The strength of AD

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Abstract

Using core model theory, we give a proof of Woodin's theorem that the consistency of AD implies the consistency of infinitely many Woodin cardinals. The proof uses a mouse set theorem which is shown by running Woodin's core model induction.

Introduction

By now it is well–known not only among set theorists that Woodin has proven AD, the full axiom of determinacy, to be equiconsistent with infinitely many Woodin cardinals (cf. for example Theorem 32.16 of [2]). In fact, if V has infinitely many Woodin cardinals, then AD holds in $L(\mathbb{R}^*)$ where $\mathbb{R}^*$ are the reals from a symmetric collapse of the Woodin cardinals to ω (cf. for example [7] Theorem 3.1; the papers [7] and [8] provide a complete proof of this fact). On the other hand, if $L(\mathbb{R})$ satisfies AD then $L(\mathbb{R})$ has a definable inner model with infinitely many Woodin cardinals.

Unfortunately, no proof of that latter direction, giving the optimal lower bound for the strength of AD, has so far found its way into the literature (although cf. [14]). We here give such a proof. Our main theorem may be stated as follows, and may be seen as a slight strengthening of Woodin's "derived model theorem" (at least of the one for $L(\mathbb{R}^*)$).

Theorem 0.1 *Suppose that $V = L(\mathbb{R}) \models AD$. There is then a generic extension of V in which there is a fine structural inner model $L[E]$ with infinitely many*

Woodin cardinals (cofinal in ω_1^V) and in which $\mathbb{R}$ (the reals of V) is the set reals of a symmetric collapse over $L[E]$ of the supremum of the Woodin cardinals of $L[E]$ to ω.

Whereas Woodin uses HOD to get the lower bound for AD, we here use core model theory, more precisely the theory developed in [13]. We shall get the Woodin cardinals from a failure of iterability of background certified core models ("K^c's") built inside certain inner models with choice. As in (one of) Woodin's proof(s), we shall stack together models with Woodin cardinals thus obtained. In order to show the the Woodin cardinals will not be collapsed in the final model we shall need a "mouse set theorem." This will be proven by using Woodin's core model induction, which in turn has many applications also at other places. (The present paper, however, seems to be the first one containing a complete sketch of that induction.)

The paper is organized as follows. Section 1 gives descriptive set theoretic preliminaries. Section 2 contains a proof of 0.1, using the "mouse set theorem". In Section 3 we prepare ourselves for running the core model induction, which is run in Section 4 in order to actually prove the "mouse set theorem".

Virtually all of the results in Sections 3 and 4 of this paper are due to Woodin (albeit Section 4 gives the second author's reconstruction of Woodin's induction). The proof of 0.1, however, contained in Section 2 is due to the second author. We remark that the aspect of 0.1, saying that $L(\mathbb{R})$ is a symmetric extension of a fine structural inner model, is what goes beyond Woodin's original theorem.

The history of this paper is as follows. In a long series of private lectures in 1997 during the 1st author's stay at Berkeley, the 2nd author explained the core model induction to the 1st author. The 1st author wrote a first and poor draft of this paper in Vienna in 1999 and placed it on his web page, all of a sudden having lost interest in improving it. Then the TeX file got lost. In 2006, Martina Pfeifer in Münster typed a new version, and interest in making it better resurrected. There is no intent to publish it.

1 Preliminaries

Recall that $y \in \mathbb{R} \cap L$ iff y is Σ^1_2 in a countable ordinal; more generally, y is a real of the least inner model containing $2n$ Woodin cardinals iff y is Σ^1_{2n+2} in a countable ordinal.

Definition 1.1 *Let Γ be a pointclass, and let x be a real. Then*

$$C_\Gamma(x) = \{y \in \mathbb{R} : \exists \xi < \omega_1 \ y \textit{ is } \Gamma(x,z) \textit{ for all } z \textit{ coding } \xi\}.$$

More precisely, $y \in C_\Gamma(x)$ iff there is a countable ordinal ξ and some $A \in \Gamma$ such that for all z coding ξ we have that $\overline{y} = y$ iff $(\overline{y}, x, z) \in A$ (iff $\forall y'[(y', x, z) \in A \Rightarrow \overline{y} = y']$; i.e., practically, if Γ is closed under $\exists^{\mathbb{R}}$ then y is $\breve{\Gamma}$ in a countable ordinal, too, and thus Δ in a countable ordinal, where $\Delta = \Gamma \cap \breve{\Gamma}$).

Let $A \subset \mathbb{R}$, say (in general, think of A as $\subset \mathbb{R}^i$ for some $i < \omega$). Then A is called (α–)Souslin if A is the projection of a tree T on $\omega \times \alpha$ (written $A = p[T]$). A is said to have a scale if there is a sequence $(\varphi_n \colon n < \omega)$ of norms on A, i.e., $\varphi_n \colon A \to OR$ for all $n < \omega$, such that whenever $(x_k : k < \omega)$ is a sequence of reals in A converging to x such that for each $n < \omega$, $\varphi_n(x_k)$ is eventually constant as $k \to \infty$, say with eventual value α_n, then $x \in A$ and $\varphi_n(x) \leq \alpha_n$ for each $n < \omega$. The proof of Lemma 1.8 below exploits a variant of the following "tree from a scale construction." If $(\varphi_n \colon n < \omega)$ is a scale on A, then we may set

$$(s, (\alpha_n \colon n < lh(s))) \in T \text{ iff } \exists x \supset s \ \forall n < lh(s) \varphi_n(x) = \alpha_n.$$

Then T witnesses that A is Souslin, i.e., $A = p[T]$, and if $x \in A$, then T_x has an honest leftmost branch f_x (i.e., $\forall g \in [T_x] \forall n < \omega f_x(n) \leq g(n)$; cf. [2] 30.2). f_x is defined just by $f_x(n) = \varphi_n(x)$ for $n < \omega$.

Let Γ be a pointclass. A is then said to have a Γ–scale if for every $x \in A$, the relation (in y, n)

$$y \in A \wedge f_y(n) \leq f_x(n)$$

is in $\Delta(x)$, uniformly in x.[1] Finally a pointclass Γ is said to have the scale property if every $A \in \Gamma$ admits a Γ–scale.

Under AD, many pointclasses do have the scale property (cf. for example [10]). We call a pointclass *good* if it is ω–parameterized, closed under recursive substitution, number quantification, and $\exists^{\mathbb{R}}$, and has the scale property. Kechris [3] has shown that (under AD) if Γ is good then $C_\Gamma(x)$ is the largest countable $\Gamma(x)$ set of reals, and that $C_\Gamma(x)$ has a $\Delta(x)$–good wellorder, which we'll denote by $<_{\Gamma,x}$. We then let $\Theta_{\Gamma,x}$ denote the order type of $<_{\Gamma,x}$; we'll have that there is an $A \in \Gamma(x)$ such that $y \in C_\Gamma(x)$ iff $\exists \xi < \Theta_{\Gamma,x}$ s.t. y is unique with $(y, z) \in A$ for all z coding ξ.

If Γ has the scale property, and if Γ is also closed under $\forall^{\mathbb{R}}$ (or alternatively $\Gamma = \exists^{\mathbb{R}}\bar{\Gamma}$, where $\bar{\Gamma}$-uniformization holds), then Γ-uniformization holds. Examples are provided by the pointclasses from Definition 1.13.

[1] I.e., there is $\leq^1 \in \Gamma$ and $\leq^2 \in \breve{\Gamma}$ such that for every $x \in A$, $\forall y \, \forall n \; ((y \in A \wedge f_y(n) \leq f_x(n)) \leftrightarrow (x, y, n) \in \leq^1)$ and $\forall y \, \forall n \; ((y \in A \wedge f_y(n) \leq f_x(n)) \leftrightarrow (x, y, n) \in \leq^2)$

Sublemma 1.2 (AD) *Let Γ be good, and let T be a tree such that $p[T]$ is a universal Γ set. Let N be admissible, let $x \in \mathbb{R}$ and suppose that $x, T \in N$. Then $C_\Gamma(x) \subset N$, and $C_\Gamma(x)$ is $\Sigma_1^N(\{T, x\})$.*

Proof Sketch: There is some $n_0 \in \omega$ with: for all $\xi < \Theta_{\Gamma,x} \cap N$, the ξ^{th} real in $<_{\Gamma,x}$ is the unique y such that for all z coding ξ, $(y, z, n_0) \in p[T]$. Thus

$$n \in y \Leftrightarrow \ \Vdash_{Col(\omega,\xi)} \exists z[z \text{ codes } \xi \wedge \exists \overline{y}((\overline{y}, z, n_0) \in p[T] \wedge n \in \overline{y})],$$

and hence $y \in N$. But then $\Theta_{\Gamma,x} < N \cap OR$ (as o.w. there'd be a partial surjection $\mathbb{R} \cap N \to N \cap OR$ which is $\Sigma_1^N(\{T, x\})$), and the result follows. $\square$

We'd like to extend $x \mapsto C_\Gamma(x)$ to countable transitive sets.

The countable transitive set a coded by the real x is the unique a such that ϕ: $(a, \in) \cong (\omega, R)$ where nRm iff $(n, m) \in x$ (if it exists). If $b \in a$ then we write b_x for the real $\phi''b$, and say that b_x codes b relative to x.

Definition 1.3 *Let Γ be a pointclass, and let a be a countable transitive set. Then $C_\Gamma(a)$ denotes the set of all $b \subset a$ such that $b_x \in C_\Gamma(x)$ for all $x \in \mathbb{R}$ coding a.*

Sublemma 1.4 *Let a be any countable set, and let $\mathcal{D}$ be a countable family of open dense subsets of $Col(\omega, a)$. Then the set of all $G \in {}^\omega a$ being $\mathcal{D}$–generic is comeager.*

Proof: Let $\mathcal{D} = \{D_i : i < \omega\}$. We may view D_i as a dense set in (the space) ${}^\omega a$ (by confusing it with $\{g \in {}^\omega a : g \restriction n \in D_i$ for all sufficiently large $n\}$. Let $C_i = {}^\omega a \backslash D_i$. Clearly, C_i is closed, i.e. $\overline{C_i} = C_i$, and so $D_i = {}^\omega a \backslash \overline{C_i}$. Thus every C_i is nowhere dense. But G is $\mathcal{D}$–generic iff $G \in \bigcap_{i<\omega} D_i$. $\square$

Sublemma 1.5 *Let $\mathcal{C} \subset {}^\omega\omega$ be comeager. Let $p_0, p_1 \in {}^{<\omega}\omega$ be such that $lh(p_0) = lh(p_1)$. Then there is $\alpha \in {}^\omega\omega$ such that $\{p_0{}^\frown\alpha, p_1{}^\frown\alpha\} \subset \mathcal{C}$.*

Proof: Let $\mathcal{D}_i, i < \omega$ be dense open sets such that $\bigcap_{i<\omega} D_i \subset \mathcal{C}$. By a simple induction, we may pick $q_0, q_0', q_1, q_1', \ldots \in {}^{<\omega}\omega$ such that for all $n < \omega$

$$\{q_0{}^\frown q_0{}^\frown q'_0{}^\frown \ldots {}^\frown q_n{}^\frown\alpha : \alpha \in {}^\omega\omega\} \subset D_n \text{ and}$$
$$\{p_1{}^\frown q_0{}^\frown q'_0{}^\frown \ldots {}^\frown q_n{}^\frown q'_n{}^\frown\alpha : a \in {}^\omega\omega\} \subset D_n.$$

But then, if we put $\alpha = q_0{}^\frown q'_0{}^\frown \ldots$, we get that $\{p_0{}^\frown\alpha, p_1{}^\frown\alpha\} \subset \bigcap_{i<\omega} D_i \subset \mathcal{C}$. $\square$

Theorem 1.6 (AD; Harrington and Kechris [1]) *Let Γ be a good pointclass, and let T be a tree witnessing Γ has the scale property. Let a be countable and transitive; then*

1. $C_\Gamma(a) = \{b \subset a$*: for comeager many x coding $a, b_x \in C_\Gamma(x)\}$, and*
2. $C_\Gamma(a) = \mathcal{P}(a) \cap L(a \cup \{T, a\})$.

Proof Sketch: For the nontrivial inclusion in 1., fix any real y coding a. By Sublemma 1.4, we can fix α such that for nonmeager many x coding a, b_x is the α^{th} real in $<_{\Gamma,x}$. Using the Γ–definability of the "nonmeager many" quantifier applied to Γ–relations, we get that b_y is $\Delta(y)$ in any code for α, so $b_y \in C_\Gamma(y)$.

