The Embedding Lemma for Pseudofinite Fields and the Completions of Psf

Jonathan Krebs

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Recap

We introduced the theory *Psf* of pseudofinite fields. We called a field K *pseudofinite* if it has the following properties:

- K is perfect;
- $\mathcal{G}al(K^{\mathrm{alg}}/K) \simeq \hat{\mathbb{Z}}$;
- K is pseudo-algebraically closed (PAC).

Lemma 1 (Facts about regular and linear disjoint extensions)

Let $K \subseteq E, F \subseteq \Omega$ be fields. Assume further that E and F are linearly disjoint over K. Then:

1. If K is perfect, then E/K is regular iff $E \cap K^{\text{alg}} = K$.
2. The natural map $E \otimes_K F \to \Omega$ given by $a \otimes b \mapsto ab$ is injective with image $E[F]$. (Conversely, this implies linear disjointness.)
3. Similarly, if $A \subseteq E$ is a ring containing K, then the natural map $A \otimes_K F \to \Omega$ is injective with image $A[F]$.
4. If F/K is algebraic, then $E[F]$ is a field (as union of finite extensions of E) and hence the image of the map $E \otimes_K F \to \Omega$ is EF. (In particular, if E/K is regular, then $E \otimes_K K^{\text{alg}} \xrightarrow{\simeq} EK^{\text{alg}} = E[K^{\text{alg}}]$.)

Lemma 2 (Embedding lemma for psudofinite fields)

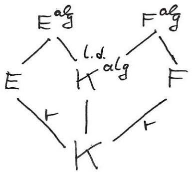
Let $K \subseteq E, F$ be perfect fields such that:

1. E/K and F/K are regular;
2. E is countable and $\mathcal{G}al(E^{\text{alg}}/E) \simeq \hat{\mathbb{Z}}$;
3. F is $\aleph_1$-saturated and pseudofinite.

Then there exists a K-embedding $\phi : E \to F$ such that F is a regular extension of $\phi(E)$.

Proof:

- Rough idea: Construct a closed subgroup of $\mathcal{G}al(E^{\text{alg}}F^{\text{alg}}/EF)$ such that the restrictions to $\mathcal{G}al(E^{\text{alg}}/E)$ and $\mathcal{G}al(F^{\text{alg}}/F)$ are isomorphisms. Then consider the fixed field of this subgroup.
- For this we want E^{alg} and F^{alg} to be linearly disjoint over K^{alg}.



- Let Ω be a common algebraically closed extension of E^{alg} and F^{alg} such that E^{alg} and F^{alg} are algebraically independent over K^{alg}.

Theorem (see [Lan02, VIII, Thm. 4.12])**.** Let $L_1, L_2 \subseteq \Omega$ be fields free over a common subfield k with L_1/k regular. Then L_1 and L_2 are linearly disjoint over k.

- It follows that E^{alg} and F^{alg} are linearly disjoint over K^{alg}.
- Notice that $E^{\text{alg}}F^{\text{alg}}/EF$ is Galois:
  - E and F are perfect, hence EF is perfect as well. (The elements of EF have the form $\dfrac{\sum e_i f_i}{\sum e_j' f_j'}$.)
  - $E^{\text{alg}}F^{\text{alg}}/EF$ is normal: Every EF-embedding of $E^{\text{alg}}F^{\text{alg}}$ in an algebraic closure is an automorphism of $E^{\text{alg}}F^{\text{alg}}$.

Claim. The map

$$\alpha : \mathcal{G}al(E^{\text{alg}}F^{\text{alg}}/EF) \to \mathcal{G}al(E^{\text{alg}}/E) \times_{\mathcal{G}al(K^{\text{alg}}/K)} \mathcal{G}al(F^{\text{alg}}/F)$$
$$\tau \mapsto (\tau\restriction_{E^{\text{alg}}}, \tau\restriction_{F^{\text{alg}}})$$

is an isomorphism (of topological groups).

Proof:

- $\tau \mapsto \tau\restriction_{E^{\text{alg}}}$ and $\tau \mapsto \tau\restriction_{F^{\text{alg}}}$ are continuous homomorphisms. By the universal property of the product, it follows that $\tau \mapsto (\tau\restriction_{E^{\text{alg}}}, \tau\restriction_{F^{\text{alg}}})$ is a continuous homomorphism.
- Injectivity is clear (consider the form of the elements of $E^{\text{alg}}F^{\text{alg}}$).

