INCIDENCE BOUNDS IN POSITIVE CHARACTERISTIC VIA VALUATIONS AND DISTALITY

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Abstract. We prove distality of quantifier-free relations on valued fields with finite residue field. By a result of Chernikov-Galvin-Starchenko, this yields Szemerédi-Trotter-like incidence bounds for function fields over finite fields. We deduce a version of the Elekes-Szabó theorem for such fields.

1. Introduction

We obtain the following incidence bound.

Theorem 1.1. *Let p be a prime, and let K be a finitely generated extension of $\mathbb{F}_p$. Let $E \subseteq K^n \times K^m$ be the zero set of a set of polynomials in $K[x_1, \ldots, x_{n+m}]$. Let $d, s \in \mathbb{N}$ and suppose E is $K_{d,s}$-free, i.e. if $A \times B \subseteq E$ then $|A| < d$ or $|B| < s$.*

Then there exists $t \in \mathbb{N}$, which can be calculated from the data of E, and $C > 0$ such that for any finite subsets $A \subseteq K^n$ and $B \subseteq K^m$,

$$|E \cap (A \times B)| \leq C(|A|^{\frac{(t-1)d}{td-1}}|B|^{\frac{td-t}{td-1}} + |A| + |B|).$$

1.1. **Background and motivation.** The Szemerédi-Trotter theorem bounds the number of point-line incidences between a set P of points and a set L of lines in the real plane. We state a version with an explicit bound, [TV06, Theorem 8.3]:

Fact 1.2. *For any finite P and L,*

$$|\{(p,l) \in P \times L : p \in l\}| \leq 4|P|^{\frac{2}{3}}|L|^{\frac{2}{3}}| + 4|P| + |L|.$$

Statements of the form of Theorem 1.1 can be seen as generalisations of this, replacing the point-line incidence relation with other algebraic binary algebraic relations. Such results were proven for characteristic 0 fields first in [ES12, Theorem 9], and subsequently strengthened in [FPS+17, Theorem 1.2]. Using such bounds for binary relations, Elekes-Szabó [ES12] obtained analogous bounds for ternary algebraic relations.

In positive characteristic, versions of Fact 1.2 have been proven ([BKT04],[SdZ17]) where one restricts to sets which are small compared to the characteristic. This is related to the sum-product phenomenon in fields, where finite fields are known to be the only obstruction ([BKT04], [TV06, Theorem 2.55]).

Meanwhile, [CGS20] and [CS18] generalise the characteristic 0 results by seeing them as a consequence of *distality* of the real field. The notion of distality and these incidence theoretic implications are summarised in Section 3.1 below. It would be surprising if the positive characteristic results mentioned above, which require an unbounded characteristic, could be seen as instances of distality. We consider instead the orthogonal situation of a function field over a finite field, and we prove Theorem 1.1 by finding sufficient distality to trigger the incidence bounds of [CGS20]. We obtain the distality using elementary notions from the model theory of valued fields, and in fact our results apply more generally to any valued field with finite residue field.

Our motivation for considering these fields is [Hru13, Section 5], which suggests a unifying explanation for all the results on existence of bounds described above:

they are all incarnations of *modularity* in the model-theoretic sense, and they are consistent with a Zilber dichotomy statement of the form "any failure of modularity arises from an infinite pseudo-finite field". In other words, finite fields should be the cause of any failure of the bounds. As a special case, this would suggest that for a field K of characteristic $p > 0$ which has finite **algebraic part** $\mathbb{K} \cap \mathbb{F}_p^{\mathrm{alg}}$, incidence bounds and Elekes-Szabó results should go through as in characteristic 0.

We partially confirm this only in the special[1] case of fields admitting finite residue field. However, in Theorem 7.1 we do confirm for such K that an Elekes-Szabó result applies: a mild strengthening of Theorem 1.1 suffices as input to the proof of one of the main results of [BB18], yielding Elekes-Szabó bounds for arbitrary arity algebraic relations in K^n which do not arise from 1-dimensional algebraic groups.

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2. Preliminaries

We use basic notions and notation from model theory.

Let $\mathcal{L}$ be a (possibly many-sorted) first order language and T a a complete $\mathcal{L}$-theory.

Notation 2.1. If $\mathcal{M} \vDash T$ and $B \subseteq \mathcal{M}$ and $x = (x_1, \ldots, x_n)$ is a tuple of variables of sorts $S_1, \ldots, S_n$, we write B^x for $\prod_i (S_i(\mathcal{M}) \cap B)$. We write $|x|$ for the length $|x| = n$ of the tuple.

For a set B, we write $B_0 \subseteq_{\mathrm{fin}} B$ to mean that B_0 is a finite subset of B.

For a formula ϕ, we define $\phi^0 := \neg\phi$ and $\phi^1 := \phi$.

If $\phi(x; y)$ is a partitioned formula and $b \in \mathcal{M}^x$ and $A \subseteq \mathcal{M}$, we set $\mathrm{tp}_\phi(b/A) := \{\phi(x, c)^\epsilon : c \in A^y;\ \epsilon \in \{0, 1\};\ \mathcal{M} \vDash \phi(b, c)^\epsilon\}$. The partitioning will often be left implicit.

