HOMOLOGY OF SPACES OF SMOOTH EMBEDDINGS

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Abstract. It is shown how the methods of the calculus of embeddings can be used to calculate, or help with the calculation of, the homology of spaces of smooth embeddings.

1. Introduction

The modest purpose of this note is to supply the proof of Lemma 5.2.1 in [8], restated below as Theorem 2.2. (It was always intended to be short, but the referee's comments have made it even shorter.) Some familiarity with [8] or [23] and [10] will be assumed.

The context of Theorem 2.2 is as follows. The calculus of embeddings as described in [22], [23], [10] was originally intended as a tool for calculating homotopy types of spaces of smooth embeddings $\mathrm{emb}(M,N)$, where M^m and N^n are smooth, without boundary for now. It aimed to describe the homotopy type of the space $\mathrm{emb}(M,N)$ in terms of the homotopy types of the spaces $\mathrm{emb}(U,N)$, where U runs through the open subsets of M which are tubular neighbourhoods of finite subsets of M. It soon became clear that there are a tangential and a nontangential part to the analysis. The tangential part is captured by the inclusion of $\mathrm{emb}(M,N)$ in the space of smooth immersions, $\mathrm{imm}(M,N)$, together with the homotopy theoretic description of $\mathrm{imm}(M,N)$ which is the main result of immersion theory [15], [12], [11]. The nontangential part aims to describe the homotopy fibers of that inclusion in terms of spaces of embeddings $\mathrm{emb}(S,N)$ where S runs through the honest finite subsets of M.

The basic 'Ansatz', suggested by Gromov's view of immersion theory [11], is to view the space $\mathrm{emb}(M,N)$ as just one value of a *good* cofunctor $V \mapsto \mathrm{emb}(V,N)$, where the variable V is an element of the poset $\mathcal{O}(M)$ of open subsets of M. In general, a cofunctor F from $\mathcal{O}(M)$ to spaces is *good* if

- it takes any inclusion $U \hookrightarrow V$ which is invertible up to smooth isotopy (as an abstract embedding) to a weak homotopy equivalence $F(V) \to F(U)$;
- for a monotone union $\bigcup V_i$ (where $V_i \subset V_{i+1}$ for $i = 0,1,2,\dots$), the canonical map from $F(\bigcup V_i)$ to $\mathrm{holim}_i\, F(V_i)$ is a weak homotopy equivalence.

For the analysis of good cofunctors on $\mathcal{O}(M)$, there is a theory of best polynomial (or Taylor) approximations. So, among the good cofunctors on $\mathcal{O}(M)$, there are some which are polynomial; and for each good cofunctor F on $\mathcal{O}(M)$ and each $r \geq 0$, there is an essentially unique best approximation $\eta_r : F \to T_rF$ of F by a cofunctor T_rF which is polynomial of degree $\leq r$. (The point is that $T_rF(V)$ can be described, by definition or otherwise, in terms of spaces $\mathrm{emb}(U,N)$ where U runs through the open subsets of M which are tubular neighbourhoods of subsets of M or cardinality $\leq r$.) When F is ρ*–analytic*, convergence takes place:

$$F(V) \xrightarrow{\ \simeq\ } \operatorname*{holim}_r \, T_rF(V)$$

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for any $V \in \mathcal{O}(M)$ which has a smooth proper Morse function with critical points of index $< \rho$ only. If $F(V)$ is based, then so is each $T_rF(V)$ and the homotopy equivalence just above implies a spectral sequence converging to $\pi_*F(V)$, with E^2–term consisting of the homotopy groups of the forgetful maps $T_rF(V) \to T_{r-1}F(V)$ for $r \geq 0$ (by convention $T_{-1}F(V) = *$).

All of this applies to the functor $V \mapsto \mathrm{emb}(V,N)$ because it is $(n-2)$–analytic by the main theorem of [10], which relies on much earlier work by Goodwillie [3], [4], [5] and forthcoming work by Goodwillie and Klein [7]. See also [9]. In view of the above explanations it should not come as a surprise that in this case the E^1–page is closely related to the homotopy groups of certain mixed configuration spaces of M and N. These are spaces of triples (R,S,f) where R and S are finite subsets of M and N respectively, of a fixed cardinality, and $f\colon R \to S$ is a bijection.

