ASSEMBLY

Michael Weiss and Bruce Williams

Abstract. The goal of assembly is to approximate homotopy invariant functors from spaces to spectra by homotopy invariant and excisive functors from spaces to spectra. We show that there exists a *best* approximation, characterized by a universal property.

1. The Ordinary Assembly Map

We adopt a very category theoretic point of view in describing assembly maps. It has been formulated explicitly by Quinn in the appendix to [Q], and more implicitly in Quinn's thesis, in [QGF], in [And] and in articles of Waldhausen, e.g. [Wa1], [Wa2]. See also [QAB]. From this point of view, the goal of assembly is: Given a homotopy invariant functor F from spaces to spectra, to approximate F from the left by an *excisive* homotopy invariant functor $F^{\%}$.

In this section, all spaces are homotopy equivalent to CW–spaces, all pairs of spaces are homotopy equivalent to CW–pairs, and all spectra are CW–spectra.

A functor $\boldsymbol{F}$ from spaces to spectra is *homotopy invariant* if it takes homotopy equivalences to homotopy equivalences. A homotopy invariant $\boldsymbol{F}$ is *excisive* if $\boldsymbol{F}(\emptyset)$ is contractible and if $\boldsymbol{F}$ preserves homotopy pushout squares (alias homotopy cocartesian squares, see [Go1], [Go2]). The excision condition implies that $\boldsymbol{F}$ preserves finite coproducts, up to homotopy equivalence. Call $\boldsymbol{F}$ *strongly excisive* if it preserves arbitrary coproducts, up to homotopy equivalence.

If $\boldsymbol{F}$ is strongly excisive, then the functor $\pi_*\boldsymbol{F}$ from spaces to graded abelian groups is a generalized homology theory—it has Mayer–Vietoris sequences, and satisfies the strong wedge axiom. Conversely, homotopy theorists know that any generalized homology theory satisfying the strong wedge axiom is isomorphic to one of the form $\pi_*\boldsymbol{F}$ where $\boldsymbol{F}(X) = X_+ \wedge \boldsymbol{Y}$ and $\boldsymbol{Y}$ is a fixed spectrum. Such an $\boldsymbol{F}$ is of course strongly excisive.

1.1. Theorem. *For any homotopy invariant $\boldsymbol{F}$ from spaces to spectra, there exist a strongly excisive (and homotopy invariant) $\boldsymbol{F}^{\%}$ from spaces to spectra and a natural transformation*

$$\alpha = \alpha_{\boldsymbol{F}} : \boldsymbol{F}^{\%} \longrightarrow \boldsymbol{F}$$

such that $\alpha : \boldsymbol{F}^{\%}(*) \to \boldsymbol{F}(*)$ is a homotopy equivalence. Moreover, $\boldsymbol{F}^{\%}$ and $\alpha_{\boldsymbol{F}}$ can be made to depend functorially on $\boldsymbol{F}$.

1991 *Mathematics Subject Classification.* Primary 55P65; Secondary 55N20, 55P42, 19D10, 19E20.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Preliminaries. We are going to use homotopy colimits in the proof. Here is a description: Let Z be a functor from a small category $\mathcal{C}$ to the category of spaces. For $n \geq 0$ let $[n]$ be the ordered set $\{0, 1, \ldots, n\}$; we view this as a category, with exactly one morphism from i to j whenever $i \leq j$, and no morphism from i to j if $i > j$. The homotopy colimit of Z, denoted $\operatorname{hocolim} Z$, is the geometric realization of the simplicial space

$$n \mapsto \coprod_{G:[n]\to\mathcal{C}} Z(G(0))$$

where the coproduct must be taken over all covariant functors G from $[n]$ to $\mathcal{C}$. We hope the face and degeneracy maps are obvious. See [BK] for more details. It is often convenient to use informal notation for a homotopy colimit, e.g.

$$\operatorname*{hocolim}_{C \text{ in } \mathcal{C}} Z(C)$$

instead of $\operatorname{hocolim} Z$. This is particularly true when the values of the functor have "names" and the functor as such has not been named.

A special case of special interest: When $Z(C)$ is a point for every C in $\mathcal{C}$, then clearly $\operatorname{hocolim} Z$ is the classifying space of $\mathcal{C}$. (We shall also say: the *nerve* of $\mathcal{C}$; strictly speaking, the nerve of $\mathcal{C}$ is a simplicial set, and the classifying space of $\mathcal{C}$ is the geometric realization of the nerve of $\mathcal{C}$.) More generally, when Z is a constant functor, then $\operatorname{hocolim} Z$ is the product of the classifying space of $\mathcal{C}$ with the constant value of Z. In some examples below, $\mathcal{C}$ is the category of faces of an incomplete simplicial set ; then the classifying space of $\mathcal{C}$ is the barycentric subdivision of the incomplete simplicial set. (An *incomplete* simplicial set is a simplicial set without degeneracy operators.)

In general, a key property of homotopy colimits is their homotopy invariance. Suppose that $f : Z \to Z'$ is a natural transformation between functors from $\mathcal{C}$ to spaces. If f_C from $Z(C)$ to $Z'(C)$ is a homotopy equivalence for every C in $\mathcal{C}$, then f_* from $\operatorname{hocolim} Z$ to $\operatorname{hocolim} Z'$ is a homotopy equivalence.

Variations: The above formula for $\operatorname{hocolim} Z$ remains meaningful when Z is a functor from $\mathcal{C}$ to spaces or spectra. Bear in mind that the geometric realization of a simplicial pointed space or simplicial spectrum $[n] \mapsto X_n$ is given by a formula of type $\left(\coprod_n \Delta^n_+ \wedge X_n\right) / \sim$ where $\sim$ stands for the usual relations.

First proof of 1.1. For a space X, let $\operatorname{simp}(X)$ be the category whose objects are maps $\Delta^n \to X$ where $n \geq 0$, and whose morphisms are commutative triangles

$$\begin{array}{ccc} \Delta^m & \xrightarrow{f_*} & \Delta^n \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

where f_* is the map induced by a monotone injection f from $\{0, 1, \ldots, m\}$ to $\{0, 1, \ldots, n\}$. Let $\boldsymbol{F}_X$ from $\operatorname{simp}(X)$ to spectra be the covariant functor sending $g : \Delta^n \to X$ to $\boldsymbol{F}(\Delta^n)$, and let

$$\boldsymbol{F}^{\%}(X) := \operatorname{hocolim} \boldsymbol{F}_X .$$

For each $g : \Delta^n \to X$ in $\mathrm{simp}(X)$ we have $g_* : \boldsymbol{F}(\Delta^n) \to \boldsymbol{F}(X)$. Letting g vary, we regard this as a natural transformation from $\boldsymbol{F}_X$ to the constant functor with value $\boldsymbol{F}(X)$. It induces

$$\alpha : \boldsymbol{F}^{\%}(X) \longrightarrow \boldsymbol{F}(X)\,.$$

Clearly α is a homotopy equivalence when X is a point. For arbitrary X, and $g : \Delta^n \to X$ in $\mathrm{simp}(X)$, we have the map $\Delta^n \to *$ which induces $\boldsymbol{F}(\Delta^n) \to \boldsymbol{F}(*)$, a homotopy equivalence. We regard this as a natural transformation from $\boldsymbol{F}_X$ to the constant functor with value $\boldsymbol{F}(*)$; by the homotopy invariance of homotopy colimits, the induced map of homotopy colimits is a homotopy equivalence

$$\boldsymbol{F}^{\%}(X) \longrightarrow |\mathrm{simp}(X)|_+ \wedge \boldsymbol{F}(*)\,.$$

It is an exercise to show that $|\,\mathrm{simp}(X)| \simeq X$. Thus $\boldsymbol{F}^{\%}(X)$ is related to $X_+ \wedge \boldsymbol{F}(*)$ by a chain of natural homotopy equivalences. $\square$