3. Distality

3.1. Distal cell decompositions.

We recall the following definition from [CGS20]:

Definition 3.1. Let A and B be sets. A binary relation $E \subseteq A \times B$ *admits a distal cell decomposition* with exponent $t \in \mathbb{R}$ if there exist $s \in \mathbb{N}$ and finitely many relations $\Delta_i \subseteq A \times B^s$ and $C \in \mathbb{R}$ such that for every $B_0 \subseteq_{\mathrm{fin}} B$, A can be written as a (not necessarily disjoint) union of $\leq C|B_0|^t$ subsets of the form $\Delta_i(c)$ for $c \in B_0^s$, each of which *cuts* no $E(b)$ for $b \in B_0$, i.e. $E(b) \subseteq \Delta_i(c)$ or $E(b) \cap \Delta_i(c) = \emptyset$.

It was proven in [CGS20] that relations admitting distal cell decompositions enjoy certain incidence bounds. For our purposes, the following version of this deduced in [CS20, Theorem 2.6,2.7(2)] is most relevant.

A binary relation $E \subseteq A \times B$ is $K_{d,s}$**-free** if it contains no subset $A_0 \times B_0$ with $|A_0| = d$ and $|B_0| = s$.

Fact 3.2. *Let $E \subseteq A \times B$ be $K_{d,s}$-free and admit a distal cell decomposition with exponent t. Then for $A_0 \subseteq_{\mathrm{fin}} A$ and $B_0 \subseteq_{\mathrm{fin}} B$,*

$$|E \cap (A_0 \times B_0)| \leq O_E(|A_0|^{\frac{(t-1)d}{td-1}} |B_0|^{\frac{td-t}{td-1}} + |A_0| + |B_0|).$$

[1]See Proposition 4.2

3.2. Distal subsets.

Definition 3.3. Let $\mathcal{M} \vDash T$. Let $\phi(x; y)$ be an $\mathcal{L}(\mathcal{M})$ formula, and let $A, B \subseteq \mathcal{M}$ be subsets.

- An $\mathcal{L}$-formula $\zeta_\phi(x; z)$ is a **uniform strong honest definition** (**USHD**) for ϕ on A over B if for any $a \in A$ and finite subset $B_0 \subseteq_{\mathrm{fin}} B$ with $|B_0| \geq 2$, there is $d \in B_0^z$ such that $\mathrm{tp}(a/B_0) \ni \zeta_\phi(x, d) \vdash \mathrm{tp}_\phi(a/B_0)$.
- We omit "on A" in the case $A = \mathcal{M}$.
- We omit "over B" in the case $B = A$.
- A is **distal in** $\mathcal{M}$ if every $\mathcal{L}(A)$-formula $\phi(x; y)$ has a USHD on A.

The notion of a strong honest definition comes from [CS15]. We work with USHDs rather than directly with distal cell decompositions in order to be able to reduce to one variable (Lemma 3.5), and because dealing with a single formula is more convenient for many purposes. As the following remark makes explicit, there is little difference between the two notions.

Remark 3.4. An $\mathcal{L}$-formula $\phi(x; y)$ has a USHD on A over B if and only if the binary relation $E := \phi(A; B) \subseteq A^x \times B^y$ admits a distal cell decomposition where the Δ_i are themselves defined by $\mathcal{L}$-formulas. The restriction $|B_0| \geq 2$ allows multiple Δ_i to be coded as one formula, a trick we will use repeatedly; explicitly, if $\delta_i(x, z_i)$ define Δ_i, then

$$\zeta(x, z_1, \ldots, z_s, w_1, \ldots, w_s, w'_1, \ldots, w'_s) := \bigwedge_i (\delta_i(x, z_i) \leftrightarrow w_i = w'_i)$$

is a USHD for ϕ on A over B.

In particular, if $A \subseteq \mathcal{M}$ is distal in $\mathcal{M}$, then the trace on A of any $\mathcal{L}(A)$-formula $\phi(x, y)$ admits a distal cell decomposition.

3.3. Reductions.

Lemma 3.5. *A subset $A \subseteq \mathcal{M}$ is distal in $\mathcal{M}$ if and only if any $\mathcal{L}$-formula $\phi(x; y)$ with $|x| = 1$ has a USHD on A.*

Proof. First, it follows by an inductive argument from the 1-variable case that any $\mathcal{L}$-formula has a USHD on A; we refer to the proof of [ACGZ20, Proposition 1.9] for this argument.

It remains to deduce that any $\mathcal{L}(A)$-formula has a USHD on A, but it follows directly from the definition that if $\phi(x; y, z)$ has a USHD on A and $a \in A^z$, then $\phi(x; y, a)$ has a USHD on A. □

Lemma 3.6. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let S and $\widetilde{S}$ be $\mathcal{L}$-sorts and let $f : \widetilde{S} \to S$ be an $\mathcal{L}$-definable function with uniformly finite fibres, say $|f^{-1}(b)| \leq N$ for all $b \in f(S)$. Suppose $B \subseteq f(S(\mathcal{M}))$, and let $\widetilde{B} := f^{-1}(B) \subseteq \widetilde{S}$.*

Let $A \subseteq \mathcal{M}^x$ and let $\phi(x, y)$ be an $\mathcal{L}$-formula such that $\phi(x; f(z))$ has a USHD on A over $\widetilde{B}$. Then $\phi(x; y)$ has a USHD on A over B.

Proof. Say $\zeta(x, w)$ is a USHD for $\phi(x; f(z))$ over $\widetilde{B}$.

Let $B_0 \subseteq_{\mathrm{fin}} B$ and $a \in A$. Then $\widetilde{B}_0 := f^{-1}(B_0)$ is a finite subset of $\widetilde{B}$, so there is $\widetilde{d}$ such that $\mathrm{tp}(a/\widetilde{B}_0) \ni \zeta(x, \widetilde{d}) \vdash \mathrm{tp}_{\phi(x;f(z))}(a/\widetilde{B}_0) \vdash \mathrm{tp}_{\phi(x;y)}(a/B_0)$.

