ACA_0, Π^1_1-CA_0, and the semantics of arithmetic, and BG, $\mathsf{BG} + \Sigma^1_1$-Ind, and the semantics of set theory

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Abstract

The truth predicate for the language of first order arithmetic is definable in the language of second order arithmetic. Whereas ACA_0 proves the Tarski schema, ACA_0 does not prove the Tarski rule of negation. However, Π^1_1-CA_0 does prove all the Tarski rules. In particular, Π^1_1-CA_0 proves the consistency of ACA_0.

Analogous results hold for set theory. The truth predicate for the language of ZF is definable in the language of BG. Whereas BG proves the Tarski schema, BG does not prove the Tarski rule of negation. However, $\mathsf{BG} + \Sigma^1_1\mathsf{Ind}$ does prove all the Tarski rules. In particular, $\mathsf{BG} + \Sigma^1_1\mathsf{Ind}$ proves the consistency of ZF.

These results must all be pretty old. The author does not know whom to give the credit, though. In any event, he doesn't claim credit for anything exposed in this note.

Let us commence with set theory. The intended model of ZF, $(V; \in)$, has a class rather than a set as its underlying universe. This paper discusses the semantics of ZF.

We let $\mathcal{L}_{\mathsf{ZF}}$ denote the language of ZF. We may enrich $\mathcal{L}_{\mathsf{ZF}}$ by adding a class of constants, $\{\dot{x} | x \in V\}$, where $\dot{x}$ is intended to denote x. We let $\underline{\mathcal{L}}_{\mathsf{ZF}}$ denote the enriched language. A formula of $\underline{\mathcal{L}}_{\mathsf{ZF}}$ comes from a formula of $\mathcal{L}_{\mathsf{ZF}}$ by replacing free occurences of variables by constants. If φ is a formula, if v is a variable, and if $x \in V$, then by φ^v_x we denote the result of replacing all free occurences of v by $\dot{x}$.

We shall also use notations like $\varphi(0)$, $\varphi(n)$, and $\varphi(n+1)$. The reader will easily figure out how these are to be understood.

If $\mathfrak{M}$ is a model, φ is a formula in which each free variable is in $\{v_0, \cdots, v_n\}$, and if $\{y_0, y_1, \cdots, y_n\} \subset |\mathfrak{M}|$ (the underlying universe of $\mathfrak{M}$) then we write

$$\mathfrak{M} \models \varphi(y_0, y_1, \cdots, y_n)$$

to express that φ holds true in $\mathfrak{M}$ with an assignment that maps v_k to y_k for $k \leq n$.

In what follows we tacitly make use of the fact that we can in ZF represent the relevant syntactical concepts of $\mathcal{L}_{\mathsf{ZF}}$ and of $\underline{\mathcal{L}}_{\mathsf{ZF}}$. If φ is a formula of $\underline{\mathcal{L}}_{\mathsf{ZF}}$ then we shall write $\ulcorner\varphi\urcorner$ for its Gödel number.

The language of $\mathcal{L}_{\mathsf{BG}}$ of BG has two sorts of variables, lower case ones for sets and upper case ones for classes. If φ is a formula of $\mathcal{L}_{\mathsf{BG}}$ then we say that φ is Σ^1_n, where $n \in \omega$, if and only if φ is provably in BG equivalent to a formula of the form

$$\exists X_1 \forall X_2 \cdots Q X_n \psi,$$

where $Q = \exists$ / $\forall$ if and only if n is even / odd and ψ does not contain any class quantifiers.

