Bounded Martin's Maximum and strong cardinals

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Abstract

We show that if Bounded Martin's Maximum (BMM) holds then for every $X \in V$ there is an inner model with a strong cardinal containing X. We also discuss various open questions which are related to BMM.

1 Introduction and statement of the result.

This paper strengthens one of the results of [6]. Bounded Martin's Maximum (BMM, for short) is the statement that whenever $\mathbb{P} \in V$ is a stationary set preserving forcing notion then

$$H_{\omega_2}^V \prec_{\Sigma_1} H_{\omega_2}^{V^{\mathbb{P}}}.$$

Bounded forcing axioms were introduced in [2] (as weakenings of the "unbounded" forcing axioms PFA and MM). Todorcevic showed (cf. [8]) that BMM implies that $2^{\aleph_0} = \aleph_2$. (This was later improved by J. Moore who showed that already the Bounded Proper Forcing Axiom implies $2^{\aleph_0} = \aleph_2$; cf. [4].) We refer the reader to [9, Section 10.3] for a discussion of BMM. We proved in [6, Theorem 1.3] that if BMM holds then for every $X \in V$, $X^{\#}$ exists. The purpose of the present note is to prove the following.

Theorem 1.1 *Suppose that* BMM *holds. Then for every $X \in V$ there is an inner model with a strong cardinal containing X.*

We do not know if Theorem 1.1 gives the optimal lower bound for the consistency strength of BMM. Woodin has shown in unpublished work (cf. [10]) that $\omega + 1$ many Woodin cardinals are an upper bound. We refer the reader to [6] for further background information.

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Whereas [6] constructs, assuming $\mathsf{BMM}+V$ is not closed under the $\#$ operator, a strictly decreasing sequence of functions from ω_1 to ω_1, the key new idea here is to construct, assuming $\mathsf{BMM}+$ there is some set X which is not in an inner model with a strong cardinal, a strictly decreasing sequence of functions from ω_1 to the set of all countable mice, where "decreasing" means "decreasing in the mouse order."

2 The proof.

Let us assume that BMM holds thruout this section. We shall prove that there is an inner model with a strong cardinal. The reader will gladly verify that the argument to follow "relativizes" to any $X \in V$, yielding a proof of Theorem 1.1.

Let us suppose that there is no inner model with a strong cardinal. We may thus let K denote the core model (cf. [3]). If $\mathcal{M}$ is a premouse and $\alpha \leq \mathcal{M} \cap \mathrm{OR}$ then we say that α is overlapped in $\mathcal{M}$ just in case there is some extender $E_\nu^{\mathcal{M}}$ such that $\mathrm{crit}(E_\nu^{\mathcal{M}}) < \alpha$ and $\nu \geq \alpha$. As there is no inner model with a strong cardinal, in K there is no E_ν^K such that $\mathrm{crit}(E_\nu^K)$ is overlapped in K.

We shall use the following notation. Let $\mathcal{M} = J_\alpha[E]$ be a premouse, and let $A \subset \bar{\alpha}$ for some $\bar{\alpha} < \alpha$. Then by $\mathcal{M}[A]$ we denote the transitive set (structure) $J_\alpha[E, A]$. Of course, in general $\mathcal{M}[A]$ will not be a premouse (or not even be a model of a reasonable fragment of ZFC). Nevertheless, models of the form $\mathcal{M}[A]$ will play a key rôle in what follows.

By $\leq^*$ we shall denote the pre-well-ordering of mice (cf. for instance [7]). I.e., if $\mathcal{M}$ and $\mathcal{N}$ are mice and $\mathcal{T}, \mathcal{U}$ is the coiteration of $\mathcal{M}$, $\mathcal{N}$ then $\mathcal{M} \leq^* \mathcal{N}$ if and only if $\mathcal{M}_\infty^{\mathcal{T}} \trianglelefteq \mathcal{M}_\infty^{\mathcal{U}}$ (in which case $[0, \infty)_{\mathcal{T}}$ contains no drop). We shall write $\mathcal{M} <^* \mathcal{N}$ iff $\mathcal{M} \leq^* \mathcal{N}$ and $\neg(\mathcal{N} \leq^* \mathcal{M})$.

