccc} T(S) & \xrightarrow{g^*} & T(S') & \xrightarrow{\mu} & C(S', X) \\ a & \longmapsto & g^*(a) & \longmapsto & \left((\mu \circ g^*)(a) : \begin{array}{ccc} S' & \longrightarrow & X \\ y' & \longmapsto & \varphi(y'^*(g^*(a))) \end{array} \right) \end{array}$$

Thus $\mu_a \circ g = (\mu \circ g^*)(a)$ and so the diagram commutes and the μ that we built is actually a natural transformation.

□

EQUIVALENT CATEGORIES

• **Proposition 2.3**: the category Cond_K is equivalent to the category of contravariant functors $T: \mathrm{CHaus}_K^{op} \longrightarrow \mathrm{Sets}$ satisfying $T(\emptyset) = *$ and (A1), (A2) in CHaus.

Proof:

The functors inducing the equivalence are given by restriction and right Kan extension. We only prove that $T: \mathrm{CHaus}_K^{op} \longrightarrow \mathrm{Sets}$ as above can be expressed in terms of its restriction to $\mathrm{Prof} \subset \mathrm{CHaus}$.

Thus, take $X \in \mathrm{CHaus}$ and find a surjection $\varphi: S \twoheadrightarrow X$ from a profinite space S (e.g. $S = \beta(X^\delta)$). Because T satisfies (A2), seeing S and X in CHaus we get:

$$T(X) \cong \{ x \in T(S) \mid p_1^*(x) = p_2^*(x) \in T(S' \times_S S') \}.$$

Since products in Prof are again in Prof, we are done.

□

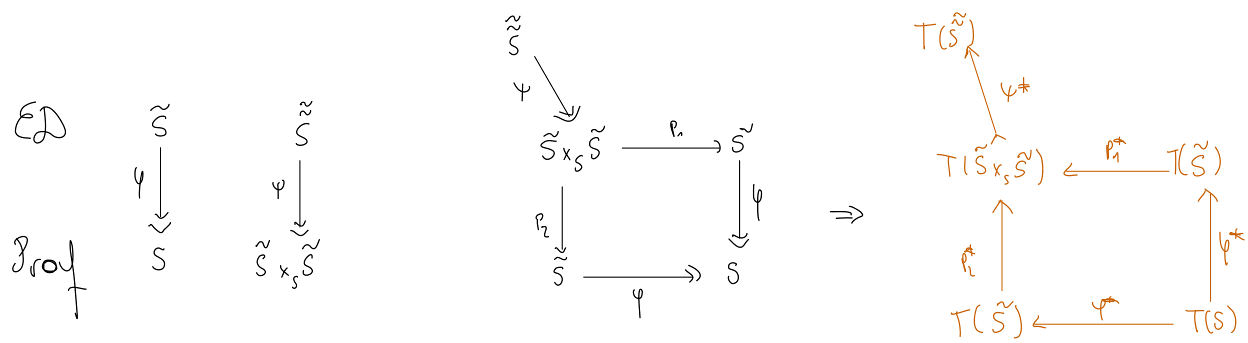
Remark: note that for the S above we have $|S| \leq 2^{2^{|S|}} < K$, so the cardinal bounds are respected.

• **Proposition 2.7**: the category Cond_K is equivalent to the category of contravariant functors $T : \mathcal{ED}_K^{op} \longrightarrow \mathcal{S}ets$ satisfying $T(\emptyset) = *$ and (A1) in $\mathcal{ED}$.
(↑ extrem. disconn.)

Proof:

The functors inducing the equivalence are given by restriction and right Kan extension. We only prove that $T : \mathrm{Prof}_K^{op} \longrightarrow \mathcal{S}ets$ in Cond_K can be expressed in terms of its restriction to $\mathcal{ED} \subset \mathrm{Prof}$.

The problem now is that products in $\mathcal{ED}$ are not in $\mathcal{ED}$ anymore. Hence, take surjections from $\mathcal{ED}$ to Prof (e.g. with Čech-Stone compactif.):



Claim: $T(\tilde{S} \times_S \tilde{S}) \longrightarrow T(\tilde{\tilde{S}})$ is injective.

Proof:
Seeing every space in Prof, by (A2) we know, as φ is surj.:

$$T(\tilde{S} \times_S \tilde{S}) \cong \{ x \in T(\tilde{S}) \mid p_1^*(x) = p_2^* \in T(\tilde{\tilde{S}} \underset{\tilde{S} \times \tilde{S}}{\times_S} \tilde{\tilde{S}}) \} \subseteq T(\tilde{S}).$$

□

Thus we have the above orange diagram and that way $T(S)$ is the equalizer $T(S) \xrightarrow{\varphi^*} T(\tilde{S}) \underset{\varphi^* \circ p_1^*}{\overset{\varphi^* \circ p_1^*}{\rightrightarrows}} T(\tilde{\tilde{S}})$ of elts in $\mathcal{ED}_K$.

Only remains to prove the following:

Claim: any contravariant functor $T: ED^{op} \longrightarrow Sets$ satisfying (A1) also satisfies (A2).

Proof:

Given a surjection $f: S' \twoheadrightarrow S$ in ED, by the 2nd talk we know that it splits: $\exists g: S \longrightarrow S'$ s.t. $f \circ g = id_S \Rightarrow g^* \circ f^* = id_{T(S)} \Rightarrow f^*: T(S) \longrightarrow T(S')$ is injective. Thus, $p_1^* \circ f^*(x) = p_2^* \circ f^*(x) \iff p_1^*(x) = p_2^*(x)$, for $x \in T(S')$ and $p_1^*, p_2^*: T(S') \to T(S' \times_S S')$, and hence:

$$f^*(T(S)) \subset \{ x \in T(S') \mid p_1^*(x) = p_2^*(x) \in T(S' \times_S S') \} =: A.$$

Now let's see that f^* surjects onto A:

Let $x \in A$. Then:

$$((g \circ f)^* \times_S id_{S'})(p_1^*(x)) = ((g \circ f)^* \times_S id_{S'})(p_2^*(x)) \rightsquigarrow f^*(g^*(x)) = x \Rightarrow$$

$\Rightarrow g^*(x) \in T(S)$ is a preimage of x.

$\square$

Finally, the same remark as above checks the cardinality.

$\square$

Finally, we state (without proof) a result that allows us to give a final definition of condensed sets/groups/rings... that does not present set-theoretical problems and does not depend on an auxiliary cardinal either.

• **Proposition 2.9**: Let $\kappa' > \kappa$ be uncountable strong limit cardinals. There is a natural fully faithful functor from $Cond_\kappa$ to $Cond_{\kappa'}$.

• **Definition**: the category of condensed sets is given by the filtered colimit of Cond_κ along the filtered poset of all uncountable strong limit cardinals κ. Equivalently, this is the category of functors

$$T : \mathcal{ED}^{op} \longrightarrow \mathcal{S}ets$$

sending finite disjoint unions to finite products, and such that for some uncountable strong limit cardinal κ, it is the left Kan extension of its restriction to κ-small disconnected sets.