Definition 0.1 *We abbreviate by $t(n, X)$ the following Σ^1_0 formula of $\mathcal{L}_{\mathsf{ZF}}$.*

$$n \in \omega \wedge \forall x \in X(x \text{ is a sentence of } \underline{\mathcal{L}}_{\mathsf{ZF}} \text{ of rank at most } n)\wedge$$

$$\forall x \forall y(\ulcorner x \in y\urcorner \in X \leftrightarrow x \in y)\wedge$$
$$\forall \text{ sentences } \ulcorner\varphi\urcorner \text{ of } \underline{\mathcal{L}}_{\mathsf{ZF}} \text{ of rank at most } n-1$$
$$\forall \text{ sentences } \ulcorner\psi\urcorner \text{ of } \underline{\mathcal{L}}_{\mathsf{ZF}} \text{ of rank at most } n-1$$
$$\forall \text{ variables } v$$

$$\begin{array}{rcl} [(\ulcorner\neg\varphi\urcorner \in X & \leftrightarrow & \ulcorner\varphi\urcorner \notin X) \wedge \\ (\ulcorner\varphi \wedge \psi\urcorner \in X & \leftrightarrow & \ulcorner\varphi\urcorner \in X \wedge \ulcorner\psi\urcorner \in X) \wedge \\ (\ulcorner\forall v\varphi\urcorner \in X & \leftrightarrow & \forall x \ulcorner\varphi^v_x\urcorner \in X)]. \end{array}$$

Definition 0.2 *We abbreviate by $T(x)$ the following Σ^1_1 formula of $\mathcal{L}_{\mathsf{ZF}}$.*

$$\exists n \exists X(t(n, X) \wedge x \in X).$$

Lemma 0.3 $\mathsf{BG} \vdash \exists X t(0, X)$.

Lemma 0.4 $\mathsf{BG} \vdash \forall n \in \omega(\exists X t(n, X) \rightarrow \exists X t(n+1, X))$.

Corollary 0.5 $\mathsf{BG} \vdash \forall\ulcorner\varphi\urcorner \in \underline{\mathcal{L}}_{\mathsf{ZF}} \forall\ulcorner\psi\urcorner \in \underline{\mathcal{L}}_{\mathsf{ZF}}(T(\ulcorner\varphi \wedge \psi\urcorner) \leftrightarrow T(\ulcorner\varphi\urcorner) \wedge T(\ulcorner\psi\urcorner))$.

Corollary 0.6 $\mathsf{BG} \vdash \forall\ulcorner\varphi\urcorner \in \underline{\mathcal{L}}_{\mathsf{ZF}} \forall v(T(\ulcorner\forall v\varphi\urcorner) \leftrightarrow \forall x T(\ulcorner\varphi^v_x\urcorner))$.

Lemma 0.7 *For all $n \in \omega$, $\mathsf{BG} \vdash \exists X t(n, X)$.*

Lemma 0.8 $\mathsf{BG} \vdash \forall n \forall X \forall Y(t(n, X) \wedge t(n, Y) \rightarrow X = Y)$.

Lemma 0.9 *For all sentences φ of $\mathcal{L}_{\mathsf{ZF}}$, $\mathsf{BG} \vdash T(\ulcorner\varphi\urcorner) \leftrightarrow \varphi$.*

Lemma 0.10 $\mathsf{BG} \nvdash \forall n \in \omega \exists X t(n, X)$*, unless* ZF *is inconsistent.*

PROOF. Suppose ZF to be consistent, and let $\mathfrak{M} = (M; E)$ be a model of ZF where M is a set and $E \subset M \times M$. Let the well-founded part of $\mathfrak{M}$ be transitive. We may and shall assume that $\mathfrak{M}$ contains non-standard integers, in other words that E restricted to the integers in the sense of $\mathfrak{M}$ is ill-founded. That is, $\omega \subsetneqq \omega^{\mathfrak{M}}$. Let K denote the set of all

$$\{x \in M \mid \mathfrak{M} \models \varphi(x, y_1, \cdots, y_n)\},$$

where φ is a formula of $\mathcal{L}_{\mathsf{ZF}}$ in which each free variable is in $\{v_0, \cdots, v_n\}$ and where $\{y_1, \cdots, y_n\} \subset M$. I.e., K is the set of all subsets of M which are boldface definable over $\mathfrak{M}$ by a formula of $\mathcal{L}_{\mathsf{ZF}}$. Then $\mathfrak{N} = (M, K; E)$ is a model of BG.

