Cardinal arithmetic and determinacy

Ralf Schindler

Institut für Mathematische Logik und Grundlagenforschung
WWU Münster
rds@math.uni-muenster.de

What is *cardinal arithmetic*?

It is the study of the (possible) behavior of the continuum function, $\kappa \mapsto 2^\kappa$.

More generally, it is the study of the (possible) behavior of the function λ, $\kappa \mapsto \lambda^\kappa$.

"Possible" because of the independence results of Gödel, Cohen, Solovay, Easton, and others.

Of particular interest in cardinal arithmetic are *singular* cardinals.

Typical question: Let κ be singular; given $\lambda \mapsto 2^\lambda$ for $\lambda < \kappa$, what are the possible values of 2^κ ?

For instance, what can $2^{\aleph_\omega}$ be if $\aleph_\omega$ is a strong limit cardinal?

Or: what can $2^{\aleph_{\omega_1}}$ be if $\aleph_{\omega_1}$ is a strong limit cardinal?

We say that GCH holds at κ if $2^\kappa = \kappa^+$.

Theorem (Silver; 1974). GCH cannot fail for the first time at a singular cardinal of *uncountable* cofinality.

Hence if GCH holds below $\aleph_{\omega_1}$ then it holds *at* $\aleph_{\omega_1}$, i.e., $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$.

Theorem (Galvin, Hajnal; 1975). If κ is a singular strong limit cardinal of unctble. cofinality with $\kappa = \aleph_\alpha < \aleph_\kappa$ then $2^\kappa < \aleph_{(2^{|\alpha|})^+}$.

One of the key questions, given Silver's result:

Can GCH fail for the first time at $\aleph_\omega$? More generally, can it fail for the first time at a cardinal of *countable* cofinality?

More generally, what are possible values for $2^{\aleph_\omega}$ if GCH holds below $\aleph_\omega$, say? or if $\aleph_\omega$ is a strong limit cardinal?

Surprisingly, this question leads to the area of *determinacy*.

Let $A \subset [0,1]$.

Gale/Stewart (1953): Consider the following game, G(A):

$$\begin{array}{c|ccccccc} I & i_0 & & i_2 & & \cdots & & \\ \hline II & & i_1 & & i_3 & & & \cdots \end{array}$$

$i_n \in \{0,1\}$. I.e. $\sum_{n<\infty} \frac{i_n}{2^{n+1}} \in [0,1]$.

I wins iff $\sum_{n<\infty} \frac{i_n}{2^{n+1}} \in A$, else II wins.

We say that G(A) (or, A itself) is determined iff either I or II has a winning strategy.

It is Jensen's Covering Lemma which shows there is a connection between cardinal arithmetic and determinacy, modulo a result of T. Martin.

Theorem (Jensen; 1974). If $0^{\#}$ does not exist then every uncountable set of ordinals is covered by a set *in* L of the same size.

Theorem (T. Martin; 1970). If $0^{\#}$ exists then every Π^1_1 set of reals is determined.

Corollary. If covering fails then every Π^1_1 set of reals is determined.

In particular:

Theorem (Jensen). If GCH fails first at $\aleph_\omega$ then every Π^1_1 set of reals is determined.

Here is the argument. Suppose both covering and GCH below $\aleph_\omega$ to hold. We have to count the ctble. subsets of $\aleph_\omega$ because of $2^{\aleph_\omega} = \aleph_\omega^{\aleph_0}$ by hypothesis.

Let X be a ctble. subset of $\aleph_\omega$. There is $Y \supset X$ in L of cardinality $\leq \aleph_1$. By the GCH in L there are at most (in fact exactly) $\aleph_{\omega+1}$ many such Y. Each such Y has $\aleph_1^{\aleph_0} = \aleph_1$ ctble. subsets. Hence there are at most $\aleph_1 \cdot \aleph_{\omega+1} = \aleph_{\omega+1}$ many X.

In fact more is true:

Theorem (Dodd, Jensen; 1980?). If GCH fails first at $\aleph_\omega$ then $0^\dagger$ exists.

The existence of $0^\dagger$ implies (actually is equivalent to) the determinacy of certain sets of reals in the difference hierarchy over Π^1_1.

The proof of the Dodd-Jensen result uses the core model K in much the same way as Jensen's earlier result uses the constructible universe L.

The *core model* is a model of the form $L[E]$ where E codes a sequence of (partial) extenders.