Second proof of 1.1. We compose $\boldsymbol{F}$ with the geometric realization functor from incomplete simplicial sets to spaces, and henceforth assume that $\boldsymbol{F}$ is a functor from incomplete simplicial sets to spaces. For an incomplete simplicial set X, we define $\mathrm{simp}(X)$ much as before: objects are the simplicial maps $\Delta^n \to X$, for arbitrary n. (These are in bijection with the simplices of X.) We define $\boldsymbol{F}_X$ from $\mathrm{simp}(X)$ to spectra much as before. We let

$$\boldsymbol{F}^{\%}(X) = \mathrm{hocolim}\, \boldsymbol{F}_X$$

as before, and we define $\alpha : \boldsymbol{F}^{\%}(X) \to \boldsymbol{F}(X)$ as before. Then we observe that $\boldsymbol{F}^{\%}(X)$ has a natural filtration:

$$\boldsymbol{F}^{\%}(X) = \bigcup_k \boldsymbol{F}^{\%}(X^k)$$

where X^k is the k–skeleton. Applying the homotopy invariance of $\boldsymbol{F}$ to the constant map from a simplex to a point, one finds that

$$\boldsymbol{F}^{\%}(X^k)/\boldsymbol{F}^{\%}(X^{k-1}) \simeq \bigvee_z \mathbb{S}^k \wedge \boldsymbol{F}(*)$$

where z runs over the k–simplices of X. Hence the natural filtration of $\boldsymbol{F}^{\%}(X)$ leads to a spectral sequence converging to the homotopy groups of $\boldsymbol{F}^{\%}(X)$, with

$$E^2_{p,q} = H_p(X; \pi_q \boldsymbol{F}(*))$$

as E^2–term. But if the E^2–term is already homotopy invariant, then so is the E^∞–term, which implies the homotopy invariance of $\boldsymbol{F}^{\%}$. Also, we see that $\alpha : \boldsymbol{F}^{\%}(X) \to \boldsymbol{F}(X)$ is a homotopy equivalence for $X = *$. Further, we see that the functor

$$X \mapsto \boldsymbol{F}^{\%}(X^k)/\boldsymbol{F}^{\%}(X^{k-1})$$

takes squares of simplicial sets of the form

$$\begin{array}{ccc} X_1 \cap Y_2 & \longrightarrow & X_1 \\ \big\downarrow & & \big\downarrow \\ X_2 & \longrightarrow & X_1 \cup X_2 \end{array}$$

to homotopy pushout squares, and preserves arbitrary coproducts (up to homotopy equivalence). Using induction on k, we conclude that the functors

$$X \mapsto \boldsymbol{F}^{\%}(X^k)$$

have these properties, too ; then $\boldsymbol{F}^{\%}$ itself has these properties. Together with homotopy invariance this implies that $\boldsymbol{F}^{\%}$ is strongly excisive. $\square$

1.2. Observation. *$\boldsymbol{F}^{\%}(X)$ is naturally homotopy equivalent to $X_+ \wedge \boldsymbol{F}(*)$.*

This is clear from the first proof of 1.1. We have not included it in Theorem 1.1 because it does not generalize well, as we shall see. In fact, our first proof does not generalize well ; that is why we have a second proof.

1.3. Observation. *If $\boldsymbol{F}$ is already excisive, then $\alpha : \boldsymbol{F}^{\%}(X) \to \boldsymbol{F}(X)$ is a homotopy equivalence for any X which is homotopy equivalent to a compact CW–space. If $\boldsymbol{F}$ is strongly excisive, then α is a homotopy equivalence for all X.*

Proof. By arguments going back to Eilenberg and Steenrod it is sufficient to verify that α is a homotopy equivalence for $X = *$. $\square$

We want to show that $\alpha = \alpha_{\boldsymbol{F}}$ is the "universal" approximation (from the left) of $\boldsymbol{F}$ by a strongly excisive homotopy invariant functor. Suppose therefore that

$$\beta : \boldsymbol{E} \longrightarrow \boldsymbol{F}$$

is another natural transformation with strongly excisive and homotopy invariant $\boldsymbol{E}$. The commutative square

$$\begin{array}{ccc} \boldsymbol{E}^{\%} & \xrightarrow{\alpha_{\boldsymbol{E}}} & \boldsymbol{E} \\ \big\downarrow{\scriptstyle \beta^{\%}} & & \big\downarrow{\scriptstyle \beta} \\ \boldsymbol{F}^{\%} & \xrightarrow{\alpha_{\boldsymbol{F}}} & \boldsymbol{F} \end{array}$$

in which the upper horizontal arrow is a homotopy equivalence by 1.3, shows that β essentially factors through $\alpha_{\boldsymbol{F}}$. Note that if $\beta : \boldsymbol{E}(*) \to \boldsymbol{F}(*)$ happens to be a homotopy equivalence, then $\alpha_{\boldsymbol{E}} : \boldsymbol{E}^{\%}(X) \to \boldsymbol{E}(X)$ and $\beta^{\%} : \boldsymbol{E}^{\%}(X) \to \boldsymbol{F}^{\%}(X)$ are homotopy equivalences for all X, by the usual Eilenberg–Steenrod arguments.

Applications. Carlsson and Pedersen [CaPe] have used this "universal" approximation property to identify their *forget control* map with the assembly map for linear algebraic K-theory. Similarly Rosenberg [Ro] has used the "universal" approximation property to identify the Kasparov index map β with the assembly map in L-theory after localizing at odd primes. Ranicki has a construction of an assembly map for homotopy invariant functors from simplicial complexes to spectra [Ra, 12.19]. His construction may be identified with the one above by universality.

In many applications to geometry, assembly is the passage from local to global. For example, the *normal invariant* of a surgery problem $f : M \to N$ (with closed n–manifolds M and N, where $n \geq 5$, and some bundle data which we suppress) is an element in $\pi_n \boldsymbol{F}^{\%}(N)$, where $\boldsymbol{F}$ is the functor taking a space X to the L-theory spectrum $\boldsymbol{L}(\mathbb{Z}\pi_1(X))$ (details below). The normal invariant vanishes if and only if the surgery problem is bordant to another surgery problem $f_1 : M_1 \to N$ where f_1 is a *homeomorphism.* The image of the normal invariant under assembly is the *surgery obstruction* ; it vanishes if and only if the surgery problem is bordant to another surgery problem $f_1 : M_1 \to N$ where f_1 is a *homotopy equivalence.*

For another illustration, we mention the Whitehead torsion of a homotopy equivalence $f : X \to Y$ between compact euclidean neighborhood retracts. This is an element in the cokernel of $\alpha_* : \pi_1 \boldsymbol{F}^{\%}(Y) \to \pi_1 \boldsymbol{F}(Y)$, where $\boldsymbol{F}$ is the functor taking Y to the algebraic K-theory spectrum $\boldsymbol{K}(\mathbb{Z}\pi_1(Y))$ (details below). The torsion of f depends only the homotopy class of f, and it vanishes when f is a *homeomorphism.* This is of course the topological invariance of Whitehead torsion, due to Chapman. See [Ch] and [RaYa].

