Seminar Topological Dynamics and Model Theory

notes for the second talk by Maximilian Strohmeier

Summer Semester 2022

Let $\mathcal{M}$ be a first-order $\mathcal{L}$-structure and $G := \mathrm{Aut}(M)$ a topological group.
Recall:

Theorem (∗)**.** *$(\Sigma^{\mathcal{M}}, \mathrm{tp}^{\mathrm{full}}(\bar{m}))$ is the universal right G-ambit with G-action $\mathrm{tp}^{\mathrm{full}}(\sigma(\bar{m})) \cdot g := \mathrm{tp}^{\mathrm{full}}(\sigma(g(\bar{m})))$.*

Definition.

(a) For a tuple $\bar{a}$ and $B \subseteq M$ let $\binom{B}{\bar{a}} := \{\bar{a}' \subseteq B \mid \exists f \in G : \bar{a}' = f(\bar{a})\}$ the set of embeddings *of $\bar{a}$ in B.*

(b) $\mathcal{M}$ has the embedding Ramsey property (ERP) *if*

$$\forall k, r \in \omega, B \overset{finite}{\subseteq} M, \bar{a} \in B^k : \exists C \overset{finite}{\subseteq} M : \forall c : \binom{C}{\bar{a}} \to r :$$
$$\exists B' \in \binom{C}{B} : \left| c\left[\binom{B'}{\bar{a}}\right]\right| = 1$$

(c) A topological group is extremely amenable *if every flow has a fixed point.*

Theorem. *$\mathcal{M}$ has the* ERP *iff G is extremely amenable.*

Proof. We will structure this proof as a series of equivalences for $\mathcal{M}$.

$$\text{ERP} \tag{1}$$

$$\Leftrightarrow$$

$$\begin{aligned}&\forall k, r \in \omega, B \overset{finite}{\subseteq} M, \bar{a} \in B^k, c : \binom{M}{\bar{a}} \to r :\\ &\exists B' \in \binom{M}{B} : \left| c\left[\binom{B'}{\bar{a}}\right]\right| = 1\end{aligned} \tag{2}$$

$$\Leftrightarrow$$

$$\begin{aligned}&\forall k, n, r \in \omega, \bar{a} \in M^k, (g_i)_{i\in n} \in G^n, \{\varphi_i((x_i)_{i\in k}) \mid i \in r\} \subseteq \mathcal{L}^{\mathrm{full}}((x_i)_{i\in k}) :\\ &\exists \sigma \in G : \bigwedge_{i<r} \bigwedge_{j<n} \left(\varphi_i(\sigma(g_j(\bar{a}))) \leftrightarrow \varphi_i(\sigma(\bar{a}))\right)\end{aligned} \tag{3}$$

$$\Leftrightarrow$$

$$\exists \sigma \in G^* : \forall g \in G : \mathrm{tp}^{\mathrm{full}}(\sigma(g(\bar{m}))) = \mathrm{tp}^{\mathrm{full}}(\sigma(\bar{m})) \tag{4}$$

$$\Leftrightarrow$$

$$\exists p \in \Sigma^{\mathcal{M}} : p \cdot G = \{p\} \tag{5}$$

$$\Leftrightarrow$$

$$G \text{ is extremely amenable} \tag{6}$$

(b) For any finite tuple from a finite set there is a finite superset such that for any coloring of all embeddings of the tuple in the superset we find an embedding of the original set monochromatic regarding the coloring.

(2) For any coloring of all embeddings of a finite tuple from a finite set we find an embedding of the set monochromatic regarding the coloring.

"(1) ⇒ (2)" Clear.

"(2) ⇒ (1)" In order to prove by contraposition we assume ¬ERP.

$$\exists k, r \in \omega, B \overset{finite}{\subseteq} M, \bar{a} \in B^k : \forall C \overset{finite}{\subseteq} M : \exists c : \binom{C}{\bar{a}} \to r :$$
$$\neg \exists B' \in \binom{C}{B} : \left| c\left[\binom{B'}{\bar{a}}\right] \right| = 1.$$

Fix such $\bar{a}$, B and r. For $C \overset{finite}{\subseteq} M$ define

$$K_C := \left\{ c : \binom{C}{\bar{a}} \to r \mid \neg \exists B' \in \binom{C}{B} : \left| c\left[\binom{B'}{\bar{a}}\right] \right| = 1 \right\} \neq \emptyset.$$

For $C \subseteq C' \overset{finite}{\subseteq} M$

$$K_C = \left\{ c \restriction_{\binom{C}{\bar{a}}} \mid c \in K_{C'} \right\}.$$

Hence we can fix in the resulting profinite space $\eta :\in \varprojlim_{C} K_C \neq \emptyset$ and define $c : \binom{M}{\bar{a}} \to r$; $c(\bar{a}') := \eta(B \cup \bar{a}')(\bar{a}')$ Since $\forall C \overset{finite}{\subseteq} M : c \restriction_{\binom{C}{\bar{a}}} \in K_C$

$$\exists k, r \in \omega, B \overset{finite}{\subseteq} M, \bar{a} \in B^k, c : \binom{M}{\bar{a}} \to r :$$
$$\neg \exists B' \in \binom{M}{B} : \left| c\left[\binom{B'}{\bar{a}}\right] \right| = 1.$$