For 2., note $\supset$ is trivial by [1]. As to $\subset$, let $b \in C_\Gamma(a)$; then for comeager many x coding $a, b_x \in L[T, x]$ by 1. and [1]. We can then find x_0 and x_1 pairwise generic over $L[a \cup \{T, a\}]$ such that $b \in (L(a \cup \{T, a\})[x_0] \cap L(a \cup \{T, a\})[x_1])$, so $b \in L(a \cup \{T, a\})$. $\square$

We have the following generalization of 1.2

Lemma 1.7 (AD) *Let Γ be good, and let T be a tree such that $p[T]$ is a universal Γ set. Let a be countable and transitive, and suppose N is an admissible set such that $T, a \in N$. Then $C_\Gamma(a) \subset N$, and in fact $C_\Gamma(a)$ is $\Sigma_1^N(\{T, a\})$. Moreover, if $\pi : M \to N$ is elementary and $\pi(\langle \overline{T}, \overline{a} \rangle) = (\langle T, a \rangle)$, then $\pi^{-1}{}'' C_\Gamma(a)$ is an initial segment of $C_\Gamma(\overline{a})$ under its canonical prewellorder.*

Lemma 1.8 *Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ be such that every $A \in \mathcal{A}$ admits a scale ($\leq_n$: $n < \omega$) s.t. each individual $\leq_n$ belongs to $\mathcal{A}$, too, and such that $A \in \mathcal{A} \Rightarrow \mathbb{R} \backslash A \in \mathcal{A}$. Let N and M be transitive models of a sufficiently large fragment of ZFC such that $N \in M$. Let $\mathcal{C} \subset {}^\omega N$ be a comeager set of $Col(\omega, N)$–generics over M (in particular, N is countable) and suppose that for each $A \in \mathcal{A}$ there is a term $\tau_A \in M$ such that whenever $G \in \mathcal{C}$ then*

$$\tau_A^G = A \cap M[G].$$

Let $\pi : \overline{M} \to M$ be elementary with $\{N\} \cup \{\tau_A : A \in \mathcal{A}\} \subset ran(\pi)$. Let $\pi(\overline{N}, \overline{\tau}_A) = N, \tau_A$. THEN whenever g is $Col(\omega, \overline{N})$–generic over $\overline{M}$, for all $A \in \mathcal{A}$,

$$\overline{\tau}_A^g = a \cap \overline{M}[g].$$

Proof: To commence, fix any $A \in \mathcal{A}$ for a while, and let $(\psi_n : n < \omega)$ be a scale on A such that for every $n < \omega$, if $\leq_n$ is the prewellorder on $\mathbb{R}$ given by ψ_n then

$\leq_n \in \mathcal{A}$. Let $\tau_n \in M$ be such that $\tau_n^G = \leq_n \cap M[G]$ for all $G \in \mathcal{C}$. Let ϕ_n be a term in M such that for every G being $Col(\omega, N)$–generic over M, ϕ_n^G is the norm on $A \cap M[G]$ given by τ_n^G. Let U_n be a term for the n^{th} level of the tree associated to these norms, i.e., for all G being $Col(\omega, N)$–generic over M,

$$\dot{U}_n^G = \{(x \restriction n, (\phi_0^G(x), \ldots, \phi_{n-1}^G(x))) : x \in A \cap M[G]\}.$$

Now let $G_h, h = 0, 1,$ be any $Col(\omega, N)$–generics over M. Then for any appropriate $\vec{a}$ we have $\vec{a} \in \dot{U}_n^{G_h}$ iff there is some $p_h \in G_h$ forcing $\vec{a} \in \dot{U}_n$. W.l.o.g., $lh(p_0) = lh(p_1)$. Hence by 1.5 we may choose $G_h^* \in \mathcal{C}$ such that for some real α, for every $n < \omega, p_0^\frown \alpha \restriction n \in G_0^*$ and $p_1^\frown \alpha \restriction n \in G_1^*$. In particular, we have $M[G_0^*] = M[G_1^*]$, which implies $\tau_n^{G_0^*} = \leq_n \cap M[G_0^*] = \leq_n \cap M[G_1^*] = \tau_n^{G_1^*}$, and so $\dot{U}_n^{G_0^*} = \dot{U}_n^{G_1^*}$. Hence $\vec{a} \in \dot{U}_n^{G_0}$ iff $\vec{a} \in \dot{U}_n^{G_0^*}$ iff $\vec{a} \in \dot{U}_n^{G_1^*}$ iff $\vec{a} \in \dot{U}_n^{G_1}$.

This means that $\dot{U}_n^G$ is independent from G and in the ground model. I.e., there are $U_n \in M$ such that $U_n = \dot{U}_n^G$ for all G being $Col(\omega, N)$–generic over M. Let U be the tree whose n^{th} level is U_n. (Of course, possibly $U \notin M$.)

Claim: Whenever G is $Col(\omega, N)$–generic over $M, A \cap M[G] \subset p[U] \subset A$.

Proof: $A \cap M[G] \subset p[U]$ is obvious from the definition of U. Let $(x, f) \in [U]$. Let G be $Col(\omega, N)$–generic over M. Let $n < \omega$; then the n^{th} level of U is U_n^G, and so we can find a real $x_n \in A$ with $x_n \restriction n = x \restriction n$ and $\forall i < n(\phi_i^G(x_n) = f(i))$. So for any $i, \phi_i^G(x_n)$ is eventually constant as $n \to \omega$. Hence $\psi_i(x_n)$ is eventually constant as $n \to \omega$. But $(\psi_i : i < \omega)$ is a scale on A, thus $x \in A$. This shows $p[U] \subset A$.

We now in particular have that

$$\Vdash_{Col(\omega,N)} \forall x[x \in \tau_A \to (x \restriction n, (\phi_0(x), \ldots, \phi_{n-1}(x))) \in U_n].$$

The elementarity of π gives that

$$\Vdash_{Col(\omega,N)} \forall x[x \in \overline{\tau}_A \to (x \restriction n, (\pi^{-1}(\phi_0)(x), \ldots, \pi^{-1}(\phi_{n-1})(x))) \in \overline{U}_n],$$

where $\overline{U}_n = \pi^{-1}(U_n)$. Let $\overline{U}$ be the tree whose n^{th} level is $\overline{U}_n$. It is easy to see that $p[\overline{U}] \subset p[U]$ using π. But now if $x \in \overline{\tau}_A^g$ for a $Col(\omega, \overline{N})$–generic g then $x \in p[\overline{U}] \subset p[U] \subset A$, by the above Claim. So $\overline{\tau}_A^g \subset A$.

However, the same reasoning with $\mathbb{R}\backslash A \in \mathcal{A}$ and $\tau_{\mathbb{R}\backslash A}$ instead of A and τ_A shows that $\overline{\tau}_{\mathbb{R}\backslash A}^g \subset \mathbb{R}\backslash A$, and thus in fact $\overline{\tau}_A^g = A \cap \overline{M}[g]$, as $\overline{\tau}_{\mathbb{R}\backslash A}^g = (\mathbb{R} \cap \overline{M}[g]) \setminus \overline{\tau}_A^g$. □

Applying this to $\pi = id$ gives at once

Corollary 1.9 *Let $\mathcal{A}$, etc. be as in 1.8. Then in fact $\tau_A^G = A \cap M[G]$ for all $A \in \mathcal{A}$ and for all G being $Col(\omega, N)$–generic over M (not just for $G \in \mathcal{C}$).*

We want to point out that, in the above proof, if we even have $(\tau_n : n < \omega) \in M$ (rather than just $\{\tau_n : n < \omega\} \subset M$) then actually $U \in M$.

Definition 1.10 *We'll say that $\tau \in M^{Col(\omega,N)}$ (weakly) captures $A \subset \mathbb{R}$ (over M) if for all (for comeager many) G being $Col(\omega, N)$–generic over M, $\tau^G = A \cap M[G]$.*

1.8 gives a condensation result for such terms.

Lemma 1.11 *Let $\mathcal{A}$, etc. be as in 1.8. Suppose further that $a \in N$ is transitive (and thus countable), that $a \in ran(\pi)$, and $\pi(\overline{a}) = a$. Let Γ be a good pointclass, and assume that $\mathcal{A}$ contains a universal Γ set, and that $C_\Gamma(a) \subset M$. THEN $C_\Gamma(\overline{a}) \subset \overline{M}$. Moreover, if $C_\Gamma(\overline{a}) \in \overline{M}$ then in fact*

$$\pi(C_\Gamma(\overline{a})) = C_\Gamma(a) \in M.$$

Proof: The set $\{(x, y) : \forall w \in C_\Gamma(x) \exists i \in \omega \; w = (y)_i\}$ is in $\breve{\Gamma}$, and so there are k_0 and $A \in \mathcal{A}$ such that

$$(k_0, x, y) \in A \Leftrightarrow \forall w \in C_\Gamma(x) \exists i \in \omega w = (y)_i.$$

Let us pick $\sigma, \rho \in M^{Col(\omega,N)}$ such that

$$\Vdash^M_{Col(\omega,N)} \sigma \in \mathbb{R} \text{ codes } a \wedge \{(\rho)_i : i < \omega\} = \mathbb{R} \cap N[\sigma].$$

We'll then have that, for every $n < \omega$,

$$\Vdash^M_{Col(\omega,N)} ((k_0, \sigma, \rho), \phi_0(k_0, \sigma, \rho), \dots , \phi_{n-1}(k_0, \sigma, \rho)) \in U_n,$$

where $\phi_0, \dots,$ and $U_0, \dots$ are as in the proof of 1.8. Let $\pi(\overline{\sigma}, \overline{\rho}, \overline{\phi}_i, \overline{U}_i) = \sigma, \rho, \phi_i, U_i$. Then by the elementarity of π,

$$\Vdash^{\bar{M}}_{Col(\omega,\overline{N})} ((k_0, \overline{\sigma}, \overline{\rho}), \overline{\phi}_0(k_0, \overline{\sigma}, \overline{\rho}), \dots , \overline{\phi}_{n-1}(k_0, \overline{\sigma}, \overline{\rho})) \in \overline{U}_n.$$

Let $\overline{U}$ be the tree whose n^{th} level is $\overline{U}_n$. Thus if g is $Col(\omega, \overline{N})$–generic over $\overline{M}$ then

$$(k_0, \overline{\sigma}^g, \overline{\rho}^g) \in p[\overline{U}] \subset p[U] \subset A$$

(c.f. the Claim in the proof of 1.8). Therefore $C_\Gamma(\overline{\sigma}^g) \subset \overline{M}[g]$. This means that if $b \in C_\Gamma(\overline{a})$ then $b_g \in C_\Gamma(\overline{\sigma}^g) \subset \overline{M}[g]$ (where b_g codes b relative to g), and hence

$b \in \overline{M}[g]$. This is true for all g which are $Col(\omega, N)$–generic over $\overline{M}$, and thus $C_\Gamma(\overline{a}) \subset \overline{M}$. The rest is easy. $\square$

Note that by 1.7, $C_\Gamma(a) \subset M$ follows from $T \in M$ where G projects to a universal Γ set. However, we'll be able to derive $C_\Gamma(a) \subset M$ from different assumptions; c.f. 3.9 below.

The following is an easy corollary to the preceeding proof.

Lemma 1.12 *Let Γ be good, and let T and U be trees projecting to a universal Γ set and its complement. Let a be countable and transitive, and let M be a Σ_2-admissible set such that T, $U \in M$. Let $\pi\colon \bar{M} \to M$ be elementary with $\pi(\bar{a}) = a$ and T, $U \in ran(\pi)$. Then $C_\Gamma(\bar{a}) \in \bar{M}$, and $\pi(C_\Gamma(\bar{a})) = C_\Gamma(a)$.*

We now want to state the mouse set theorem.

Definition 1.13 *We call a pointclass $\Gamma \subset \mathcal{P}(\mathbb{R})$ a* scaled Σ–pointclass *if it is one of the following.*

(1) $\Gamma = \Sigma^1_{2n+2}$ *for some $n < \omega$, or else*

(2) $[\alpha, \beta]$ *is a Σ_1–gap with $\alpha > 1$ and*

(2a) $\alpha < \beta$ *and $[\alpha, \beta]$ is strong and $\Gamma = \Sigma_1(J_\alpha(\mathbb{R}))$, or*

(2b) $\alpha < \beta$ *and $[\alpha, \beta]$ is weak and either $\Gamma = \Sigma_1(J_\alpha(\mathbb{R}))$ or $\Gamma = \Sigma_{n+2i}(J_\beta(\mathbb{R}))$ where $i < \omega$ is arbitrary and $n < \omega$ is least s.t. $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$, or*

(2c) $\beta = \alpha$ *and α is inadmissible and $\Gamma = \Sigma_{2i+1}(J_\alpha(\mathbb{R}))$ where $i < \omega$ is arbitrary or*

(2d) $\beta = \alpha$ *and α is admissible and $\Gamma = \Sigma_1(J_\alpha(\mathbb{R}))$.*

It is well–known that the projective pointclasses Σ^1_{2n+2} for some $n < \omega$ have the scale property. Moreover, the pointclasses listed under (2) above are exactly the ones shown in [10] to have the scale property. Hence the name.

There is a natural well–order of the scaled Σ–pointclasses. We'll say that Γ shows up *earlier* than Γ' if $\Gamma < \Gamma'$ according to this well–order. Also, if $[\alpha, \beta]$ is a gap it's clear what we mean by "Γ shows up in $[\alpha, \beta]$". In particular, if Γ shows up in $[\alpha, \beta]$ and Γ' shows up in $[\alpha', \beta']$ with $\beta < \alpha'$ then Γ shows up earlier than Γ'.

Definition 1.14 *Let $\mathcal{O} : HC \to V$ such that $\forall a \in HC$ $\mathcal{O}(a) \subset \mathcal{P}(a)$. Then $\mathcal{O}$ is called a mouse operator if $\forall a \in HC$ $\mathcal{O}(a) = \mathcal{P}(a) \cap \mathcal{M}_a$ for some $\omega_1 + 1$ iterable a–premouse $\mathcal{M}_a$. We say that $\mathcal{O}$ is captured by mice if $\forall a \in HC$ $\mathcal{O}(a) \subset \mathcal{M}_a$ for some $\omega_1 + 1$ iterable a–premouse.*

Lemma 1.15 (AD; Woodin) *Let Γ be a scaled Σ–pointclass. Then C_Γ is captured by mice with an ω_1 iteration strategy which is projective in Γ.*

If $\Gamma = \Sigma_1(J_\alpha(\mathbb{R}))$, where either α is a limit ordinal or else $\alpha - 1$ is critical [2]*, then C_Γ is in fact a mouse operator.*

We'll prove 1.15 in Section 4.

2 The derived model theorem

Lemma 2.1 (AD; Kechris) *Let $S \subset OR$. For an S–cone of reals x we have*

$$L[S,x] \models OD_S\text{–}determinacy.$$

In particular, $\omega_1^{L[S,x]}$ is measurable in $HOD_S^{L[S,x]}$.

Proof. Let us first assume that there is no S–cone of reals x such that in $L[S,x]$ all OD_S–sets of reals are determined. Define $x \mapsto A_x$ by letting A_x be the least $OD_S^{L[S,x]}$–set of reals which is not determined. I.e., if $\mathcal{G}_{A_x}$ is the usual game (in which I, II alternate playing integers) with payoff A_x, then $\mathcal{G}_{A_x}$ is not determined in $L[S,x]$. Notice that A_x only depends on the S–constructibility degree of x. Also, by hypothesis, A_x is defined for an S–cone $\mathcal{C}$ of x.

Let $\mathcal{G}$ be the game in which I, II alternate playing integers so that I produces the reals x, a, II produces the reals y, b, and I wins iff $a \oplus b \in A_{x \oplus y}$. Let us suppose that I has a winning strategy, τ, in $\mathcal{G}$. Let $\tau \in L[S,z]$, where z is in $\mathcal{C}$. Let τ^* be a strategy for I in $\mathcal{G}_{A_z}$ so that if II produces the real b, and if τ calls for I to produce the reals a, x in a play of $\mathcal{G}$ in which II plays b, $z \oplus b$, then τ^* calls for I to produce the real a. Then for every $b \in L[S,z]$, if $a = \tau^*(b)$, in fact if a, $x = \tau(b, z \oplus b)$, then

$$a \oplus b \in A_{x \oplus (z \oplus b)} = A_z.$$

So τ^* is a winning strategy for I in the game $\mathcal{G}_{A_z}$ played in $L[S,z]$. Contradiction! We may argue similarily if II has a winning strategy in $\mathcal{G}$.