2. Examples

2.1. Linear K-theory. Recall that Quillen has defined a functor $\boldsymbol{K} : \mathit{Exact} \to \mathit{Spectra}$ where *Exact* is the category of exact categories. Alternatively, one can note that an exact category $\mathcal{M}$ determines a category with cofibrations and weak equivalences in the sense of Waldhausen by letting the cofibrations be the admissible monomorphisms and letting the isomorphisms be the weak equivalences. Then Waldhausen's $S_\bullet$ construction yields a functor $\boldsymbol{K}$ which is naturally homotopy equivalent to Quillen's $\boldsymbol{K}$. Let Spaces_* be the category of spaces homotopy equivalent to CW-spaces which are equipped with nondegenerate base points. Then $\boldsymbol{K}(\mathbb{Z}\pi_1(X, *))$ is a functor from Spaces_* to *Spectra.* In order to apply the construction of the assembly map from section 1 we have to show that this functor factors through the functor $\mathit{Spaces}_* \to \mathit{Spaces}$ which forgets basepoints. The point of view is due to Quinn [QA], but the language we use is that of Lück and tom Dieck, [Lü, ch. II], [tD]. See also [Mitch].

Following a suggestion of MacLane [MaL], we use the word *ringoid* to mean a small category in which all morphism sets come equipped with an abelian group structure, and composition of morphisms is bilinear. Notice that a ringoid with one object is just a ring.

Any small category $\mathcal{C}$ gives rise to a ringoid $\mathbb{Z}\mathcal{C}$ having the same objects as $\mathcal{C}$. The set of morphisms from x_0 to x_1 in $\mathbb{Z}\mathcal{C}$ is the free abelian group generated by the set of morphisms from x_0 to x_1 in $\mathcal{C}$.

In particular, taking $\mathcal{C}$ to be the fundamental *groupoid* $\pi_1(X)$ of a space X, as in [Spa], we obtain a ringoid $\mathbb{Z}\pi_1(X)$. Objects in $\mathbb{Z}\pi_1(X)$ are points of X, and a morphism from y_0 to y_1 is a finite formal linear combination $\Sigma n_g \cdot g$, where the g are path classes beginning in y_0 and ending in y_1, and the n_g are integers.

Let $\mathcal{R}$ be a ringoid. A *left $\mathcal{R}$–module* is a covariant functor from $\mathcal{R}$ to abelian groups which is homomorphic on morphism sets; a *right $\mathcal{R}$–module* is a left $\mathcal{R}^{\mathrm{op}}$–module. A left $\mathcal{R}$–module is *free on one generator* if it is representable (that is, isomorphic to a morphism functor $\hom(x, -)$ for some object x in $\mathcal{R}$). It is *finitely generated free* if it is isomorphic to a finite direct sum of representable ones, and just *free* if it is isomorphic to an arbitrary direct sum of representable ones. It is *projective* if it is a direct summand of a free one, and *finitely generated projective* if it is a direct summand of a f. g. free one.

Left $\mathcal{R}$–modules form an abelian category in which the morphisms are natural transformations. Exercise for the reader: prove that a left $\mathcal{R}$–module P is projective if and only if any $\mathcal{R}$–module epimorphism with target P splits. The subcategory $\mathcal{P}_{\mathcal{R}}$ of finitely generated projective modules is then an exact category. For a space X, let $\boldsymbol{K}(X) = \boldsymbol{K}(\mathcal{P}_{\mathcal{R}})$ where $\mathcal{R} = \mathbb{Z}\pi_1(X)$. Since a homotopy equivalence between spaces induces an equivalence between their fundamental groupoids, our functor $\boldsymbol{K}$ is a homotopy functor and section 1 yields an assembly map for linear algebraic K-theory.

2.2. ***A*-theory.** Since Waldhausen has shown that his functor $X \mapsto \boldsymbol{A}(X)$ is a homotopy functor [Wa1, Prop. 2.1.7] we can directly apply Section 1 to get an assembly map for A-theory. (We use boldface notation, $\boldsymbol{A}(X)$, for the spectrum associated with the infinite loop space $A(X)$.)

2.3. ***L*-theory.** Recall that Ranicki [Ra, Ex. 13.6] [Ra, Ex. 1.3] has defined functors

$$\mathbb{L}_\bullet : \{\text{additive categories with chain duality}\} \longrightarrow \textit{Spectra}\ ,$$
$$\{\text{rings with involution}\} \longrightarrow \{\text{additive categories with chain duality}\}\,.$$

We write $\boldsymbol{L}$ for the first functor, rather than $\mathbb{L}_\bullet$, to be consistent. The second functor sends a ring R with involution j to the triple $(\mathcal{P}_R, T, e)$ where

- $\mathcal{P}_R$ is the category of f.g. projective left R-modules;
- T is the functor $\mathcal{P}_R \to \mathcal{P}_R$ which sends a module M to $\hom_R(M, R)$ where the involution j is used to convert this right R-module to a left R-module; and
- e is the inverse to the natural equivalence $\eta : \mathrm{id} \to T^2$ that maps a module M to $T^2(M)$ by taking the adjoint of the pairing

  $$\hom_R(M, R) \times M \to R$$

  which maps (f, m) to $j(f(m))$.

If X is a space with base point $*$, then $Z\pi_1(X, *)$ is equipped with the standard involution that takes an element $g \in \pi_1(X, *)$ to g^{-1}. Thus we again get a functor $\textit{Spaces}_* \to \textit{Spectra}$ which we have to factor through the forgetful functor $\textit{Spaces}_* \to \textit{Spaces}$.

A *ringoid with involution* is a ringoid $\mathcal{R}$ together with a ringoid isomorphism

$$j : \mathcal{R} \longrightarrow \mathcal{R}^{\mathrm{op}}$$

such that the composite functor $\mathcal{R} \xrightarrow{j} \mathcal{R}^{\mathrm{op}} \xrightarrow{j^{\mathrm{op}}} \mathcal{R}$ is the identity. Notice that a ringoid with involution, with one object, is just a ring with involution.

For any space X, the ringoid $\mathbb{Z}\pi_1(X)$ has a standard involution. The involution is trivial on objects, and maps $\sum n_g g : x_0 \longrightarrow x_1$ (a typical morphism) to

$$\sum n_g g^{-1} : x_1 \longrightarrow x_0.$$

Thus we are done if we can show Ranicki's functor

$$\{\text{rings with involution}\} \to \{\text{additive categories with chain duality}\}$$

factors through the category of ringoids with involution.

Henceforth we assume the ringoid $\mathcal{R}$ comes equipped with an involution j. Then a left $\mathcal{R}$–module P can also be regarded as a right $\mathcal{R}$–module P^t (compose with $j^{-1} = j^{\mathrm{op}}$). Similarly a right $\mathcal{R}$–module P can also be regarded as a left $\mathcal{R}$–module.

Notice that for any object x in $\mathcal{R}$, the functor $\hom_{\mathcal{R}}(x, -)$ is a left $\mathcal{R}$-module, and $\hom_{\mathcal{R}}(-, x)$ is a right $\mathcal{R}$-module. For any two left $\mathcal{R}$-modules, M and N, let $\mathrm{HOM}_{\mathcal{R}}(M, N)$ be the abelian group of natural transformations from M to N.

For any left $\mathcal{R}$-module M, consider the contravariant functor from $\mathcal{R}$ to abelian groups which sends an object x to $\mathrm{HOM}_{\mathcal{R}}(M, \hom_{\mathcal{R}}(x, -))$. We let $T(M)$ be the left module obtained by using j to make this functor covariant. Notice that if $M = \hom_{\mathcal{R}}(y, -)$, then the Yoneda lemma implies $T(M)$ is just $\hom_{\mathcal{R}}(-, y)$ converted into a left module via j. Explicitly, $T(M)(x) \cong \hom(j(x), y) \cong \hom(j(y), x)$. Thus T sends f.g. free modules to f.g. free modules and f.g. projective modules to f.g. projective modules.

