Core models in the presence of Woodin cardinals

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Abstract

Let $0 < n < \omega$. If there are n Woodin cardinals and a measurable cardinal above, but $M_{n+1}^{\#}$ doesn't exist, then the core model K exists in a sense made precise. An Iterability Inheritance Hypothesis is isolated which is shown to imply an optimal correctness result for K.

0 Introduction.

In this paper we show that the core model K might exist even if there are Woodin cardinals in V (cf. Theorem 0.5). This observation is not new. Woodin [12], in his proof that $\mathsf{AD}_{\mathbb{R}}$ implies that there is an inner model satisfying the $\mathsf{AD}_{\mathbb{R}}$ hypothesis, constructed models of ZFC in which there are fully iterable extender models with Woodin cardinals which satisfy (among other things) a weak covering property. More related to our work here, though, is Steel's proof that M_n (where $n \in \omega$) satisfies $V = HOD$ (cf. [11] and Lemma 1.1). Steel gave an argument which appears to be a special case of what we shall do in this paper. However, the general method for constructing the core model in the theory ZFC + "there is a measurable cardinal above n Woodin cardinals" + "$M_{n+1}^{\#}$ does not exist" (again, $n < \omega$) which we shall present here does seem to be new. It is to be emphasized that in contrast to the extender models which Woodin constructs in the context of $AD_{\mathbb{R}}$, our core models will not be fully iterable "at their Woodin cardinals."

This method might in turn admit generalizations, but we do not know how to achieve them.[1] We shall indicate that our method might have applications; in particular, our results should be relevant to the problem (cf. [7, Problem 9]) of proving that K is Σ^1_{n+3} correct if V is closed under the operation $X \mapsto M_n^{\dagger}(X)$ but $M_{n+1}^{\#}$ does not exist (where $n \in \omega$). We shall isolate an "Iterability Inheritance

[1]Since this paper has been written, Steel and the author have produced results in this direction; cf. [6].

Hypothesis" which we find plausible and which we show to yield said correctness of K (cf. Lemma 2.4). (That such a hypothesis should give the right correctness of K was known earlier by Woodin, Steel, Neeman, and others; cf. [7, Problem 8].)

We shall extend [5] to the new core models, and we shall verify that if M is an ultrapower of V by an extender with countably closed support, then K^M is an iterate of K (even though for $n > 0$, K will not be fully iterable).

The reader should be familiar to a certain extent with the notation and key concepts and results of the paper [10].

Definition 0.1 *Let $\mathcal{M}$ be a k-sound premouse, and let $\delta \in \mathcal{M}$. We say that $\mathcal{M}$ is* countably iterable above δ *if for all weak k-embeddings $\pi : \bar{\mathcal{M}} \to \mathcal{M}$ with $\delta \in \mathrm{ran}(\pi)$ and $\mathrm{Card}(\bar{\mathcal{M}}) = \aleph_0$, $\bar{\mathcal{M}}$ is $(k, \omega_1 + 1)$ iterable above $\pi^{-1}(\delta)$ (i.e., with respect to iteration trees in which all extenders used have critical point $\geq \pi^{-1}(\delta)$). If $\delta = 0$ then we omit "above 0."*

Definition 0.2 *We let Σ denote the following partial function. $\Sigma(\mathcal{T}) = b$ if and only if $\mathcal{T}$ is an iteration tree of limit length on a (lightface) premouse and either*

(a) $\mathrm{cf}(\mathrm{lh}(\mathcal{T})) > \omega$ and b is the unique cofinal branch through $\mathcal{T}$, or else

(b) b is the unique cofinal branch through $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T})$ exists and is countably iterable.[2]

We say that an iteration tree $\mathcal{T}$ is according to Σ *if and only if $[0, \lambda)_T = \Sigma(\mathcal{T} \restriction \lambda)$ for every limit ordinal $\lambda < \mathrm{lh}(\mathcal{T})$.*

We have to introduce a new kind of iterability in order to be able to formulate to which extent K will be iterable after all.

Definition 0.3 *Let $\mathcal{M}$ be a premouse, and let $\vec{\delta} = (\delta_1 < \delta_2 < ... < \delta_n)$ be a sequence of inaccessible cardinals. We say that $\mathcal{M}$ is $\vec{\delta}$* iterable via Σ *provided the following holds true.*

Suppose that $\mathcal{T}$ is a putative iteration tree on $\mathcal{M}$ which is according to Σ. Suppose also that we can write

$$\mathcal{T} = \mathcal{T}_0 {}^\frown \mathcal{T}_1 {}^\frown \dots {}^\frown \mathcal{T}_m,$$

where $m \leq n$ and the following assumptions are met.

(a) For each $k \leq m$, $\mathcal{T}_k$ is an iteration tree of length θ_k (possibly $\theta_k = 1$, i.e., $\mathcal{T}_k$ is trivial),

(b) for each $k < m$, θ_k is a successor ordinal and $\mathcal{T}_{k+1}$ is an iteration tree on $\mathcal{M}^{\mathcal{T}_k}_{\theta_k - 1}$, the last model of $\mathcal{T}_k$,

[2]Cf. [10, Definition 2.3] on the definition of $\mathcal{Q}(b, \mathcal{T})$. There is a failure of $\delta(\mathcal{T})$ to be Woodin definable over $\mathcal{Q}(b, \mathcal{T})$.

(c) for each $k \leq m$, if $k > 0$, then $\mathrm{crit}(E_\alpha^{\mathcal{T}_k}) > \delta_k$ *whenever* $\alpha + 1 < \mathrm{lh}(\mathcal{T}_k)$, *and if* $k < n$ *then* $\mathrm{lh}(E_\alpha^{\mathcal{T}_k}) < \delta_{k+1}$ *whenever* $\alpha + 1 < \mathrm{lh}(\mathcal{T}_k)$, *and*

(d) for each $k \leq m$ such that $k < n$, either $\theta_k < \delta_{k+1}$ or else $\theta_k \in \{\delta_{k+1}, \delta_{k+1}+1\}$ and there is an unbounded $A_k \subset \delta_{k+1}$ and a non-decreasing sequence $(\Omega_i : i \in A_k)$ of inaccessible cardinals below δ_{k+1} converging to δ_{k+1} such that if $j \leq i$, $j, i \in A_k$, then $j \leq_{T_k} i$, $\pi_{0j}^{\mathcal{T}_k}$ " $\Omega_j \subset \Omega_j$, and $\pi_{ji}^{\mathcal{T}_k} \restriction \Omega_j = \mathrm{id}$.

