Seminar on Condensed Groups

The aim of the seminar is to learn the basics about condensed mathematics, following notes by Clausen and Scholze. A condensed set is a contravariant functor t on the category prof of profinite spaces (totally disconnected compact spaces) to the category of sets/groups/rings satisfying the following two axioms.

(S1) For all profinite spaces X, Y, the natural map

$$t(X \sqcup Y) \to t(X) \times t(Y)$$

is an isomorphism, i.e. t turns finite coproducts into finite products.

(S2) If $f : X \to B$ is a surjective map of profinite spaces, with pullback diagram

$$\begin{array}{ccc} X \times_B X & \xrightarrow{p_2} & X \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & B, \end{array}$$

then

$$\begin{array}{ccc} t(X \times_B X) & \xleftarrow{t(p_2)} & t(X) \\ {\scriptstyle t(p_1)}\big\uparrow & & \big\uparrow{\scriptstyle t(f)} \\ t(X) & \xleftarrow{t(f)} & t(B) \end{array}$$

is an equalizer diagram.

An example is the functor $\underline{Z} = C(-, Z)$ which assigns, for a fixed Hausdorff space Z, to each profinite space X the set of continuous maps $X \to Z$. For example, if $X = \mathbb{N} \cup \{\infty\}$, then $C(X, Z)$ is the set of all convergent sequences in Z. A continuous map $f : X \to Y$ of profinite spaces induces a map $C(X, Z) \leftarrow C(Y, Z)$ by precomposition, hence $\underline{Z}$ is a contravariant functor. If Z happens to be a topological group, then $\underline{Z}(X) = C(X, Z)$ is again a group.

The definition given above poses some set-theoretic problems, since the category of profinite spaces is not small. One therefore restricts the construction to the subcategories prof_κ consisting of profinite spaces of cardinality $< \kappa$. Here κ is an uncountable strong limit cardinal ($2^\lambda < \kappa$ for every $\lambda < \kappa$). These categories prof_κ are equivalent to small categories.

Aside from the set-theoretic issues, axiom (S2) is somewhat tedious. This axiom can be avoided altogether if one works in the category proj of projective compact spaces.

A compact space P is projective (in the category comp of compact spaces) if for every surjective map $K \to L$ and every map $P \to L$ in comp, the lifting problem

$$\begin{array}{ccc} & & P \\ & \swarrow & \downarrow \\ K & \longrightarrow & L \end{array}$$

has a solution. It turns out that the projective compact spaces are precisely the extremally disconnected compact spaces. A Hausdorff space is extremally disconnected if the closure of every open set is open. The Čech-Stone compactification βD of any discrete space D has this property, and in fact every compact extremally disconnected space E is a retract of such a βD. These retracts are precisely the projective objects in comp. One can show that a condensed set t is determined (up to equivalence) by its restriction to the subcategory $\mathsf{proj} \subseteq \mathsf{prof}$. Even better, a functor t on proj which satisfies (S1) extends to a condensed set on prof. Hence it suffices (in principle) to study functors t on proj which satisfy (S1) in order to study condensed sets.

Condensed topological groups have several good properties The category of locally compact abelian groups lca is not an abelian category. It lacks, among other things, cokernels, as the following example shows. If $\mathbb{R}$ is the additive group of the reals, with the usual topology, and if $\mathbb{Q}^\delta$ is the additive group of the rationals, with the discrete topology, then both $\mathbb{Q}^\delta$ and $\mathbb{R}$ are second countable locally compact abelian groups, and the inclusion $\mathbb{Q}^\delta \to \mathbb{R}$ is a continuous morphism. Nevertheless, the cokernel $\mathbb{R}/\mathbb{Q}$ is not a locally compact group. The passage to condensed groups fixes this. If X is any profinite set, then there is a short exact sequence of abelian groups

$$0 \to C(X, \mathbb{Q}^\delta) \to C(X, \mathbb{R}) \to t(X) \to 0,$$

where t is indeed a condensed group. Now let lca_κ denote the subcategory of lca that consists of all groups H whose topology is determined by its restriction to all compact subsets of cardinality $< \kappa$. The functor which assigns to H the condensed group $\underline{H}$ embeds the category lca_κ fully faithfully into a larger abelian category. This is of interest for cohomology, because homological algebra usually requires an abelian category to work with.