Fix a left $\mathcal{R}$-module M. Then for any pair of objects x and y in $\mathcal{R}$, we get the pairing

$$\mathrm{HOM}_{\mathcal{R}}\big(M, \hom_{\mathcal{R}}(y, -)\big) \times M(x) \to \hom_{\mathcal{R}}(x, y)$$

which sends (f, m) to $j(f(m))$. The adjoints of these pairings determine a natural transformation from the identity functor to T^2. If we restrict this natural transformation to the category of f.g. projective modules it is a natural equivalence, and we let e be the inverse natural transformation. Then $(\mathcal{P}_{\mathcal{R}}, T, e)$ is an additive category with (0-dimensional) chain duality and we are done.

2.4. The Novikov Conjecture. The Novikov conjecture, for a homotopy invariant functor $\boldsymbol{F}$ from spaces to spectra and a discrete group π, is the hypothesis that

$$\alpha_* : \pi_* \boldsymbol{F}^{\%}(B\pi) \otimes \mathbb{Q} \longrightarrow \pi_* \boldsymbol{F}(B\pi) \otimes \mathbb{Q}$$

is injective. It was originally formulated by Novikov for the L-theory functor, 2.3 above, and for all groups. The L-theory Novikov conjecture has been verified for many groups with a finite dimensional classifying space. See [RaNo] for details. Bökstedt, Hsiang and Madsen [BHM] proved the Novikov conjecture for the algebraic K-theory functor, 2.1 above, and all groups π such that $H_i(B\pi; \mathbb{Z})$ is finitely generated for all i.

3. Easy Variations

3.1. Variation. We can still do assembly when the functor $\boldsymbol{F}$ is defined on the category of spaces *over* a reference space B. (For example, B could be BG, the classifying space for stable spherical fibrations.) By abuse of notation, a map between spaces over B is a *homotopy equivalence* if it becomes a homotopy equivalence when the reference maps to B are omitted. A square of spaces over B is a *homotopy pushout square* if it becomes a homotopy pushout square when the reference maps are omitted. We call $\boldsymbol{F}$ homotopy invariant if it takes homotopy equivalences (over B) to homotopy equivalences. We call a homotopy invariant $\boldsymbol{F}$ *excisive* if it takes the empty set to a contractible spectrum and if it takes homotopy pushout squares (over B) to homotopy pushout squares. We call it *strongly excisive* if in addition it preserves arbitrary coproducts up to homotopy equivalence. — For any homotopy invariant $\boldsymbol{F}$ defined on spaces over B we have

$$\alpha : \boldsymbol{F}^{\%} \longrightarrow \boldsymbol{F}\,,$$

natural in $\boldsymbol{F}$, where $\boldsymbol{F}^{\%}$ is homotopy invariant, strongly excisive and

$$\alpha : \boldsymbol{F}^{\%}(* \hookrightarrow B) \quad \longrightarrow \quad \boldsymbol{F}(* \hookrightarrow B)$$

is a homotopy equivalence for any point $*$ in B. If $\boldsymbol{F}$ is already strongly excisive, then α is a homotopy equivalence for all spaces over B. Prove this using the methods developed in the second proof of 1.1.

Example: Classical twisted L-theory. Let $B = K(\mathbb{Z}/2, 1)$. Then a map $X \to B$ determines a double covering $w : X^{\natural} \to X$. Unfortunately w does not, as one might expect, determine an involution on the ringoid $\mathbb{Z}\pi_1(X)$. But it does determine an involution on an equivalent category $\mathbb{Z}^w\pi_1(X)$. The objects of $\mathbb{Z}^w\pi_1(X)$ are the points of $X^{\natural}$, not X ; a morphism from x_0 to x_1 in $\mathbb{Z}^w\pi_1(X)$ is the same as a morphism from $w(x_0)$ to $w(x_1)$ in $\mathbb{Z}\pi_1(X)$. The involution is trivial on objects, and maps $\sum n_g g : x_0 \longrightarrow x_1$ (a typical morphism) to

$$\sum \mathrm{sign}(g) \cdot n_g g^{-1} : x_1 \longrightarrow x_0,$$

where the *sign* of a path class g from $w(x_0)$ to $w(x_1)$ is $+1$ if g lifts to a path class from x_0 to x_1 in $X^{\natural}$, and -1 otherwise. — Refining 2.3 we let $\boldsymbol{L}(X \to B)$ be the L-theory spectrum of the ringoid with involution $\mathbb{Z}^w\pi_1(X)$.

Example: Tate Cohomology and the Ξ transformation. Let $B = BG$, the classifying space for stable spherical fibrations. Any map $X \to B$ determines an action of $\mathbb{Z}/2$ on a spectrum $\boldsymbol{A}(X \to B)$ which is homotopy equivalent to Waldhausen's A-theory spectrum $\boldsymbol{A}(X)$. See [Vog3] and [WW2]. Thus we can consider the functor sending X to the Tate cohomology spectrum

$$\widehat{\boldsymbol{H}}^{\blacktriangledown}(\mathbb{Z}/2; \boldsymbol{A}(X \to B))$$

(see [WW2] for details). In [WW2] we construct a natural transformation

$$\Xi : \boldsymbol{L}(X \xrightarrow{c_1} B_1) \quad \longrightarrow \quad \widehat{\boldsymbol{H}}^{\blacktriangledown}(\mathbb{Z}/2; \boldsymbol{A}(X \xrightarrow{c} B))$$

where c_1 is the composition of c with the Postnikov projection $B \to B_1 = K(\mathbb{Z}/2, 1)$. Together with the appropriate assembly maps, Ξ is used to study automorphisms of manifolds. See [WW1], [WW3] for the manifolds.

3.2. Example. Let $\mathcal{G}$ be a topological group with classifying space $B\mathcal{G}$, and suppose that $\mathcal{G}$ acts on a spectrum $\boldsymbol{T}$. For a space over $B\mathcal{G}$, say $f : X \to B\mathcal{G}$, let X^f be the pullback of

$$X \xrightarrow{f} B\mathcal{G} \leftarrow E\mathcal{G}.$$

The functor from spaces over $B\mathcal{G}$ to spectra given by

$$(f : X \to B\mathcal{G}) \quad \mapsto \quad X^f_+ \wedge_{\mathcal{G}} \boldsymbol{T}$$

is strongly excisive. (The example is "typical", but we shall not go into details.)

3.3. Variation. There is a variant of assembly which applies to functors defined on *pairs* of spaces. Let $\boldsymbol{F}$ be such a functor, from pairs (X,Y) to spectra. We call $\boldsymbol{F}$ *homotopy invariant* if it takes homotopy equivalences of pairs to homotopy equivalences. We call a homotopy invariant $\boldsymbol{F}$ *excisive* if it takes the empty pair to a contractible spectrum, and if it takes homotopy pushout squares of pairs to homotopy pushout squares. (A square of pairs

$$\begin{array}{ccc} (X_1,Y_1) & \longrightarrow & (X_2,Y_2) \\ \big\downarrow & & \big\downarrow \\ (X_3,Y_3) & \longrightarrow & (X_4,Y_4) \end{array}$$

is a homotopy pushout square if the two squares made from the X_i and the Y_i, respectively, are homotopy pushout squares.) Finally $\boldsymbol{F}$ is *strongly excisive* if it is excisive and respects arbitrary coproducts, up to homotopy equivalence. — For any homotopy invariant $\boldsymbol{F}$ from pairs of spaces to spectra, there exist a strongly excisive (and homotopy invariant) $\boldsymbol{F}^{\%}$ from pairs of spaces to spectra and a natural transformation

$$\alpha = \alpha_{\boldsymbol{F}} : \boldsymbol{F}^{\%} \longrightarrow \boldsymbol{F}$$

such that

$$\alpha : \boldsymbol{F}^{\%}(*,\emptyset) \to \boldsymbol{F}(*,\emptyset)\,, \qquad \alpha : \boldsymbol{F}^{\%}(*,*) \longrightarrow \boldsymbol{F}(*,*)$$

are homotopy equivalences. Moreover, $\boldsymbol{F}^{\%}$ and $\alpha_{\boldsymbol{F}}$ can be made to depend functorially on $\boldsymbol{F}$. If $\boldsymbol{F}$ is already strongly excisive, then α is a homotopy equivalence for every pair (X,Y). Here is a brief description of $\boldsymbol{F}^{\%}$: For a pair (X,Y) we have $\mathrm{simp}(Y) \subset \mathrm{simp}(X)$, and we define $\boldsymbol{F}^{\%}(X,Y)$ as the homotopy pushout (double mapping cylinder) of

$$\operatorname*{hocolim}_{g:\Delta^n\to X} \boldsymbol{F}(\Delta^n,\emptyset) \leftarrow \operatorname*{hocolim}_{g:\Delta^n\to Y} \boldsymbol{F}(\Delta^n,\emptyset) \to \operatorname*{hocolim}_{g:\Delta^n\to Y} \boldsymbol{F}(\Delta^n,\Delta^n)$$

where the homotopy colimits are to be taken over $\mathrm{simp}(X)$, $\mathrm{simp}(Y)$ and $\mathrm{simp}(Y)$, respectively.

