FORCING AXIOMS AND PROJECTIVE SETS OF REALS

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Abstract. This paper is an introduction to forcing axioms and large cardinals. Specifically, we shall discuss the large cardinal strength of forcing axioms in the presence of regularity properties for projective sets of reals.

The new result shown in this paper says that ZFC + the bounded proper forcing axiom (BPFA) + "every projective set of reals is Lebesgue measurable" is equiconsistent with ZFC + "there is a Σ_1 reflecting cardinal above a remarkable cardinal."

1. Introduction.

The current paper* is in the tradition of the following result.

Theorem 1. ([HaSh85, Theorem B]) *Equiconsistent are:*
(1) ZFC + *there is a weakly compact cardinal, and*
(2) ZFC + *Martin's axiom* (MA) + *every projective set of reals is Lebesgue measurable.*

This theorem links a *large cardinal concept* with a *forcing axiom* (MA) and a regularity property for the *projective sets of reals.* We shall be interested in considering forcing axioms which are somewhat stronger than MA. In particular, we shall produce the following new result.†

Theorem 2. *Equiconsistent are:*
(1) ZFC + *there is a Σ_1 reflecting cardinal above a remarkable cardinal, and*
(2) ZFC + *the bounded proper forcing axiom* (BPFA) + *every projective set of reals is Lebesgue measurable.*

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We don't expect the reader to be familiar with the concepts appearing in the previous theorems, as we are going to explain them here. We do have to presuppose, though, a basic understanding of *forcing* and *constructibility theory*; this material is covered by the text books [Je78] and [Ku80].

Large cardinals are ubiquitous in set theory. In [Hau14], Hausdorff had asked whether there is a regular fixed point of the aleph function $i \mapsto \aleph_i$. We know today that the existence of such a fixed point cannot be shown in ZFC, because if κ is regular and $\kappa = \aleph_\kappa$ then $L_\kappa \models \mathsf{ZFC}$. (Here and in what follows we assume that ZFC is consistent.) We may say that a formula $\phi(-)$ defines a large cardinal concept if ZFC proves that $\forall\kappa(\phi(\kappa) \Rightarrow L_\kappa \models \mathsf{ZFC})$. A large cardinal axiom is a statement of the form $\exists\kappa\ \phi(\kappa)$ for some large cardinal concept given by $\phi(-)$. Any large cardinal axiom is thereby independent from ZFC.

Forcing axioms are also independent from ZFC. The prototype of all forcing axioms, Martin's axiom, was isolated from the Solovay-Tennenbaum proof of the consistency of the non-existence of Suslin trees on ω_1 (cf. [Je78, Theorem 50] or [Ku80, II, §4]). A forcing axiom asserts the existence of reasonably generic filters for posets from a fixed class of forcings. One may often view forcing axioms as a substitute for CH (the continuum hypothesis) in contexts where CH fails.

Projective sets of reals are primarily studied in descriptive set theory. A projective set comes from a closed set via a finite process of taking complements and projections (cf. [Je78, p. 500]). ZFC typically decides questions about "simple" projective sets, but it leaves the arbitrary case open. For instance, every analytic set of reals is Lebesgue measurable in ZFC, whereas the Lebesgue measurability of all $\mathbf{\Delta}^1_2$ sets of reals is independent from ZFC. Nevertheless, most descriptive set theorists believe it is *true* that every projective set of reals is Lebesgue measurable.

We are now going to study the Lebesgue measurability of all projective sets of reals in the context of forcing axioms.

Set theorists obtain insight into the mathematical structure provided by various hypotheses by determining their *consistency strength*. We'll write $\mathsf{Con}(\Sigma) \Rightarrow \mathsf{Con}(\Sigma')$ for the statement that the consistency of the theory Σ implies that of Σ'. We say that Σ and Σ' are *equiconsistent* just in case $\mathsf{Con}(\Sigma) \Leftrightarrow \mathsf{Con}(\Sigma')$. The consistency strength (or, large cardinal strength) of a given hypothesis Ψ is that large cardinal axiom $\exists\kappa\ \phi(\kappa)$ such that $\mathsf{ZFC} + \Psi$ is equiconsistent with $\mathsf{ZFC} + \exists\kappa\ \phi(\kappa)$. It is an empirical fact (and one of the miracles of set theory) that "natural" hypotheses do have a consistency strength; this paper in fact gives examples. We refer the reader to [Ka94] for extensive information about this phenomenon.

The current paper can also be read as an introduction to a method for proving equiconsistency results. Typically, the direction $\mathsf{Con}(\mathsf{ZFC} + \Psi) \Rightarrow \mathsf{Con}(\mathsf{ZFC} + \exists\kappa\ \phi(\kappa))$ uses constructibility theory, whereas $\mathsf{Con}(\mathsf{ZFC} + \exists\kappa\ \phi(\kappa)) \Rightarrow \mathsf{Con}(\mathsf{ZFC} + \Psi)$ is an application of the method of forcing.

2. MA and the Harrington-Shelah theorem.

Let $\mathbb{P}$ be a poset, and let $\mathfrak{D}$ be a collection of sets such that every $D \in \mathfrak{D}$ is a subset of $\mathbb{P}$. We say that a filter $F \subset \mathbb{P}$ is **$\mathfrak{D}$-generic** just in case $F \cap D \neq \emptyset$ for every $D \in \mathfrak{D}$. $D \subset \mathbb{P}$ is **predense** (in $\mathbb{P}$) if every $p \in \mathbb{P}$ is compatible with some $q \in D$. $D \subset \mathbb{P}$ is **dense** (in $\mathbb{P}$) if for every $p \in \mathbb{P}$ there is some stronger $q \in D$. If $\mathfrak{D}$ is the collection of all dense subsets of $\mathbb{P}$ then a $\mathfrak{D}$-generic filter cannot exist, unless $\mathbb{P}$ contains atoms. On the other hand, if $\mathfrak{D}$ is countable then a $\mathfrak{D}$-generic filter always exists. In particular, if M is a countable transitive model of ZFC with $\mathbb{P} \in M$ and if $\mathfrak{D} \in M$ is the collection of all the dense subsets of $\mathbb{P}$ which exist in M then there is a $\mathfrak{D}$-generic filter; it is one of the key building blocks of the theory of forcing that such a filter G can then be "adjoined" to M to produce another model of ZFC, $M[G]$.