An *extender* codes an elementary embedding of the form

$$\pi : \mathcal{M} \to \mathcal{N}$$

where $\mathcal{M}$, $\mathcal{N}$ are models of weak fragments of ZFC.

The core model K is constructed recursively in such a way that all the resonable extenders which can be found in the universe are thrown into E.

The construction of K has to be done carefully; it is hard to prove the construction converges.

In fact, in order to construct K we need an *anti large cardinal assumption*; if this assumption is fairly weak then in addition we need an assuption about V being closed under certain operators in order to prove that the construction converges.

Example: Dodd and Jensen showed that if $0^\dagger$ does not exist then K exists.

Theorem (Dodd, Jensen; 1980?). Suppose that $0^\dagger$ does not exist. Then K exists. Moreover, K satisfies (a strong form of) GCH. Also, one of the following holds true.

(1) Every uncountable set of ordinals is covered by a set *in* K of the same size.

(2) (1) is false. There is then a measurable cardinal κ in K, and there is a ctble. sequence $C \subset \kappa$ (which is unique modulo finite changes) which is Prikry generic over K such that every uncountable set of ordinals is covered by a set *in* $K[C]$ of the same size.

Notice that $K[C]$ inherits the GCH from K (if we are in case (2)). Therefore, we may run exacly the same argument as before with L replaced by K or $K[C]$ to deduce that if GCH first fails at $\aleph_\omega$ then $0^\dagger$ exists.

M. Gitik has shown the optimal result, building upon work of Jensen, Mitchell, and Shelah:

Theorem (Gitik; 1989). If GCH first fails at $\aleph_\omega$ then there is an inner model with a measurable cardinal κ of Mitchell order $o(\kappa) = (2^\kappa)^+$.

What about the consistency of $\aleph_\omega$ being the first place where GCH fails?

Theorem (Magidor; 1977). If there is a supercompact cardinal then there is a transitive model of ZFC in which GCH first fails at $\aleph_\omega$.

Again, M. Gitik has shown the optimal result:

Theorem (Gitik; 1989). If there is a measurable cardinal κ of Mitchell order $o(\kappa) = (2^\kappa)^+$ then there is a generic extension of V in which GCH first fails at $\aleph_\omega$.

I. Neeman has analysed the amount of determinacy which can be derived from the sharp for an inner model with a measurable cardinal κ of Mitchell order $o(\kappa) = (2^\kappa)^+$. It is still well below Π^1_2 determinacy.

In particular: The fact that GCH first fails at $\aleph_\omega$ does not imply Π^1_2 determinacy.

Are there any statements of cardinal arithmetic which imply Π^1_2 determinacy, or projective determinacy?

What if we not only require $\aleph_\omega$ to be the first place where GCH fails, but we also demand that $2^{\aleph_\omega} > \aleph_{\omega+1}$?

Theorem (Gitik, Mitchell; 1996). Suppose κ to be a singular strong limit cardinal with $2^\kappa \geq \lambda > \kappa^+$ where λ is not the successor of a cardinal of cofinality $< \kappa$. Then one of the following holds (provided $0^\P$ doesn't exist).

(1) $o(\kappa) \geq \lambda$ in K.

(2) κ has ctble. cofinality and for each $n < \omega$, $\{\mu < \kappa : o(\mu) \geq \mu^{+n}\}$ is unbounded in κ.

Again, Gitik has shown that this is optimal, so nothing like this gives even a strong cardinal.

It doesn't help considering gaps at singular cardinals different from $\aleph_\omega$.

It even doesn't help considering a singular cardinal κ which is a fixed point of the $\aleph$ function, i.e., $\kappa = \aleph_\kappa$ (cf. Galvin-Hajnal), by results of Gitik.

But we may look at singular cardinals of unctble. cofinality and ask a different question.

M. Gitik asked the following.

Can you have a singular (strong limit) cardinal κ of unctble. cofinality such that the set of all points below κ at which GCH holds is stationary as well as costationary, i.e., such that

$$\{\mu < \kappa\colon 2^{\mu} = \mu^{+}\}$$

is stationary and costationary.

It was known for quite a while that if this were to be consistent then it would be difficult to arrange it by forcing.

In particular, known methods using strong cardinals did not work. There is a reason for this.

Theorem (Gitik, Sch; 2001). Suppose that κ is a singular (strong limit) cardinal of unctble. cofinality such that

$$\{\mu < \kappa \colon 2^{\mu} = \mu^{+}\}$$

is stationary and costationary. Then every projective set of reals is determined.