Let $d := f(\widetilde{d})$. Then $|f^{-1}(d)| \leq N^{|w|}$, and so there is $M < N^{|w|}$ and $\overline{\epsilon}_0 \in \{0, 1\}^M$ and $\overline{b}_0 \in (B_0)^M$ such that $\theta_{M,\overline{\epsilon}_0}(w, d, \overline{b}_0)$ has the minimal number of realisations amongst the formulas

$$\theta_{n,\overline{\epsilon}}(w, d, \overline{b}) := (f(w) = d \wedge \forall x.(\zeta(x, w) \to \bigwedge_{i=1}^{n} \phi(x, b_i)^{\epsilon_i}))$$

which hold of $\widetilde{d}$, with $n \in \mathbb{N}$ and $\bar{\epsilon} \in \{0,1\}^n$ and $\bar{b} \in (B_0)^n$. The bound $M < N^{|w|}$ follows from the observation that if such a formula does not have the minimal number of realisations, then a single new instance of ϕ can be added to reduce the number of realisations. By the minimality, we have for any $b \in B_0$ that $\theta_{M,\bar{\epsilon}_0}(w, d, \bar{b}_0) \vdash \forall x.(\zeta(x,w) \to \phi(x,b)^{\epsilon})$ for some $\epsilon \in \{0,1\}$.

So $\mathrm{tp}(a/B_0) \ni \exists w.(\theta_{M,\bar{\epsilon}_0}(w, d, \bar{b}_0) \wedge \zeta(x,w)) \vdash \mathrm{tp}_{\phi(x;y)}(a/B_0)$. Coding the finitely many such formulas with $M < N^{|w|}$ and $\bar{\epsilon}_0 \in \{0,1\}^M$ into a single formula, we therefore obtain a USHD for $\phi(x;y)$ on A over B.

□

Remark 3.7. The finiteness assumption in Lemma 3.6 is necessary. Consider for example the structure $(X, O_X; <)$ where X is a set, O_X is the set of linear orders on X, and $x <_o x'$ is the corresponding ternary relation. Let $\pi_1 : X \times O_X \to X$ be the projection. As one may see by considering automorphisms, the induced structure on X is trivial, so $x = y$ has no USHD on X over X. But $x = \pi_1(z)$ has a USHD on X over $X \times O_X$ (since if $X_0 \subseteq_{\mathrm{fin}} X$ and $o \in O_X$, then $\mathrm{tp}_=(x/X_0)$ is implied by the $<_o$-cut of x in X_0).

3.4. **Remarks.** We add some further remarks concerning these definitions, which will not be used subsequently.

Remark 3.8. Suppose B is distal in an $\mathcal{L}$-structure $\mathcal{M}$. Then this is expressed in the $\mathcal{L}_P$-theory of $(\mathcal{M}; B)$, where P is a new predicate interpreted as B; i.e. if $(\mathcal{M}'; B') \equiv (\mathcal{M}; B)$, then B' is distal in $\mathcal{M}'$.

Remark 3.9. By [CS15, Theorem 21], $\mathrm{Th}(\mathcal{M})$ is distal if and only if $\mathcal{M}$ is distal in $\mathcal{M}$. (No saturation assumption is needed here, thanks to Remark 3.8.)

Remark 3.10. Distality in $\mathcal{M}$ of a subset $B \subseteq \mathcal{M}$ is equivalent to distality of the induced structure $(B; (\phi(B)_{\phi \text{ an } \mathcal{L}\text{-formula}})$ if this structure admits quantifier elimination, but in general is much weaker. We could say that distality of a subset means that it has "quantifier-free distal induced structure".

Example 3.11. If $B = (b_i)_i \subseteq \mathcal{M}$ is an $\emptyset$-indiscernible sequence which is not totally indiscernible, and this is witnessed by an $\mathcal{L}$-formula $\theta_<$ with $\mathcal{M} \vDash \theta_<(b_i, b_j) \Leftrightarrow i < j$, then B is distal in $\mathcal{M}$.

Remark 3.12. The argument of [CS15] to obtain uniformity of honest definitions goes through in this setting. Namely, if B is a subset of a model $\mathcal{M}$ of a complete NIP $\mathcal{L}$-theory T, and the $\mathcal{L}_P$-structure $(\mathcal{M}; B)$ is $|T|^+$-saturated, then B is distal in $\mathcal{M}$ if and only if for any singleton $b \in B$ and any subset $A \subseteq B$, $\mathrm{tp}^{\mathcal{M}}(b/A)$ is compressible in the sense of [Sim20]. This follows from a "(p, q)-argument" and transitivity of compressibility.

It follows in particular that Example 3.11 can be generalised slightly when $\mathcal{M}$ is NIP: any $\emptyset$-indiscernable sequence which is not totally indiscernable is distal in $\mathcal{M}$.

Question 3.13. The following question was asked by Hrushovski and Pillay. By a result of Simon, an NIP theory is distal if and only if every generically stable Keisler measure is smooth. Does a version of this result go through for distality of subsets of NIP structures? Is B distal in $\mathcal{M}$ if and only if every generically stable Keisler measure on $\mathrm{Th}_{\mathcal{L}_P}(\mathcal{M}, B)$ with $\mu(\neg P) = 0$ is smooth? This might provide an alternative route to Theorem 5.6.