Then either $\mathcal{T}$ has successor length and its last model is well-founded, or else $\mathcal{T}$ has limit length and $\Sigma(\mathcal{T})$ is well-defined.

An iteration tree as above is one which is the stack of iteration trees living between the "Woodin cardinals" $\vec{\delta}$ such that it will be guaranteed that there are cofinal well-founded branches at the "Woodin cardinals."

Even if $\mathcal{M} \cap \mathrm{OR} < \delta_n$ then it makes sense to say that $\mathcal{M}$ is $\vec{\delta}$ iterable. If $\mathcal{M} \cap \mathrm{OR} \geq \delta_k$ and if $\mathcal{T}$ is as above then it is easy to see that δ_1, ..., δ_k are not moved by the relevant embeddings in $\mathcal{T}$. If $\mathcal{T}_k \restriction \theta_k$ has length δ_{k+1} then there is a unique cofinal branch through $\mathcal{T}_k \restriction \theta_k$ by Definition 0.3 (d), and hence $\Sigma(\mathcal{T}_k \restriction \delta_k)$ is then automatically well-defined.

Definition 0.4 *Let Ω be a measurable cardinal. We say that* the core model K *(*of height Ω*)* exists *provided that there is a sequence $\vec{\delta} = (\delta_1 < \cdots < \delta_n)$ of inaccessible cardinals*[3] *with $\delta_n < \Omega$ and there is a premouse $\mathcal{M}$ of height Ω such that (setting $\delta_0 = 0$) for every measurable cardinal $\Omega_0 \leq \Omega$, if $k \leq n$ is maximal with $\delta_k < \Omega_0$, then $\mathcal{M}||\Omega_0 = K(K||\delta_k)$, where $K(K||\delta_k)$ is the unique $K||\delta_k$-weasel*[4] *of height Ω_0 whose proper initial segments above $K||\delta_k$ are very sound (cf. [10, Definition 2.19]).*

The following theorem will be proven in the first section. The second section will be devoted to a discussion of the correctness of K.

Theorem 0.5 *Let $n < \omega$. Suppose that $\vec{\delta} = (\delta_1 < \ldots < \delta_n)$ is a sequence of Woodin cardinals, and suppose further that $\Omega > \delta_n$ is a measurable cardinal. Suppose also that $M_{n+1}^{\#}$ does not exist.*

Then the core model K exists as being witnessed by a $\vec{\delta}$ iterable premouse of height Ω, which we denote by K.

It can be shown that the K of this theorem satisfies $\mathrm{cf}(\beta^{+K}) \geq \mathrm{Card}(\beta)$ for all $\beta \geq \aleph_2$ (cf. [6]).

The author wishes to thank the referee for his very helpful report.

[3] We allow $n = 0$ in which case the sequence $\vec{\delta}$ is empty.

[4] If $\mathcal{M} = (J_\alpha[E]; \in, E_\alpha)$ is a premouse and $\beta \leq \mathcal{M} \cap \mathrm{OR}$, then $\mathcal{M}||\beta = (J_\beta[E \restriction \beta]; \in, E_\beta)$.

1 The construction of K.

Fix $n < \omega$. Throughout this section we shall assume that $\delta_1 < \delta_2 < ... < \delta_n$ are Woodin cardinals, that $\Omega > \delta_n$ is measurable, and that $M_{n+1}^{\#}$ does not exist. We aim to construct K, "the core model of height Ω," i.e., we aim to prove Theorem 0.5.

We sometimes write $\delta_0 = 0$ and $\delta_{n+1} = \Omega$. We shall recursively construct $K||\delta_k$, $k \leq n+1$, in such a way that the following assumptions will hold inductively.

$\mathbf{A}_{1,k}$. Suppose that $k > 0$. If κ is measurable with $\delta_{k-1} < \kappa \leq \delta_k$, then $K||\kappa$ is the core model over $K||\delta_{k-1}$ of height κ in the sense of [10, Definition 2.19]; moreover, $K||\kappa$ is $\kappa + 1$ iterable above $K||\delta_{k-1}$.[5]

$\mathbf{A}_{2,k}$. Suppose that $k > 1$. Then δ_1, ..., δ_{k-1} are Woodin cardinals in $K||\delta_k$.

$\mathbf{A}_{3,k}$. $K||\delta_k$ is $\vec{\delta}$ iterable.

Now suppose that $K||\delta_k$ has already been contructed, where $k < n+1$. Suppose also that $\mathbf{A}_{1,k}$, $\mathbf{A}_{2,k}$, and $\mathbf{A}_{3,k}$ hold. We aim to construct $K||\delta_{k+1}$ in such a way that $\mathbf{A}_{1,k+1}$, $\mathbf{A}_{2,k+1}$, and $\mathbf{A}_{3,k+1}$ hold.

Let us first run the $K^c(K||\delta_k)$ construction inside $V_{\delta_{k+1}}$. Let $\mathcal{N}_\xi$ and $\mathcal{M}_\xi = \mathfrak{C}_\omega(\mathcal{N}_\xi)$ be the models of this construction. We would let the construction break down just if we reached some $\xi < \delta_{k+1}$ with $\rho_\omega(\mathcal{N}_\xi) < \delta_k$. We'll show in a minute that this will not be the case.

We shall prove the following statements under the assumption that $\xi \leq \delta_{k+1}$ and $\mathcal{N}_\xi$ is defined.

