Projective Uniformization Revisited

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Abstract

We give an optimal lower bound in terms of large cardinal axioms for the logical strength of projective uniformization (i.e., the assumption that for any projective set in the real plane there exists a projectively definable function selecting an element from each section of the given set) in conjuction with other regularity properties of projective sets of real numbers, namely Lebesgue measurability and its dual in the sense of category (the property of Baire). Our proof uses a projective computation of the real numbers which code inital segments of a core model and answers a question in Hauser (1995).

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0 Introduction.

A fundamental question was posed by Lusin [11] in 1930: Given a set A in the real plane, find a subset A^* of A which *uniformizes* A, i.e., A^* is the graph of a function with domain the projection of A onto the real line picking for each x in the projection of A a witness y such that $(x, y) \in A$. This selection problem is solved trivially with the axiom of choice: Fix a well-ordering of the real numbers and always choose the least witness in that well-ordering. It becomes highly non-trivial, however, when one seeks definable solutions. A natural framework for such investigations is provided by the *projective hierarchy*, the smallest class of sets containing all Borel sets and closed under the operations of forming complements and continuous images. Lusin (cf. [11]) verified that *analytic sets*, i. e., continuos images of Borel sets,

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admit projective uniformizations. Eight years later Kondo [10] building on work of Novikov established the *uniformization property for co-analytic sets*, i.e., every complememt of an analytic set in the real plane admits a uniformization which is the complement of an analytic set. An immediate consequence is that continuous images of co-analytic sets have the uniformization property, but that is all that can be proved within the standard axiom system of set theory ZFC. It took another thirty years before Moschovakis (cf. [14] for a comprehensive historical account) completed this program by isolating the concept of a *scale*, a sequence of pre-wellorderings satisfying a lower semicontinuity property. He derived the uniformization property for projective sets (*projective uniformization* PU) from projective scales (PS) [14] whose existence he obtained under the hypothesis that certain infinite games with projective pay-off sets are determined.

Identify the real numbers with infinite sequences of integers and endow them with the product topology coming from the discrete topology on the integers. This yields a topological space which is homeomorphic to the irrational numbers as a subspace of the Eucledian space by a well-known theorem of Hardy (cf. [14]). The advantage of this identification is that it accomodates in a natural way the notion of a *game* G_A associated with each set A of *reals* - as the elements of the above space are customarily called in descriptive set theory. In this game two players alternatively play integers until an infinite sequence, i.e., a real x, has been produced. Player I wins G_A if x belongs to A, otherwise II wins. A *strategy* for one of the players is an algorithm telling him which number to play at each round, and it is *winning* for that player if following it results in a win for him regardless of the other player's moves. The game G_A, or simply the set A, is *determined* if one of the players posesses a winning strategy. *Projective determinacy* (PD) states that all games with projective pay-off set are determined; it is a strong set theoretic assumption (cf. [20]) giving a complete structure theory for the projective sets generalizing what is provable in ZFC about the initial levels of the projective hierarchy (cf. [14]).

Other consequences of PD besides PS include the Lebesgue measurability of projective sets (LM) and its dual in category: Every projective set has the property of Baire (PB), i.e., it is "almost" equal to some open set in the sense that their symmetric difference is *meager*. A natural question arises: How much of PD is needed to guarantee the above three regularity properties for the projective sets? In 1982 Woodin [19] made a conjecture to the effect that *all* of PD is necessary (i.e., in ZFC, PM + PC + PU together imply PD) which marked the start of the so-called Delfino Project (cf. [9]). In this crude form, Woodin's conjecture is refuted by a recent theorem of Steel.