[2] I.e., infinitely many of $\Sigma_n(J_{\alpha-1}(\mathbb{R}))$ are scaled Σ-pointclasses

Now let $L[S,x] \models OD_S$–determinacy. Working inside $L[S,x]$, we may then define a filter μ on $\omega_1^{L[S,x]}$ as follows.

For reals x, let $|x| = sup\{||y||\colon y \equiv_T x \wedge y \in WO\}$. Let $S = \{|x|\colon x \in \mathbb{R}\}$. Let $\pi\colon \omega_1 \to S$ be the order isomorphism. Now if $A \subset \omega_1$, then we put $A \in \mu$ iff

$$\{x\colon |x| \in \pi'' A\}$$

contains an S–cone of reals. It is easy to verify that $\mu \cap HOD_S$ witnesses that ω_1 is measurable in HOD_S. $\square$

Fix $T = T_1^2$, a tree obtained from the scale property of Σ_1^2. So for any real x, the model $L[T,x]$ is Σ_1^2–correct. Moreover, with $\delta = \boldsymbol{\delta}_1^2$ we have

$$L_\delta[T,x] = V_\delta^{HOD_x} \models \omega_1^V \text{ is measurable,}$$

so that ω_1^V is measurable in $L[T,x]$, which can be seen to imply that

$$HOD_{T,\vec{Q}}^{L[T,x]} \models \omega_1^V \text{ is measurable}$$

for $\vec{Q} \in L[T,x]$, as $L[T,x]$ is a size $< \omega_1$ forcing extension of $HOD_{T,\vec{Q}}^{L[T,x]}$. We may thus try to isolate K of height ω_1^V inside various $HOD_{T,\vec{Q}}^{L[T,x]}s$. We write $\Omega = \omega_1^V$.

Lemma 2.2 *Let $P \in HC$ be transitive, and suppose that*

$$W_x = (K^c(P))^{HOD_{T,P}^{L[T,x]}},$$

constructed with height Ω, exists for a cone of x. Then there is a cone of x such that W_x cannot be $\Omega + 1$ iterable above P inside $HOD_{T,P}^{L[T,x]}$.

Proof: Suppose otherwise. By 2.1, there is then a $T \oplus P$–cone $\mathcal{C}$ so that for all $x \in \mathcal{C}$ we have $L[T,x] = L[T,P,x]$,

$$\omega_1^{L[T,x]} \text{ is measurable in } HOD_{T,P}^{L[T,x]},$$

and we may isolate

$$K_x = (K(P))^{HOD_{T,P}^{L[T,x]}}.$$

Let us fix an $x \in \mathcal{C}$, and let us write K for K_x. By "cheapo" covering and the fact that $L[T,x]$ is a size $< \Omega$ forcing extension of its $HOD_{T,P}$, we may pick some $\lambda < \Omega$ s.t.

$$\lambda^{+K} = \lambda^{+L[T,x]}.$$

Let $g \in V$ be a $Col(\omega, \lambda)$–generic over $L[T, x]$, and let $y \in V$ be a real coding (g, x). Thus

$$\omega_1^{L[T,y]} = \lambda^{+L[T,x]} = \lambda^{+K}.$$

As we also have $y \in \mathcal{C}$,

$$\omega_1^{L[T,y]} \text{ is measurable in } HOD_{T,P}^{L[T,y]}.$$

We hence get a contradiction if we can show:

Claim. $K \in HOD_{T,P}^{L[T,y]}$.

Proof: K is still fully iterable inside $L[T, y]$ by [13] Thm. 2.18. This means that K is the core model above P of $L[T, y]$ in the sense of [13] 2.17; i.e., from the point of view of $L[T, y]$, it is the common transitive collapse of $Def(W', S)$ for any W', S s.t. W' is $\Omega + 1$ iterable and Ω is S–thick. But this characterization clearly establishes $K \in HOD_{T,P}^{L[T,y]}$. □ (Claim)

□ 2.2

Using [13] Cor. 2.11 and Thm. 2.8 then immediately give:

Corollary 2.3 *In the situation of 2.2, there is a $T \oplus P$–cone of x such that for each x from that cone, there is $Q \rhd P$ together with $\delta \in Q \backslash P$ such that*

$$Q \in HOD_{T,P}^{L[T,x]}$$
$$HOD_{T,P}^{L[T,x]} \models Q \text{ is excellent, and}$$
$$Q \models \delta \text{ is Woodin.}$$

For our purposes, in 2.3 and in the following, we may let "Q is excellent" mean that Q is excellent in the sense of [13] *and* Q has a largest cardinal, denoted by $\delta = \delta(Q)$, such that $Q \models \delta$ is Woodin.

Lemma 2.4 *Let Q be excellent in $HOD_{T,P}^{L[T,x]}$, and suppose that*

$$OD_Q \cap \mathcal{P}(\delta(Q)) \subset Q.$$

Then

$$(K^c(Q))^{HOD_{T,Q}^{L[T,y]}} \text{ exists}$$

for a cone of y.

[N.b.: "$K^c(Q)$ exists" is supposed to imply $Q \cap OR = \delta(Q)^{+K^c(Q)}$, c.f. [13] §1.]

Proof: Deny. Then let $\mathcal{C}$ be a cone such that for all $y \in \mathcal{C}$,

$$(K^c(Q))^{HOD_{T,Q}^{L[T,y]}} \text{ does not exist.}$$

Consider $y \in \mathcal{C}$. As $K^c(Q)$ does not exist in $HOD_{T,Q}^{L[T,y]}$, there is a least $\mathcal{N}_\xi$ from the $K^c(Q)$–construction (inside $HOD_{T,Q}^{L[T,y]}$) with $\rho_\omega(\mathcal{N}_\xi) \leq \delta = \delta(Q)$. But then if $A \in (\Sigma_\omega(\mathcal{N}_\xi) \cap \mathcal{P}(\delta)) \setminus Q$, we have that $A \in OD_{T,Q}^{L[T,y]} \cap \mathcal{P}(\delta)$.

We may thus define $f \colon \mathcal{C} \to \mathcal{P}(\delta)$ by letting $f([y]_T)$ be the $<_{HOD_{T,Q}^{L[T,y]}}$–least $X \in (OD_{T,Q}^{L[T,y]} \cap \mathcal{P}(\delta)) \setminus OD_Q$. We have $f \in OD_Q$ (notice $T \in OD$), and f is constant on a cone. Setting $A =$ the $f([y]_T)$ for a cone of y's, we then get $A \in OD_Q$. Contradiction! $\square$ (2.4)

Definition 2.5 *Let M be a premouse with largest cardinal $\alpha \in M$. Then M is called* full *if for all $N \rhd \mathcal{J}_\alpha^M$ s.t. $\mathcal{J}_\alpha^M$ is a cutpoint in N and N is $\Omega+1$ iterable above α we have that $\mathcal{J}_{\alpha^{+N}}^N \trianglelefteq M$.*

We shall need the following key consequence of the mouse set theorem 1.15.

Lemma 2.6 *Let M be full with largest cardinal α. Then $OD_M \cap \mathcal{P}(\alpha) \subset M$.*

Lemma 2.7 *Let $P \in HC$ be such that for a cone $\mathcal{C}$, if $x \in \mathcal{C}$, then $P \in L[T,x]$ and $K^c(P)^{HOD_{T,P}^{L[T,x]}}$ exists. Let $\mathcal{C}' \subset \mathcal{C}$ be given by 2.2. Pick $x \in \mathcal{C}'$, and let $Q = Q_x \rhd P$ be as in 2.3. Then Q is full.*

Proof: Suppose not. Write $\delta = \delta(Q)$. Let $N \rhd \mathcal{J}_\delta^Q$ be s.t. $\mathcal{J}_\delta^Q$ is a cutpoint in N, N is $\Omega+1$ iterable, and $(\Sigma_\omega(N) \cap \mathcal{P}(\delta)) \setminus Q \neq \emptyset$. By Σ_1^2–correctness of $L[T,x]$, there is one such N in $L[T,x]$, and the least one such is in fact in $HOD_{T,P}^{L[T,x]}$ (recall that $Q \in HOD_{T,P}^{L[T,x]}$).

Let Σ be N's (unique) Ω–iteration strategy. By uniqueness, we have that $\Sigma \restriction HOD_{T,P}^{L[T,x]} \in HOD_{T,P}^{L[T,x]}$, so that N is iterable inside $HOD_{T,P}^{L[T,x]}$. But this gives a contradiction with the universality of $K^c(Q)$ inside $HOD_{T,P}^{L[T,x]}$. $\square$ (2.7)

We have therefore established the following.

Corollary 2.8 *Let $P \in HC$ be such that*

$$K^c(P)^{HOD_{T,P}^{L[T,x]}}$$

exists for a cone of x. There is then a full $Q \triangleright P$ such that Q's largest cardinal is Woodin in Q and

$$K^c(Q)^{HOD^{L[T,y]}_{T,Q}}$$

exists for a cone of y.

Definition 2.9 *Let $k < \omega$. We set*

$$([x_0]_T, \dots, [x_{k-1}]_T) \in A_k$$

if there exists a sequence

$$(Q_i \colon i \in \{-1\} \cup k)$$

and some x_k such that $Q_{-1} = \emptyset$, x_0 is a base for the cone from 2.1 (with $S = T$), and for all integers $j < k$, Q_j is the $<_{HOD^{L[T,x_j]}_{T,Q_{j-1}}}$–least Q such that $Q \triangleright Q_{j-1}$, Q's largest cardinal is Woodin in Q, and

$$K^c(Q)^{HOD^{L[T,x]}_{T,Q_{j-1}}}$$

exists for all x in the cone above x_{j+1}.

Notice that if $([x_0], \dots, [x_{k-1}]_T) \in A_k$, then there is a unique "Q-sequence" $(Q_i \colon i < k)$ witnessing this.

We let μ_T denote Martin's measure on the T–degrees.

Definition 2.10 *$(p, U) \in \mathbb{P}$ iff U is a subtree of $\bigcup_k A_k$ with stem p and for all $q \in U$ with $q \supset p$ we have that*

$$\{r \in D_T : q^\frown r \in U\} \in \mu_T.$$

$(p', U') \leq_{\mathbb{P}} (p, U)$ iff $p' \supset p$, and $U' \subset U$.

As any element of A_k comes with its unique "Q-sequence" $(Q_0, \dots, Q_{k-1})$ of Q's, forcing with $\mathbb{P}$ will produce an infinite sequence $\vec{Q} = (Q_0, Q_1, \dots)$ of Q's, to which we'll refer as the "Q–sequence" corresponding to the generic filter.

Lemma 2.11 *Let G be $\mathbb{P}$–generic over V, and let $\vec{Q}$ be the corresponding Q–sequence. Then $\mathcal{P}(\delta(Q_k)) \cap L[\vec{Q}] \subset Q_k$ for all $k < \omega$.*

This immediately gives:

Corollary 2.12 *If G and $\vec{Q}$ are as in 2.11 then*

$$L[\vec{Q}] \models \text{ there are } \omega \text{ many Woodin cardinals.}$$

Proof of 2.11. Let $k < \omega$. By 2.6, 2.7, and the definition of $\mathbb{P}$, in order to show that $\mathcal{P}(\delta(Q_k)) \cap L[\vec{Q}] \subset Q_k$ it is enough to verify that $\mathcal{P}(\delta(Q_k)) \cap L[\vec{Q}] \subset OD_{Q_j}$ for some $j \geq k$.

Let $\delta = \delta(Q_k)$, and let $X \in \mathcal{P}(\delta) \cap L[\vec{Q}]$. Let $\dot{X}$ be a name for X; we may in fact assume $\dot{X}$ is OD. [X is ordinal definable from $\vec{Q}$, which in turn is definable from the generic filter G. We therefore have a name for X which is ordinal definable from a name for G, i.e, a name for X which is just ordinal definable.] Well, by the Prikry lemma there is some $(p, U) \in G$ deciding all $\check{\alpha} \in \dot{X}$ for $\alpha < \delta$.

Claim. $\alpha \in X$ iff $\exists q \in A_{\mathrm{dom}(p)} \exists W (q$ gives $(Q_0, \ldots, Q_{\mathrm{dom}(p)-1})$ and $(q, W) \Vdash \check{\alpha} \in \dot{X})$.

Proof: "$\Rightarrow$": trivial.
"$\Leftarrow$": Notice $(p, U \cap W)$ and $(q, U \cap W)$ are both conditions, and we may find $\mathbb{P}$–generics G' and G'' both giving the same Q–sequence and s.t. $(p, U \cap W) \in G'$ and $(q, U \cap W) \in G''$. But then

$$\dot{X}^{G'} = \dot{X}^{G''},$$

as this interpretation only depends on the Q–sequence, and hence

$$(p, U) \Vdash \check{a} \in \dot{X} \Leftrightarrow (q, W) \Vdash \check{a} \in \dot{X}.$$

$\square$ (Claim)

But this shows $X \in OD_{Q_{\mathrm{dom}(p)-1}}$, and thus the lemma. $\square$ (2.10)

Lemma 2.13 *There is Q as in 2.3 s.t. moreover, setting $W = (K^c(Q))^{HOD^{L[T,P]}_{T,P}}$ the real x is $\mathbb{P}^W_{\delta(Q)}$–generic over W.*

Corollary 2.14 *If in 2.9 we replace "2.3" by "2.13" then still*

$$L[\vec{Q}] \models \textit{ there are } \omega \textit{ many Woodins,}$$

but also there is G^* being $Col(\omega, \Omega)$–generic over $L[\vec{Q}]$ s.t.

$$\mathbb{R}^V = \bigcup_i \mathbb{R}^{L[\vec{Q}][G^* \restriction \delta(Q_i)]},$$

i.e., $V = L(\mathbb{R})$ is a derived model of $L[\vec{Q}]$.

Proof of 2.13. Set $W = (K^c(P))^{HOD_{T,x}^{L[T,P]}}$.

Case 1. $W \models$ there is a Woodin $> P \cap OR$.

Let δ be such a Woodin. Then $\delta \in Q \backslash P$, and x is $\mathbb{P}^W_{\delta(Q)}$–generic over W, which easily follows from the fact that we require extenders with critical point κ to have certificates when being put onto the K^c–sequence.