$\mathbf{A}_{1,k,\xi}$. Suppose that $\xi > 0$. Then δ_1, ..., δ_k are Woodin cardinals in $\mathcal{N}_\xi$.

Proof of $\mathbf{A}_{1,k,\xi}$. By $\mathbf{A}_{2,k}$, it suffices to show that δ_k is a Woodin cardinal in $\mathcal{N}_\xi$, where we assume that $k > 0$. Let $f \colon \delta_k \to \delta_k$, $f \in \mathcal{N}_\xi$, and let $g \colon \delta_k \to \delta_k$ be defined by $g(\xi) =$ the least V-inaccessible strictly above $f(\xi) \cup \delta_{k-1}$. As δ_k is Woodin in V, there is some $\kappa < \delta_k$ and some V-extender $E \in V_{\delta_k}$ with critical point κ such that if $\pi \colon V \to_E M$, then M is transitive, $V_{\pi(g)(\kappa)} \subset M$, and $\pi(K||\delta_k) \cap V_{\pi(g)(\kappa)} = K||\delta_k \cap V_{\pi(g)(\kappa)}$. $\mathbf{A}_{1,k}$ then gives, by standard arguments (cf. [9, §8]), a total extender $F \in K||\delta_k$ with critical point κ such that if $\sigma \colon K||\delta_k \to_F K'$, then K' is transitive and $K \models V_{\sigma(f)(\kappa)} \subset K'$. $\square$ ($\mathbf{A}_{1,k,\xi}$)

$\mathbf{A}_{2,k,\xi}$. $\mathcal{N}_\xi$ is countably iterable above δ_k.

[5] I.e., with respect to iteration trees in which all extenders used have critical point $\geq \delta_{k-1}$.

PROOF of $\mathbf{A}_{2,k,\xi}$. Cf. [9, §§1, 2, and 9]. $\square$ ($\mathbf{A}_{2,k,\xi}$)

$\mathbf{A}_{3,k,\xi}$. Suppose that there is no $\delta > \delta_k$ which is definably Woodin in $\mathcal{N}_\xi$. Then $\mathcal{N}_\xi$ is $\vec{\delta}$ iterable via Σ.

PROOF of $\mathbf{A}_{3,k,\xi}$. Suppose not. Let

$$\mathcal{T} = \mathcal{T}_0 {}^\frown \mathcal{T}_1 {}^\frown \ldots {}^\frown \mathcal{T}_m,$$

where $m \leq k$, be a putative iteration tree on $\mathcal{N}_\xi$ according to Σ and as in Definition 0.3 such that either $\mathrm{lh}(\mathcal{T})$ is a successor ordinal and $\mathcal{T}$ has a last ill-founded model or else $\mathrm{lh}(\mathcal{T})$ is a limit ordinal and $\Sigma(\mathcal{T})$ is undefined.

Suppose for a second that $\mathrm{lh}(\mathcal{T})$ is a limit ordinal. Then $\mathrm{lh}(\mathcal{T}) \notin \{\delta_1, ..., \delta_n\}$, as otherwise $\Sigma(\mathcal{T})$ would clearly be defined by Definition 0.3 (d). Set $\delta = \delta(\mathcal{T})$, and let W be the last "$\mathcal{N}$ model" from the $K^c(\mathcal{M}(\mathcal{T}))$ construction (built, say, as in [2]) such that δ is a Woodin cardinal in W. If W has height OR, then by $\mathbf{A}_{1,k,\xi}$ and by the proof of [1, Theorem 11.3], we have that δ_1, ..., δ_n (as well as $\delta \notin \{\delta_1, ..., \delta_n\}$, of course) are Woodin cardinals in W, and Ω will be measurable in W; therefore, as $M_{n+1}^\#$ does not exist, we then know that $W||\Omega^{++}$ can't be countably iterable.

If $\mathrm{lh}(\mathcal{T})$ is a successor ordinal then we let δ and W be undefined.

Now let Θ be regular and large enough, and pick

$$\bar{H} \xrightarrow{\pi} H' \xrightarrow{\sigma} H_\Theta$$

such that $\bar{H}$ and H' are transitive, $\mathrm{Card}(\bar{H}) = \aleph_0$, and $\{\mathcal{N}_\xi, \mathcal{T}\} \subset \mathrm{ran}(\sigma \circ \pi)$. If $k > 0$, then we shall also assume that $\mathrm{Card}(H') < \delta_k$ and $\mathrm{ran}(\sigma) \cap \delta_k \in \delta_k$. (If $k = 0$, then we allow $\sigma = \mathrm{id}$.) Set $\bar{\mathcal{N}} = (\sigma \circ \pi)^{-1}(\mathcal{N}_\xi)$ and $\bar{\mathcal{T}} = (\sigma \circ \pi)^{-1}(\mathcal{T})$. If $\mathrm{lh}(\mathcal{T})$ is a limit ordinal, then we shall also assume that $W||\Omega^{++} \in \mathrm{ran}(\sigma \circ \pi)$. Setting $\bar{W} = (\sigma \circ \pi)^{-1}(W||\Omega^{++})$, if W has height OR, then we may and shall assume also that $\bar{W}$ is not $\omega_1 + 1$ iterable. We also write $\mathcal{N}' = \pi(\bar{\mathcal{N}}) = \sigma^{-1}(\mathcal{N}_\xi)$ and $\delta' = \sigma^{-1}(\delta_k)$ ($= \mathrm{crit}(\sigma)$ if $k > 0$).

Claim 1. Suppose that $k > 0$, and let $\kappa < \delta_k$ be measurable with $\mathcal{N}' \cap \mathrm{OR} < \kappa$. Then $\mathcal{N}'$ is coiterable with $K||\kappa$ via the strategy Σ.