Theorem (Steel) *Suppose that V is the minimal, iterable inner model with infinitely many cardinals that are each strong up to the cardinal successor of their supremum, λ. Then in $V^{Coll(\omega,\lambda)}$ PM + PB + PU holds.*

(It was already known that all projective sets in such a generic extension must be Lebesgue measurable and have the property of Baire as a consequence of a theorem of Woodin (see below).) *Strong cardinals* generalize the concept of measurability: a cardinal κ is α *strong* if there is an embedding of V into some transitive inner model M with *critical point* (the least ordinal being moved) equal κ such that the power set of α is contained in M. κ is $< \alpha$ *strong* if it is β strong for each $\beta < \alpha$. Under the assumption that strong cardinals exist one can define a canonical model for them which is minimal among all such models that are *iterable.* (cf. [2]) Iterability is a technical concept saying roughly that one can repeatedly use the embeddings associated to strong cardinals to form images of the model, iterate this process into the transfinite and end up with transitive (standard) models each time. In particular Steel's theorem establishes an upper bound for the logical strength of the assumptions PM + PB + PU far below PD since strong cardinals are much weaker than projective determinacy (cf. [20]). On the other hand, it provides no information about *effective* ("lightface") versions of the Delfino problem. (For more on the current status of this question the reader is refered to [3].) Steel does, however, obtain a little more than PU in his model, namely PS. Here we establish a converse to Steel's theorem:

Theorem 4.2 *(ZFC +$\mathbb{R}^\sharp$ exists.) If PM + PB + PU hold there is a transitive model M of ZFC containing all ordinals with infinitely many cardinals that are each strong up to the cardinal successor in M of their supremum.*

The existence of $\mathbb{R}^\sharp$ means that there is a truth definition for the smallest model of set theory containing all reals, $L(\mathbb{R})$ together with a class of indiscernibles. Instead of the full axiom of choice weaker choice principles like DC (dependent choices) suffice for the argument. There is also a choice-less companion of 4.2 corresponding to the variant of Steel's theorem saying that in the *symmetric collapse* of the supremum of the strong cardinals over the minimal iterable model with infinitely many cardinals strong up to their supremum, PM + PB + PU (and in fact PS) is true.

Theorem 4.6 *(ZF +$\mathbb{R}^\sharp$ exists.) If PM + PB + PU hold then there is a transitive model of ZFC containing all ordinals with infinitely many cardinals that are each strong up to their supremum.*

It is somewhat annoying that we have to include the existence of $\mathbb{R}^\sharp$ as an extra assumption in the statements of these theorems. This is due to a defect of the construction of the models in which we realize the large cardinal configuration from the conclusion of the theorems. These models are so-called *core models* - generalizations of Gödel's constructible universe, L of a less restrictive kind which come equipped with a *fine structure* and approximate various large cardinal axioms. According to the informed opinion of the cognoscenti, the extra assumption that $\mathbb{R}^\sharp$ exists can almost certainly be discarded from the construction of the core models needed for the above theorems. Modulo this technicality, theorems 4.2 and 4.6 together with the above result of Steel determine the exact position of PM + PB + PU in the large cardinal hierarchy. Moreover, they show that PU and PS are of the same logical strength in the presence of PM and PB.

One of the main ingredients in the proofs of 4.2. and 4.6 is a projective computation of the real numbers coding initial segments of core models. It turns out that the complexity of this computation is closely related to the number of strong cardinals in the core model, K up to the first uncountable cardinal ω_1^V of the outside universe. In the special case that this number equals 0 (and ω_1^V is inaccessible in K), [4] gave a definition of the countable initial segments of K at a low level of the "lightface" (i.e., effectivized) projective hierarchy, namely Δ^1_5. This strengthened an earlier result in [1] where it was also conjectured that with each additional strong cardinal up to ω_1 in K the complexity of the definition would increase by a finite amount. In other words, with at most finitely many strong cardinals below ω_1^V in K, the definition should remain projective. This is confirmed by

Theorem 3.4. *If ω_1^V is inaccessible in K and $\mathcal{J}^K_{\omega_1} \models$ "There are at most n strong cardinals," then $\mathcal{J}^K_{\omega_1^V}$ is (lightface) Δ^1_{n+5} in the codes.*

If ω_1^V fails to be inaccessible in K one obtains a "boldface" version of this:

Theorem 3.6. *If ω_1^V is a successor cardinal in K and $\mathcal{J}^K_{\omega_1^V} \models$ "There are at most n strong cardinals", then $\mathcal{J}^K_{\omega_1^V}$ is $\Delta^1_{n+4}(x)$ in the codes where x is a real coding an initial segment of $\mathcal{J}^K_{\omega_1^V}$.*

Theorems 3.4 and 3.6 are due to the second author. In analogy with [4], a uniform computation of K up to its nth strong cardinal can be given. As in that paper additional background assumptions are needed: The universe is a generic extension turning an inaccessible cardinal of the ground model into the first uncountable cardinal. (Which is in fact less than what was required in the corresponding place in [4].)