- Surjectivity:
  - Let $\sigma_1 \in \mathcal{Gal}(E^{\text{alg}}/E)$, $\sigma_2 \in \mathcal{Gal}(F^{\text{alg}}/F)$ with $\sigma_1\restriction_{K^{\text{alg}}} = \sigma_2\restriction_{K^{\text{alg}}}$.
  - Since E^{alg} and F^{alg} are linearly disjoint over K^{alg}, we have

$$E^{\text{alg}} \otimes_{K^{\text{alg}}} F^{\text{alg}} \xrightarrow{\simeq} E^{\text{alg}}[F^{\text{alg}}]$$
$$a \otimes b \mapsto ab.$$

  - Then $a \otimes b \mapsto \sigma_1(a) \otimes \sigma_2(b)$ defines a ring automorphism of $E^{\text{alg}} \otimes_{K^{\text{alg}}} F^{\text{alg}}$ and hence of $E^{\text{alg}}[F^{\text{alg}}]$, which fixes E and F.
  - It extends to an EF-automorphism of the quotient field $E^{\text{alg}}F^{\text{alg}}$.

□ (Claim)

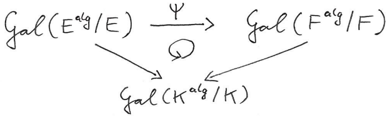
Recall: $\mathcal{G}al(E^{\text{alg}}/E) \simeq \hat{\mathbb{Z}} \simeq \mathcal{G}al(F^{\text{alg}}/F)$. We want to consider the graph of an isomorphism $\mathcal{G}al(E^{\text{alg}}/E) \xrightarrow{\simeq} \mathcal{G}al(F^{\text{alg}}/F)$ as a closed subgroup of $\mathcal{G}al(E^{\text{alg}}/E) \times_{\mathcal{G}al(K^{\text{alg}}/K)} \mathcal{G}al(F^{\text{alg}}/F)$.

Remark. Let G, H be topological groups and $f : G \xrightarrow{\simeq} H$ an isomorphism. Then the map

$$\begin{aligned} G &\to G \times H \\ g &\mapsto (g, f(g)) \end{aligned}$$

defines an isomorphism of topological groups between G and $\operatorname{graph}(f) = \{(g, f(g)) \mid g \in G\} \subseteq G \times H$ (the latter endowed with the subspace topology). If G is Hausdorff, then $\operatorname{graph}(f) \subseteq G \times H$ is a closed subgroup.

We need an isomorphism $\Psi : \mathcal{G}al(E^{\text{alg}}/E) \xrightarrow{\simeq} \mathcal{G}al(F^{\text{alg}}/F)$, whose graph lies in $\mathcal{G}al(E^{\text{alg}}/E) \times_{\mathcal{G}al(K^{\text{alg}}/K)} \mathcal{G}al(F^{\text{alg}}/F)$. In other words:



Facts about $\hat{\mathbb{Z}}$ (see [Cha05, Sec. 3]).

1. If G is a profinite group, $f : \hat{\mathbb{Z}} \to G$ a continuous epimorphism and $\sigma \in G$ a topological generator of G (i.e. $\langle \sigma \rangle$ is dense in G), then $f^{-1}(\sigma)$ contains a topological generator of $\hat{\mathbb{Z}}$.
2. Let $a, b \in \hat{\mathbb{Z}}$ be topological generators. Then $a \mapsto b$ extends to an automorphism of $\hat{\mathbb{Z}}$.

We use this to define Ψ:

- Let $\sigma_E \in \mathcal{G}al(E^{\text{alg}}/E) \simeq \hat{\mathbb{Z}}$ be a topological generator.
- The restriction $\sigma_E \restriction_{K^{\text{alg}}} \in \mathcal{G}al(K^{\text{alg}}/K)$ is a topological generator, i.e. $\overline{\langle \sigma_E \restriction_{K^{\text{alg}}} \rangle} = \mathcal{G}al(K^{\text{alg}}/K)$:

  By continuity, the preimage of $\overline{\langle \sigma_E \restriction_{K^{\text{alg}}} \rangle}$ is closed (and it contains $\langle \sigma_E \rangle$), hence it is identical to $\overline{\langle \sigma_E \rangle} = \mathcal{G}al(E^{\text{alg}}/E)$. The result follows by surjectivity of the restriction map $\mathcal{G}al(E^{\text{alg}}/E) \to \mathcal{G}al(K^{\text{alg}}/K)$ (using regularity of E/K).
- By Fact (1), $\sigma_E \restriction_{K^{\text{alg}}}$ extends to a topological generator of $\mathcal{G}al(F^{\text{alg}}/F) \simeq \hat{\mathbb{Z}}$, call it σ_F.