Let $n \in M$ and $X \in K$ be such that

$$\mathfrak{N} \models t(n, X).$$

Let $m \in \omega$ be least such that X is boldface definable over $\mathfrak{M}$ by a Σ_m formula of $\mathcal{L}_{\mathsf{ZF}}$. It is easy to see that we must have

$$\mathfrak{M} \models n \leq m.$$

That is, n must be a standard integer.

Now let $n \in \omega^{\mathfrak{M}} \setminus \omega$ be a non-standard integer of $\mathfrak{M}$. We have shown that

$$\mathfrak{N} \models \neg\exists X t(n, X).$$

$\square$

Corollary 0.11 $\mathsf{BG} \nvdash \forall\ulcorner\varphi\urcorner \in \mathcal{L}_{\mathsf{ZF}}(T(\ulcorner\neg\varphi\urcorner) \leftrightarrow \neg T(\ulcorner\varphi\urcorner))$*, unless* ZF *is inconsistent.*

Definition 0.12 *We let* $\Sigma^1_1\mathsf{Ind}$ *denote the schema which asserts that for every* Σ^1_1 *formula* Φ *of* $\mathcal{L}_{\mathsf{BG}}$ *with free variables in* $\{v_0, v_1, \cdots, v_n\}$[a],

$$\forall v_1 \cdots \forall v_n[(\Phi(0) \wedge \forall n \in \omega(\Phi(n) \rightarrow \Phi(n+1)) \rightarrow \forall n \in \omega\Phi(n)].$$

Lemma 0.13 $\mathsf{BG} + \Sigma^1_1\mathsf{Ind} \vdash \forall n \in \omega \exists X t(n, X)$.

[a] In particular, all free variables are set variables.

Corollary 0.14 $\mathsf{BG} + \Sigma^1_1\mathsf{Ind} \vdash \forall \ulcorner\varphi\urcorner \in \underline{\mathcal{L}}_{\mathsf{ZF}}(T(\ulcorner\neg\varphi\urcorner) \leftrightarrow \neg T(\ulcorner\varphi\urcorner))$.

If T is a (recursively enumerable) theory then $\mathrm{Bew}_{\mathsf{T}}$ denotes the formal representation of the provability predicate.

Lemma 0.15 $\mathsf{BG} + \Sigma^1_1\mathsf{Ind} \vdash \forall \ulcorner\varphi\urcorner \in \mathcal{L}_{\mathsf{ZF}}(\mathrm{Bew}_{\mathsf{ZF}}(\ulcorner\varphi\urcorner) \rightarrow T(\ulcorner\varphi\urcorner))$.

Proof. Let us work in $\mathsf{BG} + \Sigma^1_1\mathsf{Ind}$.

Let us first prove that every axiom of ZF satisfies T. Let us consider an instance of the separation schema as a typical example where (for notational convenience) the separating formula φ doesn't allow parameters. As we have the Tarski rules at hand (cf. Corollaries 0.5, 0.6, and 0.14), our task quickly reduces to having to show:

$$\exists y \forall z(z \in y \leftrightarrow z \in x \wedge T(\ulcorner\varphi^v_x\urcorner)).$$

Let n be the rank of φ, and let X be unique such that $t(n, X)$. Then

$$T(e) \leftrightarrow e \in X$$

for all e of rank $\leq n$. However,

$$\exists y \forall z(z \in y \leftrightarrow z \in x \wedge \ulcorner\varphi^v_x\urcorner \in X)$$

holds by the separation axiom of BG.

Now let $(\varphi_n | n \leq N)$ be a proof in ZF. By $\Sigma^1_1\mathsf{Ind}$ we may assume that $T(\ulcorner\varphi_n\urcorner)$ for every $n < N$. If φ_N is an axiom of ZF then $T(\ulcorner\varphi_N\urcorner)$ holds by the preceding paragraph. Otherwise there are j, $k \leq n - 1$ such that

$$\varphi_k \equiv \varphi_j \rightarrow \varphi_N.$$

But then an application of the Tarski rules given by Corollaries 0.5 and 0.14 readily implies that $T(\ulcorner\varphi_N\urcorner)$ holds. $\square$

Corollary 0.16 $\mathsf{BG} + \Sigma^1_1\mathsf{Ind} \vdash \mathsf{Con}(\mathsf{ZF})$.