Theorem 3.11 *For each integer $n \geq 1$ there is a Δ^1_{n+5} formula $\Phi^n(v_0)$ such that whenever the universe is of the form $V^{Coll(\omega,<\kappa)}$ with κ an inaccessible cardinal of the ground model and K exists and has at least $n+1$ cardinals strong up to κ, for all reals x,*

$$\Phi^n(x) \text{ iff } x \text{ codes a strict initial segment of } \mathcal{J}^K_{\kappa_{n+1}}$$

where κ_{n+1} denotes the $n+1$st ordinal strong in K up to κ.

Moreover, in the situation of the previous theorem, the question whether there are at least n countable ordinals which are strong up to true ω_1 in K is projective.

Theorem 3.12 *For each integer $n \geq 0$ there is a Σ^1_{n+6} sentence ψ^{n+1} such that in any universe of the form $V^{Coll(\omega,<\kappa)}$ with κ an inaccessible cardinal of the ground model where K exists,*

$$\psi^{n+1} \text{ iff there are at least } n+1 \text{ countable ordinals strong up to } \kappa \text{ in } K.$$

Theorems 3.11 and 3.12 are due to the first author. At the cost of a slight increase in complexity and with a little more work, the extra assumption about the shape of the universe can be eliminated from 3.12 and one obtains.

Theorem 3.13 *For each integer $n \geq 0$ there is a Π^1_{n+7} sentence Ψ^n such that (provably) in the theory ZFC + K exists + ω_1^V is inaccessible in K,*

$$\Psi^n \leftrightarrow \text{There are at most } n \text{ countable ordinals strong in } K \text{ up to true } \omega_1.$$

We also want to point out that all of the above computations transfer routinely to higher order analogs of Steel's one Woodin K. Moreover, these results are optimal by a theorem of Woodin about the invariance of projective truth under further forcing in the generic collapse of infinitely strong cardinals (cf. [20]) and weak covering for K [12].

The proofs in section 3 parallel and generalize the arguments from [4]. The authors of that paper were unable to implement this generalization because they could not get over the first hurdle. This consisted, among other things, of expressing in a projective way for a given countable premouse $\mathcal{M}$ with a pre-specified ordinal κ, that $\mathcal{M}$ can be extended to a universal weasel in which κ ends up as a strong cardinal up to true ω_1. The second author of the present paper noticed that this problem can be solved by employing a generalization of the *beaver* machinery of [6] and switching to the Friedman-Jensen indexing of extender sequences [7]. Equipped with these tools, he devised a method for constructing Steel's preliminary core model $K^c(\mathcal{M})$ allowing extenders with critical point in $\mathcal{M}$ on the sequence provided they have "beaver-witnesses". In previous situations requiring relativized versions of K^c (including [4]) this was forbidden, but it becomes crucial for the present paper. The details of this modified K^c construction are presented in section 2. It is curious to note that so far we have been unable to reproduce the arguments in sections 2 and 3 using the more traditional mice in [13]. In section 1 we review - primarily as a service to the reader who does not have a copy of the privately circulated notes [7] at hand - some basic facts about the Jensen-Friedman organization of K. We sketch a proof that the final output of our projective computations of initial segments of K can be translated into the set-up of [13] at the end of section 3.