PROOF OF CLAIM 1. Suppose not. Let $\mathcal{W}$ and $\mathcal{V}$ denote the putative iteration trees arising from the coiteration of $\mathcal{N}'$ with $K||\kappa$. $\mathcal{W}$ is above δ' and $\mathcal{V}$ is above δ_{k-1}. We assume that $\mathcal{W}$ has a last ill-founded model or else that $\Sigma(\mathcal{W})$ is not defined. By $\mathbf{A}_{2,k,\xi}$, we must have that $\mathcal{W}$ has limit length and $\Sigma(\mathcal{W})$ is not defined. By $\mathbf{A}_{3,k}$, $\Sigma(\mathcal{V})$ is well-defined, i.e., $\mathcal{Q}(\mathcal{V}) = \mathcal{Q}(\Sigma(\mathcal{V}), \mathcal{V})$ does exists.

Let us pick

$$\tau : H'' \to H_\Theta$$

such that H'' is transitive, $\mathrm{Card}(H'') = \aleph_0$, and $\{\mathcal{N}', \mathcal{W}, \mathcal{Q}(\mathcal{V})\} \subset \mathrm{ran}(\tau)$. Set $\bar{\mathcal{N}}' = \tau^{-1}(\mathcal{N}')$ and $\bar{\mathcal{W}} = \tau^{-1}(\mathcal{W})$.

As $\mathcal{N}_\xi$ is countably iterable above δ_k, we certainly have that $\bar{\mathcal{N}}'$ is ω_1+1 iterable above $\tau^{-1}(\delta')$. In particular, $\mathcal{Q}(\bar{\mathcal{W}})$ exists, and $\mathcal{Q}(\bar{\mathcal{W}}) \trianglelefteq \mathcal{M}^{\bar{\mathcal{W}}}_{\Sigma(\bar{\mathcal{W}})}$. However, we must have that $\mathcal{Q}(\bar{\mathcal{W}}) = \tau^{-1}(\mathcal{Q}(\mathcal{V}))$. In particular, $\mathcal{Q}(\bar{\mathcal{W}}) \in H''$, and therefore by standard arguments $\Sigma(\mathcal{W})$ is well-defined after all. $\square$ (Claim 1)

Claim 2. $\bar{\mathcal{N}}$ is $\omega_1 + 1$ iterable.

Proof of Claim 2. Well, if $k = 0$ then this readily follows from $\mathbf{A}_{2,k,\xi}$. Let us now assume that $k > 0$. Let $\kappa < \delta_k$ be measurable with $\mathcal{N}' \cap \mathrm{OR} < \kappa$. By Claim 1, we may let $\mathcal{W}$ and $\mathcal{V}$ denote the iteration trees arising from the (successful) coiteration of $\mathcal{N}'$ with $K||\kappa$ via Σ. By $\mathbf{A}_{1,k}$, we shall have that $\pi^{\mathcal{W}}_{0\infty} \colon \mathcal{N}' \to \mathcal{M}^{\mathcal{V}}_\infty||\gamma$ for some γ, where $\pi^{\mathcal{W}}_{0\infty} \restriction \delta' = \mathrm{id}$. We then have that

$$\pi^{\mathcal{W}}_{0\infty} \circ \pi : \bar{\mathcal{N}} \to \mathcal{M}^{\mathcal{W}'}_\infty||\gamma$$

witnesses, by $\mathbf{A}_{3,k}$, that $\bar{\mathcal{N}}$ is $\omega_1 + 1$ iterable via Σ. $\square$ (Claim 2)

In particular, $\mathcal{T}$ must have limit length. By Claim 2, $\mathcal{Q}(\bar{\mathcal{T}})$ exists (and is ω_1+1 iterable). Set $\mathcal{Q} = \mathcal{Q}(\bar{\mathcal{T}}) \trianglelefteq \mathcal{M}^{\bar{\mathcal{T}}}_{\Sigma(\bar{\mathcal{T}})}$, and set $\bar{\delta} = (\sigma \circ \pi)^{-1}(\delta)$. We have that $\bar{\delta}$ is a cutpoint in $\bar{W}$, $\bar{\delta}$ is a Woodin cardinal in $\bar{W}$, and $\bar{W}$ is ω_1+1 iterable above $\bar{\delta}$. Of course, $\mathcal{Q}||\bar{\delta} = \bar{W}||\bar{\delta}$. Let $\mathcal{V}'$ and $\mathcal{V}$ denote the iteration trees arising from the (successful) comparison of $\mathcal{Q}$ with $\bar{W}$.

If W has height OR, then we shall have that $\pi^{\mathcal{V}}_{0\infty} \colon \bar{W} \to \mathcal{M}^{\mathcal{V}'}_\infty||\gamma'$ for some γ'. By Claim 2, this embedding witnesses that $\bar{W}$ is $\omega_1 + 1$ iterable. Contradiction!

Therefore, W defines a witness to the non-Woodinness of δ. But then $\mathcal{V}'$ and $\mathcal{V}$ are both trivial and $\bar{W} = \mathcal{Q}(\bar{\mathcal{T}})$. Standard arguments then yield that $\Sigma(\mathcal{T})$ would be defined after all. Contradiction! $\square$ ($\mathbf{A}_{3,k,\xi}$)

$\mathbf{A}_{4,k,\xi}$. $\rho_\omega(\mathcal{N}_\xi) \geq \delta_k$.

Proof of $\mathbf{A}_{3,k,\xi}$. We may trivially assume that $k > 0$. Suppose that $\rho = \rho_\omega(\mathcal{N}_\xi) < \delta_k$. Let Θ be large enough, and pick $\pi \colon \bar{H} \to H_\Theta$ such that $\bar{H}$ is transitive, $\mathrm{Card}(\bar{H}) < \delta_k$, $\mathrm{ran}(\pi) \cap \delta_k \in (\delta_k \setminus (\rho+1))$, and $\{\mathcal{N}_\xi, \delta_k\} \subset \mathrm{ran}(\pi)$. Set $\bar{\mathcal{N}} = \pi^{-1}(\mathcal{N}_\xi)$ and $\bar{\delta} = \pi^{-1}(\delta_k)$. Let $\kappa < \delta_k$ be measurable such that $\kappa > \bar{\mathcal{N}} \cap \mathrm{OR}$. By $\mathbf{A}_{3,k,\xi}$ (or rather by the proof of Claim 1 in the proof of $\mathbf{A}_{3,k,\xi}$), we may now successfully coiterate $\bar{\mathcal{N}}$ with $K||\kappa$. $\mathbf{A}_{1,k}$ implies that any witness to $\rho = \rho_\omega(\bar{\mathcal{N}})$ is an element of $K||\kappa$. But any such witness is also a witness to $\rho = \rho_\omega(\mathcal{N}_\xi)$. Contradiction!