This was the first example of a statement of cardinal arithmetic which gives more than a strong cardinal.

This result is shown by an induction along the levels of the projective hierarchy, using locally built core models plus pcf theory.

The induction is Woodin's *core model induction*.

Suppose that κ is as in the hypothesis of the theorem, i.e., κ is singular and of unctble. cofinality and

$$\{\mu < \kappa \colon 2^\mu = \mu^+\}$$

is stationary and costationary.

We have to show by induction on $n < \omega$ that Π^1_n determinacy holds in every $V^{Col(\omega,\mu)}$ for $\mu < \kappa$.

We first have to reformulate, exploiting work of Martin, Steel, and Woodin, our goal in terms of inner model theory.

The fact that for each $n < \omega$, Π^1_n determinacy holds in very $V^{Col(\omega,\mu)}$ for $\mu < \kappa$ is equivalent to the fact that for each $n < \omega$ and for each bounded $X \subset \kappa$, $M_n^{\#}(X)$ exists.

$M_0^{\#}(\emptyset)$ is just $0^{\#}$. In general, $M_n^{\#}(X)$ is a canonical model containing X with a measurable cardinal above n Woodin cardinals.

The first step of the induction is basically the same one as the proof that if GCH fails first at $\aleph_\omega$ then $0^\#$ exists.

Let us now assume that we have shown that for each bounded $X \subset \kappa$, $M_n^\#(X)$ exists. Let us show that $M_{n+1}^\# = M_{n+1}^\#(\emptyset)$ exists.

We assume that this is *not* the case and work towards a contradiction.

Under our current hypotheses, we can build a "core model" K of height κ by piecing together locally defined core models.

Mitchell, Schimmerling, and Steel proved a covering lemma for this K similar in flavor to (but weaker than) the one Jensen has shown for L and Dodd and Jensen have shown for their K.

We need the *proof* of the result of Mitchell, Schimmerling, and Steel, though, and we have to combine it with results from pcf theory.

Shelah's *pcf theory* also has to be exploited in the proof of the results of Gitik and Gitik-Mitchell above which derive strength from assertions of cardinal arithmetic.

pcf$(a) = \{$cf$(\prod a/U)\colon U$ an ultrafilter on $a\}$.

Lemma (Gitik). In our current situation, for all increasing and continuous $(\kappa_i\colon i <$ cf$(\kappa))$ converging to κ, there is a limit point i such that

$$(*)\ \max(\text{pcf}(\{\kappa_j^+\colon j < i\})) > \kappa_i^+.$$

We use the covering argument for the current K to show $(*)$ must fail. Contradiction!

M. Gitik has recently shown the consistency of the hypothesis of the previous result.

Theorem (Gitik; 2004). Suppose that there is a supercompact cardinal. Then there is a forcing extension in which there is a singular (strong limit) cardinal κ of unctble. cofinality such that

$$\{\mu < \kappa \colon 2^\mu = \mu^+\}$$

is stationary and costationary.

Here is another statement of cardinal arithmetic which implies projective determinacy.

Theorem (Gitik, Sch; 2001). Suppose that $\aleph_\omega$ is a strong limit cardinal, and $2^{\aleph_\omega} > \aleph_{\omega_1}$. Then every projective set of reals is determined.

The proof of this needs a technically quite involved result to the effect that above $\aleph_2$, collapsing mice for initial segments of K are themselves initial segments of K.

Theorem (Shelah). If $\aleph_\omega$ is a strong limit cardinal then

$$2^{\aleph_\omega} < \aleph_{\omega_4}.$$

As indicated above, Gitik has improved Magidor's result to show that we can have that GCH holds below $\aleph_\omega$, but $2^{\aleph_\omega}$ is very large. But always $2^{\aleph_\omega} < \aleph_{\omega_1}$ in all of these models.

It is open, though, if it is consistent that $\aleph_\omega$ is a strong limit, but $2^{\aleph_\omega} > \aleph_{\omega_1}$.

If $\aleph_\omega$ is a strong limit, but $2^{\aleph_\omega} > \aleph_{\omega_1}$, then

$$pcf(\{\aleph_n \colon n < \omega\})$$

is bigger than $\aleph_0$.

It is not known if there can be an interval a of regular cardinals such that $pcf(a)$ is bigger than a.

This is one of the big open problems of pcf theory.