The approach of forcing axioms is to ask for more than countably many dense sets to be met at once, where $\mathbb{P}$ will have to belong to a specific class of posets. In what follows we shall focus on the case that no more than $\aleph_1$ many dense sets are to be met.

Definition 3. *Let $\Gamma \subset V$ be a class of posets. We say that $\mathsf{MA}(\Gamma)$ holds if for every $\mathbb{P} \in \Gamma$ and for every collection $\mathfrak{D}$ of sets such that*

- $\mathrm{Card}(\mathfrak{D}) \leq \aleph_1$, *and*
- *D is dense in $\mathbb{P}$ for every $D \in \mathfrak{D}$*

there is some $\mathfrak{D}$-generic filter.

Let $\mathsf{c.c.c.}$ denote the class of all posets which have the countable chain condition. We take **Martin's Axiom**, which is abbreviated by MA and which was mentioned in Theorem 1, to say that $\mathsf{MA}(\mathsf{c.c.c.})$ holds. In particular, MA – as we define it – implies that $2^{\aleph_0} \geq \aleph_2$ (i.e., that the continuum hypothesis, CH, is false).

The reader may consult [Je78, §§22, 23, 44] or [Ku80, Chapters II and VII] on MA. One can construe MA as a generalization of the Baire category theorem (cf. [Ku80, Theorem II.2.22]). The reader might also enjoy the discussion of the plenty consequences of MA that can be found in [Fr84].

It can be shown that $\mathsf{Con}(\mathsf{ZFC}) \Rightarrow \mathsf{Con}(\mathsf{ZFC} + \mathsf{MA})$ (cf. the paragraph right after Theorem 5 below). It is actually consistent to require that κ many dense sets can be met for any $\kappa < 2^{\aleph_0}$, but it is inconsistent to require $2^{\aleph_0}$ many dense sets to be met.

By [Ku80, Theorem II.3.4], MA is equivalent to the fact that for every $\mathbb{P} \in \mathsf{c.c.c.}$ which is of size $\leq \aleph_1$ and for every collection $\mathfrak{D}$ of sets such that $\mathrm{Card}(\mathfrak{D}) \leq \aleph_1$ and D is dense in $\mathbb{P}$ for every $D \in \mathfrak{D}$ there is some $\mathfrak{D}$-generic filter. This readily implies that MA is equivalent to $\mathsf{BMA}(\mathsf{c.c.c.})$ according to the following definition.

Definition 4. *Let $\Gamma \subset V$ be a class of posets. We say that $\mathsf{BMA}(\Gamma)$ holds if for every $\mathbb{P} \in \Gamma$ and for every collection $\mathfrak{D}$ of sets such that*
- $\mathrm{Card}(\mathfrak{D}) \leq \aleph_1$*, and*
- *D is predense in $\mathbb{P}$ and has size $\leq \aleph_1$ for every $D \in \mathfrak{D}$*

there is some $\mathfrak{D}$-generic filter.

In order to uncover the relation of MA with the Lebesgue measurability of projective sets of reals we need a first large cardinal concept. We say that a cardinal κ is **inaccessible** if $\kappa > \aleph_0$, κ is regular, and $2^\gamma < \kappa$ for all cardinals $\gamma < \kappa$. We say that ω_1 is **inaccessible to the reals** if $\omega_1^{L[x]}$ is countable for every real x. Notice that if ω_1 is inaccessible to the reals then ω_1 is an inaccessible cardinal from the point of view of L.

Theorem 5. ([Sh84]) *If every projective set of reals is Lebesgue measurable then ω_1 is inaccessible to the reals. In fact, if every $\mathbf{\Sigma}^1_3$ set of reals is Lebesgue measurable then ω_1 is inaccessible to the reals.*

There is a poset $\mathbb{P} \in \mathsf{c.c.c.}$ such that MA provably holds in $V^{\mathbb{P}}$ (cf. [Je78, §23] or [Ku80, VIII. §6]). Let us assume that $V = L$. We shall then have that $\omega_1^{V^{\mathbb{P}}} = \omega_1^V = \omega_1^L$, and we may apply Theorem 5 inside $V^{\mathbb{P}}$ to get that in $V^{\mathbb{P}}$ there is a projective set of reals which is not Lebesgue measurable. In particular, MA does not imply that all projective sets of reals are Lebesgue measurable:

Corollary 6. (Folklore) *Equiconsistent are:*

(1) ZFC*, and*

(2) $\mathsf{ZFC} + \mathsf{MA}$ + *there is a projective set of reals which is not Lebesgue measurable.*

We now want to approach Theorem 1. A cardinal κ is called **weakly compact** if κ is inaccessible and there is no Aronszajn tree on κ, i.e., for every tree T of height κ such that each level of T is of size $< \kappa$ there is a branch through T of length κ. Equivalently, κ is weakly compact if κ is Π^1_1 indescribable in that for every $A \subset \kappa$ and for every

Π_1 formula $\phi(-)$, if $V_{\kappa+1} \models \phi(A)$ then there is some $\lambda < \kappa$ with $V_{\lambda+1} \models \phi(A \cap \lambda)$ (cf. [Je78, p. 386]). The proof of $\mathsf{Con}(2) \Rightarrow \mathsf{Con}(1)$ needs an improvement of Theorem 5 in the presence of MA.

Lemma 7. ([HaSh85, p. 184f.]) *Assume that* MA *holds. If* $\aleph_1$ *is inaccessible to the reals then in fact* $\aleph_1$ *is weakly compact in* L*.*

Proof. Let us suppose that MA holds and that $\aleph_1$ is not weakly compact in L. We shall produce a real x such that $\omega_1^{L[x]} = \omega_1$.