4. Fields admitting valuations with finite residue field

By classical results in valuation theory, a valuation on a field K can be extended to any finite extension of K with a finite extension of the residue field [EP05, Theorem 3.1.2, Corollary 3.2.3], and can be extended to the transcendental extension $K(X)$ without extending the residue field [EP05, Corollary 2.2.3]. Since $\mathbb{F}_p$ and $\mathbb{Q}$ admit valuations with finite residue field (respectively trivial and p-adic), we inductively obtain:

Lemma 4.1. *Let K be a finitely generated field. Then K admits a valuation with finite residue field.*

If K is a valued field of characteristic $p > 0$, then the induced valuation on the algebraic part $K \cap \mathbb{F}_p^{\mathrm{alg}}$ is trivial. So a positive characteristic field which admits a valuation with finite residue field has finite algebraic part. However, the converse fails.

Proposition 4.2. *For any prime p, there exists an algebraic extension $L \geq \mathbb{F}_p(t)$ such that $L \cap \mathbb{F}_p^{\mathrm{alg}} = \mathbb{F}_p$ but no valuation on L has finite residue field.*

Proof. We work in an algebraic closure $\mathbb{F}_p(t)^{\mathrm{alg}}$ of $\mathbb{F}_p(t)$. Let $\wp : \mathbb{F}_p(t)^{\mathrm{alg}} \to \mathbb{F}_p(t)^{\mathrm{alg}}$ be the Artin-Schreier map $\wp(x) := x^p - x$, an additive homomorphism with kernel $\mathbb{F}_p$.

Claim. $\deg(\mathbb{F}_p(t, (\wp^{-1}(t^a))_{a>0})/\mathbb{F}_p(t))$ *is infinite.*

Proof. By [Lan02, Theorem 8.3], it suffices to see that $\{t^a | a > 0\}$ is not contained in any finite union of additive cosets of $\wp(\mathbb{F}_p(t))$. Let $a, b \in \mathbb{N} \setminus p\mathbb{N}$ be distinct. Let $\beta_{a,b} := \sum_{i \geq 0}(t^{ap^i} - t^{bp^i}) \in \mathbb{F}_p[[t]]$. Then $\wp(\beta_{a,b}) = t^b - t^a$. Now $\beta_{a,b} \notin \mathbb{F}_p(t)$, since there are arbitrarily long intervals between exponents with non-zero coefficient in this power series. So $(t^a)_{a \in \mathbb{N} \setminus p\mathbb{N}}$ lie in distinct cosets of $\wp(\mathbb{F}_p(t))$. □

We write res for the residue field map associated to a chosen valuation v on a field K, and $\mathrm{res}(K)$ for the corresponding residue field.

Claim. *Let $K' \geq K \geq \mathbb{F}_p(t)$ be finite field extensions, and suppose $K' \cap \mathbb{F}_p^{\mathrm{alg}} = \mathbb{F}_p$. Let v be a valuation on K with $\mathrm{res}(K)$ finite.*

Then there exists a finite field extension $K'' \geq K'$ such that $K'' \cap \mathbb{F}_p^{\mathrm{alg}} = \mathbb{F}_p$ but for any extension of v to K'', $\mathrm{res}(K'') \gneq \mathrm{res}(K)$.

Proof. The valuation v is non-trivial, so say $v(s) > 0$. So v induces the s-adic valuation on $\mathbb{F}_p(s) \leq K$. Now s is transcendental, so t is algebraic over s, so K is also a finite extension of $\mathbb{F}_p(s)$. So we may assume without loss that v restricts to the t-adic valuation on $\mathbb{F}_p(t)$.

Since $\mathrm{res}(K)$ is finite, it is not Artin-Schreier closed; say $\alpha \in \mathrm{res}(K) \setminus \wp(\mathrm{res}(K))$. Say $\mathrm{res}(\bar{\alpha}) = \alpha$.

Since $\deg(K'/\mathbb{F}_p(t))$ is finite, it follows from the above Claim that

$$\deg(K'(\wp^{-1}(\bar{\alpha}), (\wp^{-1}(\bar{\alpha} + t^a))_{a>0})/K')$$

is infinite. So say $a > 0$ is such that $K'' := K(\wp^{-1}(\bar{\alpha} + t^a)) \not\subseteq K'(\mathbb{F}_{p^p})$. Then by considering degrees, $K'' \cap \mathbb{F}_p^{\mathrm{alg}} = \mathbb{F}_p$. But for any extension of v to K'',

$$\wp(\mathrm{res}(\wp^{-1}(\bar{\alpha} + t^a))) = \mathrm{res}(\bar{\alpha} + t^a) = \alpha \notin \wp(\mathrm{res}(K)),$$

so $\mathrm{res}(K'') \gneq \mathrm{res}(K)$. □

Now we recursively construct a chain $K_0 := \mathbb{F}_p(t) \leq K_1 \leq \ldots$ of finite extensions of $\mathbb{F}_p(t)$. Let $\eta : \omega \times \omega \to \omega$ be a bijection such that $\eta(i, j) \geq i$ for all i, j.

Note that $\mathbb{F}_p(t)$ admits only countably many valuations (identifying a valuation with its valuation ring); indeed, as above, each non-trivial valuation is a finite extension of the s-adic valuation on some $\mathbb{F}_p(s) \leq \mathbb{F}_p(t)$; there are only countably many choices for s, and only finitely many ways to extend a valuation to a finite extension ([EP05, Theorem 3.2.9]). Hence also there are also only countably many valuations on each K_i. Once K_i is constructed, let $\{v_{i,j} : j \in \omega\}$ be the set of valuations on K_i with finite residue field.