"(2) ⇒ (3)" For any $k, n, r \in \omega$, $\bar{a} \in M^k$, $(g_i)_{i \in n} \in G^n$, $\{\varphi_i((x_i)_{i \in k}) \mid i \in r\} \subseteq \mathcal{L}^{\text{full}}((x_i)_{i \in k})$ take the coloring $c : \binom{M}{\bar{a}} \to 2^r$; $c(\bar{a}')(i) := \chi_{\varphi_i(\bar{a}')}$. Then for a $B \subseteq M$, $\{\bar{a}, g_0(\bar{a}), ..., g_{n-1}(\bar{a})\} \subseteq B$ by statement (2)

$$\exists B' \in \binom{M}{B} : \left| c\left[\binom{B'}{\bar{a}}\right] \right| = 1.$$

Therefore we have a $\sigma \in G$ with $\sigma[B] = B'$ and since $\sigma(\bar{a}), \sigma(g_0(\bar{a})), ..., \sigma(g_{n-1}(\bar{a})) \in \binom{B'}{\bar{a}}$ we conclude statement (3).

"(3) ⇒ (2)" For any $k, r \in \omega, B \overset{finite}{\subseteq} M, \bar{a} \in B^k, c : \binom{M}{\bar{a}} \to r$ the fibers of c are subsets of M^k and hence can be defined by formulas $\{\varphi_i((x_i)_{i \in k}) \mid i \in r\} \subseteq \mathcal{L}^{\text{full}}((x_i)_{i \in k})$. Take $(g_i)_{i \in n} \in G^n$ such that $\{g_0(\bar{a}), ..., g_{n-1}(\bar{a})\} = \binom{B}{\bar{a}}$. From statement (3) we get

$$\exists \sigma \in G : \left| c\left[\binom{\sigma[B]}{\bar{a}}\right] \right| = 1$$

which yields (2).

"(3) ⇔ (4)" By $|M|^+$-saturation of $\mathcal{M}^*$.

"(4) ⇔ (5)" Trivial.

"(5) ⇔ (6)" Follows from Theorem (∗).

□

Example (Kechris-Pestov-Todorcevic)**.** *Some examples of extremely amenable automorphism groups:*

(a) $\mathrm{Aut}(\mathbb{Q}, <)$.

(b) $\mathrm{Aut}([0,1], \lambda)$.

(c) The automorphism group of the random ordered graph.

(d) The automorphism group of the random ordered hypergraph of a type.

There is a similar Theorem with weaker equivalent properties:

Definition.

(a) For subsets $B \subseteq C \subseteq M$ *and a enumeration* $\bar{b} \overset{bijective}{\in} B^{|B|}$ *let*

$$\left\langle \binom{B}{\bar{a}} \right\rangle := \left\{ \sum_{i\in k} \lambda_i \bar{b}_i \mid k \overset{\neq 0}{\in} \omega, \forall i \in k : \bar{b}_i \in \binom{C}{\bar{b}}, \forall i \in k : \lambda_i \in [0,1], \sum_{i\in k} \lambda_i = 1 \right\}$$

be the set of affine combinations of copies of $\bar{b} \in C$.

Also for $k \in \omega$, $\bar{a} \in B^k$, $\bar{a}' \in \binom{\bar{a}}{B}$, $v = \sum_{i\in k} \lambda_i \bar{b}_i \in \left\langle \binom{B}{\bar{a}} \right\rangle$, $\sigma_0, ..., \sigma_{k-1} \overset{\sigma_i(\bar{b})=\bar{b}_i}{\in} G$ *write*

$$v \circ \bar{a}' := \sum_{i\in k} \lambda_i \sigma_i(\bar{a}') \in \left\langle \binom{B}{\bar{a}} \right\rangle .$$

For $c : \binom{C}{\bar{a}} \to 2^r$

$$c(v \circ \bar{a}') := \sum_{i\in k} \lambda_i c(\bar{a}'_i) \in [0,1]^r .$$

(b) $\mathcal{M}$ *has the* embedding convex Ramsey property (ECRP) *if*

$$\forall \epsilon \in \mathbb{R}_{>0}, k, r \in \omega, B \overset{finite}{\subseteq} M, \bar{b} \overset{bijective}{\in} B^{|B|}, \bar{a} \in B^k : \exists C \overset{finite}{\subseteq} M : \forall c : \binom{C}{\bar{a}} \to 2^r :$$
$$\exists v \in \left\langle \binom{C}{\bar{b}} \right\rangle : \forall \bar{a}', \bar{a}'' \in \binom{B}{\bar{a}} : \| c(v \circ \bar{a}') - c(v \circ \bar{a}'') \|_{\sup} \leq \epsilon .$$

$\mathcal{M}$ *has the* strong ECRP *if the above holds with* $\epsilon = 0$.

(c) A topological group is amenable *if on every flow there is a G-invariant Borel probability measure.*

Theorem. *The following are equivalent:*

(a) $\mathcal{M}$ *has the* ECRP.

(b) $\mathcal{M}$ *has the strong* ECRP.

(c) G *is amenable.*

The proof is similarly structured to before.