Case 2. $W \models$ there is no Woodin $> P \cap OR$.

In this case we have to go a bit deeper into [13].

Inside $HOD_{T,x}^{L[T,P]}$, let Σ be the following strategy for the good player in the iteration game on $K^c(P)^{HOD_{T,x}^{L[T,P]}}$ above P; if $\mathcal{T}$ has limit length then pick a cofinal branch coming with a weakly iterable Q–structure, i.e., pick b s.t. there is $\mathcal{M}(\mathcal{T}) \trianglelefteq Q \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$ bith all collapses of countable substructures of Q being $\omega_1 + 1$ iterable above $\delta(\mathcal{T})$. Standard arguments show that in fact there is nothing to pick, i.e., there is always only at most one such branch.

By 2.2, however, Σ cannot be an iteration strategy for $K^c(P)^{HOD_{T,x}^{L[T,P]}}$ above P (inside $HOD_{T,x}^{L[T,P]}$). A few more standard arguments then show that there is an iteration tree $\mathcal{T} \in HOD_{T,x}^{L[T,P]}$ on $W, \mathcal{T}$ being above P, s.t. $\mathcal{T}$ was formed by following $\Sigma, \mathcal{T}$ has limit length, and there is no weakly iterable Q–structure for $\mathcal{M}(\mathcal{T})$. This of course implies that

$$K^c(\mathcal{M}(\mathcal{T})) \models \delta(\mathcal{T}) \text{ is Woodin,}$$

as initial segments of $K^c(\mathcal{L}(\mathcal{T}))$ are $\omega_1 + 1$ iterable.

Set $\mathcal{M} = \mathcal{M}(\mathcal{T})$. Working inside $L[T, x]$, we now define a simple iteration tree $\mathcal{U}$ on $\mathcal{M}$ as follows. (A) At successor steps, hit the least extender ($>$ the largest Woodin below, if there is one) s.t. there is a real which doesn't satisfy the associated axiom. (B) At limit stages we pick the (unique!) cofinal branch coming with a weakly iterable Q–structure.

Notice that $\mathcal{U} \in HOD_{T,x}^{L[T,P]}$. Let's work inside $HOD_{T,x}^{L[T,P]}$.

Case 2a. $\mathcal{M}_\alpha^{\mathcal{U}}$ exists, but there is no extender as in (A).

Let $\delta' = \mathcal{M}_\alpha^{\mathcal{U}} \cap OR$. In this case, to prove 2.13 it clearly suffices to verify the

Claim. $K^c(\mathcal{M}_\alpha^{\mathcal{U}})$ exists and $\models \delta'$ is Woodin.

Proof: Let Θ be large enough, and pick an elementary $\pi : N \to V_\Theta$ with N countable and transitive and all sets of current interest are in $ran(\pi)$.

In V (which is $HOD_{T,x}^{L[T,P]}$ for the moment!) there is a maximal branch b thru $\overline{\mathcal{T}}$ together with a realization map $\sigma : C_k(\mathcal{N}_\xi)$, with $C_k(\mathcal{N}_\xi)$ being from the $K^c(P)$–construction. Let $\overline{Q} \trianglelefteq \mathcal{M}_b^{\overline{\mathcal{T}}}$ be the Q–structure for $\mathcal{M}(\overline{\mathcal{T}} \restriction \sup(b))$ provided by $\mathcal{M}_b^{\overline{\mathcal{T}}}$.

By the usual argument of comparing $\overline{Q}$ with the Q–structure provided by $\mathcal{M}_{\sup(b)}^{\overline{\mathcal{T}}}$ is $\sup(b) < lh(\overline{\mathcal{T}})$ we in fact get that b is cofinal thru $\overline{\mathcal{T}}$.

We may now view $\overline{\mathcal{U}}$ as a tree acting on $\overline{Q}$ (instead of just on $\mathcal{M}(\overline{\mathcal{T}})$, so that we get a map

$$\tilde{\pi}\overline{Q} \to \tilde{Q}$$

extending the iteration map

$$\pi_{0\overline{\alpha}}^{\overline{U}} : \mathcal{M}(\overline{\mathcal{T}}) \to \mathcal{M}_{\overline{\alpha}}^{\overline{\mathcal{U}}},$$

together with a realization $\sigma' : \tilde{Q} \to C_k(\mathcal{N}_\xi)$ with $\sigma \restriction \overline{Q} = \sigma' \circ \tilde{\pi}$.

Now let $\mathcal{N}_\eta$ be the least model from the $K^c(\mathcal{M}_\alpha^{\mathcal{U}})$–construction with the property that $\rho_\omega(\mathcal{N}_\eta) < \delta'$ or δ' is not definably Woodin over $\mathcal{N}_\eta$. Then, as usual $\tilde{Q} = \pi^{-1}(\mathcal{N}_\eta)$, so that $\tilde{Q} \in N$. But then

$$\overline{Q} \simeq h^{\tilde{Q}}(ran(\pi_{0\overline{\alpha}}^{\overline{\mathcal{U}}}) \cup \{p\}), \text{ some } p,$$

and $\tilde{\pi}$ is the inverse of the transitive collapse. Hence both $\overline{Q}$ and $\tilde{\pi}$ are elements of N.

We have shown that $\tilde{\pi} : \overline{Q} \to \tilde{Q}$ exists in N. Moreover $\overline{Q}$ is weakly iterable in N (as $\mathcal{N}_\eta$ is weakly iterable in V). This implies $\overline{Q}$ is weakly iterable in N. By elementarity, then, $\pi(\overline{Q})$ is weakly iterable in V, so that $\mathcal{M}(\mathcal{T})$ admits a weakly iterable Q–structure.

We have reached a contradiction! $\square$ (Claim)

Case 2b. V (which is still $HOD_{T,x}^{L[T,P]}$ here) doesn't see a Q–structure for the common part model.

We then have

$$W' = K^c(\mathcal{M}(\mathcal{T})) \models \delta(\mathcal{T}) \text{ is a Woodin cardinal.}$$

But then by the construction of $\mathcal{U}$, the extenders of W' witnessing Woodinness of $\delta(\mathcal{T})$ in W' all satisfy the desired axiom. $\square$

3 Coarse Γ Woodin mice

In the next section we'll aim to prove 1.15 by using Woodin's core model induction. The rest of the current section will prepare ourselves for the task.

We first show that it is enough to get "Γ–Woodin premice".

Definition 3.1 *Let a be countable and transitive; then a is Γ–amenable if whenever $x \in a$ and $b \in C_\Gamma(a)$, then $b \cap x \in a$.*

Γ–amenability is sometimes called Γ–completeness or Γ–fullness.

Definition 3.2 *Let N be countable and transitive; then N is a coarse Γ–Woodin premouse if*

1. *$N \models ZFC$*
2. *N is Γ–amenable, and*
3. *letting $\delta = OR \cap N$, for any $f : \delta \to \delta$ such that $f \in C_\Gamma(N)$, there is a $\kappa < \delta$ such that $f''\kappa \subseteq \kappa$ and an E such that $N \models E$ is an extender with critical point κ and $V_{i_E(f)(\kappa)} \subseteq Ult(V, E)$.*

Let T be the tree of a Γ scale on a universal Γ set, where Γ is good, and assume AD. Let N be countable transitive and $\delta = OR \cap N$. It is easy to see then that N is a coarse Γ–Woodin premouse iff $N = V_\delta^{L(N \cup \{T,N\})}$ and $L(N \cup \{T, N\}) \models \delta$ is Woodin.

Our above condensation results easily yields the following lemma.

Lemma 3.3 *Let Γ_0 and Γ_1 be good pointclasses such that $\Gamma_0 \subseteq \Delta_1$. Let N be a coarse Γ_1–Woodin premouse; then for some $\eta < OR \cap N$, V_η^N is a coarse Γ_0–Woodin premouse.*

Proof: Let T_1 be the tree of a Γ_1 scale on a universal Γ_1 set, and let $T, U \in L(N \cup \{T_1, N\})$ be trees projecting to the universal Γ_0 set and its complement. Let $\delta = OR \cap N$, and let M be a Σ_2 admissible set of the form $L_\alpha(N \cup \{T_1, N\})$ such that $T, U \in M$. As δ is strongly inaccessible in $L(N \cup \{T_1, N\})$ we can, working in the universe, form a hull of M whose intersection with $V_\delta^M (= N)$ is of the form V_η^M for some $\eta < \delta$. Letting $\overline{M}$ be the collapse of this hull, we have $C_{\Gamma_0}(V_\eta^N) \subseteq \overline{M}$ by 1.12. On the other hand, δ is Woodin in M, so η is Woodin in $\overline{M}$. Therefore V_η^N is a coarse Γ_0–Woodin premouse. □

In what follows, we shall actually need coarse Γ-Woodin premice which are iterable:

Lemma 3.4 *Let Γ_i, $i < 5$, be good pointclasses such that $\Gamma_i \subset \Delta_{i+1}$ for all $i < 4$. Let y be a real, and P be a coarse Γ_4-Woodin premouse with $y \in P$. Then there is a coarse Γ_0-Woodin premouse M such that $y \in M$, and M has an ω_1-iteration strategy Σ in Γ_2 such that every Σ-iterate of M is again a coarse Γ_0-Woodin premouse.*

Proof: Let $\eta < P \cap OR$ be least such that V_η^P is a Γ_0-Woodin premouse. By 3.3, there is no $\xi \leq \eta$ such that V_ξ^P is a Γ_1-Woodin premouse. There is thus a function $f : \xi \to \xi$ in $C_{\Gamma_1}(V_\xi^P)$ witnessing non-Woodinness of ξ. Let T, U be trees for the universal Γ_1 set and its complement which are constructible from T_4 (the tree for a universal Γ_4 set). Let N be a Σ_2 admissible set of the form $L_\alpha(V_\eta^P \cup \{T, U\})$, and let $\pi : \bar{N} \to N$ be elementary, with π, $\bar{N}$ in P and countable there, and $\pi(\bar{\eta}, \bar{T}, \bar{U}) = \eta, T, U$.

Claim. In P, $\bar{N}$ is ω_1-iterable (w.r.t. trees living on $V_{\bar{\eta}}^{\bar{N}}$) via the strategy of choosing the unique cofinal π-realizable branch.

Proof: Work in P. By [5], there is always a maximal such branch. We therefore just have to see that if $\mathcal{T}$ is on $\bar{N}$ has cofinal π-realizable branches, then $b = c$. Well, let $\delta = \delta(\mathcal{T})$, and $M = V_\delta^{\mathcal{M}_b^\mathcal{T}} = V_\delta^{\mathcal{M}_c^\mathcal{T}}$. Let σ, τ be the realizing maps for b, c respectively. Since T, $U \in ran(\sigma)$, 1.12 gives that $C_{\Gamma_1}(M) \in \mathcal{M}_b^\mathcal{T}$ and $\sigma(C_{\Gamma_1}(M)) = C_{\Gamma_1}(V_\xi^N)$, where $\xi = \sigma(\delta) \leq \eta$. By the choice of η, this gives that there is an $f \in C_{\Gamma_1}(M)$ witnessing non-Woodinness for M. Since $C_{\Gamma_1}(M) \in \mathcal{M}_c^\mathcal{T}$ by the same argument, $f \in \mathcal{M}_b^\mathcal{T} \cap \mathcal{M}_c^\mathcal{T}$. But then [5] yields that $b = c$. □

This proof easily yields the following.

Claim. In P, if b is a cofinal π-realizable branch thru a countable $\mathcal{T}$ on $V_{\bar{\eta}}^{\bar{N}}$, then setting $M = V_{\delta(\mathcal{T})}^{\mathcal{M}_b^\mathcal{T}}$, b is the unique cofinal branch d thru $\mathcal{T}$ such that $C_{\Gamma_1}(M) \subset \mathcal{M}_d^\mathcal{T}$.

It easily follows from the claims that P satisfies that $\bar{N}$ has an ω_1-iteration strategy which is in Γ_2. But P is Γ_4-correct, so this is indeed true. □

We now borrow from the next section:

Lemma 3.5 *Let $[\alpha, \beta]$ be a gap, and suppose that for all scaled Σ–pointclasses $\overline{\Gamma}$ which show up earlier than $\Sigma_1(J_\alpha(\mathbb{R}))$ we have that $C_{\overline{\Gamma}}$ is caputured by mice* *with an ω_1 iteration strategy which is projective in $\overline{\Gamma}$. Then for all scaled Σ–pointclasses Γ which show up in $[\alpha,\beta]$ there is a coarse Γ–Woodin mouse with an ω_1–iteration strategy which is projective in Γ.*

Proof of 1.15 from 3.5. Notice first that if for all scaled Σ–pointclasses $\overline{\Gamma}$ which show up earlier that $\Sigma_1(J_\alpha(\mathbb{R}))$, $C_{\overline{\Gamma}}$ is captured by mice with an ω_1 iteration strategy which is projective in $\overline{\Gamma}$, and if α is a limit ordinal or else if $\alpha-1$ is critical, then $C_{\Sigma_1(J_\alpha(\mathbb{R}))}$ is easily seen to be a mouse operator. This already gives the second part of 1.15.

We may assume from now on that either α is inadmissible and $\Gamma=\Sigma_{2n+1}(J_\alpha(\mathbb{R}))$ for some n, or else α is admissible and $\Gamma=\Sigma_{n+2i}(J_\beta(\mathbb{R}))$ for some i, where n is least with $\rho_n(J_\beta(\mathbb{R}))=\mathbb{R}$.

Let us turn towards the first part of 1.15. It suffices to prove the first part of 1.15 for reals. We first prove it "on a cone". An operator $\mathcal{O}:\mathbb{R}\mapsto\mathcal{P}(\mathbb{R})$ is called "fine structural" if it is a mouse operator on a cone, that is, if for a cone of $x, \mathcal{O}(x)=\mathbb{R}\cap\mathcal{M}_x$ for some ω_1+1 iterable x–premouse $\mathcal{M}_x$. [This terminology might be a bit awkward.]

Claim. $x\mapsto C_\Gamma(x)$ is fine structural. Moreover, for a cone of $x, \mathcal{O}(x)=\mathbb{R}\cap\mathcal{M}_x$ for some ω_1+1 iterable x–premouse $\mathcal{M}_x$ with an ω_1–iteration strategy which is projective in Γ.