Suppose $k = \eta(i,j)$ and K_k has been constructed. Let $K_{k+1} \geq K_k$ be an extension as in the second Claim for the extensions $K_k \geq K_i \geq \mathbb{F}_p(t)$ and the valuation $v_{i,j}$ on K_i.

Now let $K_\omega := \bigcup_{k<\omega} K_k$. We have $K_\omega \cap \mathbb{F}_p^{\mathrm{alg}} = \mathbb{F}_p$ since this holds for each K_k.

Suppose v is a valuation on K_ω with finite residue field. Then $\mathrm{res}(K_\omega) = \mathrm{res}(K_i)$ say, and the restriction of v to K_i is $v_{i,j}$ say. Then $\mathrm{res}(K_{\eta(i,j)+1}) = \mathrm{res}(K_i)$, contradicting the construction. □

5. Distality in ACVF of subfields with finite residue field

5.1. **Uniform Swiss cheese decompositions.** Let L be a non-trivially valued algebraically closed field. Write v for the valuation map and res for the residue field map. We consider L as an $\mathcal{L}_{\mathrm{div}} := \{+, -, \cdot, |, 0, 1\}$-structure, where $x|y \Leftrightarrow v(x) \leq v(y)$; by a result of Robinson, L has quantifier elimination in this language. An open resp. closed **ball** in L is a definable set of the form $\{x : v(x-a) > \alpha\}$ resp. $\{x : v(x-a) \geq \alpha\}$, where $a \in L$ and $\alpha \in v(L) \cup \{-\infty, +\infty\}$.

Fact 5.1 (Canonical Swiss cheese decomposition). *Any boolean combination of balls can be represented as a finite disjoint union of "Swiss cheeses" $\dot{\bigcup}_{i<k}(b_i \setminus \dot{\bigcup}_{j<k_i} b_{ij})$, where the b_i are balls, each b_{ij} is a proper sub-ball of b_i, for each i the b_{ij} are disjoint, and no b_i is equal to any $b_{i'j}$. This representation is unique up to permutations.*

We call the b_i the "rounds" and the b_{ij} the "holes" of a Swiss cheese decomposition, and we say such a decomposition has **complexity** $\leq N$ if there are $k \leq N$ rounds each with $k_i \leq N$ holes. Let $\phi(x, y)$ be an $\mathcal{L}_{\mathrm{div}}$-formula with $|x| = 1$. For any $a \in L$, it follows directly from quantifier elimination that $\phi(L, a)$ is a boolean combination of balls. We will need the following form of uniformity in a of the Swiss cheese decompositions.

Lemma 5.2. *There are N and d depending only on ϕ such that for all $a \in L^y$, $\phi(L, a)$ has a Swiss cheese decomposition of complexity $\leq N$, each round and each hole of which contains a point in a field extension of the subfield generated by a of degree dividing d.*

Proof. By quantifier elimination, $\phi(x, y)$ is equivalent to a boolean combination of formulas of the form $\phi_i(x, y) := v(f_i(x, y)) < v(g_i(x, y))$ for polynomials $f_i, g_i \in \mathbb{Z}[x, y]$.

Given i and $a \in L$, let α_j resp. β_j be the roots of $f_i(x, a)$ resp. $g_i(x, a)$ in L. Then $\phi_i(x, a) \Leftrightarrow \sum_j v(x - \alpha_j) < \sum_j v(x - \beta_j)$.

Claim 5.3. *$\phi_i(x, a)$ is a boolean combination of balls centred at the α_j and β_j.*

Proof. We show more generally, by induction on s, that any affine linear constraint $\sum_{i=1}^s n_i v(x - \gamma_i) < \nu$ is equivalent to a boolean combination of balls centred at the γ_i. Conditioning on the finitely many cases, it suffices to show this for a given order type of $\epsilon := v(x - \gamma_1)$ over $\{v(\gamma_1 - \gamma_i) : i > 1\}$. If $\epsilon = v(\gamma_1 - \gamma_i)$ for some $i > 1$, we conclude by the inductive hypothesis. Otherwise, by the ultrametric triangle inequality, $v(x - \gamma_i) = \epsilon$ if $\epsilon < v(\gamma_1 - \gamma_i)$, and $v(x - \gamma_i) = v(\gamma_1 - \gamma_i)$ otherwise, so the affine constraint is equivalent to $\epsilon < \nu'$ or $\epsilon > \nu'$ for some ν'. □

We may assume L is $\aleph_0$-saturated, and so by compactness we obtain a bound on the number of balls involved in this boolean combination which is uniform in a.

So $\phi(x,a)$ is a boolean combination of boundedly many balls each having a point in an extension of the subfield generated by a of degree dividing

$$d := \mathrm{lcm}_i(\mathrm{lcm}(\deg_x f_i, \deg_x g_i))$$

, and it follows that the rounds and holes in the Swiss cheese decomposition also have this property and are bounded in number. □

5.2. **Compressing cheeses.** Let B be the imaginary sort of L consisting of balls, both open and closed, including the empty ball and its complement. We write $x \in b$ for the corresponding $\emptyset$-definable (in $\mathcal{L}^{\mathrm{eq}}$) element relation $(\in) \subseteq L \times B$.