Using the Dodd-Jensen Lemma and the fact that there are no degenerate iterations of mice, one can show that $\leq^*$ is indeed a pre-well-ordering. The following Lemma will thus give the desired contradiction.

Lemma 2.1 *($\mathsf{BMM}+$ there is no inner model with a strong cardinal.) There is a sequence $\mathcal{S} = (A_n, C_n, (\mathcal{N}_{n,\alpha} \colon \alpha \in C_n) : n < \omega)$ such that for every $n < \omega$, $A_n \subset \omega_1$, C_n is a club subset of ω_1, $C_n \subset C_{n-1}$ if $n > 0$, and for every $\alpha \in C_n$, $\mathcal{N}_{n,\alpha}$ is a sound mouse with $\rho_\omega(\mathcal{N}_{n,\alpha}) = \kappa \geq \alpha$, where κ is the largest cardinal of $\mathcal{N}_{n,\alpha}$, and κ is not overlapped in $\mathcal{N}_{n,\alpha}$, $(A_n \cap \alpha)_{\mathrm{odd}}$ codes the mouse $\mathcal{N}_{n,\alpha}||\kappa$,[2] and $\mathcal{N}_{n,\alpha}[A_n \cap \alpha] \models$*

[2] If A is a set of ordinals then by A_{odd} we mean the set $\{\alpha \colon 2\alpha + 1 \in A\}$. The coding here and in what follows is to be understood as being according to some standard soft coding device. For instance, if M is transitive and of size ζ then there is some $E \subset \zeta^2$ and some isomorphism $\sigma \colon (\zeta; E) \cong (M; \in)$; via Gödel's pairing function, E may be construed as a subset of ζ which codes M.

"α is countable"; moreover, $\mathcal{N}_{n,\alpha} <^* \mathcal{N}_{n-1,\alpha}$ if $n > 0$ and $\alpha \in C_n$.

The proof of Lemma 2.1 exploits a version of the "faster reshaping forcing" which we had introduced in [6].

Let $n \in \omega$, and let us assume that $\mathcal{S} \restriction n$ has been constructed. We aim to construct A_n, C_n, and $(\mathcal{N}_{n,\alpha} \colon \alpha \in C_n)$.

By BMM, it suffices to force the existence of these objects with the desired properties by a stationary set preserving forcing notion.

By [5], there is a σ-closed forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$, CH holds, there is some $A^+ \subset \omega_1$ with $H_{\omega_2} = K||\omega_2[A^+]$, and $K||\omega_2$ has a largest cardinal κ which is not overlapped in $K||\omega_2$. We may assume that A^+ is chosen such that $H_{\omega_1} = K||\omega_1[A^+]$.

Let us for the rest of this argument assume that $n > 0$. The case $n = 0$ is an easier variant of what is to come. We thus may and shall assume that A^+_{odd} is the join of A_{n-1} and a code of $K||\kappa$. Let $C \subset C_{n-1}$ be club and such that there is a sequence $(\mathcal{M}_\alpha \colon \alpha \in C)$ of mice such that for every $\alpha \in C$, $(A^+ \cap \alpha)_{\text{odd}}$ codes the join of $A_{n-1} \cap \alpha$ and a code of $\mathcal{M}_\alpha$. Let κ_α denote the height of $\mathcal{M}_\alpha$ for $\alpha \in C$. The following will be crucial.

Claim 1. Let $\pi \colon \bar{K}[A^+ \cap \alpha] \cong X \prec (K||\omega_2[A^+]; \in, A^+)$, where X is countable and $\alpha = X \cap \omega_1 \in C$. Then $\mathcal{M}_\alpha \lhd \bar{K}$, $\kappa_\alpha = \pi^{-1}(\kappa)$ is the largest cardinal of $\bar{K}$, and $\bar{K} \leq^* \mathcal{N}_{n-1,\alpha}$.

Proof of Claim 1. Everything except for $\bar{K} \leq^* \mathcal{N}_{n-1,\alpha}$ easily follows by the elementarity of π. Let us write $\mathcal{N} = \mathcal{N}_{n-1,\alpha}$. Suppose Claim 1 to be false; i.e., $\mathcal{N} <^* \bar{K}$. We shall argue that $\bar{K}[A^+ \cap \alpha] \models$ "α is countable," which is of course nonsense.