$\square$ ($\mathbf{A}_{3,k,\xi}$)

We have shown that $\mathcal{N}_{\delta_{k+1}}$ exists, is $\vec{\delta}$ iterable, and that δ_1, ..., δ_k are Woodin cardinals in $\mathcal{N}_{\delta_{k+1}}$. Let us write $K^c(K||\delta_k)$ for $\mathcal{N}_{\delta_{k+1}}$.

Now let Ω_0 be a measurable cardinal with $\delta_k < \Omega_0 < \delta_{k+1}$. Using $K^c(K||\delta_k)||\Omega_0$ we may define K_{Ω_0} as the core model over $K||\delta_k$ of height Ω_0. If Ω_0 and Ω_1 are measurable cardinals with $\delta_k < \Omega_0 \leq \Omega_1 \leq \delta_{k+1}$ then $K_{\Omega_0} \trianglelefteq K_{\Omega_1}$ by local definability. Let us define $K||\delta_{k+1}$ as the "union" of the K_{Ω_0}, where Ω_0 is a measurable cardinal with $\delta_k < \Omega_0 \leq \delta_{k+1}$. It is now easy to see that $\mathbf{A}_{1,k+1}$, $\mathbf{A}_{2,k+1}$, and $\mathbf{A}_{3,k+1}$ hold true.

In the end, we therefore have $\mathbf{A}_{1,n+1}$, $\mathbf{A}_{2,n+1}$, and $\mathbf{A}_{3,n+1}$. This finishes the proof of Theorem 0.5.

The following result is due to John Steel; it is included here with his permission.

Lemma 1.1 (Steel) *Let $n < \omega$. Then $M_n \models$ "$V = K$." In particular, $M_n \models$ "$V = HOD$."*

PROOF. We may assume that $n > 0$. Let $\delta_1 < \cdots < \delta_n$ be the Woodin cardinals of M_n. Set $\delta_{-1} = 0$. Write $K = K^{M_n}$. Suppose that $M_n \models$ "$V \neq K$," so that there is a least $k < n$ and some Ω with $\delta_k < \Omega < \delta_{k+1}$ such that Ω is measurable in M_n, but $K||\Omega \neq M_n||\Omega$. Because $K||\delta_k = M_n||\delta_k$, we may coiterate $K||\Omega$ with $M_n||\Omega$ inside M_n, and in fact by the proof of [4, Theorem 2.3] $M_n||\Omega$ is a normal iterate of $K||\Omega$ with iteration map $\pi\colon K||\Omega \to M_n||\Omega$, say. But then π must be the identity, as otherwise K has an extender on its sequence which does not exist in M_n, which is absurd. Contradiction! $\square$ (Lemma 1.1)

2 From iterability to correctness.

Definition 2.1 *Let $n \geq 1$. We say that K is Σ^1_{n+1} correct if for every nonempty $A \subset {}^\omega\omega$ which is Π^1_n, there is some (countable) $\omega_1 + 1$ iterable premouse $\mathcal{M}$ such that $A \cap \mathcal{M} \neq \emptyset$.*

By Shoenfield absoluteness, K is always Σ^1_2 correct. Steel has shown in [9] that if there are two measurable cardinals but no inner model with a Woodin cardinal then K is Σ^1_3 correct.

We could have defined "K is Σ^1_{n+1} correct" to mean that $K \prec_{\Sigma^1_{n+1}} V$, but we prefer a formulation which does not presuppose the existence of the core model K.

If K exists then K is Σ^1_{n+1} correct (in the sense of definition 2.1) if and only if $K \prec_{\Sigma^1_{n+1}} V$.

The purpose of this section is to isolate a sufficient criterion for the fact that K is Σ^1_{n+1} correct. Such criteria may also implicitly be found in work of Woodin, Steel, Neeman, and others, but the exact way in which we bring the previous section of the current paper into play should be new.

Definition 2.2 *Let $x \in {}^\omega\omega$, and let $\mathcal{D}$ be an x premouse. Let $n \in \omega$. We say that $\mathcal{D}$ is an n* large dagger *(over x) if and only if, for*

$$\varphi \equiv \text{ there are two measurable cardinals above } n \text{ Woodin cardinals,}$$

we have that $\mathcal{D} \models \varphi$ (as being witnessed by the extender sequence of $\mathcal{D}$), but for every proper initial segment $\bar{\mathcal{D}}$ of $\mathcal{D}$, $\bar{\mathcal{D}} \models \neg\varphi$.

Let $\mathcal{D}$ be an n large dagger. We let $\delta_1 = \delta_1^{\mathcal{D}} < \ldots < \delta_n = \delta_n^{\mathcal{D}}$ denote the Woodin cardinals of $\mathcal{D}$, we let $\kappa = \kappa^{\mathcal{D}} < \Omega = \Omega^{\mathcal{D}}$ denote the two largest measurable cardinals of $\mathcal{D}$. The following definition is basically borrowed from [3, p. 330].