The other main ingredient in the proofs of the equiconsistence results 4.2 and 4.6 is a technology developed by Woodin in [19] for building models of set theory whose projective theories agree with the one in V (*correctness*) and are stable under forcing (*absoluteness*) up to a certain degree. By packing enough absoluteness into such a model one obtains arbitrarily many strong cardinals in its K below ω_1 via [1, theorem 3.10]. Correctness together with 3.13 (or the methods of 3.11) allows to deduce the existence of infinitely many strong cardinals in true K below ω_1^V which proves 4.2.

In the absence of the axiom of choice we consider a pieced-together version K^* of the core model coming from local computations inside sufficiently absolute and correct models. An adaption of an argument in [3] which relies on the representability of the projective theory of V as the asymptotic limit of the projective theory of sufficiently absolute and correct models, together with 3.11 yields that K^* is a rank initial segment of true K. From this 4.6 follows.

In the meantime, some of the results in this paper have been generalized by the second author beyond the projective hierarchy (cf. [17]).

1 Preliminaries.

Thruout sections 1, 2, and 3 we shall assume that the following two assumptions are met.

A 1. ZFC + there is no inner model with a Woodin cardinal, and

A 2. Ω is a fixed measurable cardinal.

These assumptions allow one to build Steel's core model K of height Ω, as in Steel's book [18]. In fact, neither **A 1** nor **A 2** is really needed in what follows. We could replace their conjunction by the assumption that there is a transitive model $\mathcal{M}$ of ZFC having enough of the properties of the K of [18] so that the arguments to follow were to go thru with K replaced by $\mathcal{M}$. It is conjectured that **A 1** alone suffices for the existence of such an $\mathcal{M}$. It is known that the existence of such $\mathcal{M}$ is provable under more general assumptions than **A 1** and **A 2**.

For example, if **A 1** holds and instead of **A 2** we know that the universe is closed under sharps then we can form such an $\mathcal{M}$, the global version of the core model, by knitting together various locally computed core models. In this case as far as $K \cap HC$ is concerned, it suffices to consider the core model built locally inside an inner model of the form $(L(V_\kappa)[U])^{Coll(\omega,\kappa)}$ (where V_κ denotes the rank initial segment of the ground model up to $\kappa > \omega$ and U is a normal measure above κ.) It is actually this latter scenario which will play a role in the equiconsistency proofs of the last section.

Instead of [18] (and its fine structure theory [13]) however we here build upon Jensen's handwritten notes [7]. Jensen in [7] presents a core model theory for Woodin cardinals (as has previously been done in [13] and [18]), but using his Σ^*-fines tructure (which already had appeared in his [5] and [6], say; cf. also [8]) and a new organization of premice dating back to an idea of Sy Friedman, according to which a "classical" extender E is replaced by a fragment of the map itself which E codes.

It will be the case that if $\mathcal{M}$ is a premouse and $F = E_\nu^{\mathcal{M}} \neq \emptyset$, then letting $\mathcal{N} = J_\nu^{\mathcal{M}}$ there are $\kappa < \nu$ and $\tilde{F} \supset F$ such that

$$\tilde{F} \colon J^{\mathcal{M}}_{\kappa^{+\mathcal{N}}} \to \mathcal{N}$$

is a Σ_0-cofinal elementary embedding with critical point κ, and $F = \tilde{F} \cap \mathcal{P}(\kappa)$. Such an F is now also called an extender (at κ, ν, with length $F(\kappa)$). Note that $F(\kappa)$ is determined as the largest cardinal in $J_\nu^{\mathcal{M}}$.

We shall not give the official definition of a "new" extender here (cf. [7] §1). However, we note the following fact. As already indicated in the last paragraph, any new extender F is a map $F : U \to \mathcal{P}(\lambda)$ where $U \subset \mathcal{P}(\kappa)$ and $\kappa < \lambda$. If we let

$$X \in E_{\{\alpha\}} \text{ iff } \alpha \in F(X)$$

for $X \in U$ and $\alpha < \lambda$, then $E = (E_{\{\alpha\}} : \alpha < \lambda)$ is the hypermeasure associated to F. Moreover, as κ and λ are always supposed to be p.r. closed, we may derive from F a partial map, also called F, from $\mathcal{P}([\kappa]^{<\omega})$ to $\mathcal{P}([\lambda]^{<\omega})$ by first coding by a subset of κ, applying F, and then decoding. On the other hand, any classical extender E gives a new one by just restricting the induced map to the subsets of the critical point being measured.