To repeat, this theory was originally developed with the good cofunctor $V \mapsto \mathrm{emb}(V,N)$ in mind. Comparison with the somewhat different–looking, but equally calculus–inspired work of Vassiliev [16], [17], [18], [19] and Kontsevich [13] on the *homology* of spaces of embeddings $\mathrm{emb}(\mathbb{S}^1,\mathbb{R}^n)$ suggested however that $V \mapsto \Omega^\infty(\mathrm{emb}(V,N)_+ \wedge \mathbf{H}\mathbb{Z})$ might be another cofunctor from $\mathcal{O}(M)$ to spaces worth looking at. Here $\mathbf{H}\mathbb{Z}$ is the Eilenberg–MacLane spectrum associated with $\mathbb{Z}$, so that $\pi_*\Omega^\infty(\mathrm{emb}(V,N)_+ \wedge \mathbf{H}\mathbb{Z})$ is the integer homology of $\mathrm{emb}(V,N)$. This led to the question: if F is a ρ–analytic cofunctor from $\mathcal{O}(M)$ to spaces, what are the goodness and analyticity properties of the cofunctor $\lambda_{\mathbf{J}}F$ given by $V \mapsto \Omega^\infty(F(V)_+ \wedge \mathbf{J})$ where $\mathbf{J}$ is a fixed spectrum, bounded from below ?

It turns out that $\lambda_{\mathbf{J}}F$ is only 'half' good — it does take isotopy equivalences to weak homotopy equivalences, but does not behave well with respect to monotone unions. To fix this one can use the taming of $\lambda_{\mathbf{J}}F$, a good cofunctor which agrees with $\lambda_{\mathbf{J}}F$ up to natural homotopy equivalence on *tame* elements of $\mathcal{O}(M)$ (those which are interiors of compact smooth codimension zero submanifolds of M). See [8, §4.1] for the details. Theorem 2.2 below states that the taming of $\lambda_{\mathbf{J}}F$ has good analyticity properties if F does, and if the 'first few' Taylor approximations to F vanish. The example one should have in mind is

$$F(V) := \mathrm{hofiber}\,[\,\mathrm{emb}(V,N) \to \mathrm{imm}(V,N)\,]$$

where imm(...) denotes spaces of smooth immersions. (We assume that a base point in $\mathrm{imm}(M,N)$ has been selected.) Here T_1F vanishes and F is $(n-2)$–analytic with excess $3-n$. Theorem 2.2 implies that the taming of $\lambda F := \lambda_{\mathbf{H}\mathbb{Z}}F$ is $(n/2-1/2)$–analytic provided $n/2-1/2 > m$. If M is the interior of a compact smooth manifold, there is no need to distinguish between $\lambda F(M)$ and the tame version. Hence the Taylor tower leads in this case to a second quadrant spectral sequence of the form

$$E^1_{-p,q} = \pi_{q-p}(L_p(\lambda F)(M)) \quad \Rightarrow \quad H_{q-p}F(M) = H_{q-p}(\mathrm{emb}(M,N))$$

where $L_p(\lambda F)$ is the p–th homogeneous layer of the taming of λF. There is a very explicit description of $E^1_{-p,q}$ in the case where M is closed and oriented: $E^1_{-p,q} = 0$ for $p < 0$ and

$$E^1_{-p,q} \cong H_{pm+q}(X_p, Y_p; \mathbb{Z}^\pm)$$

for $p \geq 0$, where X_p is the space of subsets S of M having cardinality p, and Y_p is the space of pairs (S,z), with $S \in X_p$ and $z \in \mathrm{hocolim}_{\emptyset \neq R \subset S} F(R)$. Here $F(R)$ is an abbreviation for $\mathrm{hofibre}\,[\mathrm{emb}(R,N) \to \mathrm{imm}(R,N)]$. Although Y_p is not a subspace of X_p, it maps forgetfully to X_p and so can be viewed as a subspace of a mapping cylinder homotopy equivalent to X_p; hence the "pair" notation. The coefficients are untwisted integer coefficients $\mathbb{Z}^+$ when m is odd. When m is even use $\mathbb{Z}^-$, integer coefficients twisted by means of the composition

$$\pi_1 X_p \to \Sigma_p \to \mathbb{Z}/2 = \mathrm{aut}(\mathbb{Z}).$$

This example is discussed in somewhat greater generality in [8, 5.2.2]. (Unfortunately some errors appear there in the explicit description of $E^1_{-p,q}$.) It is also explained in [8, §5] how the above spectral sequence can be seen as a "twice generalized" Eilenberg–Moore spectral sequence, and how it appears to agree with the spectral sequences found by Vassiliev and Kontsevich in the case where $\dim(M) = 1$. This suspected agreement has recently been confirmed by Volic [20], [21].