We finally want to study the relation of $\mathsf{BG} + \Sigma^1_1\mathsf{Ind}$ with "$\mathsf{Tr}(\mathsf{ZF})$." We consider the Feferman-style "ordinary truth theory" for ZF which we call $\mathsf{Tr}(\mathsf{ZF})$. To get $\mathsf{Tr}(\mathsf{ZF})$, we extend the language $\mathcal{L}_{\mathsf{ZF}}$ by adding constants for all elements of V, plus we add a primitive truth predicate $\dot{T}(v)$ governed by Tarski's four axioms:

$\mathsf{T}_{\mathrm{Atom}}$ $\forall \ulcorner x \in y\urcorner \in \underline{\mathcal{L}}_{\mathsf{ZF}}(\dot{T}(\ulcorner x \in y\urcorner) \leftrightarrow x \in y)$,
$\mathsf{T}_{\mathrm{Neg}}$ $\forall \ulcorner\varphi\urcorner \in \underline{\mathcal{L}}_{\mathsf{ZF}}(\dot{T}(\ulcorner\neg\varphi\urcorner) \leftrightarrow \neg\dot{T}(\ulcorner\varphi\urcorner))$,

$\mathsf{T}_{\mathrm{Conj}}$ $\forall\ulcorner\varphi\urcorner \in \underline{\mathcal{L}}_{\mathsf{ZF}}\forall\ulcorner\psi\urcorner \in \underline{\mathcal{L}}_{\mathsf{ZF}}(\dot{T}(\ulcorner\varphi\wedge\psi\urcorner) \leftrightarrow \dot{T}(\ulcorner\varphi\urcorner)\wedge\dot{T}(\ulcorner\psi\urcorner))$,
$\mathsf{T}_{\mathrm{Quant}}$ $\forall\ulcorner\varphi\urcorner \in \underline{\mathcal{L}}_{\mathsf{ZF}}\forall v(\dot{T}(\ulcorner\forall v\varphi\urcorner) \leftrightarrow \forall x\dot{T}(\ulcorner\varphi^v_x\urcorner))$

(cf. Corollaries 0.5, 0.6, and 0.14), and we replace the separation and replacement schemas of ZF by the following separation and replacement axioms:

$\mathsf{T}_{\mathrm{Sep}}$ $\forall\ulcorner\varphi\urcorner \in \underline{\mathcal{L}}_{\mathsf{ZF}}\forall x\{y\in x \mid \dot{T}(\ulcorner\varphi^v_y\urcorner)\}$ is a set, and
$\mathsf{T}_{\mathrm{Repl}}$ $\forall\ulcorner\varphi\urcorner \in \underline{\mathcal{L}}_{\mathsf{ZF}}$, if $\dot{T}(\ulcorner\varphi\urcorner)$ defines a function then
$\forall x\exists y\forall u\in x\exists u'\in y\dot{T}(\ulcorner(\varphi^v_u)^{v'}_{u'}\urcorner)$.

Lemma 0.17 $\mathsf{BG}+\Sigma^1_1\mathsf{Ind}$ *and* $\mathsf{Tr}(\mathsf{ZF})$ *prove the same sentences in the language* $\mathcal{L}_{\mathsf{ZF}}$*.*

Proof. The above results clearly imply that if $\mathsf{Tr}(\mathsf{ZF})\vdash\varphi$, where φ is a sentence of the language $\mathcal{L}_{\mathsf{ZF}}$, then $\mathsf{BG}+\Sigma^1_1\mathsf{Ind}\vdash\varphi$ as well.

