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Universität Münster
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Geometric Group Theory I
Exercise Sheet 10

Exercise 1. Let C be a countable group with $(c_i)_{i\geq 1}$ an enumeration of elements in C. Let $F(\{a,b\})$ be the free group with generators $\{a,b\}$ and let $H := C * F(\{a,b\})$ be the free product.

a) Show that $A_0 := \{a, a^b, \ldots, a^{b^n}, \ldots\}$ and $B_0 := \{b, c_1 b^a, \ldots, c_n b^{a^n}, \ldots\}$ generate free subgroups A and B in H respectively. Find an isomorphism $\phi : A \to B$.
Hint: You only need to show B is free, since we have already shown A is free in Exercise Sheet 3, Exercise 4.

b) Let ι_A, ι_B be the inclusions of A and B into H respectively. Then
$$\eta := \iota_B \circ \phi : A \to H$$
defines an embedding of A into H. Consider the HNN-extension $G := \mathrm{HNN}(H, A, \iota_A, \eta)$. Show that G is generated by two elements. Conclude that any countable group embeds into a group generated by two elements.
Hint: Consider the presentation $G = \langle H, t \mid \cdots \rangle$, show that G is generated by $\{a,t\}$

(6 Points)

Exercise 2. Let $(\mathcal{G}, Y)$ be a graph of groups with G_x trivial for all $x \in Y^0$.
In this exercise we will show that the universal cover of $(\mathcal{G}, Y)$ is the universal cover of the graph Y, which gives an alternative proof of the existence of universal covers for connected non-empty graphs.
Let $T \subseteq Y$ be a maximal subtree. Let $G := \pi_1(Y, T)$. Recall that the universal cover of $(\mathcal{G}, Y)$ with trivial vertex groups is defined as the oriented graph $\tilde{Y}$:

$$\begin{gathered}\tilde{Y}^0 := \{(g,x) : g \in G, x \in Y^0\};\\ \tilde{Y}^+ := \{(g,e) : g \in G, e \in Y^+\};\\ \alpha((g,e)) = (g, \alpha(e));\\ \omega((g,e)) = (ge, \omega(e)).\end{gathered}$$

Let $p : \tilde{Y} \to Y$ be the map defined as $p((g,x)) := x$ and $p((g,e)) := e$.

a) Show that p is locally bijective.

b) Show that $\tilde{Y}$ is connected.
You could copy the proof of connectedness in Theorem 7.32.

c) Show that $\tilde{Y}$ is acyclic.
You cannot copy the proof of acylicity in Theorem 7.32, since it uses the existence of universal covers. You should find a direct proof instead.

(6 Points)

Exercise 3. Let G be a non-inversive group action on a tree X. We denote X^G to be the fixed subgraph of G, namely $(X^G)^0 := \{x \in X^0 : g(x) = x, \text{ for all } g \in G\}$ and $(X^G)^1 := \{e \in X^1 : g(e) = e, \text{ for all } g \in G\}$. We say a group G has *property (FA)* if whenever G acts non-inversively on a non-empty tree X, then $X^G \neq \emptyset$.

a) Let $G = \pi_1(\mathcal{G}, Y, x)$ where $(\mathcal{G}, Y)$ is a graph of groups. Suppose G has property (FA). Show that $G = G_y$ for some $y \in Y^0$.

b) Let G be a group which has property (FA). Show that G is not isomorphic to any HNN-extension.

(4 Points)

Exercise 4. Deduce/verify the normal forms of the following groups from the normal form of $\pi_1(\mathcal{G}, Y, x)$ for a graph of groups $(\mathcal{G}, Y)$ (see Corollary 3.34 (ii) in the lecture notes).

a) The amalgamated free product $G_1 *_A G_2$;

b) The HNN-extension $\mathrm{HNN}(G, A, \phi_1, \phi_2)$;

c) The fundamental group $\pi_1(Y, x)$ where Y is a connected graph and $x \in Y^0$.
Hint: $\pi_1(Y, x)$ is a special case of $\pi_1(\mathcal{G}, Y, x)$ where $\mathcal{G}$ contains only trivial groups.

(4 Points)

Exercise 5. Let G be a group acting non-inversively on a tree X. Let $Y := {}_G\backslash^X$ be the quotient graph. For $x \in X^0$, define G_x to be the stabilizer of the vertex x, namely $G_x := \{g \in G : g(x) = x\}$. Let $(\mathcal{G}, Y)$ be the quotient graph of groups (see Definition 7.36 in the lecture notes).

Show that the following are equivalent:

a) Y is a tree;

b) G is generated by $\{G_x : x \in X^0\}$;

c) G is generated by $\{G_y \in \mathcal{G} : y \in Y^0\}$.

(4 Points)

Submission by ***Wednesday*** *morning 11:00, 11.01.2023, in Briefkasten 161.*
The exercise sheets should be solved and submitted in pairs.
Tutorial: Fridays 12:00-14:00, in room SR1d.
If you have questions about the problem sheet, please write to Tingxiang: tingxiangzou@gmail.com.