- By Fact (2), $\sigma_E \mapsto \sigma_F$ extends to an isomorphism $\Psi : \mathcal{G}al(E^{\text{alg}}/E) \xrightarrow{\simeq} \mathcal{G}al(F^{\text{alg}}/F)$, which is as required:
  - By definition, we have $\sigma_E \restriction_{K^{\text{alg}}} = \Psi(\sigma_E) \restriction_{K^{\text{alg}}}$.
  - Obviously, this extends to the generated subgroups, i.e. for $\sigma \in \langle \sigma_E \rangle$, we have $\sigma \restriction_{K^{\text{alg}}} = \Psi(\sigma) \restriction_{K^{\text{alg}}}$.
  - By continuity of Ψ, this property extends to the closure $\overline{\langle \sigma_E \rangle} = \mathcal{G}al(E^{\text{alg}}/E)$.

Using the Remark, we get that

$$\begin{aligned} \text{graph}(\Psi) &= \{(\sigma, \Psi(\sigma)) \mid \sigma \in \mathcal{G}al(E^{\text{alg}}(E)\} \\ &\subseteq \mathcal{G}al(E^{\text{alg}}/E) \times_{\mathcal{G}al(K^{\text{alg}}/K)} \mathcal{G}al(F^{\text{alg}}/F) \end{aligned}$$

is a closed subgroup isomorphic to $\hat{\mathbb{Z}}$ with topological generator (σ_E, σ_F). Set

$$\begin{aligned} H_\Psi &:= \alpha^{-1}\big(\text{graph}(\Psi)\big) \subseteq \mathcal{G}al(E^{\text{alg}}F^{\text{alg}}/EF), \\ \tau_\Psi &:= \alpha^{-1}\big((\sigma_E, \sigma_F)\big) \in H_\Psi. \end{aligned}$$

(By definition of α, we have $\tau_\Psi \restriction_{E^{\text{alg}}} = \sigma_E$ and $\tau_\Psi \restriction_{F^{\text{alg}}} = \sigma_F$.)

Let $M \subseteq E^{\mathrm{alg}} F^{\mathrm{alg}}$ be the fixed field of τ_Ψ (which is identical to the fixed field of $H_\Psi = \overline{\langle \tau_\Psi \rangle}$).

Claim.

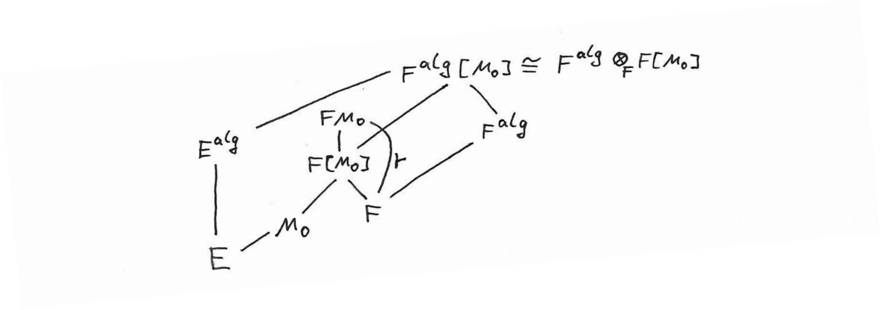
1. M/E and M/F are regular extensions.
2. $E^{\mathrm{alg}} F^{\mathrm{alg}} = MF^{\mathrm{alg}} = M[F^{\mathrm{alg}}]$. (In particular, $E^{\mathrm{alg}} \subseteq M[F^{\mathrm{alg}}]$.)