Definition 2.3 *Let $n \in \omega$, and let $\mathcal{D}$ be an n large dagger. Let $k \leq n$. We say that $\mathcal{D}$ is 0-iterable iff $\mathcal{D}$ is (linearily) iterable by the (unique) measure on $\kappa^{\mathcal{D}}$ and its images. For $k > 0$, we say that $\mathcal{D}$ is k iterable iff for every iteration tree $\mathcal{T}$ on $\mathcal{D}$ of length ω which uses extenders below $\delta^{\mathcal{D}}_{n-k+1}$ and above $\delta^{\mathcal{D}}_{n-k}$ (if $k < n$) there is a cofinal branch b such that the limit model $\mathcal{M}^{\mathcal{T}}_b$ is well-founded and $k-1$ iterable.*

The significance of n iterability for our purposes is the following. The result is due to W. Hugh Woodin (cf. [8, Corollary 4.4]), modulo [3, Corollary 1.8].[6]

Lemma 2.4 *Let $n \in \omega$, and let $\mathcal{D}$ be a countable n large dagger. If $\mathcal{D}$ is n iterable then $\mathcal{D}$ is Σ^1_{n+2} correct, i.e., $\mathcal{D} \prec_{\Sigma^1_{n+2}} V$.*

Proof sketch. We prove by induction on $k \leq n$ that the k iterability of $\mathcal{D}'$ implies $\mathcal{D}' \prec_{\Sigma^1_{k+2}} V$ for all countable $\mathcal{D}'$ which look like an n large dagger cut off at $\Omega + n - k$.[7] Well, the 0 iterability of such a $\mathcal{D}'$ easily gives $\mathcal{D}' \prec_{\Sigma^1_2} V$ by Shoenfield absoluteness. Now let $k > 0$, let $\vec{a} \in {}^\omega\omega \cap \mathcal{D}'$, let $\varphi(v, \vec{w})$ be Π^1_{k+1}, and assume that $\exists v \varphi(v, \vec{a})$ holds in V. We aim to prove that $\mathcal{D}' \models \exists v \varphi(v, \vec{a})$.

[6] I basically learned the proof to follow from M. Zeman.

[7] If $\mathcal{D}$ is an n large dagger, then $\mathcal{D}$ cut off at $\Omega + n - k$ is $\mathcal{D}||\Omega + n - k$, where of course $\Omega = \Omega^{\mathcal{D}}$. Notice that the concept of k iterability still makes perfect sense for such structures $\mathcal{D}'$.

Pick $y \in {}^\omega\omega \cap V$ such that $\varphi(y, \vec{a})$ holds. Let $\mathcal{T}$ be an iteration tree on $\mathcal{D}'$ of length ω which uses extenders below $\delta^{\mathcal{D}'}_{n-k+1}$ and above $\delta^{\mathcal{D}'}_{n-k}$ (if $k < n$), and let b be a cofinal branch b such that the limit model $\mathcal{M}^{\mathcal{T}}_b$ is well-founded and $k-1$ iterable and such that y is $Col(\omega, \pi^{\mathcal{T}}_{0b}(\delta_{n-k+1}))$-generic over $\mathcal{M}^{\mathcal{T}}_b$. Such $\mathcal{T}$, b exist by [3, Corollary 1.8]. Our induction hypothesis easily gives that

$$\Vdash^{Col(\omega,\pi^{\mathcal{T}}_{0b}(\delta_{n-k+1}))}_{\mathcal{M}^{\mathcal{T}}_b} \exists v \varphi(v, \check{\vec{a}}),$$

and therefore via $\pi^{\mathcal{T}}_{0b}$,

$$\Vdash^{Col(\omega,\delta_{n-k+1})}_{\mathcal{D}'} \exists v \varphi(v, \check{\vec{a}}).$$

Pick $\sigma \colon \mathcal{P} \rightarrow \mathcal{D}'||(\Omega + n - k - 1)$ with $\vec{a} \in \mathcal{P}$, where $\sigma \in \mathcal{D}'$ and $\mathcal{P}$ is transitive and countable in $\mathcal{D}'$.[8] Let $g \in \mathcal{P}$ be $Col(\omega, \sigma^{-1}(\delta_{n-k+1}))$-generic over $\mathcal{P}$. We have that $\mathcal{P}[g] \models \exists v \varphi(v, \vec{a})$, so that (as $\mathcal{P}[g]$ looks like an initial segment of a $k-1$ large dagger which, from the point of view of $\mathcal{D}'$, is $k-1$ iterable) by the induction hypothesis (applied inside $\mathcal{D}'$), $\mathcal{D}' \models \exists v \varphi(v, \vec{a})$.

In particular, $\mathcal{D}||\Omega \prec_{\Sigma^1_{n+2}} V$, and hence $\mathcal{D} \prec_{\Sigma^1_{n+2}} V$ as well. □

The key idea for proving that K is Σ^1_{n+3} correct is to show that K contains an n large dagger $\mathcal{D}$ which certifies a given Σ^1_{n+3} statement φ, i.e., such that $\mathcal{D} \models \varphi$ and $\mathcal{D}$ is n iterable. The n iterability of $\mathcal{D}$ should follow from a certain amount of iterability of $K^{\mathcal{D}}$, the core model of $\mathcal{D}$ in the sense of the previous section. In order to make this precise we shall now formulate a hypothesis.

Definition 2.5 *Let $n \in \omega$, and let $\mathcal{D}$ be an n large dagger. If $\mathcal{D} \models$ "$M^{\#}_{n+1}$ doesn't exist," we let $K^{\mathcal{D}}$ be the core model from the point of view of $\mathcal{D}$ and of height $\Omega^{\mathcal{D}}$, constructed in the fashion of the previous section.*

Iterability Inheritance Hypothesis (IIH). Let $n \in \omega$, and let $\mathcal{D}$ be an n large dagger such that $\mathcal{D} \models$ "$M^{\#}_{n+1}$ doesn't exist." If $K^{\mathcal{D}}$ is countably iterable, then $\mathcal{D}$ is n iterable.[9]

Steel showed (IIH) for $n = 0$, cf. [9, §7 D.]. We shall discuss the general (IIH) at the end of this paper.[10]

Theorem 2.6 *Assume (IIH). Let $n \in \omega$. Assume that V is closed under $X \mapsto M^{\dagger}_n(X)$, but $M^{\#}_{n+1}$ doesn't exist. Then K is Σ^1_{n+3} correct.*

[8] $\Omega + n - k - 1$ is the largest ordinal of $\mathcal{D}'$.