As $\aleph_1$ is not weakly compact in L, there is a tree $T \in L$ such that (in V!) T is an Aronszajn tree on ω_1 (this is due to Silver, cf. [HaSh85, Claim 5]). We may and shall assume that T has infinitely many nodes of height 0. For a sequence $\vec{d} = (d_i \colon i < \omega_1)$ of subsets of ω we define $\mathbb{P}(\vec{d})$ by setting $p \in \mathbb{P}(\vec{d})$ iff p is a finite order preserving partial function from T into $\mathbb{Q}$ (the rationals) such that if $a \in dom(p)$ has height $\omega \cdot i$ in T and $p(a) \in \mathbb{N} \subset \mathbb{Q}$ then $p(a) \in d_i$. We set $p \leq_{\mathbb{P}(\vec{d})} q$ iff $p \supset q$. It is not hard to verify that $\mathbb{P}(\vec{d}) \in \mathsf{c.c.c.}$

Now let $\vec{b} = (b_i \colon i < \omega_1)$ be such that $b_i \subset \omega$ and $\omega_1^{L[b_i]} > i$ for each $i < \omega_1$. It suffices to show that there is some $c \subset \omega$ with $\vec{b} \in L[c]$. For $i < \omega_1$ let us write $T \upharpoonright i$ for the restriction of T to nodes of height $< i$.

We shall define $\vec{b^n} = (b_i^n \colon i < \omega_1)$ and F_n by induction on $n < \omega$. Let $b_i^0 = b_i$ for $i < \omega_1$. Given $\vec{b^n}$, let F_n be, by an application of MA, a $\mathfrak{D}_n$-generic filter on $\mathbb{P}(\vec{b^n})$, where $\mathfrak{D}_n$ is some appropriate collection of dense sets with $\mathrm{Card}(\mathfrak{D}_n) = \aleph_1$ such that the following will hold true:
$(1)_n$ $m \in b_i^n$ iff there is an a of height $\omega \cdot i$ in T with $F_n(a) = m$, and
$(2)_n$ F_n is continuous at limit ordinals.
Let $b_i^{n+1} \subset \omega$ be a canonical code for $F_n \upharpoonright (T \upharpoonright \omega \cdot (i+1))$, where $i < \omega_1$. Finally, let $c \subset \omega$ be a code for $(b_0^n \colon n < \omega)$.

It is now easy to verify that $(b_i^n \colon n < \omega, i < \omega_1) \in L[c]$: Let $i < \omega_1$, and suppose that we already know $(b_j^n \colon n < \omega, j < i)$. As b_j^{n+1} codes $F_n \upharpoonright (T \upharpoonright \omega \cdot (j+1))$ for every $j < i$, we can thus compute $F_n \upharpoonright (T \upharpoonright \omega \cdot i)$. But $(2)_n$ then gives us $F_n \upharpoonright (T \upharpoonright \omega \cdot i + 1)$, and hence by $(1)_n$ we can compute b_i^n.

$\square$ (Theorem 7)

Let us now turn towards $\mathsf{Con}(1) \Rightarrow \mathsf{Con}(2)$. The key concept here is $L(\mathbb{R})$-absoluteness for a class of posets. Let $\Gamma \subset V$ be a class of posets. We say that **$L(\mathbb{R})$-absoluteness for Γ holds** (in V) iff for all $\mathbb{P} \in \Gamma$ we have that the first order theories (with names for reals from V) of $L(\mathbb{R})$ and $L(\mathbb{R}^{V^{\mathbb{P}}})$ agree with each other.

Lemma 8. (Folklore) *Let* $\Gamma \subset V$ *be a definable class of posets. Let* κ *be an inaccessible cardinal, and suppose that for all* $\dot{\mathbb{P}} \in \Gamma^{V^{Col(\omega,<\kappa)}}$ *and* *for all reals* $x \in V^{Col(\omega,<\kappa)\star\dot{\mathbb{P}}}$ *we have that there is some poset* $\mathbb{Q} \in V_\kappa$ *with* $x \in V^{\mathbb{Q}}$. *Then* $L(\mathbb{R})$*-absoluteness for* Γ *holds in* $V^{Col(\omega,<\kappa)}$.

Proof. Let G be $Col(\omega, < \kappa)$-generic over V, and let H be $\dot{\mathbb{P}}^G$-generic over $V[G]$. Finally, let E be $Col(\omega, (2^{\aleph_0})^{V[G][H]})$-generic over $V[G][H]$. Let $(e_i \colon i < \omega) \in V[G][H][E]$ be such that $\{e_i \colon i < \omega\} = \mathbb{R} \cap V[G][H]$. By working inside $V[G][H][E]$ we may use the hypothesis of Lemma 8 to construct $(\alpha_i, G_i \colon i < \omega)$ such that for all $i < \omega$ we shall have that $\alpha_i < \alpha_{i+1} < \kappa$, G_i is $Col(\omega, < \alpha_i)$-generic over V, $G_{i-1} \subset G_i$ (with the convention that $G_{-1} = \emptyset$), $G_i \in V[G][H]$, and $e_i \in V[G_i]$. Set $G' = \bigcup_i G_i$. Because $Col(\omega, < \kappa)$ has the κ-c.c., G' is $Col(\omega, < \kappa)$-generic over V, and every real in $V[G']$ is in $V[G_i]$ for some $i < \omega$. We thus get that $\mathbb{R} \cap V[G'] = \mathbb{R} \cap V[G][H]$.

This yields that the first order theories of $L(\mathbb{R}^{V[G]})$ and $L(\mathbb{R}^{V[G][H]})$ agree with each other: because $Col(\omega, < \kappa)$ is homogeneous (cf. [Je78, Theorem 63]), we get that for any formula $\phi(\vec{v})$ and for any $\vec{x} \in \mathbb{R} \cap V[G]$ we have that $L(\mathbb{R}^{V[G]}) \models \phi(\vec{x})$ iff $\Vdash^{V[\vec{x}]}_{Col(\omega,<\kappa)}$ "$L(\dot{\mathbb{R}}) \models \phi(\check{\vec{x}})$" iff $L(\mathbb{R}^{V[G][H]}) \models \phi(\vec{x})$.