Because $\bar{K}$ does not have any active extenders with indices between $\pi^{-1}(\kappa)$ and its height, it is easy to see that there must be now some $\xi < \bar{K} \cap \mathrm{OR}$ such that $\mathcal{N} <^* \bar{K}||\xi$. Let $\mathcal{T}$, $\mathcal{U}$ denote the coiteration of $\mathcal{N}$, $\bar{K}||\xi$. We know that $[0, \infty)_{\mathcal{T}}$ contains no drops, and that $\mathcal{M}^{\mathcal{T}}_\infty \trianglelefteq \mathcal{M}^{\mathcal{U}}_\infty$.

Let $\rho_\omega(\mathcal{N}) = \eta$. Because $(A_{n-1} \cap \alpha)_{\text{odd}}$ codes $\mathcal{N}||\eta$, $\mathcal{N}||\eta \in L[A_{n-1} \cap \alpha] \subset L[A^+ \cap \alpha]$.

Let $\bar{\mathcal{T}}, \bar{\mathcal{U}}$ denote the coiteration of $\mathcal{N}||\eta$, $\bar{K}||\xi$. Clearly, $\bar{\mathcal{T}}$ is an "initial segment" of $\mathcal{T}$, and $\bar{\mathcal{U}}$ is an "initial segment" of $\mathcal{U}$. Moreover, $\mathcal{N}||\eta <^* \bar{K}||\xi$, so that $[0, \infty)_{\bar{\mathcal{T}}}$ contains no drops and $\mathcal{M}^{\bar{\mathcal{T}}}_\infty \trianglelefteq \mathcal{M}^{\bar{\mathcal{U}}}_\infty$. If we had $\mathrm{lh}(\mathcal{T}) > \mathrm{lh}(\bar{\mathcal{T}})$ then, as η is not overlapped in $\mathcal{N}$ and $\rho_\omega(\mathcal{N}) = \eta$, $\mathcal{M}^{\mathcal{T}}_\infty$ would be non-sound, although $\mathcal{N} <^* \bar{K}$. Therefore, $\mathrm{lh}(\mathcal{T}) = \mathrm{lh}(\bar{\mathcal{T}})$. But then we must also have that $\mathrm{lh}(\mathcal{U}) = \mathrm{lh}(\bar{\mathcal{U}})$, as η is the largest cardinal of $\mathcal{N}$.

Because $\mathcal{N}||\eta \in L[A^+ \cap \alpha]$ and $\bar{K}||\xi \in L[\bar{K}||\xi]$, we know that $\pi^{\bar{\mathcal{T}}}_{0\infty}$ as well as $\mathcal{M}^{\bar{\mathcal{U}}}_\infty = \mathcal{M}^{\mathcal{U}}_\infty$ are elements of $L[A^+ \cap \alpha, \bar{K}||\xi]$.

As $\mathcal{M}^{\mathcal{T}}_\infty \trianglelefteq \mathcal{M}^{\mathcal{U}}_\infty$, we thus have that $\mathcal{M}^{\mathcal{T}}_\infty \in L[A^+ \cap \alpha, \bar{K}||\xi]$. But

$$\mathcal{N} \cong h^{\mathcal{M}^{\mathcal{T}}_\infty}(\mathrm{ran}(\pi^{\bar{\mathcal{T}}}_{0\infty}) \cup \{\pi^{\mathcal{T}}_{0\infty}(p_{\mathcal{N}})\}),$$

where $p_{\mathcal{N}}$ is the standard parameter of $\mathcal{N}$ and $h^{\mathcal{M}^{\mathcal{T}}_\infty}$ is an appropriate Skolem hull operator. This yields that in fact $\mathcal{N} \in L[A^+ \cap \alpha, \bar{K}||\xi]$.