Proof:

1. Since $\tau_\Psi \restriction_{E^{\mathrm{alg}}} = \sigma_E$ and $\tau_\Psi \restriction_{F^{\mathrm{alg}}} = \sigma_F$, we have $M \cap E^{\mathrm{alg}} = E$ and $M \cap F^{\mathrm{alg}} = F$.
   (Notice that the fixed field of a topological generator is the ground field.)
2. The second equality follows from F^{alg} being algebraic over M. For the first, it suffices to show that $\mathcal{Gal}(E^{\mathrm{alg}} F^{\mathrm{alg}} / MF^{\mathrm{alg}}) = \{\mathrm{id}\}$:
   - Let $\tau \in \mathcal{Gal}(E^{\mathrm{alg}} F^{\mathrm{alg}} / MF^{\mathrm{alg}})$.
   - Then $\tau \in \mathcal{Gal}(E^{\mathrm{alg}} F^{\mathrm{alg}} / M) = H_\Psi \simeq \mathrm{graph}(\Psi)$ and $\tau \restriction_{F^{\mathrm{alg}}} = \mathrm{id}_{F^{\mathrm{alg}}} \in \mathcal{Gal}(F^{\mathrm{alg}} / F)$.
   - It follows that $\tau \restriction_{E^{\mathrm{alg}}} = \Psi^{-1}(\mathrm{id}_{F^{\mathrm{alg}}}) = \mathrm{id}_{E^{\mathrm{alg}}}$.
   - Consequently, $\tau = \mathrm{id}_{E^{\mathrm{alg}} F^{\mathrm{alg}}}$.

□ (Claim)

- By countability of E, let $E \subseteq M_0 \subseteq M$ be a countable intermediate field such that $E^{\mathrm{alg}} \subseteq F^{\mathrm{alg}}[M_0]$.
- Since M/F is regular, FM_0/F is regular.



Lemma from last week – statement (2) (see [Cha05, 6.7])**.** Let F be a perfect $\aleph_1$-saturated PAC field and A a countable subset of some field containing F such that $F(A)/F$ is regular. Then there exists an F-homomorphism $F[A] \to F$.

- It follows that there is an F-homomorphism $F[M_0] \to F$, which extends to an F^{alg}-homomorphism $\phi : F^{\text{alg}}[M_0] \to F^{\text{alg}}$.
- We show that this is the ϕ we are looking for (more precisely, $\phi\restriction_E$). Notice: $\phi\restriction_E$ is a K-embedding $E \to F$ (since $K \subseteq F^{\text{alg}}$ and $E \subseteq M_0$).
- It remains to show that $F/\phi(E)$ is regular.

Claim. For $a \in E^{\mathrm{alg}}$, we have $\phi(\sigma_E(a)) = \sigma_F(\phi(a))$.

Proof:

- Write $a = \sum_i m_i b_i$ with $m_i \in M_0$ and $b_i \in F^{\mathrm{alg}}$.
- Firstly:

$$\begin{aligned}
\phi(\sigma_E(a)) &= \phi\Big(\sigma_E\Big(\sum_i m_i b_i\Big)\Big) \\
&= \phi\Big(\tau_\Psi\Big(\sum_i m_i b_i\Big)\Big) && \Big[\tau_\Psi\restriction_{E^{\mathrm{alg}}} = \sigma_E\Big] \\
&= \phi\Big(\sum_i \tau_\Psi(m_i)\tau_\Psi(b_i)\Big) \\
&= \phi\Big(\sum_i m_i \sigma_F(b_i)\Big) && \Big[\tau_\Psi\restriction_M = \mathrm{id}_M \text{ and } \tau_\Psi\restriction_{F^{\mathrm{alg}}} = \sigma_F\Big] \\
&= \sum_i \phi(m_i)\sigma_F(b_i) && \Big[\phi\restriction_{F^{\mathrm{alg}}} = \mathrm{id}_{F^{\mathrm{alg}}}\Big]
\end{aligned}$$

- Secondly:

$$\begin{aligned}\sigma_F(\phi(a)) &= \sigma_F\left(\phi\left(\sum_i m_i b_i\right)\right)\\ &= \sigma_F\left(\sum_i \phi(m_i) b_i\right)\\ &= \sum_i \phi(m_i)\sigma_F(b_i) \qquad \left[\phi(M_0) \subseteq F \text{ and } \sigma_F\restriction_F = \mathrm{id}_F\right]\end{aligned}$$

□ (Claim)

We conclude that $F/\phi(E)$ is regular. It suffices to show that $\phi(E^{\mathrm{alg}}) \cap F = \phi(E)$. Let $a \in E^{\mathrm{alg}}$ with $\phi(a) \in F$. Then:

$$\begin{aligned}&\sigma_F(\phi(a)) = \phi(a)\\ \Rightarrow\ &\phi(\sigma_E(a)) = \phi(a) \qquad [\text{Claim}]\\ \Rightarrow\ &\sigma_E(a) = a\\ \Rightarrow\ &a \in E \qquad \left[\overline{\langle\sigma_E\rangle} = \mathcal{G}al(E^{\mathrm{alg}}/E)\right].\end{aligned}$$