$\square$ (Lemma 8)

Theorem 9. ([So70]) *Let* κ *be inaccessible. Then in* $V^{Col(\omega,<\kappa)}$ *we have that every projective set of reals is Lebesgue measurable.*

Corollary 10. *Let* $\Gamma \subset V$ *be a definable class of posets. Suppose that* κ *is an inaccessible cardinal and* ϕ *is a statement such that*

(a) *for all* $\dot{\mathbb{P}} \in \Gamma^{V^{Col(\omega,<\kappa)}}$ *and for all reals* $x \in V^{Col(\omega,<\kappa)\star\dot{\mathbb{P}}}$ *we have that there is some poset* $\mathbb{Q} \in V_\kappa$ *with* $x \in V^{\mathbb{Q}}$,

(b) ϕ *holds in* $V^{Col(\omega,<\kappa)}$, *and*

(c) *provably in* ZFC + ϕ, *there is some* $\mathbb{P} \in \Gamma$ *such that* $\mathsf{MA}(\Gamma)$ *holds in* $V^{\mathbb{P}}$.

There is then a poset $\mathbb{S}$ *such that in* $V^{\mathbb{S}}$ *we have that:* $\mathsf{MA}(\Gamma)$ *holds and all projective sets of reals are Lebesgue measurable.*

The same result holds with "$\mathsf{MA}(\Gamma)$*" being replaced by "*$\mathsf{BMA}(\Gamma)$*."*

Proof. We set $\mathbb{S} = Col(\omega, < \kappa) \star \dot{\mathbb{P}}$ for some $\dot{\mathbb{P}}$ which is provided by (b) and an application of (c) inside $V^{Col(\omega,<\kappa)}$. The result follows by Lemma 8 and Theorem 9, via (a).

$\square$ (Corollary 10)

We are now in a position to be able to give a proof for Con(2) $\Rightarrow$ Con(1) of Theorem 1. As it is provable in ZFC that there is a poset $\mathbb{P} \in$ c.c.c. such that MA holds in $V^{\mathbb{P}}$, we may apply Corollary 10 with ϕ being some logical truth, say. It remains to see that (a) of Corollary 10 holds for $\Gamma =$ c.c.c. and for κ being weakly compact:

Lemma 11. (Kunen, [HaSh85, p. 186]) *Let κ be weakly compact. Suppose that $V^{Col(\omega,<\kappa)} \models \dot{\mathbb{P}}$ is a poset which has the c.c.c. Then for all reals $x \in V^{Col(\omega,<\kappa)\star\dot{\mathbb{P}}}$ we have that there is some poset $\mathbb{Q} \in V_\kappa$ with $x \in V^{\mathbb{Q}}$.*

Proof. It is easy to verify that $Col(\omega,<\kappa)\star\dot{\mathbb{P}} \in$ c.c.c.. Let us work with Boolean algebras, and set $\mathfrak{B} = \mathrm{r.o.}(Col(\omega,<\kappa)\star\dot{\mathbb{P}})$. It suffices to show that any $X \subset \mathfrak{B}$ of size $<\kappa$ generates a complete subalgebra of $\mathfrak{B}$ of size $<\kappa$.

Well, there is clearly a complete subalgebra $\bar{\mathfrak{B}}$ of $\mathfrak{B}$ with $X \subset \bar{\mathfrak{B}} \subset \mathfrak{B}$ and $\mathrm{Card}(\bar{\mathfrak{B}}) \leq \kappa$. W.l.o.g., $\bar{\mathfrak{B}} \subset \kappa$. Let $A \subset [\kappa]^{<\kappa}$ be the set of all maximal antichains of $\bar{\mathfrak{B}}$. As κ is Π^1_1 indescribable, there is some $\lambda < \kappa$ such that $X \subset \lambda$, $\bar{\mathfrak{B}} \cap \lambda$ is a $<\lambda$-complete Boolean algebra, and $A \cap [\lambda]^{<\lambda}$ is the set of all maximal antichains of $\bar{\mathfrak{B}} \cap \lambda$. But then $\bar{\mathfrak{B}} \cap \lambda$ is a complete Boolean algebra, and hence a complete subalgebra of $\mathfrak{B}$.

$\square$ (Lemma 11)

3. PFA and projective sets.

If X is a set then $[X]^\omega$ denotes the set of all subsets of X of size $\aleph_0$. Let $\alpha \geq \aleph_1$. A set $S \subset [\alpha]^\omega$ is called **stationary** iff for all models $\mathfrak{M} = (\alpha; ...)$ of finite type and with universe α there is some $(X; ...) \prec \mathfrak{M}$ such that $X \in S$. A poset $\mathbb{P}$ is called **proper** if for all $\alpha \geq \aleph_1$, whenever $S \subset [\alpha]^\omega$ is stationary in $[\alpha]^\omega$ then

$$V^{\mathbb{P}} \models \check{S} \text{ is stationary.}$$

We let proper denote the class of all posets which are proper. The **Proper Forcing Axiom**, abbreviated by PFA, says that MA(proper) holds.

The reader may find information on PFA in [Je86] and [Bau84]. In particular, if there is a supercompact cardinal then there is a proper poset $\mathbb{P}$ such that PFA holds in $V^{\mathbb{P}}$ (cf. [Je86, Theorem 6.2]).

In this section we shall see that PFA implies that all projective sets of reals are Lebesgue measurable. It is not known how to prove this directly, though. We have to show a *transfer theorem* of the following form: if PFA holds then there are certain inner models for such-and-such large cardinals, which in turn implies Lebesgue measurability for projective sets. The argument will in fact give that PFA implies much more, namely Projective Determinacy.

Here is the key concept which links PFA with inner model theory. Let κ be an infinite cardinal. We say that $\square_\kappa$ holds if there is a sequence $(C_i \colon \kappa < i < \kappa^+ \wedge \lim(i))$ such that each C_i is a closed unbounded subset of i of order type $\leq \kappa$ and $C_j = C_i \cap j$ whenever j is a limit point of i. The principle $\Box_\kappa$ was isolated by Jensen in the context of his seminal work on Gödel's constructible universe.

Lemma 12. ([To84]) $\mathsf{PFA} \Rightarrow \forall\kappa \geq \aleph_1\ (\neg\Box_\kappa)$.

PROOF. Let T be the set of all limit ordinals i with $\kappa < i < \kappa^+$. Suppose $(C_i \colon i \in T)$ witnesses that $\Box_\kappa$ holds. For $j < i \in T$ let us write $j \prec i$ iff j is a limit point of C_i (iff $j < i$ and $C_j = C_i \cap j$). Then $(T, \prec)$ is a tree, which we shall denote by $\mathfrak{T}$.