Now let $a \in {}^\omega\omega \cap \mathcal{N}[A_n \cap \alpha]$ code a well-order of ω of order-type α. We have shown that $a \in L[A^+ \cap \alpha, \bar{K}||\xi]$. By [6], $V^{\mathbb{P}}$ is closed under the # operator. We therefore have that $(A^+ \cap \alpha, \bar{K}||\xi)^{\#}$ exists and $a \in (A^+ \cap \alpha, \bar{K}||\xi)^{\#}$. But $A^+ \cap \alpha$ and $\bar{K}||\xi$ are both elements of $\bar{K}[A^+ \cap \alpha]$. Due to π, $\bar{K}[A^+ \cap \alpha]$ is closed under the # operator, so that in fact $a \in \bar{K}[A^+ \cap \alpha]$. But then α is countable in $\bar{K}[A^+ \cap \alpha]$, a contradiction. □ (Claim 1)

In $V^{\mathbb{P}}$, we shall now consider the following forcing notion, denoted by $\mathbb{Q}$. We let $(c, p) \in \mathbb{Q}$ if and only if c is a countable closed subset of C, $p\colon \max(c) \to 2$, and for every $\alpha \in c$ there is some sound mouse $\mathcal{N} \trianglerighteq \mathcal{M}_\alpha$ such that κ_α is the largest cardinal of $\mathcal{N}$, κ_α is not overlapped in $\mathcal{N}$, $\rho_\omega(\mathcal{N}) = \kappa_\alpha$, $\mathcal{N}[A^+ \cap \alpha, p \restriction \alpha] \models$ "α is countable," and $\mathcal{N} <^* \mathcal{N}_{n-1,\alpha}$. A condition (c', q) is stronger than (c, p) iff $\max(c') \geq \max(c)$, $c' \cap (\max(c) + 1) = c$, and $q \restriction \max(c) = p$.

Claim 2. ("Extendability Lemma") If $(c, p) \in \mathbb{Q}$ and $\alpha < \omega_1$ then there is some $(c', q) \leq (c, p)$ such that $\max(c') \geq \alpha$.

Proof of Claim 2. Given (c, p) and α, let us assume w.l.o.g. that $\alpha \geq \max(c) + \omega$, $\alpha \in C$, and $\alpha = X \cap \omega_1$ for some $X \prec K||\omega_2[A^+]$, so that $X \cong \bar{K}$ for some $\bar{K} \triangleright \mathcal{M}_\alpha$ and $\bar{K} \leq^* \mathcal{N}_{n-1,\alpha}$ by Claim 1. Pick a code $x \in {}^\omega\omega$ for the ordinal α, and let $\mathrm{dom}(q) = \alpha$, where $q \restriction \mathrm{dom}(p) = p$, $q(\max(c) + n) = x(n)$ for all $n < \omega$, and $q(\gamma) = 0$ for all $\gamma \geq \mathrm{dom}(c) + \omega$. There will be some $\mathcal{P}$ with $\bar{K}||\pi^{-1}(\kappa) \triangleleft \mathcal{P} \triangleleft \bar{K}$ such that $\rho_\omega(\mathcal{P}) = \pi^{-1}(\kappa)$, $\pi^{-1}(\kappa)$ is the largest cardinal of $\mathcal{P}$, and $\mathcal{P}[A^+ \cap \alpha, q] \models$ "α is countable." Therefore, $(c \cup \{\alpha\}, q)$ is as desired. □ (Claim 2)

In order to finish the proof of Lemma 2.1 (and hence of Theorem 1.1) it now obviously suffices to verify the following.

Claim 3. $\mathbb{Q}$ is stationary set preserving.

Proof of Claim 3. Let $S \subset \omega_1$ be stationary, and let $(c, p) \Vdash$ "$\dot{C}$ is a club subset of $\check{\omega}_1$." We aim to construct some $(c', q) \leq (c, p)$ such that $(c', q) \Vdash$ "$\dot{C} \cap \check{S} \neq \emptyset$."