Lemma 3 (Embedding lemma – 2nd version)

Let $K \subseteq E$ and $K' \subseteq F$ be perfect fields such that:

1. $\phi : K \xrightarrow{\simeq} K'$ is an isomorphism;
2. E/K and F/K' are regular;
3. E is countable and $\mathcal{Gal}(E^{\text{alg}}/E) \simeq \hat{\mathbb{Z}}$;
4. F is $\aleph_1$-saturated and pseudofinite.

Then there exists an embedding $\phi' : E \to F$, which extends ϕ and such that F is a regular extension of $\phi'(E)$.

Proof:

Extend ϕ to an embedding ϕ_0 with domain E and apply the Embedding Lemma to $\phi_0(E)/K'$.

□

Proposition 4

Let E and F be pseudofinite fields, which are regular extensions of a common perfect subfield K. Then $E \equiv_K F$.

Proof:

WLOG we may assume:

- E and F are $\aleph_1$-saturated. (Otherwise consider $\aleph_1$-saturated elementary extensions. They are also pseudofinite and – by being regular extensions of E and F – regular extensions of K.)
- K is countable, otherwise:
  - Show $E \equiv_A F$ for all countable subsets $A \subseteq K$.
  - By Löwenheim-Skolem, let $A \subseteq K' \preceq K$ be a countable elementary substructure.
  - K' is perfect, since K is. Furthermore, K/K', E/K and F/K being regular implies that E/K' and F/K' are regular.

We build recursively sequences of partial K-isomorphisms $(\phi_i : E \dashrightarrow F)_{i<\omega}$ and $(\psi_i : F \dashrightarrow E)_{i<\omega}$ with the following properties:

- $\mathrm{dom}(\phi_i)$ and $\mathrm{dom}(\psi_i)$ are countable subfields containing K.
- $\mathrm{dom}(\phi_i) \preceq E$ and $F/\mathrm{im}(\phi_i)$ is regular.
- $\mathrm{dom}(\psi_i) \preceq F$ and $E/\mathrm{im}(\psi_i)$ is regular.
- ψ_i extends ϕ_i^{-1} and ϕ_{i+1} extends ψ_i^{-1}.

Then $\bigcup \phi_i$ is a K-isomorphism between $E' := \bigcup_{i<\omega} \mathrm{dom}(\phi_i) \preceq E$ and $F' := \bigcup_{i<\omega} \mathrm{dom}(\psi_i) \preceq F$. Hence $E' \equiv_K F'$ and so $E \equiv_K F$.

As for the construction:

- ϕ_0: Let $E_0 \preceq E$ be countable containing K. We have that E_0/K is regular and E_0 is pseudofinite. By the embedding lemma, there exists a K-embedding $\phi_0 : E_0 \to F$, such that $F/\mathrm{im}(\phi_0)$ is regular.
- ψ_0: Let $F_0 \preceq F$ be countable containing $\mathrm{im}(\phi_0)$. We have that $F_0/\mathrm{im}(\phi_0)$ is regular and F_0 is pseudofinite. By the embedding lemma (version 2), there exists an extension $\psi_0 : F_0 \to E$ of ϕ_0^{-1}, such that $E/\mathrm{im}(\psi_0)$ is regular.
- For the inductive step, proceed as for ψ_0.

Corollary 5

Let $E \subseteq F$ be pseudofinite fields. Then $E \preceq F$ iff F/E is regular (i.e. $E^{\text{alg}} \cap F = E$).

Proof:

"$\Rightarrow$": Elementary substructures are relatively algebraically closed.
"$\Leftarrow$": Apply Proposition 4 to $K := E$.

□

Theorem 6

Let E and F be pseudofinite fields and K a common subfield. Then

$$E \equiv_K F \iff E \cap K^{\mathrm{alg}} \simeq_K F \cap K^{\mathrm{alg}}.$$

(“$\Rightarrow$” holds for arbitrary fields, see [Cha05, remark after (6.13)]).