3.4. Remark. Let $\boldsymbol{T}$ be a spectrum ; then the functor

$$X \mapsto X_+ \wedge \boldsymbol{T}$$

is homotopy invariant and strongly excisive. Any homotopy invariant and strongly excisive functor $\boldsymbol{F}$ from spaces to spectra has this form, up to a chain of natural homotopy equivalences (observations 1.2 and 1.3). The appropriate $\boldsymbol{T}$ is of course $\boldsymbol{F}(*)$. Next, let $f : \boldsymbol{T}_1 \to \boldsymbol{T}_2$ be a map of spectra. Then the functor

$$(X,Y) \mapsto \text{ homotopy pushout of } \Big(Y_+ \wedge \boldsymbol{T}_2 \xleftarrow{f_*} Y_+ \wedge \boldsymbol{T}_1 \hookrightarrow X_+ \wedge \boldsymbol{T}_1 \Big)$$

is strongly excisive. Any strongly excisive functor $\boldsymbol{F}$ from pairs of spaces to spectra has this form, up to a chain of natural homotopy equivalences. The appropriate $\boldsymbol{T}_1$ is $\boldsymbol{F}(*,\emptyset)$, the appropriate $\boldsymbol{T}_2$ is $\boldsymbol{F}(*,*)$, and the appropriate f is induced by the inclusion of $(*,\emptyset)$ in $(*,*)$.

It follows that a strongly excisive $\boldsymbol{F}$ defined on pairs need not take every collapse map $(X,Y) \to (X/Y,*)$ to a homotopy equivalence. It does, however, if $\boldsymbol{F}(*,*)$ is contractible ; then $\boldsymbol{F}$ has the form $(X,Y) \mapsto (X/Y) \wedge \boldsymbol{F}(*,\emptyset)$ up to a chain of natural homotopy equivalences.

Equivariant versions of assembly are currently being developed by J. Davis and W. Lück [DaLü].

4. Assembly with Control

For the purposes of this section, a *control space* is a pair of spaces $(\bar{X}, X)$ where $\bar{X}$ is compact Hausdorff, X is open dense in $\bar{X}$, and X is an ENR. Informally, the set $\bar{X} \smallsetminus X$ is the *singular set*, whereas X is the *nonsingular set.* A *morphism* of control spaces is a continuous map of pairs $f : (\bar{X}, X) \to (\bar{Y}, Y)$ such that $f^{-1}(Y) = X$.

It seems that the use of *control* in topology began with Connell and Hollingsworth [CoHo]. For a survey of applications until 1986, see [QLA]. Through the influence of [Q], controlled topology led to *bounded algebra* and *controlled algebra*, [QA], [PW1], [PW2], [ACFP], and a plethora of functors from control spaces to spectra. Most of these have some homotopy invariance properties, i. e., they take homotopy equivalences to homotopy equivalences ; some of them also have excision properties [PW1], [PW2] [Vog1], [Vog2]. For applications, see also [CaPe] and [DWW], and many others.

Our goal here is roughly the following. Suppose that $\boldsymbol{F}$ is a homotopy invariant functor (details follow) from control spaces to spectra. We want to construct another functor $\boldsymbol{F}^{\%}$ from control spaces to spectra, homotopy invariant and excisive (details follow), and a natural transformation

$$\alpha : \boldsymbol{F}^{\%}(\bar{X}, X) \longrightarrow \boldsymbol{F}(\bar{X}, X)$$

which is a homotopy equivalence for $(\bar{X}, X) = (*, *)$. Moreover we would like to say that $\boldsymbol{F}^{\%}(\bar{X}, X)$ is related to $X^{\bullet} \wedge \boldsymbol{F}(*, *)$ by a chain of (weak) homotopy equivalences. Here $X^{\bullet}$ is the one–point compactification, usually not homotopy equivalent to a CW–space, so that $X^{\bullet} \wedge \boldsymbol{F}(*, *)$ is usually not homotopy equivalent to a CW–spectrum. (Hence we must allow *weak* homotopy equivalences in the chain.)

4.1. Terminology. Two morphisms $f_0, f_1 : (\bar{X}, X) \to (\bar{Y}, Y)$ between control spaces are *homotopic* if they agree on $\bar{X} \smallsetminus X$ and if they extend to a continuous one–parameter family of morphisms $f_t : (\bar{X}, X) \to (\bar{Y}, Y)$, where $0 \le t \le 1$, and all f_t agree on $\bar{X} \smallsetminus X$. A morphism $f : (\bar{X}, X) \to (\bar{Y}, Y)$ is a *homotopy equivalence* if there exists another morphism $g : (\bar{Y}, Y) \to (\bar{X}, X)$ such that gf and fg are homotopic to the identity. Note that a homotopy equivalence restricts to a *homeomorphism* of the singular sets.

A commutative square in the category of control spaces is a *homotopy pushout square* if the underlying square of nonsingular sets is a proper homotopy pushout square (details follow) in the category of locally compact spaces. *Details*: Recall that a map between locally compact spaces is *proper* if it extends to a continuous map between their one–point compactifications. A commutative square of locally compact spaces and proper maps

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \big\downarrow & & \big\downarrow \\ X_3 & \longrightarrow & X_4 \end{array}$$

is a *proper homotopy pushout square* if the resulting proper map from the homotopy pushout of $X_3 \leftarrow X_1 \to X_2$ to X_4 is a proper homotopy equivalence.

4.2. More terminology. A covariant functor $\boldsymbol{F}$ from control spaces to CW–spectra is *homotopy invariant* if it takes homotopy equivalences to homotopy equivalences. A homotopy invariant $\boldsymbol{F}$ is *excisive* if it takes homotopy pushout squares of control spaces to homotopy pushout squares of spectra, and $\boldsymbol{F}(\emptyset, \emptyset)$ is contractible.

Suppose that $\boldsymbol{F}$ is homotopy invariant and excisive, and let $(\bar{X}, X)$ be a control space with *discrete* but possibly infinite X. For any $y \in X$, we have a homotopy equivalence

$$\boldsymbol{F}(\bar{X} \smallsetminus y, X \smallsetminus y) \vee \boldsymbol{F}(y, y) \longrightarrow \boldsymbol{F}(\bar{X}, X)$$

by excision, and hence a projection $\boldsymbol{F}(\bar{X}, X) \to \boldsymbol{F}(y, y)$, well defined up to homotopy. We call $\boldsymbol{F}$ *pro–excisive* if these projection maps induce an isomorphism

$$\pi_n \boldsymbol{F}(\bar{X}, X) \longrightarrow \prod_{y \in X} \pi_n \boldsymbol{F}(y, y) \qquad (n \in \mathbb{Z}).$$

In the following example, let $(\text{—})_{CW}$ be the standard CW approximation procedure replacing arbitrary spectra by CW–spectra. In detail, if $\boldsymbol{Y} = \{Y_n \mid n \in \mathbb{Z}\}$ is a spectrum with structure maps $\Sigma Y_n \to Y_{n+1}$, then the geometric realizations of the singular simplicial sets of the Y_n form a CW–spectrum $(\boldsymbol{Y})_{CW}$. Note that the functor π_* does not distinguish between $\boldsymbol{Y}$ and $(\boldsymbol{Y})_{CW}$.