Let $X \prec (H_{\omega_2}; \in, A^+, \mathbb{Q}, (c,p))$ be transitive and of size $\aleph_1$, and let $(X_i: i \leq \omega_1)$ be a continuous chain of countable elementary submodels of X approaching X (i.e., $X = X_{\omega_1}$). Recall that $H_{\omega_2} = K||\omega_2[A^+]$ (in $V^{\mathbb{P}}$). Let

$$\pi: \bar{K}[A^+ \cap \alpha] \cong Y \prec (H_{\omega_2}; \in, A^+, \mathbb{Q}, (c,p), (X_i: i \leq \omega_1)),$$

where Y is countable and $\alpha = Y \cap \omega_1 = \omega_1^{\bar{H}[A^+\cap\alpha]} < \omega_1$. We also may and shall assume that $\alpha \in S$. Let us write $\alpha_i = X_i \cap \omega_1 < \omega_1$ for $i < \omega_1$. Of course $(\alpha_i: i < \alpha) \in \bar{K}[A^+ \cap \alpha]$, where $(\alpha_i: i < \alpha)$ is cofinal in α. Let us pick (externally, i.e., in $V^{\mathbb{P}}$) a sequence $(i_n: n < \omega)$ which is cofinal in α; hence $(\alpha_{i_n}: n < \omega)$ is cofinal in α as well. As $H_{\omega_1} = K||\omega_1[A^+]$, $(H_{\omega_1})^{\bar{K}[A^+\cap\alpha]} \subset \bigcup_{i<\alpha} X_i$, so that we may assume that in fact $(c,p) \in X_{i_0}$.

We shall now recursively construct sequences $((c_n,p_n): n < \omega)$ and $((c'_n,p'_n): n < \omega)$ of conditions in $\mathbb{Q}$ with the following properties.

(1) $(c_0,p_0) = (c,p)$,

(2) $(c'_n,p'_n) \leq (c_n,p_n)$, and $(c'_n,p'_n) \in X_{i_{n+1}}$,

(3) for all $\xi \in (\mathrm{dom}(p'_n) \setminus \mathrm{dom}(p_n)) \cap \{\alpha_i: i < \alpha\}$, $p'_n(\xi) = 1$ if and only if $\xi = \alpha_{i_n}$,

(4) $(c_{n+1},p_{n+1}) \leq (c'_n,p'_n)$, and $(c_{n+1},p_{n+1}) \in X_{i_{n+1}}$,

(5) there is some $\beta > \alpha_{i_n}$ such that $(c_{n+1},p_{n+1}) \Vdash$ "$\check{\beta} \in \dot{C}$, and

(6) $\max(c_{n+1}) > \alpha_{i_n}$.

In the light of (the proof of) Claim 2, there is no problem with this recursion.

Let us now set $c^* = \bigcup_{n<\omega} c_n \cup \{\alpha\}$ and $p^* = \bigcup_{n<\omega} p_n$. We're done if we can show that (c^*,p^*) is a condition, because then $(c^*,p^*) \Vdash$ "$\check{\alpha} \in \check{S} \cap \dot{C}$."

Well, we have that $\bar{K}[A^+ \cap \alpha, p^*] \models$ "α is countable," because for $\xi \in \{\alpha_i: i_0 \leq i < \alpha\}$, $p^*(\xi) = 1$ if and only if $\xi = \alpha_{i_n}$ for some $n < \omega$, so that $(\alpha_{i_n}: n < \omega) \in \bar{K}[A^+ \cap \alpha, p^*]$.

By Claim 1, $\bar{K} \triangleright \mathcal{M}_\alpha = \bar{K}||\pi^{-1}(\kappa)$. Moreover, there will again certainly be some $\mathcal{P}$ with $\bar{K}||\pi^{-1}(\kappa) \triangleleft \mathcal{P} \triangleleft \bar{K}$ such that $\rho_\omega(\mathcal{P}) = \pi^{-1}(\kappa)$, $\pi^{-1}(\kappa)$ is the largest cardinal of $\mathcal{P}$, and $\mathcal{P}[A^+ \cap \alpha, p^*] \models$ "α is countable." By Claim 1, $\mathcal{P} <^* \mathcal{N}_{n-1,\alpha}$, so that (c^*,p^*) is really a condition. $\square$ (Claim 3)

$\square$ (Lemma 2.1, Theorem 1.1)

3 Some problems.

Let $(f_\alpha: \alpha < \omega_2)$ denote "the" sequence of canonical functions from ω_1 to ω_1. I.e., $f_\alpha(\nu) = \mathrm{otp}\ g_\alpha{}''\nu$, where $g_\alpha: \omega_1 \to \alpha$ is bijective. The Club Bounding Principle, CBP for short, says that for every $f: \omega_1 \to \omega_1$ there is some $\alpha < \omega_2$ such that $f < f_\alpha$ on a club, i.e., such that $\{\nu: f(\nu) < f_\alpha(\nu)\}$ contains a club subset of ω_1.