Our motivation for working with the premice of Jensen's [7] here is not religious. Instead, at the time of writing we do not see how to run the main arguments of sections 1, 2, and 3 using the Mitchell-Steel organization of premice of [13]. What makes the Friedman-Jensen organization of premice amenable to our purposes is provided by a simple lemma, 1.1.

The general policy here is that if a theorem of [7] is cited then we also give in brackets a reference to the analoguous theorem in [13] or [18], so that a reader not having a copy of [7] available (but familiar to a certain extent with [13] and [18]) should nevertheless have no problem following our string of argumentation using the information provided below. Moreover, we'll stick to the terminology of [18] whenever possible. From now on, "extender" always means "new extender."

A pre-premouse is an acceptable J-model of the form $\mathcal{M} = (J_\alpha[E]; \in, E, E_\alpha)$, also written $J_\alpha[E]$, where

$$E = \{(\nu, \xi, X) : \xi \leq \nu < \alpha \text{ and } \xi \in E_\nu(X)\}$$

codes a sequence of extenders, with the following two properties:

(a) If $E_\nu \neq \emptyset$ for $\nu \leq \alpha$, then E_ν is an extender at κ, ν, some $\kappa < \nu$, whose domain is just $\mathcal{P}(\kappa) \cap J_\nu[E]$, and if

$$i_{E_\nu} : J_\nu[E] \rightarrow_{E_\nu} \mathcal{P} = Ult(J_\nu[E], E_\nu)$$

is the (Σ_0-) ultrapower map then $J_\nu[E]$ (without top extender) is an initial segment of $\mathcal{P}$ and $E_\nu^{\mathcal{P}} = \emptyset$, and

(b) Proper initial segments of $\mathcal{M}$ are sound.

Condition (a) is often referred to as "coherency." The ultrapower formed in (a) is according to the "upward extension of embeddings" technique using E_ν as a fragment of the i_{E_ν} to be formed. We always suppose the well-founded part of a model to be transitive. The concept of "soundness" in (b) is according to Jensen's fine structure (cf. [8]).

Let $\mathcal{M}$ be a pre-premouse as above. Let $F = E_\nu \neq \emptyset$ be an extender at κ, ν for some $\kappa < \nu$ and $\nu \leq \alpha$, and let $\bar{\nu} \leq \nu$ be such that $\bar{\nu} \geq \kappa^{+J_\nu[E]}$. Set $\beta = \bar{\nu}$ if $\bar{\nu} = \kappa^{+J_\nu[E]}$ or $\bar{\nu}$ is a limit cardinal in $J_\nu[E]$, and let otherwise be $\beta =$ the cardinal predecessor of $\bar{\nu}$ in the sense of $J_\nu[E]$. We may then define an extender $F \upharpoonright\upharpoonright \bar{\nu}$ at κ, β with the same domain as F by setting

$$(F \upharpoonright\upharpoonright \bar{\nu})(X) = F(X) \cap \beta \quad \text{for} \quad X \in dom(F).$$

A pre-premouse $\mathcal{M}$ as above is now called a premouse if whenever F and $\bar{\nu}$ are as in the previous paragraph such that $(J_{\bar{\nu}}[E], F \upharpoonright\upharpoonright \bar{\nu})$ is a pre-premouse then actually $E_{\bar{\nu}} \neq \emptyset$.

This condition is called the "initial segment condition" and plays a similar role as the condition under the same heading in [13] (cf. Definition 1.0.4). In particular, it is needed in the proof of the comparability for sufficiently iterable premice (cf. [7] §4 Lemma 2). We shall also use it in some of our proofs (cf. 1.4, 1.6).