Example. The functor $(\bar{X}, X) \mapsto (X^\bullet \wedge \boldsymbol{S}^0)_{CW}$ is homotopy invariant and pro–excisive. Here $\boldsymbol{S}^0$ is the sphere spectrum. *Proof:* Transversality and Thom–Pontryagin construction lead to an interpretation of $\pi_n(X^\bullet \wedge \boldsymbol{S}^0) = \pi_n^s(X^\bullet)$ as the bordism group of stably framed smooth n–manifolds equipped with a *proper* map to X. This in turn leads to Mayer–Vietoris sequences from homotopy pushout squares of control spaces. Excision follows, and then pro–excision is clear. *Warning:* Be sure to use the correct topology on $X^\bullet$. Note that X could be any ENR, such as the universal cover of a wedge of two circles, or a countably infinite discrete set.

4.3. Proposition. *Suppose that $\boldsymbol{Y}$ is a CW–spectrum. Suppose also that $\boldsymbol{Y}$ is an Ω–spectrum (details below), or the suspension spectrum of a CW–space. Then the functor $(\bar{X}, X) \mapsto (X^\bullet \wedge \boldsymbol{Y})_{CW}$ is homotopy invariant and pro–excisive.*

Proof. First suppose that $\boldsymbol{Y}$ is a suspension spectrum $\Sigma^\infty Y_0$. If the CW–space Y_0 is finite–dimensional, then we can use the preceding example and induction on the dimension of Y_0 to prove that $(\bar{X}, X) \mapsto (X^\bullet \wedge \boldsymbol{Y})_{CW}$ is homotopy invariant and pro–excisive. If Y_0 has infinite dimension, we reduce to the finite dimensional case by observing that

$$\pi_n(X^\bullet \wedge \boldsymbol{Y}) \cong \pi_n^s(X^\bullet \wedge Y_0) \cong \pi_n^s(X^\bullet \wedge Y_0^{n+1})$$

where Y_0^{n+1} is the $(n+1)$–skeleton of Y_0.

Now suppose that $\boldsymbol{Y}$ is an arbitrary CW–spectrum. Then

$$\pi_n(X^\bullet \wedge \boldsymbol{Y}) := \operatorname*{colim}_k \pi_{n+k}(X^\bullet \wedge Y_k) \cong \operatorname*{colim}_k \pi_{n+k}(X^\bullet \wedge \Sigma^\infty Y_k).$$

Using the suspension spectrum case of 4.3, which we have established, we deduce immediately that the functor $(\bar{X}, X) \mapsto (X^\bullet \wedge \boldsymbol{Y})_{CW}$ is homotopy invariant and excisive. Furthermore, for a control space $(\bar{X}, X)$ with discrete X, we have

$$\pi_n(X^\bullet \wedge \boldsymbol{Y}) = \operatorname*{colim}_k \pi_{n+k}(X^\bullet \wedge Y_k) \cong \operatorname*{colim}_k \prod_{x \in X} \pi_{n+k}(Y_k).$$

Here we want to exchange direct limit and product to get

$$\prod_{x\in X} \operatorname*{colim}_k \pi_{n+k}(Y_k) \cong \prod_{x\in X} \pi_n(\boldsymbol{Y})\,.$$

In general this is not permitted. But it is clearly permitted if $\boldsymbol{Y}$ is an Ω–spectrum—the adjoints of the structure maps $\Sigma Y_k \to Y_{k+1}$ are homotopy equivalences $Y_k \to \Omega Y_{k+1}$. $\square$

4.4. Theorem. *Suppose that $\boldsymbol{F}$ is a homotopy invariant functor from control spaces to CW–spectra. Suppose also that $\boldsymbol{F}$ behaves like a pro–excisive functor on the category of control spaces $(\bar{X}, X)$ with discrete X (details follow). Then there exists a pro–excisive functor $\boldsymbol{F}^{\%}$ from control spaces to CW–spectra, and a natural transformation $\alpha = \alpha_{\boldsymbol{F}} : \boldsymbol{F}^{\%} \to \boldsymbol{F}$ such that*

$$\alpha : \boldsymbol{F}^{\%}(*,*) \to \boldsymbol{F}(*,*)$$

is a homotopy equivalence. The construction can be made natural in $\boldsymbol{F}$.

Details. The extra hypothesis on $\boldsymbol{F}$ means that $\boldsymbol{F}$ takes a homotopy pushout square of control spaces with *discrete* nonsingular sets to a homotopy pushout square of spectra, and that, for any $(\bar{X}, X)$ with discrete X, the homomorphisms

$$\pi_n \boldsymbol{F}(\bar{X}, X) \longrightarrow \prod_{y\in X} \pi_n \boldsymbol{F}(y, y)$$

(defined as in 4.2) are isomorphisms. Carlsson [Car] has shown that functors of type "controlled algebraic K-theory" satisfy this condition. (Carlsson seems to have been the first to realize that this requires proof.)

4.5. Construction. The following *teardrop construction* will be needed in the proof of 4.4. Let $f : X \to Y$ be a proper map of ENR's, where Y is the nonsingular set of a control space $(\bar{Y}, Y)$. We note that the diagram of control spaces

$$(\bar{Y}, Y) \xrightarrow{\text{collapse}} (Y^{\bullet}, Y) \xleftarrow{f} (X^{\bullet}, X)$$

has a limit (=pullback) in the category of control spaces ; its nonsingular set is canonically identified with X, and we denote it by $(\bar{X}, X)$.

4.6. Notation. Suppose that X is the geometric realization of an incomplete simplicial set (simplicial set without degeneracies). Then

- X_n is the set of n–simplices in X.
- X^n is the n–skeleton.
- For each monotone injection $f : [m] \to [n]$, we write X_f to mean $\Delta^m \times X_n$. There is a *characteristic map* from X_f to X, via $\Delta^n \times X_n$. Note that this depends on f, not just on m and n. When f equals $\mathrm{id} : [n] \to [n]$, we write $X_{[n]}$ instead of X_f.

Proof of 4.4. Let $\mathcal{C}$ be the category of all control spaces. A key observation is that $\boldsymbol{F}$ is sufficiently determined by its restriction to a certain subcategory $\mathcal{C}'$, which we now describe. An object in $\mathcal{C}'$ is a control space $(\bar{X}, X)$ where X is the geometric realization of an incomplete simplicial set. Then X is a CW–space, and we require additionally that X have *small cells*, which means the following: For every $z \in \bar{X} \smallsetminus X$ and neighbourhood U of z in $\bar{X}$, there exists another neighborhood W of z in $\bar{X}$ such that any (open) cell of X intersecting W is contained in U. Note also that since X is an ENR, the underlying incomplete simplicial set must be locally finite, finite dimensional and countably generated. A morphism in $\mathcal{C}'$, say from $(\bar{X}, X)$ to $(\bar{Y}, Y)$, is a morphism of control spaces whose restriction to nonsingular sets is given by a simplicial map. Note that any finite diagram (=finitely generated simplicial subset of the nerve) in $\mathcal{C}'$ has a colimit.