To prove the converse, let $\mathfrak{M}=(M;\in,T)$ be a model of $\mathsf{Tr}(\mathsf{ZF})$. Let K denote the set of all $X\subset M$ such that for some $\ulcorner\varphi\urcorner\in M$,

$$\forall x\in M(x\in X \Leftrightarrow \mathfrak{M}\models\dot{T}(\ulcorner\varphi^v_x\urcorner).$$

It is straightforward to verify that $\mathfrak{N}=(M,K;\in)$ is then a model of $\mathsf{BG}+\Sigma^1_1\mathsf{Ind}$. $\Box$

Let us now turn to arithmetic. The situation here is entirely analogous. We may leave the details to the reader. We may define predicates $t(n,X)$ (saying that X is a set of integers containg exactly the Gödel numbers of true first order statements of arithmetic which have rank at most n) and $T(x)$ (saying that $\exists n\exists X(t(n,X)\wedge x\in X))$ in much the same way as in the case of set theory. Let $\mathcal{L}_{\mathsf{PA}}$ denote the language of first order arithmetic. We get:

Lemma 0.18 $\mathsf{ACA}_0\vdash\exists Xt(0,X)$.

Lemma 0.19 $\mathsf{ACA}_0\vdash\forall n\in\omega(\exists Xt(n,X)\to\exists Xt(n+1,X))$.

Corollary 0.20 $\mathsf{ACA}_0\vdash\forall\ulcorner\varphi\urcorner\in\mathcal{L}_{\mathsf{PA}}\forall\ulcorner\psi\urcorner\in\mathcal{L}_{\mathsf{PA}}(T(\ulcorner\varphi\wedge\psi\urcorner)\leftrightarrow T(\ulcorner\varphi\urcorner)\wedge T(\ulcorner\psi\urcorner))$.

Corollary 0.21 $\mathsf{ACA}_0\vdash\forall\ulcorner\varphi\urcorner\in\mathcal{L}_{\mathsf{PA}}\forall v(T(\ulcorner\forall v\varphi\urcorner)\leftrightarrow\forall xT(\ulcorner\varphi^v_x\urcorner))$.

Lemma 0.22 *For all* $n\in\omega$, $\mathsf{ACA}_0\vdash\exists Xt(n,X)$.

Lemma 0.23 $\mathsf{ACA}_0\vdash\forall n\forall X\forall Y(t(n,X)\wedge t(n,Y)\to X=Y)$.

Lemma 0.24 *For all sentences* φ *of* $\mathcal{L}_{\mathsf{PA}}$, $\mathsf{ACA}_0\vdash T(\ulcorner\varphi\urcorner)\leftrightarrow\varphi$.

Lemma 0.25 $\mathsf{ACA}_0 \nvdash \forall n \in \omega \exists X t(n, X)$, *unless* PA *is inconsistent.*

Corollary 0.26 $\mathsf{ACA}_0 \nvdash \forall \ulcorner \varphi \urcorner \in \mathcal{L}_{\mathsf{PA}}(T(\ulcorner \neg\varphi \urcorner) \leftrightarrow \neg T(\ulcorner \varphi \urcorner))$, *unless* PA *is inconsistent.*

Lemma 0.27 $\Pi^1_1\text{-}\mathsf{CA}_0 \vdash \forall n \in \omega \exists X t(n, X)$.

Corollary 0.28 $\Pi^1_1\text{-}\mathsf{CA}_0 \vdash \forall \ulcorner \varphi \urcorner \in \mathcal{L}_{\mathsf{PA}}(T(\ulcorner \neg\varphi \urcorner) \leftrightarrow \neg T(\ulcorner \varphi \urcorner))$.

Lemma 0.29 $\Pi^1_1\text{-}\mathsf{CA}_0 \vdash \forall \ulcorner \varphi \urcorner \in \mathcal{L}_{\mathsf{PA}}(\mathrm{Bew}_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \rightarrow T(\ulcorner \varphi \urcorner))$.

Corollary 0.30 $\Pi^1_1\text{-}\mathsf{CA}_0 \vdash \mathsf{Con}(\mathsf{PA})$.

Finally, let Tr(PA) denote the theory which comes from PA in exactly the same way as Tr(ZF) comes from ZF.

Lemma 0.31 $\Pi^1_1\text{-}\mathsf{CA}_0$ *and* Tr(PA) *prove the same sentences in the language* $\mathcal{L}_{\mathsf{PA}}$.

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