Proof: By a theorem of Rudominer and Steel, it is enough to find a fine structural inner model operator which is above $x\mapsto C_\Gamma(x)$ in $\leq_m$ (here, $\leq_m$ denotes the prewellorder of inner model operators; cf. [9]). By the comparability of inner model operators, this follows if we show that for any real y, there is an $x\geq_T y$, an ω_1–iterable x–mouse $\mathcal{R}$, and a real $z\in\mathcal{R}$ such that $z\notin C_\Gamma(x)$. So fix a real y.

Let Γ_0 be a good pointclass such that $\Gamma\subset\Delta_0$, and Γ_0 shows up in $[\alpha,\beta]$. Let M be a coarse Γ_0–Woodin premouse which has an ω_1–iteration strategy projective in Γ_0 (in fact, projective in Γ) and is such that $y\in M$; the existence of M is guaranteed by the proof of 3.5. Let $\Omega=OR\cap M$, and $\langle\mathcal{N}_\eta|\eta\leq\Omega\rangle$ be the levels of the $L[\vec{E},y]$ construction of [6] done inside M. (So y is thrown in at the bottom, and we use full background extenders.) Since M is fully iterable, all $\mathcal{N}_\eta$ are fully iterable, and the construction never breaks down. (Cf. [6].) As Ω is Woodin in $L(M\cup\{M\})$, Ω is Woodin in $\mathcal{Q}$, where $\mathcal{Q}$ is the premouse of height OR whose Ω^{th} level $\mathcal{J}^{\mathcal{Q}}_\Omega$ is $\mathcal{N}_\Omega$. Since $\mathcal{Q}$ has an ω_1–iteration strategy in close to Γ_0, Ω is tame. It follows that for all sufficiently large $\eta<\Omega$, η is not Woodin in $\mathcal{Q}$.

There is a club $B\subseteq\Omega$ in $L(M\cup T_0,M)$ such that for all $\eta\in B$, V^M_η is

Γ–Woodin and $\mathcal{N}_\eta = J_\eta^{\mathcal{Q}}$. (Here T_0 is the tree of a Γ_0 scale on a universal Γ_0 set. We actually use here the proof of 3.3, and not just its statement.) Fix $\eta \in B$ such that η is not Woodin in $\mathcal{Q}$. Now $J_\eta^{\mathcal{Q}}$ is the output of the $L[\vec{E}, y]$–construction done up to η in $L(V_\eta^M \cup \{T\})$, where T is the tree of a scale for Γ, and η is Woodin in this universe. It follows from [6] that if $f : \eta \mapsto \eta$ is in $C_\Gamma(V_\eta^M)$ and amenable to $J_\eta^{\mathcal{Q}}$, then $J_\eta^{\mathcal{Q}}$ is Woodin with respect to f. Since η is not Woodin in $\mathcal{Q}$, we can fix a subset b of η which is in $\mathcal{Q}$ but not in $C_\Gamma(V_\eta^M)$. Let $\mathcal{P} = J_{\xi+1}^{\mathcal{Q}}$, where $b \in (J_{\xi+1}^{\mathcal{Q}} \backslash J_\xi^{\mathcal{Q}})$.

Now let $g : \omega \to J_\eta^{\mathcal{Q}}$ be $\mathcal{Q}$–generic for $Col(\omega, J_\eta^{\mathcal{Q}})$ and such that, setting $x = x_g$, we have $b_x \notin C_\Gamma(x)$; there are in fact comeager many such g. Clearly, $y \leq_T x$ and $b_x \in \mathcal{P}[x]$. It remains only to show that $\mathcal{P}[x]$ can be re–arranged as an x–mouse $\mathcal{R}$. We define $\mathcal{R}$ by adding E to the $\mathcal{R}$–sequence with index α just in case $\eta < \alpha$, α indexes an extender F on the $\mathcal{P}$–sequence, and E is the canonical extension of F to $J_\alpha^{\mathcal{R}} = J_\alpha^{\mathcal{P}}[x]$ determined by the fact that that this structure is a small forcing extension of $J_\alpha^{\mathcal{P}}$. One can prove by induction on β, using the quantifier–by–quantifier definability of the forcing relation over $J_{\eta+\beta}^{\mathcal{P}}$, that $J_\beta^{\mathcal{R}}$ has the same projecta and standard parameters as $J_{\eta+\beta}^{\mathcal{P}}$, and hence is ω–sound. We leave the details to the reader. $\square$

We now want to show that if $z \in C_\Gamma$ then there is a (fine structural lightface) premouse $\mathcal{Q}$ such that $z \in \mathcal{Q}$ and $\mathcal{Q}$ has an ω_1–iteration strategy which is projective in Γ. This will suffice, as we may as well choose $\mathcal{Q} = \mathcal{Q}_z$ so that it projects to ω and is ω–sound. Letting $\eta = OR \cap \mathcal{Q}$, we then have $\mathcal{Q}$ is definable as the unique ω_1–iterable, ω–sound premouse of height η projecting to ω. If we then let $\mathcal{M}$ be the premouse whose proper initial segments are precisely the $\mathcal{Q}$'s, we'll have that $C_\Gamma \subset \mathcal{M}$. The proof will straightforwardly relativize to any real.

So fix $z \in C_\Gamma$. Let $B \in \Gamma$ and $\xi < \omega_1$ be such that z is unique with $(z, z^*) \in B$ for any z^* coding ξ. We'll have that for all reals x, $(C_\Gamma(x); B \cap C_\Gamma(x))$ is a Σ_1–elementary substructure of $(\mathbb{R}; B)$. [Here we use that Γ has the scale property, so that each relation in Γ can be uniformized by a function whose graph is in Γ, and hence each non-empty $\Gamma(z)$-set has a member u such that $\{u\}$ is in $\Gamma(z)$.] Let $\Gamma = \Sigma_n(J_\gamma(\mathbb{R}))$ (where $\gamma \in \{\alpha, \beta\}$ and $\rho_n(J_\gamma(\mathbb{R})) = \mathbb{R}$). We then may and shall in fact assume that B codes the n^{th} reduct, call it M^n, of $J_\beta(\mathbb{R})$. Notice that we can express in a Π_2 fashion that $\mathbb{R} =$ the reals of the transitive collapse of $(\mathbb{R}; B)$. Thus $(C_\Gamma(x), B \cap C_\Gamma(x)) \prec_{\Sigma_1} (\mathbb{R}, B)$ will give an embedding $\overline{\pi} : \overline{M} \to_{\Sigma_1} M^n$, which lifts to

$$\pi : J_{\overline{\beta}}(C_\Gamma(x)) \to_{\Sigma_n} J_\beta(\mathbb{R})$$

for some $\overline{\beta} \leq \beta$. Then our given z is definable over $J_\beta(C_\Gamma(x))$ from a countable ordinal. We'll use this fact below.

By the above Claim, the operator $x \mapsto C_\Gamma(x)$ is fine structural; in fact, we may fix y so that whenever $x \geq_T y$ there is an ω_1–iterable x–mouse $\mathcal{N}_x$ whose ω_1 iteration strategy is projective in Γ and whose reals are those in $C_\Gamma(x)$.

Let Γ_0 be a good pointclass such that $\Gamma \subset \Delta_0$ and Γ_0 shows up in $[\alpha, \beta]$, and let M be a coarse Γ_0 Woodin mouse having an ω_1–iteration strategy projective in Γ_0 and such that $y \in M$. Let $\langle \mathcal{N}_\eta | \eta \leq \Omega \rangle$ be the models of the $L[\vec{E}]$ construction (as in [6], and done over the real $\emptyset$) of M. Just as in the proof of the above Claim, we can fix an $\eta < \Omega$ projecting to η is not in $C_\Gamma(V_\eta^M)$. Let us choose $\mathcal{Q}$ to be the first level $\mathcal{P}$ of $\mathcal{N}_\Omega$ such that $\mathcal{P}$ projects to η and $\mathcal{P} \notin C_\Gamma(\mathcal{N}_\eta \cup \{y, \mathcal{N}_\eta\})$. Notice that $\mathcal{Q} \models$ "η is Woodin".

Let $\mathbb{P}$ be the every–real–generic poset of $\mathcal{Q}$ (up to η). Here we only use extenders from the $J_\eta^\mathcal{Q}$–sequence which are total and strong out to their lengths to define the identities. Since the $J_\eta^\mathcal{Q}$–sequence has background extenders from V_η^M (which haven't been collapsed in the construction) for these extenders on the $J_\eta^\mathcal{Q}$ sequence, every real in M is $\mathbb{P}$–generic over $\mathcal{Q}$. In particular, y is so generic.

By our choice of $\mathcal{Q}$ there are comeager many $y : \omega \to J_\eta^\mathcal{Q} \cup \{y\}$ such that $\mathcal{Q}$ is not coded by any real in $C_\Gamma(x_f)$. We can therefore fix such an f which is $Col(\omega, J_\eta^\mathcal{Q} \cup \{y\})$ generic over $\mathcal{Q}[y]$. Let $x = x_f$. Clearly, $y \leq_T x$. Also x is $\mathcal{Q}$–generic over a poset of size η in $\mathcal{Q}$, and x codes $J_\eta^\mathcal{Q}$, so by the level–by–level definability of forcing we can find an x–premouse $\mathcal{R}$ whose universe is $\mathcal{Q}[x]$. The iterability of $\mathcal{Q}$ guarantees that of $\mathcal{R}$. Since $\mathcal{Q}$ projects to η, $\mathcal{R}$ projects to ω. By our choice of x, the real canonically coding $\mathcal{R}$, its first order theory with parameter x is not in $C_\Gamma(x)$. On the other hand, every proper initial segment of $\mathcal{Q}$ projecting to η is in $C_\Gamma(J_\eta^\mathcal{Q} \cup \{y\})$, and therefore every proper initial segment of $\mathcal{R}$ with $\mathcal{N}_x$ we see easily that $\mathbb{R} \cap \mathcal{R} = C_\Gamma(x)$.

We now show that z is ordinal definable over $\mathcal{R}$. This will suffice to finish the proof, since $\mathcal{R}$ is a homogeneous forcing extension of $\mathcal{Q}$ (being an extension via a poset of size η which collapses η to ω), so that we have $z \in \mathcal{Q}$ as desired.

Now recall that z is definable over $J_{\overline{\beta}}(C_\Gamma(x))$ from a countable ordinal. W.l.o.g., $\overline{\beta} \in \mathcal{R}$; this is because the extender sequence of $\mathcal{R}$ is nonempty (since $\Pi_2^1 \subset \Gamma$). Since $C_\Gamma(x)$ is ordinal definable over $\mathcal{R}$, as its set of reals, we have that z is ordinal definable over $\mathcal{R}$, as desired. $\square$

Definition 3.6 *Let M be a (countable) premouse, and let $\delta \in M$ be s.t. $M \models \delta$ is Woodin.*

Let $A \subset \mathbb{R}^n$, some $n < \omega$, and $\tau \in m^{Col(\omega,\delta)}$. Then τ captures A over M if the following holds true. There is an ω_1 iteration strategy Σ for M s.t.: if M^*

is a (countable) simple Σ–iterate of M with iteration map π_{MM^*} and $g \in V$ is $Col(\omega, \pi_{MM^*}(\delta))$–generic over M^* then

$$(\pi_{MM^*}(\tau))^g = A \cap M^*[g].$$

Moreover, let Γ be a pointclass. Then Γ is captured over M if for any $A \in \Gamma$ there is $\tau \in M^{Col(\omega,\delta)}$ capturing A over M.

Lemma 3.7 *Let M be a countable premouse, and let $\delta < \eta \in M$ be s.t. $M \models$ both δ and η are Woodin cardinals. Let $B \subset \mathbb{R} \times \mathbb{R}$. Suppose that $\tau \in M^{Col(\omega,\eta)}$ captures B over M. Then there is $\sigma \in M^{Col(\omega,\delta)}$ capturing $\exists^{\mathbb{R}} B$ over M.*

Proof: Let Σ be an ω_1 iteration strategy for M witnessing τ captures B over M. Given ρ, a name for a real in $M^{Col(\omega,\delta)}$ s.t. for all simple Σ–iterates M^* of M with iteration map π_{MM^*} and for all $g \in V$ being $Col(\omega, \pi_{MM^*}(\delta))$–generic over M^* we have that $\pi_{MM^*}(\overline{\rho})^g \in (M^*[g])^{Col(\omega,\pi_{MM^*}(\eta))}$ is a name for the real ρ^g and $\pi_{MM^*}(\nu)^g \in (M^*[g])^{Col(\omega,\pi_{MM^*}(\eta))}$ is a name capturing B over $M^*[g]$.

Now let us define $\sigma \in M^{Col(\omega,\delta)}$ as follows. We put $(p, \rho) \in \sigma$ iff $p \in Col(\omega, \delta), \rho$ is a name for a real in $M^{Col(\omega,\delta)} \cap H^M_{\delta+}$, and

$$p \Vdash \text{“}\exists q \in Col(\check{\omega}, \check{\eta}) q \Vdash \exists y(\overline{\rho}, y) \in \overline{\tau}\text{''}.$$

We claim that Σ also witnesses σ captures $\exists^{\mathbb{R}} B$ over M.

Let M^* be a countable simple Σ–iterate of M with iteration map π_{MM^*}, and let $g \in V$ be $Col(\omega, \pi_{MM^*}(\delta))$–generic over M^*. As usual Σ induces an ω_1 strategy for iterating $M^*[g]$ above δ, which we'll also denote by Σ.

First let $x \in \exists^{\mathbb{R}} B \cap M^*[g]$. We aim to show $x \in \pi_{MM^*}(\sigma)^g$.

Pick y_0 s.t. $(x, y_0) \in B$. By Woodin's genericity theorem there is a countable iterate M of $M^*[g]$ with iteration map $\pi_{M^*[g]\tilde{M}}$ s.t., setting $\tilde{\eta} = \pi_{M^*[g]\tilde{M}} \circ \pi_{MM^*}(\eta)$, y_0 is $\mathbb{P}^{\tilde{M}}_{\eta}$–generic over $\tilde{M}$. Moreover $\mathbb{P}^{\tilde{M}}_{\tilde{\eta}}$ has the $\tilde{\eta}$–c.c., so that in fact $\tilde{M}[y]$ is a $\tilde{\eta}$–c.c. extension of a Σ–iterate M' of M. Thus g, y_0 can be absorbed by some $G \in V$ being $Col(\omega, \tilde{\eta})$–generic over M', and we may also write $\tilde{M} = M'[G] = M'[g][\overline{G}]$ where $\overline{G}$ is $Col(\omega, \tilde{\eta})$–generic over $M'[g]$.