The standard way to attempt recovery of a functor from its restriction to a subcategory is by *Kan extension*, here: *homotopy Kan extension.* Hence the following claim: for every $(\bar{Y}, Y)$ in $\mathcal{C}$, the canonical map

$$\operatorname*{hocolim}_{\substack{(\bar{X},X)\to(\bar{Y},Y)\\ (\bar{X},X) \text{ in } \mathcal{C}'}} \boldsymbol{F}(\bar{X}, X) \quad \longrightarrow \quad \boldsymbol{F}(\bar{Y}, Y)$$

is a homotopy equivalence. The homotopy colimit is taken over the category whose objects are objects in $\mathcal{C}'$ with a reference morphism to $(\bar{Y}, Y)$, and whose morphisms are morphisms in $\mathcal{C}'$, over $(\bar{Y}, Y)$. We denote this category by $(\mathcal{C}' \downarrow (\bar{Y}, Y))$.

To prove this claim, we observe that the canonical map in question is a natural transformation of functors in the variable $(\bar{Y}, Y)$. Since every $(\bar{Y}, Y)$ in $\mathcal{C}$ is a retract of some object $(\bar{X}, X)$ in $\mathcal{C}'$ (with a retraction morphism $(\bar{X}, X) \to (\bar{X}, X)$ which need not belong to $\mathcal{C}'$), it is enough to check the claim when $(\bar{Y}, Y)$ is already in $\mathcal{C}'$. Since any finite diagram in $(\mathcal{C}' \downarrow (\bar{Y}, Y))$ has a colimit, we have, almost from the definition,

$$\operatorname*{colim}_{(\bar{X},X)\to(\bar{Y},Y)} \pi_* \boldsymbol{F}(\bar{X}, X) \quad \xrightarrow{\cong} \quad \pi_*\big(\operatorname*{hocolim}_{(\bar{X},X)\to(\bar{Y},Y)} \boldsymbol{F}(\bar{X}, X)\big)\,.$$

Hence our claim is proved if we can show that the canonical homomorphism

$$\operatorname*{colim}_{(\bar{X},X)\to(\bar{Y},Y)} \pi_* \boldsymbol{F}(\bar{X}, X) \quad \longrightarrow \quad \boldsymbol{F}(\bar{Y}, Y)$$

is an isomorphism. But this is obvious. We conclude that homotopy invariant functors on $\mathcal{C}$ are sufficiently determined by, and can be recovered from, their restriction to $\mathcal{C}'$. From now on we regard 4.4 as a statement about functors on $\mathcal{C}'$.

For $(\bar{X}, X)$ in $\mathcal{C}'$ and a monotone injection $f : [m] \to [n]$, we have the characteristic map $X_f \to X$ which we can use to compactify X_f (teardrop). This compactification is understood in the following definition:

$$\boldsymbol{F}^{\%}(\bar{X}, X) := \operatorname*{hocolim}_{f} \boldsymbol{F}(\bar{X}_f, X_f)\,.$$

The homotopy colimit is taken over the category whose objects are monotone injections $f : [m] \to [n]$, with arbitrary $m, n \geq 0$; a morphism from f to g is a commutative square of monotone injections

$$\begin{tikzcd} {[m]} \arrow[r, "f"] \arrow[d] & {[n]} \\ {[p]} \arrow[r, "g"] & {[q]}\,. \arrow[u] \end{tikzcd}$$

We can now proceed as in the second proof of 1.1. The filtration of X by skeletons X^k leads to a filtration of $\boldsymbol{F}^{\%}(\bar{X},X)$ by subspectra $\boldsymbol{F}^{\%}(\bar{X}^k,X^k)$. Here another teardrop construction is understood. By inspection,

$$\boldsymbol{F}^{\%}(\bar{X}^k,X^k)/\boldsymbol{F}^{\%}(\bar{X}^{k-1},X^{k-1}) \quad\simeq\quad \mathbb{S}^k\wedge\boldsymbol{F}(\bar{X}_{[k]},X_{[k]})\,.$$

From our extra hypothesis on $\boldsymbol{F}$, we then get isomorphisms

$$\pi_n\big(\boldsymbol{F}^{\%}(\bar{X}^k,X^k)/\boldsymbol{F}^{\%}(\bar{X}^{k-1},X^{k-1})\big) \quad\simeq\quad \prod_{x\in X_k}\pi_{n-k}\boldsymbol{F}(*,*)$$

for $n\in\mathbb{Z}$, and this shows immediately that $\boldsymbol{F}^{\%}$ is homotopy invariant and excisive, and even pro–excisive. (Imitate the second proof of 1.1 ; use homology with locally finite coefficients to describe the E^2–term of the appropriate spectral sequence converging to $\pi_*\boldsymbol{F}^{\%}(\bar{X},X)$.) Finally the assembly map

$$\alpha:\boldsymbol{F}^{\%}(\bar{X},X)\longrightarrow\boldsymbol{F}(\bar{X},X)$$

is obvious, and it is an isomorphism when $(\bar{X},X)=(*,*)$. $\square$

4.7. Observation. *If $\boldsymbol{F}$ in 4.4. is already pro–excisive, then the assembly α from $\boldsymbol{F}^{\%}(\bar{X},X)$ to $\boldsymbol{F}(\bar{X},X)$ is a homotopy equivalence for every $(\bar{X},X)$.*

Proof. Fix $\boldsymbol{F}$, homotopy invariant and pro–excisive. We lose nothing by restricting $\boldsymbol{F}$ to $\mathcal{C}'$ (see proof of 4.4). When $(\bar{X},X)=(*,*)$, the assembly α is an isomorphism by 4.4. By pro–excision, assembly is then an isomorphism for any $(\bar{X},X)$ where X is discrete. For arbitrary $(\bar{X},X)$ in $\mathcal{C}'$, we can argue by induction on skeletons: X is the strict and homotopy pushout of a diagram

$$X^{k-1}\leftarrow\partial\Delta^k\times X_k\xrightarrow{\subset}\Delta^k\times X_k\,.$$

Each of the spaces in this diagram has a canonical (teardrop) compactification ; two of the spaces in the diagram have dimension $<k$, the third is homotopy equivalent (with control) to a discrete space. Note that we use the condition on small cells at this point. $\square$

4.8. Corollary. *If $\boldsymbol{F}$ in 4.4. is pro–excisive then there exists a chain of natural weak homotopy equivalences*

$$\boldsymbol{F}(\bar{X},X)\simeq\ \dots\ \simeq X^{\bullet}\wedge\boldsymbol{F}(*,*)_{\Omega}$$

where $\boldsymbol{F}(*,*)_{\Omega}$ is an Ω–spectrum envelope of $\boldsymbol{F}(*,*)$.

Proof. We may restrict to $\mathcal{C}'$. We may also assume $\boldsymbol{F}$ is a functor from control spaces to CW–Ω–spectra. Here it is understood that the morphisms in the category of CW–Ω–spectra are *functions*, not *maps*, in the language of [Ad, III§2]. Reason for making this technical assumption: the category of CW–Ω–spectra has arbitrary and well-behaved products whereas the category of CW–spectra does not. Writing

$\langle\simeq$ and $\simeq\rangle$ for weak homotopy equivalences going in the direction indicated, we have

$$\begin{aligned}
\boldsymbol{F}(\bar{X}, X) &\langle\simeq \boldsymbol{F}^{\%}(\bar{X}, X) \\
&= \operatorname*{hocolim}_{f} \boldsymbol{F}(\bar{X}_f, X_f) \\
&\simeq\rangle \operatorname*{hocolim}_{f:[m]\to[n]} \boldsymbol{F}(\bar{X}_n, X_n) \\
&\simeq\rangle \operatorname*{hocolim}_{f:[m]\to[n]} \prod_{y\in X_n} \text{ cofiber } \big[\boldsymbol{F}(\bar{X}_n \smallsetminus y, X_n \smallsetminus y) \longrightarrow \boldsymbol{F}(\bar{X}_n, X_n)\big] \\
&\langle\simeq \operatorname*{hocolim}_{f:[m]\to[n]} \prod_{y\in X_n} \boldsymbol{F}(y, y) \\
&\cong \operatorname*{hocolim}_{f:[m]\to[n]} \prod_{y\in X_n} \boldsymbol{F}(*, *)\,.
\end{aligned}$$

The first $\simeq\rangle$ is induced by the projections $p_f : X_f \to X_n$, for f from $[m]$ to $[n]$, where X_n must be compactified in such a way that p_f extends to a morphism of control spaces restricting to a homeomorphism of the singular sets. Again, this uses the small cells condition. The second of the weak homotopy equivalences labelled $\langle\simeq$ is an inclusion, and it is a weak homotopy equivalence by excision.