Literature:

[D] J. Dugundji, *Topology*

[E] R. Engelking, *General Topology*

[G] A.M. Gleason, Projective topological spaces, Illinois J. Math. 2 (1958), 482–489.

[HM] K.H. Hofmann, S. Morris, *The structure of compact groups*

[K] L. Kramer, A note on Stone duality (preprint available)

[ML] S. MacLane, *Categories for the Working Mathematician*

[R] J. Rainwater, A note on projective resolutions, Proc. Amer. Math. Soc. 10 (1959), 734–735.

[S] P. Scholze, Lectures on Condensed Mathematics (available on his web page)
[St] M. Stroppel, *Locally compact groups*
[W] C. Weibel, An *Introduction to Homological Algebra*

The talks/topics planned so far in the seminar are as follows. Additional suggestions are welcome.

1. Compact spaces (1 - 2 talks) [D,E,G,K,R].

   Introduce compact totally disconnected spaces and explain that these spaces are zero-dimensional. Explain that every totally disconnected compact space is profinite, i.e. embeds into a product of finite spaces [E]. State and explain Stone duality [K]. Introduce extremally disconnected compact spaces and show that they are totally disconnected. Explain the universal property of the Čech-Stone compactification. Theorem. *For a compact space X, the following are equivalent: (1) X is extremally disconnected, (2) X is projective in the category of compact spaces, (3) X is a retract of βD, for some discrete space D* [G,R,E]. Explain that every compact space X has a unique projective resolution $P \to X$ [G,R,E].

2. Basic category theory (1 talk) [ML].

   Briefly recall what categories, functors, small and large categories and skeletal subcategories are. Give examples. Explain equivalences of categories, with examples. Recall what (co)limits, filtered (co)limits, equalizers, epics and monics are. Explain that hom-sets preserve certain limits. Explain adjoint functors and reflexive subcategories. Show that the Čech-Stone functor β is an example of an adjoint functor. Recall that adjoints preserve certain types of limits.

3. Condensed sets (1 - 2 talks) [S].

   Following Lecture I in [S], give the definition of a condensed set as a functor on compact spaces, satisfying (S1) and (S2). (The formulations involving sites and Grothendieck topologies can be skipped here.) Explain the example $t = C(-, Z)$ for a Hausdorff space Z in detail, and explain Prop. 1.7. (The material about abelian categories can be postponed.) Explain Prop 2.3. and Prop 2.7 in detail. Explain Prop 2.9.

4. Condensed abelian groups (1 talk) [ML,S,W].

   Explain briefly what an abelian category is. Following Lecture II in [S], state and explain Thm. 2.2.

5. Locally compact abelian groups [HM,St].

   Introduce the category of lca groups. Introduce duals and Pontrjagin-Van Kampen duality. Give examples of duality: discrete vs. compact, the dual of the circle group. Explain the structure Theorem 7.57 in [HM].

6. Condensed Cohomology (1 talk) [S].

   Give the definition of condensed cohomology of compact spaces and its comparison with traditional sheaf and Čech cohomology, as described in Lecture III in [S]. Explain Thm. 3.2 and Thm. 3.3.

7. Cohomology of condensed lca groups (1 - 2 talks) [S,W].

   Give an outline of the material covered in Lecture IV in [S]. This requires the notion of a derived category. Explain in particular Prop. 4.2, and Thm. 4.3 and its consequences.

8. Further topics in [S] (solid abelian groups...) if time permits.

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