2. Estimates

We assume from now on M is smooth, possibly with boundary, and $\mathcal{O}(M)$ is the poset of open subsets of M containing ∂M. The concept of a ρ–analytic cofunctor from $\mathcal{O}(M)$ to spaces was originally defined in [10] for $\rho \in \mathbb{Z}$. In the revised definition of [8, 4.1.11], any $\rho \in \mathbb{R}$ is allowed. It is still true that, if F is ρ–analytic and V has a smooth proper Morse function with critical points of index $< \rho$ only, then $F(V) \simeq \operatorname{holim}_r T_r F(V)$. For more precise estimates see [8, 4.2.1].

Proposition 2.1. *Let F be a good cofunctor on $\mathcal{O}(M)$ and let $\mathbf{J}$ be a (-1)–connected CW–spectrum. Suppose that F is ρ–analytic with excess $c \geq 0$, where $\rho \in \mathbb{Z}$ and $\rho > m$. Then the taming of $\lambda_{\mathbf{J}} F$ is also ρ–analytic with excess c.*

Proof. This is a straightforward application of Goodwillie's dual Blakers–Massey theorem for cubes, [6, 2.6]. In detail: Suppose given a tame $V \in \mathcal{O}$ and pairwise disjoint closed subsets A_i of V, for $i \in \{1, \dots, k\}$. Suppose also that the closures of the A_i in $\bar{V}$ are disjoint smoothly embedded disks of codimension q_i, respectively, with boundary in $\partial\bar{V}$. For $U \subset S$, let A_U be the union of the A_i taken over $i \in U$. It is enough to show that the k–cube $\{F(V \smallsetminus A_U) \,|\, U \subset S\}$ is $(k - 1 + k\rho + c - \sum_i q_i)$–cocartesian. Our assumption on F implies that for nonempty $T \subset S$, the $|T|$–cube $\{F(V \smallsetminus A_U) \,|\, S \smallsetminus T \subset U \subset S\}$ is b_T–cartesian, where

$$b_T = |T|\rho + c - \sum_{i \in T} q_i.$$

According to [6, 2.6] our full k–cube is then p–cocartesian where p is the minimum of the numbers $k - 1 + \sum_\alpha b_{T(\alpha)}$, taken over the partitions of S into disjoint nonempty subsets $T(\alpha)$. Clearly the minimum is attained when the partition has only one part, and is therefore equal to $k - 1 + b_S = k - 1 + k\rho + c - \sum_i q_i$. □

Theorem 2.2. *Let F be a good cofunctor on $\mathcal{O}(M)$ and let $\mathbf{J}$ be a (-1)–connected CW–spectrum. Suppose that $T_{r-1}F \simeq *$ for some $r > 0$, and F is ρ–analytic with excess $c < 0$, where $\rho + c/r > m$. Then the taming of $\lambda_{\mathbf{J}} F$ is $(\rho + c/r)$–analytic with excess 0.*

Proof. As in the proof of proposition 2.1, select a tame $V \in \mathcal{O}$ and pairwise disjoint closed subsets A_i of V, where $i \in \{1, \dots, k\}$. Suppose again that the closures of the A_i in $\bar{V}$ are smoothly embedded disks of codimension q_i, respectively, with boundary in $\partial\bar{V}$. It suffices to show that the k–cube

$$\{F(V \smallsetminus A_U) \,|\, U \subset S\}$$

is $[\![k - 1 + k(\rho + c/r) - \sum_i q_i]\!]$–cocartesian, where $[\![a]\!] = \min\{b \in \mathbb{Z} \mid b \geq a\}$ for $a \in \mathbb{R}$. Let $S = \{1, \dots, k\}$, and for $T \subset S$ let Σ_T be the sum of all q_i for $i \in T$. By [6, 2.6] it suffices to check that, for nonempty $T \subset S$, the subcube

$$\{\, F(V \smallsetminus A_U) \quad | \quad S \smallsetminus T \subset U \subset S \,\} \tag{1}$$

is $[\![(\rho+c/r)|T|-\Sigma_T]\!]$–cartesian. Without loss of generality, $T=S$; otherwise replace V by the complement in V of a thickening of $A_{S\smallsetminus T}$ and renumber the elements of T. What we have to prove, therefore, is that

(2) *the k–cube* $\{\,F(V\smallsetminus A_U)\,|\,U\subset S\,\}$ *is* $[\![(\rho+c/r)k-\Sigma_S]\!]$*–cartesian.*

By the analyticity of F, and our assumption $c<0$, this is certainly true if $k\ge r$. We can therefore argue by downward induction on k. That is to say, we can concentrate on a particular $k<r$, and assume that statement (2) is established with $k+1$ in place of k. (At the same time we will argue by upward induction on q_1. The induction beginning is postponed, so we are reducing to the situation where $q_1=0$, and then similarly $q_i=0$ for $i=2,\dots,k$.)