We conclude that a homotopy invariant and pro–excisive functor $\boldsymbol{F}$ on $\mathcal{C}'$ is determined, up to a chain of weak homotopy equivalences, by what it does to the control space $(*, *)$. Hence such an $\boldsymbol{F}$ is related by a chain of natural weak homotopy equivalences to the functor

$$(\bar{X}, X) \mapsto X^{\bullet} \wedge \boldsymbol{F}(*, *)_{\Omega}$$

whose CW–approximation is homotopy invariant and pro–excisive by 4.3.

References

[ACFP]: D.R. Anderson, F.X. Connolly, S. Ferry and E.K. Pedersen, *Algebraic K-theory with continuous control at infinity*, J. Pure and Appl. Algebra **94** (1994), 25–47.

[Ad]: J.F.Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Math., The University of Chicago Press, Chicago and London, 1974.

[And]: D.W. Anderson, *Chain functors and homology theories.*, Proc. 1971 Seattle Algebraic Topology Symposium, Lect. Notes in Math., vol. 249, Springer-Verlag, Berlin, New York, 1971, pp. 1–12.

[BHM]: M. Bökstedt, W.-C. Hsiang and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. (1993), 465–540.

[BK]: A.K. Bousfield and D.M. Kan, *Homotopy limits, completions, and localizations*, Lect. Notes in Math., vol. 304, Springer-Verlag, Berlin, New York, 1972.

[Car]: G. Carlsson, *Bounded K-theory and the assembly map in algebraic K-theory*, these proceedings.

[CaPe]: G. Carlsson and E.K. Pedersen, *Controlled algebra and the Novikov Conjectures for K- and L-theory*, Topology (to appear).

[Ch]: T.A. Chapman, *The topological invariance of Whitehead torsion*, Amer. J. Math. **96** (1974), 488–497.

[CoHo]: E.H. Connell and J. Hollingsworth, *Geometric groups and Whitehead torsion*, Trans. Amer. Math. Soc. **140** (1969), 161–181.

[DaLü]: J.F. Davis and W. Lück, *Spaces over a category, assembly maps, and applications of the isomorphism conjecture in K- and L-theory*, in preparation.

[DWW]: W. Dwyer, M. Weiss and B.Williams, *A parametrized index theorem for the algebraic K-theory Euler class*, Preprint, Feb. 1995, 63 pp.

[Go1]: T.G. Goodwillie, *Calculus I: The First Derivative of Pseudoisotopy Theory*, K-Theory **4** (1990), 1–27.

[Go2]: T.G. Goodwillie, *Calculus II: Analytic Functors*, K-Theory **5** (1992), 295–332.

[Lü]: W. Lück, *Transformation Groups and Algebraic K-Theory*, Lect. Notes in Math., vol. 1408, Springer-Verlag, Berlin, New York, 1989.

[MaL]: S. MacLane, *Categories for the Working Mathematician*, Graduate Texts in Math., vol. 5, Springer-Verlag, Berlin, New York, 1971.

[Mitch]: B. Mitchell, *Rings with several objects*, Adv. in Math. **8** (1972), 1–161.

[PW1]: E.K. Pedersen and C. Weibel, *A nonconnective delooping of algebraic K-theory*, Proc. 1983 Rutgers Conf. on Algebraic Topology, Lecture Notes in Math., vol. 1126, Springer-Verlag, Berlin, New York, 1985, pp. 166–181.

[PW2]: E.K. Pedersen and C. Weibel, *K-theory homology of spaces*, Proc. 1986 Int. Conf. Algebraic Topology, Arcata, Lecture Notes in Math., vol. 1370, Springer-Verlag, Berlin, New York, 1989.

[Q]: F. Quinn, *Ends of maps II*, Invent. Math. **68** (1982), 353–424.

[QA]: F. Quinn, *Geometric algebra*, Proc. 1983 Rutgers Conf. on Algebraic Topology, Lecture Notes in Math., vol. 1126, Springer-Verlag, Berlin, New York, 1985, pp. 182–198.

[QAB]: F. Quinn, *Assembly maps in bordism-type theories*, these proceedings.

[QGF]: F. Quinn, *A geometric formulation of surgery*, Topology of Manifolds, Proceedings 1969 Georgia Topology Conf., Markham Press, Chicago, 1970, pp. 500–511.

[QLA]: F. Quinn, *Local algebraic topology*, Notices Amer. Math. Soc. **33** (1986), 895–899.

[Ra]: A. Ranicki, *Algebraic L-theory and Topological Manifolds*, Cambridge Tracts in Math., Cambridge Univ. Press, Cambridge, 1992.

[RaNo]: A. Ranicki, *On the Novikov Conjecture*, these proceedings.

[RaYa]: A. Ranicki and M. Yamasaki, *Controlled K-theory*, Topology and its Appl. **61** (1995), 1–59.

[Ro]: J. Rosenberg, *Analytic Novikov for topologists*, these proceedings.

[Spa]: E. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[tD]: T. tom Dieck, *Transformation Groups*, Studies in Math., vol. 8, De Gruyter, Berlin, New York, 1987.

[Vog1]: W. Vogell, *Algebraic K-theory of spaces, with bounded control*, Acta Math. **165** (1990), 161–187.

[Vog2]: W. Vogell, *Continuously controlled A-Theory*, Preprint, SFB 343 at Bielefeld University, Germany, 1992, 11 pp.

[Vog3]: W. Vogell, *The involution in the algebraic K-theory of spaces*, Proc. 1983 Rutgers Conf. on Algebraic Topology, Lecture Notes in Math., vol. 1126, Springer-Verlag, Berlin, New York, 1985, pp. 277–317.

[Wa1]: F. Waldhausen, *Algebraic K-theory of Spaces*, Proc. 1983 Rutgers Conf. on Algebraic Topology, Lecture Notes in Math., vol. 1126, Springer-Verlag, Berlin, New York, 1985, pp. 318–419.

[Wa2]: F. Waldhausen, *Algebraic K-theory of spaces, concordance, and stable homotopy theory*, Proc. 1983 Princeton Conf. on Alg. Topology and Alg. K-Theory, Annals of Math. Studies, vol. 113, Princeton Univ. Press, Princeton, 1987, pp. 392–417.

[WW1]: M. Weiss and B. Williams, *Theory: I*, K-Theory **1** (1988), 575–626.

[WW2]: M. Weiss and B. Williams, *Theory: II*, J. Pure and Appl. Algebra **62** (1989), 47–107.

[WW3]: M. Weiss and B. Williams, *Theory: Finale*, Preprint, 1994, 62pp.

Dept. of Mathematics, Univ. of Michigan, Ann Arbor, MI 48109-1003, USA
E-mail address: `msweiss@math.lsa.umich.edu`

Dept. of Mathematics, Univ. of Notre Dame, Notre Dame, IN 46556, USA
E-mail address: `bruce@bruce.math.nd.edu`