Proof:

“$\Leftarrow$”:

- WLOG $E \cap K^{\mathrm{alg}} = F \cap K^{\mathrm{alg}} =: K'$. (Otherwise, let $f : E \cap K^{\mathrm{alg}} \xrightarrow{\simeq} F \cap K^{\mathrm{alg}}$ be a K-isomorphism, consider an extension f' to E, and apply the result to $f'(E)$.)
- Since E is perfect, K' is perfect. Furthermore, it follows that E and F are regular extensions of K'.
- By Proposition 4, it follows $E \equiv_{K'} F$. In particular, $E \equiv_K F$.

"$\Rightarrow$": (We work in a common algebraically closed extension.)

Step 1. Let L be a finite Galois extension of K, then $E \cap L \simeq_K F \cap L$:

- By the primitive element theorem and separability, $E \cap L = K(\alpha)$ for some $\alpha \in E \cap L$.
- $E \equiv_K F$ implies that the minimal polynomial of α over K has a zero $\alpha' \in F$. By normality, $\alpha' \in L$ and hence $K(\alpha') \subseteq F \cap L$.
- The K-embedding $E \cap L \to F \cap L$ given by the isomorphism $K(\alpha) \simeq K(\alpha')$ implies $[E \cap K : K] \leq [F \cap K : K]$. By symmetry, we have $[E \cap K : K] = [F \cap K : K]$, and hence $F \cap K = K(\alpha')$. Thus, the embedding is an isomorphism.

Step 2. $E \cap K^{sep} \simeq_K F \cap K^{sep}$:
Let $\mathcal{N}$ be the set of all finite Galois extensions of K. For $L \in \mathcal{N}$ consider

$$S_L := \{\sigma \in \mathcal{G}al(K^{\text{sep}}/K) \mid \sigma(E \cap L) = F \cap L\}.$$

Claim: $\bigcap_{L \in \mathcal{N}} S_L \neq \emptyset$.

- By step 1, $S_L \neq \emptyset$ for all $L \in \mathcal{N}$.
- Finite intersections are non-empty: For $L \subseteq M \in \mathcal{N}$, we have $S_L \supseteq S_M$. In particular, for $L, M \in \mathcal{N}$, we have $S_L \cap S_M \supseteq S_{LM}$.
- $S_L \subseteq \mathcal{G}al(K^{\text{sep}}/K)$ is closed for all $L \in \mathcal{N}$:
    - For $\sigma \in S_L$ and $\tau \in \mathcal{G}al(K^{\text{sep}}/L)$, we have $\tau\sigma \in S_L$. Hence, S_L is a union of cosets of $\mathcal{G}al(K^{\text{sep}}/L)$.
    - Furthermore, $\mathcal{G}al(K^{\text{sep}}/L)$ is an open and hence clopen subgroup. It follows that arbitrary unions of cosets are clopen.
- The claim follows by compactness of $\mathcal{G}al(K^{\text{sep}}/K)$.

Any $\sigma \in \bigcap_{L \in \mathcal{N}} S_L$ restricts to a K-isomorphism $E \cap K^{sep} \xrightarrow{\simeq} F \cap K^{sep}$.

Step 3. $E \cap K^{\text{alg}} \simeq_K F \cap K^{\text{alg}}$:

- An isomorphism $E \cap K^{sep} \xrightarrow{\simeq} F \cap K^{sep}$ extends (uniquely) to an isomorphism $(E \cap K^{sep})^{perf} \xrightarrow{\simeq} (F \cap K^{sep})^{perf}$.
- But
$$(E \cap K^{sep})^{perf} = E^{perf} \cap (K^{sep})^{perf} = E \cap K^{\text{alg}},$$
since E is perfect; analogously for F. □

Corollary 7 (The completions of Psf)

Let E and F be pseudofinite fields with prime fields E_0 and F_0. Then:

$$E \equiv F \iff E \cap E_0^{\mathrm{alg}} \simeq F \cap F_0^{\mathrm{alg}}$$
$$\iff \{f(X) \in \mathbb{Z}[X] \mid E \models \exists x. f(x) = 0\} =$$
$$\{f(X) \in \mathbb{Z}[X] \mid F \models \exists x. f(x) = 0\}.$$

Proof:

The first equivalence follows directly from Theorem 6.

As for the second equivalence: "$\Rightarrow$" is obvious. "$\Leftarrow$" follows from the proof of the "$\Rightarrow$"-direction of Theorem 6. Notice that also the characteristic is fixed by the given set of polynomials. Notice further that for characteristic 0, the polynomials over $\mathbb{Q}$ with roots are determined by those over $\mathbb{Z}$. □