Given $N \in \mathbb{N}$, let S_N be the imaginary sort of L which codes Swiss cheese decompositions of complexity at most N. This means that we have an associated $\emptyset$-definable element relation, which we also write as $(\in) \subseteq L \times S_N$, such that $c_1 = c_2$ iff $\{x : x \in c_1\} = \{x : x \in c_2\}$, and setting

$$X_N := \{(b_i)_{i<N}, (b_{ij})_{i,j<N} : b_i, b_{ij} \in B \text{ are as in Fact 5.1}\} \subseteq B^{N(N+1)}$$

we obtain a $\emptyset$-definable surjection $f_{S_N} : X_N \twoheadrightarrow S_N$ defined by $f_{S_N}((b_i)_i, (b_{ij})_{ij}) :=$ [code of $\bigcup_i (b_i \setminus \bigcup_j b_{ij})$]. By Fact 5.1, any $c \in S_N$ has a unique-up-to-permutation representation as a Swiss cheese decomposition of complexity $\le N$, so f_{S_N} has finite fibres.

With a view to proving Theorem 5.6, for K a valued subfield of L with finite residue field and $d \in \mathbb{N}$, define $B_{K,d} \subseteq B$ to be the set of balls which contain an element of some finite field extension of K within L of degree dividing d over K. Let $X_N(B_{K,d}) := X_N \cap B_{K,d}^{N(N+1)}$ and $S_N(B_{K,d}) := f_{S_N}(X_N(B_{K,d}))$.

Lemma 5.4. *Let $N, d \in \mathbb{N}$.*

(i) $x \in y$ has a USHD over $B_{K,d}$.
(ii) $x \in f_{S_N}(y)$ has a USHD over $B_{K,d}$.
(iii) $x \in z$ has a USHD over $S_N(B_{K,d})$.

Proof. (i) By assumption, the residue field of K is a finite field, say $\mathbb{F}_q$.

Let $B_0 \subseteq_{\mathrm{fin}} B_{K,d}$. Let $B_0' := \{b \vee b' : b, b' \in B_0\}$ where the join $b \vee b'$ is the smallest ball containing both b and b'. By the ultrametric triangle inequality, B_0' is then closed under join. Note that $B_0' \subseteq B_{K,d}$, since $B_{K,d}$ is upwards-closed.

Let $p \in L$. Let $b \in B_0' \cup \{L\}$ be minimal such that $p \in b$, and let $b_1, \ldots, b_s \in B_0'$ be the maximal proper subballs (if any) of b in B_0'. Then

$$(x \in b \wedge \bigwedge_{i=1}^{s} x \notin b_i) \vdash \mathrm{tp}_{x\in y}(p/B_0),$$

and each b_i is the join of two balls in B_0, and either the same goes for b or $b = L$. So coding the finitely many possibilities yields a USHD as required if we can bound s independently of p.

Assume $s > 1$. Say $p_i \in b_i$ is of degree dividing d over K, and let $\alpha \in v(L)$ be the valuative radius of b. Then $v(p_i - p_j) = \alpha$ for $i \neq j$, since $b_i \vee b_j = b$ (in particular, $b \neq L$).

Then $i \mapsto \lambda_i := \mathrm{res}(\frac{p_i - p_1}{p_2 - p_1})$ is an injection of $\{1, \ldots, s\}$ into $\mathrm{res}(L)$. Indeed, if $\lambda_i = \lambda_j$ then $\mathrm{res}(\frac{p_i - p_j}{p_2 - p_1}) = 0$, so $v(p_i - p_j) > v(p_2 - p_1) = \alpha$, so $i = j$.

Since each λ_i is in the residue field of an extension of K of degree dividing d^3, we have $\lambda_i \in \mathbb{F}_{q^{d^3}}$ by the valuation inequality ([EP05, Corollary 3.2.3]). So $s \le q^{d^3}$.

(ii) $x \in f_{S_N}(y)$ is equivalent, by the definition of f_{S_N}, to a certain boolean combination of the formulas $(x \in y_i)_{i<N(N+1)}$. So by (i), coding these formulas yields a formula which is a USHD for $x \in f_{S_N}(y)$ over $B_{K,d}$.
(iii) Considering now X_N as a sort and y as a variable of sort X_N, it follows from (ii) that $x \in f_{S_N}(y)$ has a USHD over $X_N(B_{K,d})$. Then we conclude by Lemma 3.6.

□

5.3. **Concluding distality.**

Lemma 5.5. *Let L be a non-trivially valued algebraically closed field. Let $K \leq L$ be a subfield and suppose* $\mathrm{res}(K)$ *is finite. Let $\phi(x;y)$ be an $\mathcal{L}_{\mathrm{div}}$-formula with $|x| = 1$. Then ϕ has a USHD over K.*

Moreover, for any $r \geq 1$, ϕ has a USHD over the set $K_r \subseteq K^{\mathrm{alg}} \subseteq L$ of points with degree over K dividing r,

$$K_r := \{a \in L : \deg(K(a)/K) | r\}$$

Proof. Let N and d be as in Lemma 5.2 for ϕ. Then there is a $\emptyset$-definable function $h : L^{|y|} \to S_N$ such that $L^{\mathrm{eq}} \models \forall x, y.(\phi(x,y) \leftrightarrow x \in h(y))$, and $h(K_r) \subseteq S_N(B_{K,dr})$.

By Lemma 5.4(iii), say $\zeta(x, z'_1, \dots, z'_s)$ is a USHD for $x \in y$ over $S_N(B_{K,dr})$. Then (an $\mathcal{L}_{\mathrm{div}}$-formula equivalent to) $\zeta(x, h(z_1), \dots, h(z_s))$ is a USHD for $\phi(x;y)$ over K_r. □

Theorem 5.6. *Let K be a valued field with finite residue field.*
Let $L \geq K$ be an algebraically closed valued field extension.
Then K is distal in L, as is each K_r defined as in Lemma 5.5.