Set $\mathbb{P} = \{p \subset \kappa^+ \colon p$ is closed and countable$\}$, ordered by end-extension. Notice that $\mathbb{P}$ is ω-closed, and forcing with $\mathbb{P}$ adds a closed unbounded subset of κ^+ of order type ω_1. The proof of the following Claim makes use of all the properties of our $\Box_\kappa$-sequence.

Claim. In $V^{\mathbb{P}}$, there is no branch through $\mathfrak{T} \upharpoonright \bigcup \dot{G}$ of order type ω_1.

PROOF. Suppose that $p \in \mathbb{P}$ forces that there is some such branch. Notice that $\mathbb{P} \times \mathbb{P}$ is also ω-closed. Let $G_0 \times G_1$ be $\mathbb{P} \times \mathbb{P}$ generic over V with $p \in G_0 \cap G_1$. Let $b^h \in V[G_h]$ be a branch through $\mathfrak{T} \upharpoonright \bigcup \dot{G}^{G_h}$ of order type ω_1, for $h \in \{0, 1\}$. Let $A^h = \bigcup_{i \in b^h} C_i$ for $h \in \{0, 1\}$. As $cf^{V[G_0 \times G_1]}(\kappa^+) = \omega_1$, it is straightforward to see that we must have $A^0 = A^1$. Therefore, $A^0 \in V$. But then we must have $\mathrm{otp}^V(A^0) = \kappa^+$, and therefore there is some $i \in T$ with $\mathrm{otp}(C_i) > \kappa$. Contradiction!

$\Box$ (Claim)

Inside $V^{\mathbb{P}}$, there is hence a forcing $\dot{\mathbb{Q}}$ for adding, with finite conditions, a specializing function f for $\mathfrak{T} \upharpoonright \bigcup \dot{G}$, i.e., an order preserving $f \colon \mathfrak{T} \upharpoonright \bigcup \dot{G} \to \mathbb{Q}$ (cf. the forcing $\mathbb{P}(\vec{d})$ in the proof of Lemma 7). A Δ-system argument shows that $\dot{\mathbb{Q}}$ has the c.c.c. In particular, the iteration $\mathbb{P} \star \dot{\mathbb{Q}}$ is proper.

For $k < \omega_1$, let D_k be the set of all $(p, \dot{q}) \in \mathbb{P} \star \dot{\mathbb{Q}}$ such that $\mathrm{otp}(p) \geq k + 1$ and if i is the k^{th} element of p then $(p, \dot{q})$ decides $\dot{f}(i)$. Each D_k is dense. Let $\mathfrak{D} = \{D_k \colon k < \omega_1\}$.

By PFA, let (H_0, H_1) be a $\mathfrak{D}$-generic filter. Let $i = \sup(\bigcup H_0)$, and let f be the specializing function for $\mathfrak{T} \upharpoonright \bigcup H_0$ which is provided by H_1. Then f witnesses that $\mathfrak{T} \upharpoonright \bigcup H_0$ cannot have a branch of order type ω_1. However, $C_i \cap \bigcup H_0$ is exactly one such branch. Contradiction!

$\Box$ (Lemma 12)

The following two results were products of Jensen's fine structural analysis of L.

Lemma 13. ([Je72]) $V = L \Rightarrow \forall\kappa \geq \aleph_1\ \Box_\kappa$.

We say that $0^\#$ exists if there is an elementary embedding $\pi\colon L \to L$ with $\pi \neq \mathrm{id}$. The existence of $0^\#$ is a large cardinal axiom in that if $\pi\colon L \to L$ has critical point κ then $L_\kappa \models \mathsf{ZFC}$.

Lemma 14. (Jensen, [DeJe75]) *Suppose that $0^\#$ does not exist. If κ is a singular cardinal then $\kappa^{+L} = \kappa^+$. In fact, if $X \subset L$ then there is a set $Y \in L$ with $Y \supset X$ and $\mathrm{Card}(Y) \leq \mathrm{Card}(X) + \aleph_1$.*

Corollary 15. $\mathsf{PFA} \Rightarrow 0^\#$ *exists.*

Proof. Assume that PFA holds. Set $\kappa = \aleph_\omega$. By Lemma 12 we know that $\square_\kappa$ fails. On the other hand, $L \models V = L$, and therefore we have that $L \models \square_\kappa$ by Lemma 13. Let $(C_i\colon \kappa < i < \kappa^{+L}, \lim(i)) \in L$ be a witness. By Lemma 14, $\kappa^{+L} = \kappa^+$, and therefore $(C_i\colon \kappa < i < \kappa^+, \lim(i))$ in fact witnesses that $\square_\kappa$ holds in V. Contradiction!

$\square$ (Corollary 15)

We may look at the previous proof from a more abstract point of view. It tells us that PFA implies that there can be no pair (κ, W) such that κ is a cardinal, W is an inner model, $\kappa^{+W} = \kappa^+$, and $W \models \square_\kappa$. However, the existence of such pairs (κ, W) can be shown from weaker anti large cardinal hypotheses than the non-existence of $0^\#$. Such an analysis leads to the following remarkable theorem, via work of Martin, Mitchell, Schimmerling, Steel, and Woodin. In its statement, PD denotes Projective Determinacy (cf. [Je78] p. 560). We refer the reader to [Ka94, Chapter 6] for a thorough and nicely written discussion of the significance of PD in modern set theory.

Theorem 16. (Martin, Mitchell, Schimmerling, Steel, Todorcevic, and Woodin) $\mathsf{PFA} \Rightarrow \mathsf{PD}$.

Proof. The basic idea, which is due to Woodin, is to prove by induction on $n < \omega$ that V is closed under the operator $x \mapsto M_n^\#(x)$ for all $n < \omega$. Here, $M_n^\#(x)$ denotes a sufficiently iterable "sharp" for an inner model with n Woodin cardinals which contains x. The reader may find the details of such an induction in the paper [FoMaSch∞]. The point is that if $n < \omega$ is such that V is closed under $x \mapsto M_n^\#(x)$ but $M_{n+1}^\#(x_0)$ does not exist for some x_0 then there is a constructible inner model $L[E]$ such that $\kappa^{+L[E]} = \kappa^+$ and $L[E] \models \square_\kappa$ for all large enough singular cardinals κ. This gives a contradiction with PFA via Lemma 12.