By how $\overline{\tau}$ was chosen, we have that $\tilde{M} \models (x, y_0) \in \pi_{M^*[g]\tilde{M}}(\pi_{MM^*}(\overline{\tau})^g)^{\overline{G}}$, i.e., $\tilde{M} \models \exists y(x, y) \in \pi_{M^*[g]\tilde{M}}(\pi_{MM^*}(\overline{\tau})^g)^{\overline{G}}$. So $M'[g] \models \exists y\ q \Vdash \exists y(\check{x}, y) \in \pi_{\tilde{M}[g]\tilde{M}}(\pi_{MM^*}(\overline{\tau})^g)$, which implies $M[g] \models \exists q\ q \Vdash \exists y(\check{x}, y) \in \pi_{MM^*}(\nu)^g$. This now gives $x \in \pi_{MM^*}(\sigma)^g$.

For the other direction, let $x \in \pi_{MM^*}(\sigma)^g$. We want to see that $x \in \exists^{\mathbb{R}} B$.

By the definition of σ and the elementarity of π_{MM^*} we may pick $\rho \in (M^*)^{Col(\omega,\pi_{MM^*}(\delta))}$ and $p \in g$ s.t. $\rho^g = x$ and $p \Vdash$ “$\exists q \in Col(\check{\omega}, \pi_{MM^*}(\check{\eta}))q \Vdash$

$\exists y(\overline{\rho}, y) \in \pi_{MM^*}(\overline{\tau})''$. Hence $M^*[g] \models \exists q \in Col(\omega, \pi_{MM^*}(\eta)) q \ \Vdash \exists y(\check{x}, y) \in \pi_{MM^*}(\overline{\tau})^g$.

But then if $\overline{G}$ is $Col(\omega, \pi_{MM^*}(\delta))$–generic over $M^*[g]$ we have, by the choice of $\overline{\tau}$, $M^*[g][\overline{G}] \models \exists y(x, y) \in (\pi_{MM^*}(\overline{\tau})^g)^{\overline{G}} \subset B$. Thus $x \in \exists^{\mathbb{R}} B$. □ (3.6)

Corollary 3.8 *Let $[\alpha, \beta]$ be a gap, and let $n < \omega$. Suppose that either $\alpha < \beta$, $[\alpha, \beta]$ is weak, and n is least s.t. $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$ or else $\alpha = \beta$ is inadmissible and $n = 1$. Let M be a countable premouse, and let $\delta_0 < \delta_1 < \ldots < \delta_{2i} \in M$ be s.t. $\forall j \leq 2i M \models \delta_j$ is Woodin.*

Suppose that there is a universal $\Sigma_n(J_\beta(\mathbb{R}))$–set being captured over M. Then a universal $\Sigma_{n+2i}(J_\beta(\mathbb{R}))$–set is captured over M, too.

Proof: This easily follows from 3.7. □

The following Lemma of course also relativizes.

Lemma 3.9 *Let M be a premouse with $\rho_\omega(M) > \omega$, and let $a \in \mathbb{R}$ be s.t. $a \in C_\Gamma$. Suppose that Γ is captured over M. Then in fact $a \in M$.*

Proof: By assumption, there are $A \in \Gamma \cap \mathbb{R}^2$ and a countable ordinal α s.t. for any real code y for α we have that α is the unique x with $A(x, y)$. Let M' be the result of hitting the least total–on–M extender (and its images) $\alpha + 1$ many times, and let $\pi_{MM'}$ be the iteration map.

Claim. $n \in a \Leftrightarrow \ \Vdash^{M'}_{Col(\omega, \pi_{MM'}(\delta))} \exists x, y(x \text{ codes } \check{\alpha} \wedge (x, y) \in \tau \wedge \check{n} \in x)$.

Proof of Claim. "⇒" Let M^* be a (countable) simple Σ–iterate of M' with iteration map $\pi_{MM'}$ s.t. a is $\mathbb{P}^{M^*}_{\pi_{M'M^*} \circ \pi_{MM'}(\delta)}$–generic over M^*. By the $\pi_{M'M^*} \circ \pi_{MM'}(\delta)$–c.c. of $\mathbb{P}^{M^*}_{\pi_{M'M^*} \circ \pi_{MM'}(\delta)}$ we may pick $g \in V$ being $Col(\omega, \pi_{M'M^*} \circ \pi_{MM'}(\delta))$–generic over M^* s.t. $a \in M^*[g]$.

Let $y \in M^*[g]$ be a real code for α. We have that $\tau^g = A \cap M^*[g]$, so $(a, y) \in \tau^g$, and so

$$\Vdash^{M^*}_{Col(\omega, \pi_{M'M^*} \circ \pi_{MM'}(\delta))} \exists x, y(y \text{ codes } \check{\alpha} \wedge (x, y) \in \tau \wedge \check{n} \in x).$$

But then by elementarity

$$\Vdash^{M^*}_{Col(\omega, \pi_{MM'}(\delta))} \exists x, y(y \text{ codes } \check{\alpha} \wedge (x, y) \in \tau \wedge \check{n} \in x).$$

"⇐": Let $h \in V$ be $Col(\omega, \pi_{MM'}(\delta))$–generic over M'. If $(x, y) \in M'[h]$ are s.t. y codes α and $(x, y) \in \tau^h$ then of course $x = a$, so that $n \in x$ implies $n \in a$.

□ (Claim)

□ (3.8)

4 The core model induction

This section is entirely devoted to a

Proof of 1.15. The proof is by induction on gaps $[\alpha, \beta]$, and by working ourselves thru the gap. So let us fix a gap $[\alpha, \beta]$, and let's assume 1.15 to be already established for all previous gaps $[\overline{\alpha}, \overline{\beta}]$ for $\overline{\beta} < \alpha$. We know by now that it suffices to prove 3.5 for the scaled Σ–pointclasses which show up in $[\alpha, \beta]$.

There is bad news, namely that the proof has to be split into three main cases, the third of which consists of six subcases. We commence with listing the cases together with brief discussions. On the other hand, the good news is that there is a single common pattern according to which all of the individual cases will be settled.

Case1. ("Projective case") $\alpha = \beta = 1$.

Here we should show, by induction on $n < \omega$, that the mouse operators $M_n^\#$ are all total on $\mathbb{R}$ (cf. [11] on the definition of the $M_n^\#$'s). This case may be viewed as a degenerate case of what is going on in the "weak gap case" argument below. Moreover, it is a straightforward generalization of the proof of [12] Thm. 7.7 (which essentially gives the result for $n = 1$).

We shall hence leave that case as an exercise. (As the careful reader might notice, PD is enough for the result that all $M_n^\#$'s are all total on $\mathbb{R}$.[3])

Case 2. ("Weak gap case") $\alpha < \beta$, i.e., $[\alpha, \beta]$ is a proper gap.

Set $\Gamma = \Sigma_1(J_\alpha(\mathbb{R}))$. By [10] 2.6 and 2.9, α is admissible, which implies that

$$C_\Gamma \bigcup_{\overline{\alpha}<\alpha \text{ begins a gap}} C_{\Sigma_1(J_{\overline{\alpha}(\mathbb{R})})}.$$

This immediately gives 1.15 for $[\alpha, \beta]$, unless it is a *weak* gap. Let us assume this to be the case.

Generalizing the argument from Case 1 we shall prove, by induction on $n < \omega$, that certain mouse operators "$\tilde{M}_n^\#$" (this won't be our notation) are total on $\mathbb{R}$. What distinguishes $\tilde{M}_n^\#$ from $M_n^\#$ is that it is "hybrid" in so far as beyond an

[3] In particular, PD proves the consistency of $ZFC \cup \{$ "there are n Woodin cardinals" $: n < \omega\}$, but of course not of $ZFC \cup \{$ "there are infinitely many Woodin cardinals" $\}$ (not even AD gives the latter).

extender sequence it also contains terms capturing a universal set at that level where we have to establish 1.15 next. By 3.8 and 3.9 the $\tilde{M}_n^{\#}$'s will "almost" witness 1.15 — if it were not the case that they are hybrid. However, the desired conclusion will then follow from 3.5.

Let us be a bit more specific already at this point. Let $n < \omega$ be least s.t. $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$, and set $\Gamma^* = \Sigma_n(J_\beta(\mathbb{R}))$. Notice that Γ^* is the next pointclass for which we aim to verify C_{Γ^*} is a mouse operator.

Definition 4.1 $\mathcal{A} = \{A_i : i < \omega\}$ *is called a (Case 2–) self–justifying system (sjs) if*

(a) A_0 *is a universal* Γ*–set,*

(b) $\cup_i A_{2i+1}$ *is a universal* Γ^**–set,*

(c) for any $i < \omega$*,* A_i *has a scale whose individual norms are all* $\in \mathcal{A}$*,*

(d) $\forall i \exists j A_j = \neg A_i$*, and*

(e) $\mathcal{A} \subset J_\beta(\mathbb{R})$*.*

By [10], there is a sjs. Let us pick one, $\mathcal{A}$. We shall verify (cf. 4.5 below) that certain premice which are closed under C_Γ (i.e., closed under what we have so far!) contain terms capturing the individual elements of $\mathcal{A}$. We shall then build new, hybrid, premice, by "throwing in" (codes for) sequences of such terms. Hence, such hybrid premice will contain terms capturing a universal Γ^*–set.

By arguing similar as in the proof of [12] 7.7 (in fact, by building "hybrid K^c's" inside such hybrid premice) we shall eventually be able to construct — using 3.8 and 3.9 — coarse $\Sigma_{n+i}(J_\beta(\mathbb{R}))$–Woodin premice. Finally, 3.5 will give what we are shooting for.

Let's now see how this strategy will be adapted to the remaining cases.

Case 3. ("Improper gap case") $\alpha = \beta > 1$.

Here, we shall set $\Gamma^* = \Sigma_1(J_\alpha(\mathbb{R}))$. Notice that, except for the case that α is admissible (and in which, as above with the strong gap, we get 1.15 for $[\alpha, \alpha]$ for free), again Γ^* is the next pointclass for which we aim to verify C_{Γ^*} is a mouse operator.

However, there won't be a pointclass Γ around, so that we will have to redefine what it means to be a sjs.

Case 3.1. α is a successor, $\alpha = \gamma + 1$, say.

Definition 4.2 *$\mathcal{A} = \{A_i : i < \omega\}$ is called a (Case 3.1–) self–justifying system (sjs) if*

(a) for all $i < \omega$ there is a universal $\Sigma_i(J_\gamma(\mathbb{R}))$–set in $\mathcal{A}$,

(b) $\cup_i A_{2i+1}$ is a universal Γ^*–set,

(c) for any $i < \omega$, A_i has a scale whose individual norms are all $\in \mathcal{A}$,

(d) $\forall i \exists j A_j = \neg A_i$, and

(e) $\mathcal{A} \subset J_\alpha(\mathbb{R})$.

Notice that any set $A \in \Sigma_1(J_\alpha(\mathbb{R}))$ can be written as $\bigcup_k A_k$ with the A_k's being in $J_\alpha(\mathbb{R})$. [Let $A_k = \{x : S_{\omega\gamma+k}(\mathbb{R}) \models \exists t\phi(t,x)\}$ where $\phi \in \Sigma_0$ and $x \in A \leftrightarrow J_\alpha(\mathbb{R}) \models \exists t\phi(t,x)$.]

Case 3.1.1. The previous gap $[\overline{\gamma}, \gamma]$ is a weak (proper) gap.

Here, by [10], every set in $J_\alpha(\mathbb{R})$ has a scale in $J_\alpha(\mathbb{R})$. In particular, a sjs trivially exists.

Case 3.1.2. The previous gap $[\overline{\gamma}, \gamma]$ is a strong (proper) gap.

By [10], every set in $J_\alpha(\mathbb{R})$ has a scale in $\Sigma_1(J_\alpha(\mathbb{R}))$ all of whose individual norms are in $J_\alpha(\mathbb{R})$. In particular, still a sjs exists.

Case 3.1.3. The previous gap $[\overline{\gamma}, \gamma]$ is an improper one, i.e. $\gamma = \overline{\gamma}$.

Case 3.1.3.1. γ is inadmissible.

Here, by [10], every set in $J_\alpha(\mathbb{R})$ has a scale in $J_\alpha(\mathbb{R})$. So again, as in Case 3.1 a sjs trivially exists.

Case 3.1.3.2. γ is admissible.

This case resembles 3.1.2. By [10], every set in $J_\alpha(\mathbb{R})$ has a scale in $\Sigma_1(J_\alpha(\mathbb{R}))$ all of whose individual norms are in $J_\alpha(\mathbb{R})$. So again a sjs exists.

Case 3.2. α is a limit ordinal.

Case 3.2.1. $cf(\alpha) = \omega$.

It is easy to see that here, too, any set $A \in \Sigma_1(J_\alpha(\mathbb{R}))$ can be written as $\bigcup_k A_k$ with the A_k's being in $J_\alpha(\mathbb{R})$. [Letting $(\alpha_k : k < \omega)$ being cofinal in α we may set $A_k = \{x : J_{\alpha_k}(\mathbb{R}) \models \exists t\phi(t,x)\}$ where $\phi \in \Sigma_0$ and $x \in A \leftrightarrow J_\alpha(\mathbb{R}) \models \exists t\phi(t,x)$.]

Once more, we have to slightly redefine the concept of a sjs.

Definition 4.3 $\mathcal{A} = \{A_i : i < \omega\}$ *is called a (Case 3.2.1–) self–justifying system (sjs)*

(a) $\cup_i A_{2i+1}$ *is a universal* Γ^**–set,*

(b) for any $i < \omega$*,* A_i *has a scale whose individual norms are all* $\in \mathcal{A}$*,*

(c) $\forall i \exists j A_j = \neg A_i$*, and*

(d) $\mathcal{A} \subset J_\alpha(\mathbb{R})$.

It is clear now that by [10] there is a sjs.

We now finally turn towards our last case. Let us remark at this point that (except for the projective case) this is the only one where we won't have to produce hybrid premice as a tool for proving 1.15. It will be enough to work with ordinary fine structural premice.

Case 3.2.2. $cf(\alpha) > \omega$.