Proof. We may assume that L is non-trivially valued, as otherwise K is finite and the result is trivial.

The result then follows from Lemma 5.5 and Lemma 3.5. □

Remark 5.7. This does not reprove distality of $\mathbb{Q}_p$, because $\mathbb{Q}_p$ does not eliminate quantifiers in $\mathcal{L}_{\mathrm{div}}$.

6. Incidence theory consequences

Theorem 6.1. *Let K be a valued field with finite residue field. Let $E \subseteq K^n \times K^m$ be quantifier-free definable in $\mathcal{L}_{\mathrm{div}}(K)$. Suppose E omits $K_{d,s}$, where $d, s \in \mathbb{N}$. Then there exist t (see Remark 6.2) and $C > 0$ such that for $A_0 \subseteq_{\mathrm{fin}} K^n$ and $B_0 \subseteq_{\mathrm{fin}} K^m$,*

$$|E \cap (A_0 \times B_0)| \leq C(|A_0|^{\frac{(t-1)d}{td-1}} |B_0|^{\frac{td-t}{td-1}} + |A_0| + |B_0|).$$

The same holds if K is replaced by $K_r \subseteq K^{\mathrm{alg}}$ defined as in Lemma 5.5.

Proof. By Theorem 5.6 and Remark 3.4, E admits a distal cell decomposition, and we conclude by Fact 3.2. □

The version of this stated in the introduction, Theorem 1.1, follows by considering Lemma 4.1 and the special case that E is defined as the zero set of polynomials over K.

Remark 6.2. By examining the proof, in the case $n = 1$ one can obtain a bound on the exponent of the resulting distal cell decomposition giving $t \leq 2(q^{d^3} + 1)$ where $q = |\mathrm{res}(K)|^r$ and d is as in the proof of Lemma 5.5. Indeed, this is the exponent arising from bounding the number of balls used in Lemma 5.4(i), and neither Lemma 5.4(ii) nor Lemma 3.6 increase the exponent. So we obtain the corresponding explicit bounds in Fact 3.2. However, we have no reason to expect these bounds to be anything like optimal.

For $n > 1$, calculating explicit bounds is complicated by the fact that when reducing to one variable a USHD for a quantified formula is used, so one needs a bound on the degrees in the quantifier-free formula obtained by quantifier elimination in ACVF. This quantifier elimination is primitive recursive [Wei84], so in principle this could be done, yielding an effective algorithm for computing an exponent t for a given E. But we do not attempt to make this explicit here.

Instead, we illustrate the idea by showing that in the special case of Szemerédi-Trotter, $E = \{((x,y),(a,b)) : y = ax + b\}$, we can take $t := 4(q+1)$.

The proof of Lemma 5.4(i) in this case gives a USHD $\zeta(y, x, \overline{z})$ for $\phi(y; x, (a,b)) := (x,y)E(a,b)$ over K, expressing that y is an element of a boolean combination of the points $z_{i,1}x + z_{i,2}$ and the balls spanned by pairs of such points, with at most $2(q+1)$ such points involved. Using coding to choose the form of the boolean combination, this has exponent $\leq 2(q+1)$.

By [Wei84, Theorem 2.1], if an $\mathcal{L}_{\mathrm{div}}$ qf-formula $\psi(x, \overline{y}, \overline{z})$ is linear in $x, \overline{y}$, i.e. each polynomial has degree 1 in x and each y_i, then $\exists x \psi(x, \overline{y}, \overline{z})$ is equivalent modulo ACVF to a qf-formula linear in $\overline{y}$.

Now the formula $\zeta(y, x, \overline{z}) \to (x,y)Ew$ is linear in x, y, so $\forall y.(\zeta(y, x, \overline{z}) \to (x,y)Ew)$ is equivalent to a qf-formula which is linear in x. Similarly $\forall y.(\zeta(y, x, \overline{z}) \to \neg(x,y)Ew)$ is equivalent to a qf-formula linear in x, and the two can be coded into a single qf-formula linear in x. This then itself admits (by Remark 6.2 with $d = 1$) a USHD $\xi(x, \overline{w})$ over K of exponent $\leq 2(q+1)$. Then $\xi(x, \overline{w}) \wedge \zeta(y, x, \overline{z})$ is a USHD for E over K of exponent $\leq 2(q+1) + 2(q+1) = 4(q+1)$.

Question 6.3. Is the dependence on q in these bounds necessary? For example, does there exist $\epsilon > 0$ such that for all primes p there exists C such that for all $X, A \subseteq \mathbb{F}_p(t)^2$ we have $|\{((x,y),(a,b)) \in X \times A : y = ax + b\}| \leq C \max(|X|, |A|)^{\frac{3}{2}-\epsilon}$? (Remark 6.2 yields a bound depending on p of $\epsilon = \frac{1}{16p+14}$ in this case. Meanwhile one can obtain a lower bound exponent of $\frac{4}{3}$ by considering a rectangular example with bounded degree polynomials, $\mathbb{F}_p[t]_{<n} \times \mathbb{F}_p[t]_{<2n}$.)

7. Elekes-Szabó consequences

Elekes-Szabó [ES12] exploit incidence bounds in characteristic zero to find that commutative algebraic groups are responsible for ternary algebraic relations with asymptotically large intersections with finite grids. In [BB18], this is generalised to relations of arbitrary arity. In this section, we remark that these arguments go through in the present positive characteristic context, at least if we restrict to the 1-dimensional situation of [BB18, Theorem 1.4].