$\square$ (Theorem 16)

A classical result of Mycielski and Swierczkowski yields that PD implies that every projective set of reals is Lebesgue measurable (cf. [Je78, Theorem 102 (a)]). We therefore finally established the following.

Corollary 17. $\mathsf{PFA} \Rightarrow$ *every projective set of reals is Lebesgue measurable.*

We do not know whether there is a proof of Corollary 17 which does not go through PD.

We let stap denote the class of all posets $\mathbb{P}$ which are **stationary preserving**, i.e., such that for all $S \subset \omega_1$ we have that

$$S \text{ is stationary } \Rightarrow V^{\mathbb{P}} \models \check{S} \text{ is stationary.}$$

Every proper forcing is clearly stationary preserving, but the converse is false.

The papers [FoMaSh88] and [FoMaSh88] introduced **Martin's Maximum**, abbreviated by MM, as $\mathsf{MA}(\mathsf{stap})$. Obviously, $\mathsf{MM} \Rightarrow \mathsf{PFA}$. We refer the reader to [Je86, III §7] or [FoMaSh88] and [FoMaSh88a] for further information on MM.

4. A few remarks on OCA.

Let X be a set. By $[X]^2$ we denote the set of all $\{x, y\} \subset X$ with $x \neq y$. Let $[X]^2 = K_0 \cup K_1$ with $K_0 \cap K_1 = \emptyset$; K_0, K_1 is then called a **colouring** of $[X]^2$. For $h \in \{0, 1\}$ we say that $Y \subset X$ is h-**homogeneous** iff $[Y]^2 \subset K_h$.

Now let $X \subset \mathbb{R}$. We view X as a topological space with the topology being induced by that of $\mathbb{R}$. $[X]^2$ is a topological space as well via the product topology. We say that $\mathsf{OCA}(X)$ holds iff for every colouring K_0, K_1 of $[X]^2$ such that K_0 is open in $[X]^2$, either there is an uncountable 0-homogeneous $Y \subset X$ or else X is a countable union of 1-homogeneous sets. The **Open Colouring Axiom**, abbreviated by OCA, is the statement that $\mathsf{OCA}(X)$ holds for every $X \subset \mathbb{R}$. The reader may find background information about OCA in [To89, Chap. 8] and [FaTo93, Chap. 10].

Lemma 18. ([To89, Theorem 8.0]) $\mathsf{PFA} \Rightarrow \mathsf{OCA}$.

Proof. Let $X \subset \mathbb{R}$, and let K_0, K_1 be a colouring of $[X]^2$. Let us suppose that X is not the union of countably many 1-homogeneous sets. Then in $V^{Col(\omega,\mathbb{R})}$, X is still not the union of countably many 1-homogeneous sets. By [To89, Theorem 4.4], in $V^{Col(\omega,\mathbb{R})}$ there is therefore some uncountable $\dot{Y} \subset X$ such that

$$\dot{\mathbb{P}} = \{p \subset \dot{Y} : \text{Card}(p) < \aleph_0, [p]^2 \subset K_0\},$$

ordered by reverse inclusion, has the c.c.c. Now $Col(\omega, \mathbb{R}) \star \dot{\mathbb{P}} \in \mathsf{proper}$, and an application of PFA yields an uncountable 0-homogeneous subset of X.

□ (Lemma 18)

It can be shown, however, that $\mathsf{Con(ZFC)} \Rightarrow \mathsf{Con(ZFC + OCA)}$. In particular, OCA cannot imply that every projective set of reals is Lebesgue measurable, by Theorem 5.

We say that $\mathsf{OCA}^*(X)$ holds iff for every colouring K_0, K_1 of $[X]^2$ such that K_0 is open in $[X]^2$, either there is a perfect 0-homogeneous $Y \subset X$ or else X is a countable union of 1-homogeneous sets. For all $X \subset \mathbb{R}$, $\mathsf{OCA}^*(X)$ implies that X has the perfect subset property; therefore, $\mathsf{OCA}^*(X)$ cannot hold for all $X \subset \mathbb{R}$ (under the axiom of choice, that is).

Theorem 19. ([Fe83]) $\mathsf{PD} \Rightarrow \mathsf{OCA}^*(X)$ *for every projective* $X \subset \mathbb{R}$.

The following corollary is thus another example of a transfer theorem. It follows from Theorems 16 and 19. Again (cf. the remark right after the statement of Corollary 17) we do not know whether there is a proof of Corollary 20 which does not go through PD.

Corollary 20. $\mathsf{PFA} \Rightarrow \mathsf{OCA}^*(X)$ *for every projective* $X \subset \mathbb{R}$.

Question. Suppose that $\mathsf{OCA}^*(X)$ holds for every projective $X \subset \mathbb{R}$. Is then every projective set of reals Lebesgue measurable?

5. BPFA AND PROJECTIVE SETS.

We shall from now on be concerned with *bounded* forcing axioms, i.e., with $\mathsf{BMA}(\Gamma)$ for classes of posets Γ. It turns out that $\mathsf{BMA}(\Gamma)$ is much weaker than $\mathsf{MA}(\Gamma)$ for any reasonable class Γ of posets which is significantly larger than c.c.c..

The next lemma says that $\mathsf{BMA}(\Gamma)$ can be phrased in terms of generic absoluteness.

Lemma 21. ([Ba00, Theorem 5]) *Let* $\mathbb{P}$ *be a poset. Then* $\mathsf{BMA}(\mathbb{P})$ *holds iff for every* Σ_1 *formula* $\phi(v_1, ..., v_n)$ *and for all* $x_1, ..., x_n \in H_{\aleph_2}$,

$$\phi(x_1, ..., x_n) \Leftrightarrow V^{\mathbb{P}} \models \phi(\check{x}_1, ..., \check{x}_n).$$

The **Bounded Proper Forcing Axiom**, abbreviated by BPFA, was introduced by Goldstern and Shelah in [GoSh95] as $\mathsf{BMA}(\mathsf{proper})$. This section will produce a proof of Theorem 2.