That this case is a sort of "exception" is also highlighted by the fact that here we can't hope to get a sjs $\subset J_\alpha(\mathbb{R})$. However, we can do with $\mathbf{\Delta}_1(J_\alpha(\mathbb{R}))$:

Definition 4.4 $\mathcal{A} = \{A_i : i < \omega\}$ *is called a (Case 3.2.2–) self–justifying system (sjs)*

(a) $\exists^{\mathbb{R}} A_0$ *is a universal* Γ^**–set,*

(b) for any $i < \omega$*,* A_i *has a scale whose individual norms are all* $\in \mathcal{A}$*,*

(c) $\forall i \exists j A_j = \neg A_i$*, and*

(d) $\mathcal{A} \subset \mathbf{\Delta}_1(J_\alpha(\mathbb{R}))$.

Let us assume that α is inadmissible, as o.w. there is nothing to prove. Pick $g : \mathbb{R} \to \alpha$ cofinal, $g \in \mathbf{\Sigma}_1(J_\alpha(\mathbb{R}))$. Let $(y,(z,\phi)) \in A_0$ iff $y, z \in \mathbb{R}$, ϕ is Σ_1, and $J_{g(y)}(\mathbb{R}) \models \phi(z)$. A_0 is easily seen to be $\mathbf{\Delta}_1(J_\alpha(\mathbb{R}))$.

Moreover, $\exists^{\mathbb{R}} A_0$ is a universal Γ^*–set. [Let ϕ be Σ_1. Then $J_\alpha(\mathbb{R}) \models \phi(z)$ iff $J_{\overline{\alpha}}(\mathbb{R}) \models \phi(z)$ for some $\overline{\alpha} < \alpha$ iff $J_{g(y)}(\mathbb{R}) \models \phi(z)$ for some $y \in \mathbb{R}$ iff $(y, (z, \phi)) \in A_0$ iff $(z, \phi) \in \exists^{\mathbb{R}} A_0$.]

Now we have that $Scale(\Delta_1(J_\alpha(\mathbb{R})))$ by $Scale(\Sigma_1(J_\alpha(\mathbb{R})))$. These shows that, with our choice of A_0, we easily get a sjs in the last case.

This finishes the exposition of the various cases (together with their associated sjs's). We now turn to the Durchführung. We shall try to make it as uniform as possible. However, of course, certain arguments depend on which case we are in.

Let us first assume until further notice that we're *not* in Case 3.2.2. (This case is an exception as we won't have to build "hybrid" models here.) Hence a pointclass Γ^* is defined, and we'll have that $\mathcal{A}$ is a sjs such that $\cup_i A_{2i+1}$ is a universal Γ^* –set. We aim to construct

$$\text{coarse } \exists^{\mathbb{R}}\forall^{\mathbb{R}} \ldots \forall^{\mathbb{R}}\Gamma^* \text{ Woodin mice.}$$

We can build fine structural models which contain terms capturing the elements of $\mathcal{A}$. However, in order to get a coarse $\exists^{\mathbb{R}}\forall^{\mathbb{R}} \ldots \forall^{\mathbb{R}}\Gamma^*$ Woodin mouse we'll have to build "hybrid" models which in addition to an extender sequence are also constructed from a predicate coding *sequences of* terms capturing the elements of $\mathcal{A}$, so that the models will contain a term capturing the universal Γ^*–set $\cup_i A_{2i+1}$. We'll then use 3.8 and 3.9.

Note that $\mathcal{A} \subset J_\beta(\mathbb{R})$. For $z \in \mathbb{R}$ let us write $d \in OD_z^{<\beta}$ iff there are $\overline{\beta} < \beta$, $\vec{\gamma} < \overline{\beta}$, and Φ such that $d = \{e \in J_{\overline{\beta}}(\mathbb{R}) : J_{\overline{\beta}}(\mathbb{R}) \models \Phi(e, \vec{\gamma}, z)\}$, i.e., iff d is ordinal definable from z over some $J_{\overline{\beta}}(\mathbb{R})$ for $\overline{\beta} < \beta$. By an easy induction, for any $A \in J_\beta(\mathbb{R})$ we have that $A \in OD_z^{<\beta}$ for some $z \in \mathbb{R}$. Moreover, if $x \in \mathbb{R} \cap OD_z^{<\beta}$ then $x \in C_{\Sigma_1(J_\beta(\mathbb{R}))}(z)$; this is because we'll then have that there is some $\xi < \omega_1$ such that for all sufficiently large $\overline{\beta} < \beta, x$ is the ξ^{th} real which is ordinal definable from z over $J_{\overline{\beta}}(\mathbb{R})$ (in the canonical wellorder of such reals). If $x \in \mathbb{R} \cap OD_z^{<\beta}$ where $\beta = \overline{\beta} + 1$, then this reasoning gives that $x \in C_{\Sigma_n(J_{\overline{\beta}}(\mathbb{R}))}(x)$ for some large enough $n < \omega$. We have the following criterion for when a model contains a term (weakly) capturing an element of $J_\beta(\mathbb{R})$.

For our purposes we'll say that σ is a *$Col(\omega, \kappa)$–standard term for a real* if

$$\sigma = \bigcup_n A_n \times \{\check{n}\}, \text{ where for all } n < \omega, A_n \subset Col(\omega, \kappa).$$

Lemma 4.5 *Let M be a countable transitive model of a sufficiently large fragment of ZFC which is closed under C_Γ. Let $\kappa \in Card^M$, and let z_0 be a real which is an element of M. Then every set of reals which is in $OD_{z_0}^{<\beta}$ is weakly captured by some $\tau \in M^{Col(\omega,\kappa)}$ over M.*

Proof: Fix $A \in \mathcal{P}(\mathbb{R}) \cap OD^{<\beta}_{z_0}$. We let $(p, \sigma) \in \tau$ iff $p \in Col(\omega, \kappa), \sigma$ is a $Col(\omega, \kappa)$–standard term for a real, and for comeager many g being $Col(\omega, \kappa)$–generic over $M : p \in g \Rightarrow \sigma^g \in A$. Trivially, $\tau \subset H^M_{\kappa^+}$.

Claim 1. $\tau \in M$.

Proof: Let $x \in \mathbb{R}$ be $Col(\omega, H^M_{\kappa^+})$-generic over M, i.e., $(\omega, E_x) \cong (H^M_{\kappa^+}, \in)$. It is easy to verify that $\tau_x \in OD^{<\beta}_{x,z_0}$. But $z_0 \in OD^{<\beta}_x$: Let m be the preimage of z_0 under the isomorphism $(\omega, E_x) \cong (H^M_{\kappa^+}, \in)$. Then $k \in z_0$ iff

$$\exists a \subset \omega (a \text{ represents the set of integers in } (\omega, E_x),$$

$$\text{and if } f : (a, E_x \restriction a) \cong (\omega, \in) \text{ then } f^{-1}(k) E_x m)).$$

Hence $\tau_x \in OD^{<\beta}_x$. Therefore, $\tau_x \in C_{\Sigma_1(J_\beta(\mathbb{R}))}(x)$, and thus $\tau_x \in C_{\Sigma_1(J_\alpha(\mathbb{R}))}(x)$. This shows that $\tau_x \in M[x]$, and thus $\tau \in M[x]$.

But this is true for all x, i.e., if x, x' are mutually $Col(\omega, H^M_{\kappa^+})$-generic over M, then $\tau \in M[x] \cap M[x']$. It follows that $\tau \in M$. $\square$

Claim 2. τ weakly captures A over M.

Proof: For $p \in Col(\omega, \kappa)$ and σ a term in $M^{Col(\omega,\kappa)}$ for a real let $C_{p,\sigma} = \{G : p \in G \wedge \sigma^G \in A\}$ and $C'_{p,\sigma} = \{G : p \in G \wedge \sigma^G \notin A\}$. We have $\tau = \tau_A = \{(p, \sigma) : C_{p,\sigma}$ is comeager$\} \in M$.

We claim that for all σ, $\{p \in Col(\omega, \kappa) : C_{p,\sigma}$ or $C'_{p,\sigma}$ is comeager $\}$ is dense in $Col(\omega, \kappa)$. Fix σ. Let $q \in Col(\omega, \kappa)$. Suppose that $C_{q,\sigma}$ is not comeager. As $C_{q,\sigma}$ has the property of Baire, there is an open set $\mathcal{O}$ such that $(\mathcal{O} \setminus C_{q,\sigma}) \cup (C_{q,\sigma} \setminus \mathcal{O})$ is meager. If $\mathcal{O} = \emptyset$, then $C'_{q,\sigma}$ is comeager. Let us assume that $\mathcal{O} \neq \emptyset$. Then there is some p such that $U_p \setminus C_{q,\sigma}$ is meager, where $U_p = \{g : p \in g\}$. We must have that $p \leq q$, as otherwise $U_p \setminus C_{q,\sigma} = U_p$, which is not meager. But then $C_{p,\sigma}$ is comeager, as $g \notin C_{p,\sigma}$ iff $g \in U_p \setminus C_{p,\sigma}$.

If $C_{p,\sigma}$ or $C'_{p,\sigma}$ is comeager, then let $C^*_{p,\sigma}$ denote the comeager one of them. There are only countably many such p's and σ's so that

$$\mathcal{C} = \bigcap_{p,\sigma} C^*_{p,\sigma}$$

is a comeager set.

Now let $g \in \mathcal{C}$. Then $\sigma^g \in \tau^g \Rightarrow \exists p \in g (p, \sigma) \in \tau \Rightarrow \exists p \in g C_{p,\sigma}$ is comeager $\Rightarrow \sigma^g \in A$. On the other hand, if $\sigma^g \notin \tau^g$, then $\forall p \in g (p, \sigma) \notin \tau$, so $\forall p \in g C_{p,\sigma}$ is not comeager. By densitiy, $\exists p \in g C_{p,\sigma}$ or $C'_{p,\sigma}$ is comeager. Therefore, $\exists p \in g C'_{p,\sigma}$ is comeager, and hence $\sigma^g \notin A$. This shows that $\tau^g = A \cap M[g]$ for all $g \in \mathcal{C}$.

$\square$ (Claim 2)

$\square$ (4.5)

Let M, etc. be as above. Let $\overline{M}$ be an inner model of M. Then $\tau \cap \overline{M}$ is just the term which the proof of 4.5 would construct for $\overline{M}$ instead of M. On the other hand, we can easily amalgamate a term capturing a given set from terms capturing it over various inner models.

Now let $y \in \mathbb{R}$. As a warm up, we can build a y–premouse of height ω_1 denoted by

$$L^\alpha(y),$$

which is the least transitive fine–structural model with

- $y \in L^\alpha(y)$,
- $L^\alpha(y) \cap OR = \omega_1$, and
- $L^\alpha(y)$ is closed under $C_{\overline{\Gamma}}$ for all $\overline{\Gamma}$ which show up in $[\overline{\alpha}, \overline{\beta}]$ for some $\overline{\beta} < \alpha$, i.e., for club many points $\gamma < \omega_1$, an initial segment $J^\alpha_\xi(y)$ of $L^\alpha(y)$ with $\xi > \gamma$ witnesses that $C_{\overline{\Gamma}}(J^\alpha_\gamma(y))$ is captured by a mouse.

We could actually make $L^\alpha(y)$ independent of α and just let it be $Lp(y)$, the lower part closure of y. $Lp(y)$ stratifies into initial segments $J^{Lp(y)}_\gamma$ of height γ, where $\gamma \subset C$ for a club $C \subset \omega_1$, such that if γ, γ' are consecutive elements of C, then $J^{Lp(y)}_{\gamma'}$ is the collapsing mouse for $J^{Lp(y)}_\gamma$, i.e., $J^{Lp(y)}_{\gamma'}$ is a γ-sound $\omega_1 + 1$ iterable premouse end-extending $J^{Lp(y)}_\gamma$ with $\rho_\omega(J^{Lp(y)}_{\gamma'}) \leq \gamma$.

We had picked $\mathcal{A}$, a sjs. Let $z_0 \in \mathbb{R}$ be s.t. every $A \in \mathcal{A}$ is $OD^{<\beta}_{z_0}$.[4] Let us assume that $z_0 \leq_T y$. Then from 4.5 it follows that for any $A \in \mathcal{A}$ and $\kappa \in Card^{L^\alpha(y)}$ there is some $\tau \in (L^\alpha(y))^{Col(\omega,\kappa)}$ weakly capturing A over $L^\alpha(y)$.

We have $\mathcal{A} = \{A_n : n < \omega\}$. We may then say that for all $\kappa \in Card^{L^\alpha(y)}$ there is some $(\tau_n : n < \omega) \subset L^\alpha(y)$ such that for all n, $\tau_n \in (L^\alpha(y))^{Col(\omega,\kappa)}$ weakly captures A_n over $L^\alpha(y)$.

We'll need a revised version $L^\alpha_{hyb}(y)$ of $L^\alpha(y)$, where $(\tau_n : n < \omega) \subset L^\alpha(y)$ becomes $(\tau_n : n < \omega) \in L^\alpha_{hyb}(y)$. Our plan is to build a "hybrid y–premouse" of height ω_1, denoted by

$$L^\alpha_{hyb}(y),$$

which is the least transitive model with

[4] Notice that the choice of $\mathcal{A}$ is a serious use of dependent choice. The choice of z_0 uses AC_ω.

- $y \in L^{\alpha}_{hyb}(y)$,
- $L^{\alpha}_{hyb}(y) \cap OR = \omega_1$, and
- $L^{\alpha}_{hyb}(y)$ is closed under $C_{\overline{\Gamma}}$ for all $\overline{\Gamma}$ which show up in $[\overline{\alpha}, \overline{\beta}]$ for some $\overline{\beta} < \alpha$, and
- for all $\kappa \in Card^{L^{\alpha}_{hyb}(y)}$ there is $(\tau_n : n < \omega) \in L^{\alpha}_{hyb}(y)$ such that for all n, $\tau_n \in (L^{\alpha}_{hyb}(y))^{Col(\omega,\kappa)}$ weakly captures A_n over $L^{\alpha}_{hyb}(y)$.

We leave the details of the construction of $L^{\alpha}_{hyb}(y)$ to the reader's discretion. We'll have that $L^{\alpha}_{hyb}(y) = L_{\omega_1}[E^{\alpha}, T^{\alpha}, y] = L_{\omega_1}[E, T, y]$ where E codes a sequence of extenders (coming from closing under all $C_{\overline{\Gamma}}$'s, for $\bar{\Gamma}$ showing up in $[\bar{\alpha}, \bar{\beta}]$, $\bar{\beta} < \alpha$), and T codes a sequence $(\tau^i_n : n < \omega \wedge i \in a)$, where a is a subset of ω_1, of term sequences (coming from throwing in term sequences at certain points). In particular, we want that $\tau^i_n \in (L^{\alpha}_{hyb}(y))^{Col(\omega,\kappa)}$ weakly captures A_n over $L^{\alpha}_{hyb}(y)|\kappa$ for a certain local cardinal $\kappa < i$ depending on i.