[9] The referee points out that the proof of Theorem 2.6 only uses that if $K^{\mathcal{D}}$ is $\omega_2 + 1$ iterable, then $\mathcal{D}$ is n iterable.

[10] I first learned that some such hypothesis should prove that K is correct from Hugh Woodin.

PROOF. Under the hypothesis of Theorem 2.6, the core model K exists. The reason is that for each α, we may construct K of height κ inside $M_n^\dagger(V_\alpha)$, where κ is the least measurable cardinal of $M_n^\dagger(V_\alpha)$,[11] and that for each ξ there is some α such that

$$K^{M_n^\dagger(V_\beta)}||\xi = K^{M_n^\dagger(V_\alpha)}||\xi$$

for all $\beta \geq \alpha$, so that we may define $K||\xi$ of V as the eventual value of $K^{M_n^\dagger(V_\alpha)}||\xi$. We shall now in fact prove by induction that for each $k \leq n$, K contains a tree which projects to a universal Π^1_{k+2} set of reals.

So let us fix $k \leq n$, and let us assume that K contains a tree U which projects to a universal Π^1_{k+1} set of reals. (This is certainly true for $k = 0$.) Let $A \subset {}^\omega\omega$ be a universal Π^1_{k+2} set of reals, say $x \in A \iff \varphi(x)$, where φ is Π^1_{k+2}. It is easy to construct a tree $T \in K$ which, through finite approximations, searches for:

- a real x
- a countable k large dagger $\mathcal{D}$ over a real y such that $x \in \mathcal{D}$ and $\mathcal{D} \models \varphi(x)$
- a countable (lightface) premouse $\mathcal{R}$ together with an elementary embedding $\sigma: K^{\mathcal{D}} \to \mathcal{R}$
- a countable (lightface) premouse $\mathcal{P}$ together with an elementary embedding $\pi: \mathcal{P} \to K||\omega_2^V$
- an iteration tree $\mathcal{T}$ on $\mathcal{P}$ with last model $\mathcal{R}$ such that for each limit ordinal $\lambda < \mathrm{lh}(\mathcal{T})$, there is some $\mathcal{Q} \trianglelefteq \mathcal{M}_\lambda^\mathcal{T}$ witnessing that $\delta(\mathcal{T} \restriction \lambda)$ is not Woodin such that $\mathcal{Q}$ is Π^1_{k+1} iterable above $\delta(\mathcal{T} \restriction \lambda)$ as being witnessed by U.

We shall write $p[T]$ for the projection of T, i.e., for the set of reals x such that there are objects $\mathcal{D}$, y, $\mathcal{R}$, σ, $\mathcal{P}$, π, $\mathcal{T}$ as above. We now prove that $p[T] = A$. Our hypothesis (IIH) is only used for showing that $p[T] \subset A$.

Claim 1. Let $A \subset p[T]$.

PROOF OF CLAIM 1. Fix $x \in A$. Let $X \subset 2^{\aleph_2}$ code a well-ordering of H_{ω_3}. Let $\tau: \bar{\mathcal{D}} \to M_k^\dagger(X)$, where $\bar{\mathcal{D}}$ is countable and transitive, and $x \in \bar{\mathcal{D}}$. Let $y \in {}^\omega\omega \cap V$ code a $Col(\omega, \tau^{-1}(2^{\aleph_1}))$-generic filter over $\bar{\mathcal{D}}$. It is easy to see that $\bar{\mathcal{D}}[y]$ can be rearranged as a k large dagger over y. Set $\mathcal{D} = \bar{\mathcal{D}}[y]$. Obviously, $\mathcal{D} \models \varphi(x)$ by Lemma 2.4.

[11] $M_n^\dagger(V_\alpha)||\kappa$ will be closed under $X \mapsto M_n^\#(X)$, so that its K^c of height κ is $\kappa+1$ iterable inside $M_n^\dagger(V_\alpha)$, and we may construct K from this K^c as in [1].

Because $\tau \upharpoonright K^{\mathcal{D}}: K^{\mathcal{D}} = K^{\bar{\mathcal{D}}} \to K^{M_k^\dagger(X)}$, $K^{\mathcal{D}}$ is ω_2+1 iterable (in V as well as in $M_k^\dagger(X)$). By the universality of $K||\omega_2$ with respect to countable ω_2+1 iterable premice, there is thus an iteration tree $\mathcal{T}'$ on $K^{M_k^\dagger(X)}||\omega_2$ with a last model $\mathcal{R}'$ which is built according to the iteration strategy such that there is some $\sigma': K^{\mathcal{D}} \to \mathcal{R}'$. By taking a Skolem hull, we then see that in fact $x \in p[T]$. $\square$ (Claim 1)

Claim 2. Let $p[T] \subset A$.

Proof of Claim 2. Fix $x \in p[T]$, and let $\mathcal{D}$, y, $\mathcal{R}$, σ, $\mathcal{P}$, π, $\mathcal{T}$ be witnesses to this fact. In particular, $\mathcal{R}$ is an iterate of $\mathcal{P}$ according to the iteration strategy, so that $K^{\mathcal{D}}$ is ω_1+1 iterable. By (IIH), $\mathcal{D}$ is k iterable. But then $\mathcal{D} \models \varphi(x)$ implies $V \models \varphi(x)$ in the light of Lemma 2.4. This means that $x \in A$. $\square$ (Claim 2)

(Theorem 2.6)

It remains to discuss why we actually conjecture (IIH) to be true. The following does hold true.

Lemma 2.7 *Suppose that $\delta_1 < ... < \delta_n$ are Woodin cardinals, that $\Omega > \delta_n$ is measurable, and that $M_{n+1}^\#$ does not exists. Let $E \in V_\Omega$ be an extender with countably closed support. Let $\pi: V \to_E M$ where M is transitive.*

Let K denote the core model of height Ω constructed in the previous section. There is then an iteration tree $\mathcal{T}$ on K as in Definition 0.3 such that $\mathcal{M}_\infty^{\mathcal{T}} = \pi(K)$ and $\pi_{0\infty}^{\mathcal{T}} = \pi \upharpoonright K$.