We say that a regular cardinal κ is Σ_1 **reflecting** iff for all $a \in H_\kappa$ and for every formula $\phi(-)$, if there is some regular $\theta \geq \kappa$ with $H_\theta \models \Phi(a)$ then there is some regular $\bar{\theta} < \kappa$ with $H_{\bar{\theta}} \models \Phi(a)$ (cf. [GoSh95, Definition 2.2]). It can easily be verified that every Σ_1 reflecting cardinal is inaccessible and that if λ is a Mahlo cardinal then the set of all $\kappa < \lambda$ which are Σ_1 reflecting in V_λ is stationary in λ.

Theorem 22. ([GoSh95, Theorems 2.11 and 4.1]) *If* BPFA *holds then* $\aleph_2$ *is a* Σ_1 *reflecting cardinal in* L. *On the other hand, if* κ *is a* Σ_1 *reflecting cardinal then there is a poset* $\mathbb{P} \in$ proper *such that in* $V^{\mathbb{P}}$, BPFA *holds.*

We say that a cardinal κ is **remarkable** iff for all regular cardinals $\theta > \kappa$ there are π, M, $\bar{\kappa}$, σ, N, and $\bar{\theta}$ such that the following hold: $\pi\colon M \to H_\theta$ is an elementary embedding, M is countable and transitive, $\pi(\bar{\kappa}) = \kappa$, $\sigma\colon M \to N$ is an elementary embedding with critical point $\bar{\kappa}$, N is countable and transitive, $\bar{\theta} = M \cap \mathrm{OR}$ is a regular cardinal in N, $\sigma(\bar{\kappa}) > \bar{\theta}$, and $M = H^N_{\bar{\theta}}$, i.e., $M \in N$ and $N \models$ "M is the set of all sets which are hereditarily smaller than $\bar{\theta}$" (cf. [Sch00a, Definition 0.4]).

We are now ready to prove Theorem 2. Let us commence with proving $\mathsf{Con}(2) \Rightarrow \mathsf{Con}(1)$. By Theorems 5 and 22 it will be enough to verify that if BPFA holds and $\aleph_1$ is inaccessible to the reals then in fact $\aleph_1$ is remarkable in L. For this in turn, via Lemma 21, it suffices to prove the following.

Lemma 23. ([Sch01, §3]) *Suppose that* $\aleph_1$ *is not remarkable in* L. *There is then a poset* $\mathbb{T} \in$ proper *such that in* $V^{\mathbb{T}}$, *there is a real* x *with* $\omega_1 = \omega_1^{L[x]}$.

Proof. There is an ω-closed poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, $H_{\aleph_2} = L_{\aleph_2}[A]$ for some $A \subset \omega_1$. This part of the argument just uses the non-existence of $0^\#$. We may for instance let $\mathbb{P} = Col(\delta^+, 2^\delta) \star Col(\omega_1, \delta) \star \bar{\mathbb{P}}$ where δ is a singular cardinal of uncountable cofinality with $\delta^{\aleph_0} = \delta$ and $\bar{\mathbb{P}}$ is a forcing which codes some $D \subset \delta^+$ of the intermediate model, where $L_{\delta^+}[D] = H_{\delta^+}$, by $A \subset \omega_1$ using δ^+ many pairwise almost-disjoint subsets of δ provided by Jensen's Covering Lemma for L.

Now fix such $\mathbb{P}$ and $A \in V^{\mathbb{P}}$. In $V^{\mathbb{P}}$, we may define a poset $\dot{\mathbb{Q}}$ by setting $p \in \dot{\mathbb{Q}}$ iff there is an ordinal $\alpha < \omega_1$ such that $p\colon \alpha \to \{0,1\}$, and for all $\beta \leq \alpha$ we have that

$$L[A \cap \beta, p \restriction \beta] \models \beta \text{ is countable.}$$

It can be verified that the fact that $\aleph_1$ is not remarkable in L implies that $V^{\mathbb{P}} \models \dot{\mathbb{Q}} \in$ proper. Therefore, $\mathbb{P} \star \dot{\mathbb{Q}} \in$ proper.

Forcing with $\mathbb{P} \star \dot{\mathbb{Q}}$ adds some $B \subset \omega_1$ such that

$$L[B \cap \beta] \models \beta \text{ is countable}$$

for every $\beta < \omega_1$. Inside $V^{\mathbb{P} \star \dot{\mathbb{Q}}}$, we may therefore define a sequence $(a_i \colon i < \omega_1)$ of pairwise almost-disjoint subsets of ω such that for every $i < \omega_1$, $a_i =$ the $L[B \cap i]$-least subset of ω which is almost-disjoint from every a_j, $j < i$. We may then define a poset $\dot{\mathbb{S}} \in V^{\mathbb{P}\star\dot{\mathbb{Q}}}$ by setting $p = (l(p), r(p)) \in \dot{\mathbb{S}}$ iff $l(p)\colon n \to \{0,1\}$ for some $n < \omega$ and $r(p)$ is a finite subset of ω_1. We let $p = (l(p), r(p)) \leq_{\dot{\mathbb{S}}} q = (l(q), r(q))$ iff $l(p) \supset l(q)$, $r(p) \supset r(q)$, and for all $i \in r(q)$, if $i \in B$ then

$$\{m \in \mathrm{dom}(l(p)) \setminus \mathrm{dom}(l(q)) \ : \ l(p)(m) = 1\} \cap a_i = \emptyset.$$

The generic object will give the characteristic function of some subset a of ω such that $i \in B$ iff $a \cap a_i$ is finite. We shall have that $V^{\mathbb{P}\star\dot{\mathbb{Q}}} \models \dot{\mathbb{S}} \in \mathsf{c.c.c.}$, so that $\mathbb{T} := \mathbb{P} \star \dot{\mathbb{Q}} \star \dot{\mathbb{S}} \in \mathsf{proper}$.

However, in $V^{\mathbb{T}}$, a will be such that $H_{\aleph_2} = L_{\aleph_2}[a]$.