We shall need the following version of 2.1:

Lemma 4.6 *For all $u \in HC$, there is a cone of y,*

$$\omega_1^{L^{\alpha}_{hyb}(y)} \textit{ is measurable in } HOD^{L^{\alpha}_{hyb}(y)}_{E,T,u}.$$

The proof of 4.6 is just a straightforward variant of the proof of 2.1.

Put $H = HOD^{L^{\alpha}_{hyb}(y)}_{E,T,u}$. We want to build

$$K^c_{hyb}(u) = (K^c_{hyb}(u))^H,$$

a "hybrid" K^c over u inside H (and of height ω_1^V). It will be of the form $L_{\omega_1}[E', T', u]$ where E' codes a sequence of *background certified* extenders, and T' codes a sequence $(\tau^i_n : n < \omega \wedge i \in a)$, where a is a subset of ω_1, of term sequences (coming from throwing in term sequences at certain points). We say that τ^i_n is an "n^{th} term" of $L_{\omega_1}[E', T', u]$.

Definition 4.7 *Let $\mathcal{M} = J_{\alpha}[E, T, y]$ be a hybrid y–premouse. $\mathcal{M}$ is called $(\mathcal{A}, \Theta)$–good if there is a Θ–iteration strategy Σ for $\mathcal{M}$ such that whenever $\mathcal{M}'$ is a Σ–iterate of $\mathcal{M}$ and if τ is one of the n^{th} terms of $\mathcal{M}'$ then τ weakly captures A_n over (an initial segment of) $\mathcal{M}'$.*

We sometimes aim to say that certain models are $(\mathcal{A}, \Theta)$–good inside H, say. However, this doesn't make much sense to begin with, because H can't see $\mathcal{A}$. However, we can redefine "$(\mathcal{A}, \Theta)$–goodness" as follows, which will make sense in particular inside hybrid premice.

Definition 4.8 *Let $\mathcal{M} = J_\alpha[E,T,y]$ be a hybrid y–premouse. $\mathcal{M}$ is called $(\mathcal{A},\Theta)$–good if there is a Θ–iteration strategy for $\mathcal{M}$ such that whenever $\mathcal{M}'$ is a Σ–iterate of $\mathcal{M}$ and if τ is one of the n^{th} terms of $\mathcal{M}'$ then*

$$\Vdash \tau = \tilde{\tau} \cap \check{\mathcal{M}}',$$

where $\tilde{\tau}$ is the appropriate n^{th} term from the term predicate of the universe.

This new definition "coincides with" the old one. We now have the following.

Sublemma 4.9 *Inside H, countable substructures of models showing up in the recursive construction of $K^c_{hyb}(u)$ are $(\mathcal{A},\omega_1+1)$–good.*

Proof Sketch: This follows from the standard proof of iterability (c.f. [12]) together with 1.8. Let $\pi : \overline{\mathcal{M}} \to \mathcal{M}_\xi$, where $\mathcal{M}_\xi$ is a model from the recursive construction of $K^c_{hyb}(u)$. We iterate $\overline{\mathcal{M}}$ by choosing unique cofinal realizable branches. Thus if $\overline{\mathcal{M}}'$ is an iterate of $\overline{\mathcal{M}}$, then there is some $\overline{\xi} \leq \xi$ and some $\pi' : \overline{\mathcal{M}}' \to \mathcal{M}_{\overline{\xi}}$. But then any n^{th} term of $\overline{\mathcal{M}}'$ captures A_n over $\overline{\mathcal{M}}'$ by 1.8. $\square$

This gives:

Sublemma 4.10 *Inside H, $K^c_{hyb}(u)$ is $(\mathcal{A},\omega_1^V)$–good "above the largest Woodin cardinal", i.e., there is an ω_1^V–iteration strategy Σ for iterating $K^c_{hyb}(u)$ and if τ is one of the n^{th} terms of $\mathcal{M}'$ then τ weakly captures A_n over $\mathcal{M}'$.*

We'd in fact have to develop a theory of hybrid mice of the form $J_\alpha[\overline{E},\overline{T},u]$ where the terms from $\overline{T}$ strongly capture the sets in $\mathcal{A}$ in the above sense. This would then allow us to prove, using 4.6 above, in a fashion à la Section 2, that

$$(K^c_{hyb}(u))^H \models \text{ there is a Woodin cardinal.}$$

Let us now turn towards the real induction. We'll inductively assume that HC is closed under the operator

$$a \mapsto M^\alpha_{n,hyb}(a) = M(a),$$

Where for all transitive A, $M(a)$ is the least sound hybrid a–premouse $J_\alpha[E,T,a]$ such that

- $M(a) \models ZF^- +$ there are n Woodin cardinals and a measurable above,
- $M(a)$ is closed under all $C_{\overline{\Gamma}}$ where $\overline{\Gamma}$ shows up in $[\overline{\alpha},\overline{\beta}]$ for $\overline{\beta} < \alpha$, and

- $M(a)$ is $(\mathcal{A}, \omega_1)$–good.

We emphasize that the above arguments give that HC is closed under $a \mapsto M^{\alpha}_{n,hyb}(a) = M(a)$.

Given $y \in \mathbb{R}$, we then build the hybrid y–premouse

$$L^{\alpha}_{n,hyb}(y)$$

as the least transitive model of height ω_1 which contains y and is closed under $a \mapsto M^{\alpha}_{n,hyb}(a) = M(a)$.

Easy arguments show that $L^{\alpha}_{n,hyb}(y)$ is $\mathcal{A}$–good. (The terms for the elements of $\mathcal{A}$ are amalgamations of terms from various $M^{\alpha}_{n,hyb}(-)$.)

We then continue in a fashion as above. Let $L^{\alpha}_{n,hyb}(y) = L_{\omega_1}[E, T, y]$, and let us write $H = HOD^{L^{\alpha}_{n,hyb}(y)}_{E,T,u}$ (for some appropriate $u <_T y$). We build

$$W = K^c_{hyb}(u) = (K^c_{hyb}(u))^H.$$

We then argue that

$$W \models \text{ there is a Woodin cardinal,}$$

by combining a version of (1) above, of 4.10, and of arguments as in Section 2. Let δ be the largest Woodin cardinal of W. Let

$$\pi : \mathcal{P} \rightarrow_{\Sigma_1} M^{\alpha}_{n,hyb}(W|\delta),$$

where $ran(\pi)$ is the Σ_1 hull of $\emptyset$, formed inside $M^{\alpha}_{n,hyb}(W|\delta)$. Notice that

- $\mathcal{P} \models ZF^-+$ there are $n+1$ Woodin cardinals and a measurable above.

By 1.11,

- $\mathcal{P}$ is closed under all C_Γ where $\overline{\Gamma}$ appears in $[\overline{\alpha}, \overline{\beta}]$ for $\overline{\beta}\alpha$.

An argument actually shows that $M^{\alpha}_{n,hyb}(W|\delta)$ is an initial segment of W. [This uses the "universality" of W inside H, as well as the fact that $M^{\alpha}_{n,hyb}(W|\delta) \in H$ is sufficiently iterable there.] π thus exists inside W. The condensation lemma hence gives that in fact

$$\mathcal{P} \lhd W,$$

i.e., $\mathcal{P}$ is an initial segment of W. We have shown that W has a least initial segment, all it $\mathcal{P}'$, which is not $(n+1)$–small. Let us assume w.l.o.g. that $\mathcal{P} = \mathcal{P}'$.

The argument for 4.9 now gives that

- $H \models \mathcal{P}$ is ω_1–iterable via the "realization strategy".

Let us denote this strategy by Σ. It witnesses that $H \models \mathcal{P}$ is $(\mathcal{A}, \omega_1)$–good. It can be verified that, inside H, Σ is characterized as follows. Let $\mathcal{T}$ be a tree on $\mathcal{P}$ of limit length. Then $\Sigma(\mathcal{T})$ = the unique cofinal branch coming with a $\mathcal{Q}$–structure which is $(n+1)$–small and $(\mathcal{A}, \omega_1 + 1)$–good. Because H thinks that an $(n+1)$–small hybrid premouse is $(\mathcal{A}, \omega_1 + 1)$–good (in the sense of 4.8) then it is really $(\mathcal{A}, \omega_1)$–good (in the sense of 4.7). We'll need later that an $(n+1)$–small hybrid premouse $\mathcal{Q} \in H$ is $(\mathcal{A}, \omega_1)$–good in V iff H thinks that $\mathcal{Q}$ is $(\mathcal{A}, \omega_1 + 1)$–good. The same applies to set–generic extensions of H.

Unfortunately, H won't be sufficiently correct to immediately give that $\mathcal{P}$ is (really) $(\mathcal{A}, \omega_1)$–good. We'll have to use AD once more (or rather the fact that Martin's measure on the Turing degrees $\mathcal{D}$ is a σ–complete ultrafilter) in order to get some such $\mathcal{P}$. Let us write $\mathcal{P}(y, u)$ for the $\mathcal{P}$ isolated above. Let us consider $f : \mathcal{D} \to \mathbb{R}$ given by

$$[y] \mapsto \text{ a canonical real code for } \mathcal{P}(y, u).^5$$

For each $n < \omega$, the set $\{y \in \mathbb{R} : n \in f([y])\}$ either contains a cone or is disjoint from a cone. Let $n \in \tilde{\mathcal{P}}$ iff for a cone of $y, n \in f([y])$. Then $f([y]) = \tilde{\mathcal{P}}$ on a cone of y.

Sublemma 4.11 *$\tilde{\mathcal{P}}$ is (really) $(\mathcal{A}, \omega_1)$–good.*

Proof Sketch: Let us consider the following strategy Σ for iterating $\tilde{\mathcal{P}}$: Suppose that $\mathcal{T}$ is an iteration tree on $\mathcal{P}$ of limit strength. Then we let $\Sigma(\mathcal{T})$ = the unique cofinal branch b thru $\mathcal{T}$ such that $\mathcal{M}_b^{\mathcal{T}}$ comes with a $(\mathcal{A}, \omega_1)$–good $\mathcal{Q}$–structure ($\Sigma(\mathcal{T})$ is supposed to be undefined if there is no unique such b). A standard comparison argument shows that if there is some cofinal branch b thru $\mathcal{T}$ such that $\mathcal{M}_b^{\mathcal{T}}$ comes with a $(\mathcal{A}, \omega_1)$–good $\mathcal{Q}$–structure then there is a unique such b. We claim that Σ witnesses $\tilde{\mathcal{P}}$ is $(\mathcal{A}, \omega_1)$–good.

Suppose not. There is then a putative iteration tree $\mathcal{T}$ with $[0, \lambda)_T = \Sigma(\mathcal{T} \restriction \lambda)$ for all limit $\lambda < lh(\mathcal{T})$ such that EITHER $\mathcal{T}$ has successor length and its last model is ill–founded or its terms don't (weakly) capture the elements of $\mathcal{A}$, OR ELSE $\mathcal{T}$ has limit length and there is no cofinal branch b thru $\mathcal{T}$ such that $\mathcal{M}_b^{\mathcal{T}}$ is $(\mathcal{A}, \omega_1)$–good. Let's restrict ourselves to discussing the 2^{nd} alternative.

Let us pick y in the cone where f is constant and such that

$$\mathcal{T} \in L^{\alpha}_{n,hyb}(y),$$

[5] The fact that $\rho_\omega(\mathcal{P}(y, u)) = \omega$ means that $\mathcal{P}(y, u)$ comes with a canonical real code for itself.

and is countable there. As before, let us write

$$H = HOD_{E,T,u}^{L_{hyb}^{\alpha}(y)}.$$

By Vopěnka, then, $\mathcal{T}$ is a countable tree on $\mathcal{P}$ in $H[g]$, a generic extension of H. Now $\mathcal{T}$ is such that for all limit $\lambda < lh(\mathcal{T})$, $[0,\lambda)_T =$ the unique cofinal branch b coming with an $(n+1)$–small hybrid $(\mathcal{A},\omega_1)$–good (in the sense of 4.7) $\mathcal{Q}$–structure. By remarks above, hence, $H[g]$ thinks that $\mathcal{T}$ is such that for all limit $\lambda < lh(\mathcal{T})$, $[0,\lambda)_T =$ the unique cofinal branch b coming with an $(n+1)$–small hybrid $(\mathcal{A},\omega_1)$–good (in the sense of 4.8) $\mathcal{Q}$–structure. In other words, for all limit $\lambda < lh(\mathcal{T})$, $[0,\lambda)_T = \Sigma(y,u)(\mathcal{T} \restriction \lambda)$, i.e., $\mathcal{T}$ was built according to $\Sigma(y,u)$ inside $H[g]$. But then $\Sigma(y,u)(\mathcal{T})$ is well–defined. In other words, there is a cofinal branch b thru $\mathcal{T}$ coming with an $(n+1)$–small hybrid $(\mathcal{A},\omega_1)$–good (in $H[g]$, and hence in V) $\mathcal{Q}$–structure. $\square$

We have shown that $M_{n+1,hyb}^{\alpha}(u)$ exists.

Let us finally discuss Case 3.2.2. The good news is that we won't have to produce hybrid models here. Let $B \in \Sigma_1(J_\alpha(\mathbb{R}))$, and let $X \subset \mathbb{R}$ be countable. We are assuming that $cf(\alpha) > \omega$. Hence the set of witnesses for elementhood in $B \cap X$ is bounded below α, and thus $B \cap X \in J_\alpha(\mathbb{R})$, i.e. $B \cap X \in OD_{z_0}^{<\alpha}$ for some $z_0 \in \mathbb{R}$. Thus, going into the proof of 4.5, if $A \in \mathcal{P}(\mathbb{R}) \cap \Delta_1(J_\alpha(\mathbb{R}))$ then we now still get that $\tau_x \in OD_{x,z_0}^{<\alpha}$ (notation as there). Hence in this Case we get 4.5 where in its statement "every set of reals which is in $OD_{x,z_0}^{<\alpha}$" can be replaced by "every set of reals which is in $\Delta_1(J_\alpha(\mathbb{R}))$".

With this observation we see that we can now run the above argument where instead of producing hybrid models we always produce ordinary fine structural models. $\square$

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