Proof. This follows by the methods of [5]. Let $E \in V_\rho$, where ρ is measurable and in fact $\mathrm{crit}(E) < \delta_i \Rightarrow \rho < \delta_i$ for $i = 1, ...,n$. Let $(\Omega_i: i \leq \beta)$ be the monotone enumeration of the cardinals in the closed interval $[\rho, \Omega]$ which are either measurable or else Woodin. Notice that $\pi"\Omega_i \subset \Omega_i$ and $\pi(\Omega_i) = \Omega_i$ for all $i \leq \beta$.

Claim. For all $i \leq \beta$, the following assertion, call it $\mathbf{A}_i$, holds true. There is an iteration tree $\mathcal{T}_i$ of length $\alpha_i + 1$, some $\alpha_i \leq \Omega_i$, on $K||\Omega_i$ as in Definition 0.3 such that $\mathcal{M}_{\alpha_i}^{\mathcal{T}_i} = \pi(K||\Omega_i) = \pi(K)||\Omega_i$. Moreover, if $j \leq i$ then (we may construe $\mathcal{T}_i$ as an extension of $\mathcal{T}_j$ and) $\alpha_j \in [0, \alpha_i]_{T_i}$ (i.e., α_j is on the main branch of $\mathcal{T}_i$), and $\pi_{\alpha_j \alpha_i}^{\mathcal{T}_i} \upharpoonright \Omega_j = \mathrm{id}$.

In the light of [5], the proof of this Claim is an easy induction on $i \leq \beta$. Notice that if Ω_i is one of the Woodin cardinals then $\mathbf{A}_i$ trivially follows from $\mathbf{A}_j$ for all $j < i$. If Ω_i is not one of the Woodin cardinals then we get the first part of $\mathbf{A}_i$ from $\mathbf{A}_j$ for all $j < i$ together with the proof of [5, Theorem 2.1]. The second part of $\mathbf{A}_i$ is obtained as follows. If $\alpha_j \notin [0, \alpha_i]_{T_i}$ then $\alpha_j < \alpha_i$ and if $\xi + 1$ is least in $[0, \alpha_i]_{T_i} \setminus \alpha_j$

then $\mathrm{crit}(E_\xi^{\mathcal{T}_i}) < \Omega_j$ and $\mathrm{lh}(E_\xi^{\mathcal{T}_i}) \geq \Omega_j$; but then $\pi_{0\alpha_i}^{\mathcal{T}_i}{}''\Omega_j \not\subset \Omega_j$, although $\pi''\Omega_j \subset \Omega_j$ and $\pi_{0\alpha_i}^{\mathcal{T}_i} \restriction \Omega_i = \pi \restriction \Omega_i$. The same argument shows that $\pi_{\alpha_j\alpha_i}^{\mathcal{T}_i} \restriction \Omega_j = \mathrm{id}$. $\square$ (Claim)
$\square$ (Lemma 2.7)

The conjecture might thus be that if $\mathcal{T}$ is an iteration tree of length ω on a k large dagger $\mathcal{D}$ then there is an iteration tree $\mathcal{U}$ on $K^{\mathcal{D}}$ which "absorbs" $\mathcal{T}$, i.e., there is $\varphi\colon \omega \to \mathrm{lh}(\mathcal{U})$ such that $K^{\mathcal{M}_n^{\mathcal{T}}} = \mathcal{M}_{\varphi(n)}^{\mathcal{U}}$ and $\pi_{mn}^{\mathcal{T}} \restriction K^{\mathcal{M}_m^{\mathcal{T}}} = \pi_{\varphi(m)\varphi(n)}^{\mathcal{U}}$ whenever $m \leq_T n$ and such that the cofinal branch through $\mathcal{U}$ given by the iteration strategy for $K^{\mathcal{D}}$ induces a cofinal branch through $\mathcal{T}$ with a well-behaved limit model. It is conceivable that the methods of [5] will eventually yield a proof of (IIH).

One could also, as remarked by the referee, formulate a version of (IIH) in which $K^{\mathcal{D}}$ is replaced by $L[E]^{\mathcal{D}}$, the fully backgrounded extender model as constructed in $\mathcal{D}$. However, there is no equivalent of [5] for $L[E]$, so that this version of (IIT) for $L[E]$ might be an even harder nut to crack.

References

[1] W. Mitchell and J. Steel, *Fine structure and iteration trees*, Lecture Notes in Logic #3, Springer-Verlag 1994.

[2] W. Mitchell and R. Schindler, *K^c without large cardinals in V*, Journal Symb. Logic **69** (2004), pp. 371-386.

[3] I. Neeman, *Optimal proofs of determinacy*, Bull. Symb. Logic **1** (1995), pp. 327-339.

[4] E. Schimmerling and J. Steel, *The maximality of the core model*, Trans. Amer. Math. Soc. **351** (1999), pp. 3119-3141.

[5] R. Schindler, *Iterates of the core model*, J. Symb. Logic, to appear.

[6] R. Schindler and J. Steel, *The self-iterability of* $L[E]$, in preparation.

[7] R. Schindler and J. Steel, List of open problems in inner model theory, available at
`wwwmath1.uni-muenster.de/logik/org/staff/rds/list.html`.

[8] J. Steel, *Projectively well-ordered inner models*, Ann. Pure Appl. Logic **74** (1995), pp. 77-104.

[9] J. Steel, *The core model iterability problem* Lecture Notes in Logic #8, Springer-Verlag 1996.

[10] J. Steel, *Core models with more Woodin cardinals*, Journal Symb. Logic **67** (2002), pp. 1197-1226.

[11] J. Steel, *private communication.*

[12] W.H. Woodin, *private communication.*