Let K_0 be a field admitting a valuation with finite residue field (e.g. a function field over a finite field). Let $\mathcal{U}$ be a non-principal ultrafilter on ω. Define

$$K' := ((K_0)^{\mathcal{U}})^{\mathrm{alg}} \leq ((K_0)^{\mathrm{alg}})^{\mathcal{U}} =: K.$$

For $r \geq 1$, let

$$K_r := \{a \in (K_0)^{\mathrm{alg}} : \deg(K_0(a)/K_0) | r\}.$$

So (by Łoś's theorem) we have $K' = \bigcup_{r \in \omega} (K_r)^{\mathcal{U}}$.

We work with the setup of [BB18, 2.1], with $(K_0)^{\mathrm{alg}}$ in place of $\mathbb{C}$, and in a countable language in which each of these internal sets $(K_r)^{\mathcal{U}} \subseteq K$ is definable.

Theorem 7.1. *Let K_0 be a field admitting a valuation with finite residue field. Let $V \subseteq \mathbb{A}^n$ be an affine algebraic variety defined over K_0 of dimension d. Then at least one of the following holds:*

(i) *V admits a powersaving on K_0: there exist $C, \epsilon > 0$ such that for all $X_i \subseteq_{\text{fin}} K_0$, $i = 1, \ldots, n$, we have*

$$|V(K_0) \cap \prod_i X_i| \leq C(\max_i |X_i|)^{d-\epsilon}.$$

(ii) *V is* special*: V is in co-ordinatewise correspondence[2] with a product $\prod_i H_i \leq \prod G_i^{n_i}$ of connected subgroups of H_i of powers G_i of 1-dimensional algebraic groups.*

Proof. Let $K' \leq K$ be as above. Also let $C_0 \leq K'$ be a countable algebraically closed subfield over which V is defined.

The proof in [BB18] goes through, but using Theorem 6.1 in place of [BB18, Theorem 2.14], and with [EH91, Theorem 3.3.1] replacing [BB18, Proposition A.4]. We describe the necessary changes.

Firstly, [BB18, Theorem 2.15] goes through in the case that $X_i \subseteq ((K_r)^{\mathcal{U}})^{n_i}$ for some r ($i = 1, 2$). The proof is identical, using Theorem 6.1; the sublinearity of the dependence on s where $K_{2,s}$ is omitted, discussed after [BB18, Theorem 2.14], also holds here: this is described in [CS20, Remark 2.7(2),Corollary 2.8], and in more detail in [CPS]. (In fact this sublinearity isn't necessary for the present 1-dimensional case.)

Now [BB18, Theorem 5.9] goes through for $P \subseteq (K')^{<\omega}$. The proof is identical, except that in the proof of [BB18, Proposition 5.14], since $\overline{a}, \overline{d} \in (K')^{<\omega}$, already $\overline{a}, \overline{d} \in ((K_r)^{\mathcal{U}})^{<\omega}$ for some r, and this passes through to the types X_i since $(K_r)^{\mathcal{U}}$ is definable, so the above restricted form of [BB18, Theorem 2.15] applies.

Next, $\mathrm{End}^0_{C_0}(G)$ must be redefined as the skew-field of quotients of $\mathrm{End}_{C_0}(G)$ (this agrees with $\mathbb{Q} \otimes \mathrm{End}_{C_0}(G)$ in characteristic 0); see [EH91, 3.1] for discussion of the possibilities.

Finally, we indicate how to circumvent the use of [BB18, Proposition A.4], which is proven only in characteristic 0, in the 1-dimensional case. Where this is applied in [BB18, Proposition 6.1], we have $a_i \in K'$ ($i = 1, \ldots, n$) such that $\mathcal{G}_{\overline{a}} = \{\mathrm{acl}^0(a_i) : i\}$ embeds in a projective subgeometry of the acl^0-geometry $\mathcal{G}_K$ of K. (Here we have $a_i \in K'$ rather than $a_i \in K^{<\omega}$, as this is what arises in the proof, via [BB18, Theorem 7.4], in the 1-dimensional case corresponding to the statement of the current theorem.) By [EH91, Theorem 3.3.1], there is a 1-dimensional algebraic group G over C_0 and generic $\overline{x} \in G^m$ over C_0 (where $m = \dim(\mathcal{G}_{\overline{a}})$) and $A \in \mathrm{Mat}_{n,m}(\mathrm{End}(G))$ such that, setting $\overline{h} := A\overline{x}$, we have $\mathrm{acl}^0(h_i) = \mathrm{acl}^0(a_i)$. Then $\mathrm{loc}^0(\overline{h}) = AG^m$ is a connected algebraic subgroup of G^n, as required.

The rest of the proof goes through unchanged. □

Remark 7.2. The only obstruction to pushing this to higher dimension, i.e. to a version of [BB18, Theorem 1.11], is the need to generalise the higher dimensional version of Evans-Hrushovski [BB18, Proposition A.4] to positive characteristic.

Meanwhile, the proof of the converse direction (showing that every special variety admits no powersaving) makes essential use of the characteristic 0 assumption in [BB18, Proposition 7.10]; this may not be so easy to generalise, and the statement may need to change.

For these reasons, we leave positive characteristic analogues of [BB18, Theorem 1.11] to future work.

[2] As defined in [BB18, Definition 1.1]

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