$\square$ (Lemma 23)

Let us now turn towards proving $\mathsf{Con}(1) \Rightarrow \mathsf{Con}(2)$ of Theorem 2. We plan on using Corollary 10 for $\Gamma = \mathsf{proper}$. Fix $\kappa < \lambda$ such that κ is remarkable and λ is Σ_1 reflecting. It is easy to verify that λ is still Σ_1 reflecting in $V^{Col(\omega,<\kappa)}$. Let ϕ denote the statement that there is a Σ_1 reflecting cardinal. We then have (b) and (c) of Corollary 10, the latter one by the second part of Theorem 22 (more precisely, by the proof of [GoSh95, Theorem 4.1]). It remains to verify (a) of Corollary 10; we shall state this as a lemma.

Lemma 24. ([Sch01, Lemma 2.1]) *Let κ be a remarkable cardinal. Let G be $Col(\omega, < \kappa)$-generic over V. Let $\mathbb{P} \in V[G]$ be a proper poset, and let H be $\mathbb{P}$-generic over $V[G]$. Then for every real x in $V[G][H]$ there is a poset $\mathbb{Q}_x \in V_\kappa$ such that x is $\mathbb{Q}_x$-generic over V.*

Proof. Let $\theta > \kappa$ be a large enough regular cardinal; in particular, we want that $\mathfrak{P}(\mathbb{P}) \subset H_\theta[G]$, i.e., that the power set of $\mathbb{P}$ be contained in $H_\theta[G]$. Let $x \in \mathbb{R} \cap V[G][H]$, and let $\dot{x} \in H_\theta[G]$ be such that $\dot{x}^H = x$. The fact that κ is remarkable in V implies that in $V[G]$, there is a stationary set Σ of countable subsets of $H_\theta[G]$ such that for each $X \in \Sigma$ we have the following: there are π, α, and $\bar{\theta}$ such that

$$\pi\colon (H_{\bar{\theta}}^{V[G\cap H_\alpha^V]}; \in, H_{\bar{\theta}}^V, G \cap H_\alpha^V) \to (H_\theta^{V[G]}; \in, H_\theta^V, G)$$

is an elementary embedding with $X = \mathrm{ran}(\pi)$, α is the critical point of π, and α and $\bar{\theta}$ are regular cardinals of V (and hence of $V[G \cap H_\alpha^V]$) (this actually characterizes remarkability; cf. [Sch01, Lemma 1.6]). As $\mathbb{P}$ is proper, Σ is still stationary in $V[G][H]$.

Now consider the structure

$$\mathfrak{M} = (H_\theta[G]; \in, \mathbb{P}, \dot{x}, H).$$

Because Σ is stationary in $V[G][H]$, we may pick some $X \in \Sigma$ such that X is the universe of an elementary submodel of $\mathfrak{M}$. Let

$$\pi\colon (H_{\bar\theta}^{V[G\cap H_\alpha^V]}; \in, \bar{\mathbb{P}}, \bar{\dot x}, \bar H) \to (H_\theta^{V[G]}; \in, \mathbb{P}, \dot x, H)$$

be the inverse of the Mostowski collapse of X. By the elementarity of π, $\bar H$ is $\bar{\mathbb{P}}$-generic over $H_{\bar\theta}^{V[G\cap H_\alpha^V]}$, and hence over all of $V[G\cap H_\alpha^V]$, as $\mathfrak{P}(\bar{\mathbb{P}}) \cap V[G\cap H_\alpha^V] \subset H_{\bar\theta}^{V[G\cap H_\alpha^V]}$. Moreover, by the definability of forcing, we get that $n \in \bar{\dot x}^{\bar H}$ iff $\exists p \in \bar H\ p \Vdash \check n \in \bar{\dot x}$ iff $\exists p \in H\ p \Vdash \check n \in \dot x$ iff $n \in \dot x^H$ iff $n \in x$. So $\bar{\dot x}^{\bar H} = x$, and we may set $\mathbb{Q}_x = Col(\omega, <\alpha) \star \dot{\bar{\mathbb{P}}}$ where $\dot{\bar{\mathbb{P}}}^{\bar H} = \bar{\mathbb{P}}$. Notice finally that $\mathbb{Q}_x \in V_\kappa$.

□ (Lemma 24)

6. BMM.

This final section will be concerned with another bounded forcing axiom. **Bounded Martin's Maximum**, abbreviated by BMM, was introduced in [GoSh95] as the statement that BMA(stap) holds. There is a lengthy discussion of BMM in [Wo99, §10.3].

Lemma 25. ([Sch00]) *Suppose that $X^\#$ does not exist for some set X. Then there is a poset $\mathbb{T} \in$ stap such that in $V^{\mathbb{T}}$, there is a real x with $\omega_1 = \omega_1^{L[x]}$.*

Proof. The point is that if $X^\#$ does not exist for some set X then we can still define (a version of) the forcing $\mathbb{T}$ from the proof of Lemma 23; moreover, $\mathbb{T}$ will be stationary preserving.

□ (Lemma 25)

As a corollary to Lemma 25 do we get that BMM is much stronger than BPFA in the presence of Lebesgue measurability of projective sets.

Corollary 26. *Suppose that* BMM *holds. If ω_1 is inaccessible to the reals then V is closed under #'s, i.e., $X^\#$ exists for every set X. In particular, if* BMM *holds and every projective set of reals is Lebesgue measurable then V is closed under #'s.*

This corollary can be improved to get the following better lower bound on BMM + "every projective set of reals is Lebesgue measurable."

Theorem 27. ([Sch00]) *If* BMM *holds and every projective set of reals is Lebesgue measurable then there is an inner model with a strong cardinal.*

As an upper bound, we only have:

Theorem 28. ([Wo99, Theorem 10.99]) *Suppose that* ZFC + *"there is a proper class of Woodin cardinals" is consistent. Then so is* ZFC + BMM + *"every projective set of reals is Lebesgue measurable."*

This leads to the obvious question: what is the consistency strength of ZFC + BMM + "every projective set of reals is Lebesgue measurable"? Let me finish with a conjecture which consists of two parts.

Conjecture. (a) If BMM holds and every projective set of reals is Lebesgue measurable then there is an inner model with infinitely many strong cardinals. (b) If BMM holds and every set of reals in $L(\mathbb{R})$ is Lebesgue measurable then the axiom of determinacy holds in $L(\mathbb{R})$.

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