P. Larson has shown that CBP implies $P_2(\omega_1)^+$ (cf. [1, Fact 3.4]; cf. [1, Definition 3.1 (b)] on the definition of $P_2(\omega_1)^+$). On the other hand one has:

Lemma 3.1 *If* BMM *and* $P_2(\omega_1)^+$ *hold then* CBP *holds.*

PROOF SKETCH. Let $f: \omega_1 \to \omega_1$. If $P_2(\omega_1)^+$ holds then the natural forcing $\mathbb{P}_f$ for adding a canonical function above f is easily seen to be stationary set preserving. $\mathbb{P}_f$ is just the collection of all (c, p) such that c is a closed countable subset of ω_1 and $p: \max(c) \to \omega_2$ is such that for all $\nu \in c$, $f(\nu) < \text{otp } p"\nu$. $\square$ (Lemma 3.1)

We do not know if BMM alone implies the Club Bounding Principle. In order to answer this question we need a better understanding of how to construct models of BMM. The same remark applies to the following questions.

Woodin introduced the principle ψ_{AC} and showed that ψ_{AC} implies that $2^{\aleph_0} = \aleph_2$. (Cf. [9, Definition 5.12 and Lemma 5.15].) We do not know if BMM implies that ψ_{AC} holds (we know that BMM implies that $2^{\aleph_0} = \aleph_2$, though, cf. [8]). But we have:

Lemma 3.2 *If* BMM *and* $P_1(\omega_1)^+$ *hold then* ψ_{AC} *holds.*

PROOF SKETCH. Cf. [1, Definition 3.1 (d)] on the definition of $P_1(\omega_1)^+$. Let S, $T \subset \omega_1$ both be stationary and costationary. By $P_1(\omega_1)^+$, the canonical forcing $\mathbb{P}_{S,T}$ for adding a witness to ψ_{AC} is easily seen to be stationary set preserving. $\mathbb{P}_{S,T}$ consists of all (c, p) such that c is a closed countable subset of ω_1, $p: \max(c) \to \omega_2$, and for all $\nu \in c$, $\text{otp } p"\nu \in T \longleftrightarrow \nu \in S$. $\square$ (Lemma 3.2)

It is not known if ψ_{AC} implies $P_1(\omega_1)^+$. By [1, Fact 3.1], ψ_{AC} implies CBP.

Another interesting question is whether BMM implies that $u_2 = \omega_2$. This question makes sense by [6]. If BMM holds then every real has a #, and hence we may ask if u_2, the second uniform indiscernible, is equal to ω_2.

References

[1] Aspero, D., and Welch, Ph., *Bounded Martin's Maximum, weak Erdös cardinals, and* ψ_{AC}, Journal Symb. Logic **67** (2002), pp. 1141-1152.

[2] Goldstern, M., and Shelah, S., *The bounded proper forcing axiom*, Journal Symb. Logic **60** (1995), pp. 58-73.

[3] Jensen, R., *The core model for non-overlapping extender sequences*, handwritten notes, Oxford.

[4] Moore, J., *Set mapping reflection*, preprint.

[5] Schindler, R., *Coding into K by reasonable forcing*, Trans. Amer. Math. Soc. **353** (2000), pp. 479-489.

[6] Schindler, R., *Semi-proper forcing, remarkable cardinals, and Bounded Martin's Maximum*, Mathematical Logic Quarterly, submitted.

[7] Steel, J., and Welch, Ph., Σ^1_3 *absoluteness and the second uniform indiscernible*, Israel Journal of Mathematics **104** (1998), pp. 157-190.

[8] Todorcevic, S., *Generic absoluteness and the continuum*, Mathematical Research Letters **9** (2002), pp. 1-7.

[9] Woodin, H., *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, de Gruyter 1999.

[10] Woodin, H., private communication, Aug. 03.