Assuming that $q_1>0$, we can extend the inclusion $\bar A_1\to\bar V$ to an embedding of $\bar A_1\times[0,1]$ in $\bar V$, taking $\partial\bar A_1$ to $\partial\bar V$ and avoiding $\bar A_i$ for $i\in\{2,\dots,k\}$. Identify the image with $\bar A_1\times[0,1]$. Let $B_0=A_1\times\{0\}$, $B_1=A_1\times\{1\}$ and $B_i=A_i$ for $i\in\{2,\dots,k\}$. Let $C_1=A_1\smallsetminus(B_0\cup B_1)$ and $C_i=A_i$ for $i\in\{2,\dots,k\}$. By our standing assumption, the $(k+1)$–cube

(3) $$\{\,F(V\smallsetminus B_R)\quad|\quad R\subset\{0\}\cup S\,\}$$

is $[\![(\rho+c/r)(k+1)-\Sigma_S-q_1]\!]$–cartesian and consequently $[\![(\rho+c/r)k-\Sigma_S]\!]$–cartesian, since $\rho+c/r>m\ge q_1$. By inductive assumption, since $\operatorname{codim}(C_1)=\operatorname{codim}(A_1)-1$, the k–cube $\{\,F(V\smallsetminus(B_0\cup B_1\cup C_U))\,|\,U\subset S\,\}$ is $[\![(\rho+c/r)k-\Sigma_S+1]\!]$–cartesian. This last fact implies easily that the k–cube

(4) $$\{\,F(V\smallsetminus(B_0\cup B_U))\quad|\quad U\subset S\,\}$$

is $[\![(\rho+c/r)k-\Sigma_S]\!]$–cartesian: namely, for $U\subset\{2,\dots,k\}$ the inclusion

$$V\smallsetminus(B_0\cup B_1\cup C_{\{1\}\cup U})\longrightarrow V\smallsetminus(B_0\cup B_1\cup C_U)$$

is an isotopy right inverse for the inclusion $V\smallsetminus(B_0\cup B_{\{1\}\cup U})\to V\smallsetminus(B_0\cup B_U)$. Combining the estimates for the cubes (3) and (4), we conclude using [6, 1.6] that $\{\,F(V\smallsetminus B_U)\,|\,U\subset S\,\}$ and hence $\{\,F(V\smallsetminus A_U)\,|\,U\subset S\,\}$ are $[\![(\rho+c/r)k-\Sigma_S]\!]$–cartesian cubes.

This leaves the induction beginning, i.e., the special case of statement (2) in which $q_i=0$ for $i=1,2,\dots,k$. In this case the A_i are all m–dimensional and V is the disjoint union of some tame open $V'\subset M$ with $A_1,\dots,A_k$. We will proceed by upward induction on the number of handles in a fixed handle decomposition of the closure of V'. (Again the induction beginning is postponed, so we are reducing to the situation where $V'=\emptyset$.) Let therefore $A_0\subset V'$ be the "cocore" of one of the handles, of codimension q_0. Thus A_0 is diffeomorphic to a euclidean space and the inclusion $A_0\to V'$ extends to a smooth embedding of a disk into the closure of V'. By our standing assumption, the $(k+1)$–cube

(5) $$\{\,F(V\smallsetminus A_R)\,|\,R\subset\{0\}\cup S\,\}$$

is $[\![(\rho+c/r)(k+1)-q_0]\!]$–cartesian, hence $[\![(\rho+c/r)k]\!]$–cartesian. By the inductive assumption involving numbers of handles, the k–cube

(6) $$\{\,F((V\smallsetminus A_0)\smallsetminus A_U)\,|\,U\subset S\,\}$$

is $[\![(\rho+c/r)k]\!]$–cartesian. We combine the estimates for cubes (5) and (6) and use [6, 1.6] to deduce that the k–cube $\{\,F(V\smallsetminus A_U)\,|\,U\subset S\,\}$ is $[\![(\rho+c/r)k]\!]$–cartesian.

Finally we have to look at the special case of statement (2) in which $q_i=0$ for all i and V is equal to the (disjoint) union of the A_i. Here the hypothesis $T_{r-1}F\simeq *$ comes in: the spaces $F(V\smallsetminus A_U)=F(A_{S\smallsetminus U})$ are all contractible since $A_{S\smallsetminus U}$ is a disjoint union of at most k open balls, where $k<r$. $\square$

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