For premice $\mathcal{M} = (J_\alpha[E]; \in, E, E_\alpha)$ and $\beta \leq \alpha$ we use $\mathcal{J}_\beta^{\mathcal{M}}$ to ambiguously denote either the premouse $(J_\beta[E]; \in, E \restriction \beta, E_\beta)$, or the premouse $(J_\beta[E]; \in, E \restriction \beta)$, and we use $J_\beta^{\mathcal{M}}$ to denote the universe of $\mathcal{J}_\beta^{\mathcal{M}}$.

Now with this concept of a premouse replacing whatever the reader might have had in her/his brain under the label of "premouse" so far, she/he is now able to verify the following simple but crucial lemma. (It is false for Mitchell-Steel premice.)

Lemma 1.1 *Let $\mathcal{M}$ be a premouse, and let F be an extender at κ, ν such that $\mathcal{J}_\kappa^{\mathcal{M}} \models$ "there are only boundedly many strong cardinals." Suppose that F can be applied to $\mathcal{M}$ and that $Ult(\mathcal{M}, F)$ is transitive.*

Then there is no extender $G = E_{\tilde{\nu}}^{Ult(\mathcal{M},F)} \neq \emptyset$ with critical point μ such that $\kappa \leq \mu < F(\kappa)$ and $\tilde{\nu} \geq \nu$.

Proof. Suppose otherwise, and fix a counterexample G. Let

$$\sigma\colon \mathcal{M} \rightarrow_F \mathcal{N} = Ult(\mathcal{M}, F)$$

be the ultrapower map. Then $F \subset \sigma$, and $\sigma(\kappa^{+\mathcal{M}}) = \nu = \sigma(\kappa)^{+\mathcal{N}}$. In particular, ν is a cardinal in $\mathcal{N}$. Hence, as $\sigma(\kappa) = F(\kappa)$ is an inaccessible cardinal in $J_\nu^{\mathcal{N}} = J_\nu^{\mathcal{M}}$, we also have that $\sigma(\kappa)$ is an inaccessible cardinal in $\mathcal{N}$. Set $\lambda = \sigma(\kappa)$.

Now notice that G cannot be the top extender of $\mathcal{N}$, because otherwise certainly $\mu = c.p.(G) \in ran(\sigma)$. But then $\mu \in [\kappa, \lambda) \cap ran(\sigma) = \emptyset$. Contradiction! This means that $G \in \mathcal{N}$. It is also easy to see that G is total on $\mathcal{N}$, as $\mu < \lambda < \tilde{\nu}$ and λ is a cardinal in $\mathcal{N}$.

Now let, for $\beta \leq G(\mu)$, $G \restriction \beta$ denote the "restriction" of G to generators $< \beta$. I.e., if τ is i_G followed by the transitive collapse of the Σ_0-hull of $\beta \cup ran(i_G)$ taken inside $Ult(\mathcal{N}, G)$, then

$$G \restriction \beta = \tau \restriction \mathcal{P}(\mu) \cap \mathcal{N}.$$

Using acceptability and the inaccessibility of λ in $\mathcal{N}$ we get that $(G \restriction \beta : \beta < \lambda) \subset J_\lambda^{\mathcal{N}}$ witnesses that

$$\mathcal{J}_\lambda^{\mathcal{N}} \models \text{"}\mu \text{ is a strong cardinal."}$$

So if $\alpha < \kappa$,

$$\mathcal{J}_\lambda^{\mathcal{N}} \models \text{"}\exists \xi > \alpha\ \xi \text{ is a strong cardinal."}$$

Hence, using σ,

$$\mathcal{J}_\kappa^{\mathcal{M}} \models \text{"}\forall \alpha\ \exists \xi > \alpha\ \xi \text{ is a strong cardinal,"}$$

contradicting our assumption that there are only boundedly many strong cardinals in $\mathcal{J}_\kappa^{\mathcal{M}}$.

$\square$ (1.1)

This means that if $\sigma: \mathcal{M} \to_F \mathcal{N}$ as in this proof were to appear in a comparison, then the rest of the comparison would not use critical points in the half-open interval $[\kappa, F(\kappa))$.

We now have to reflect on the hull- and definability property. We first need:

Lemma 1.2 *Let W, W' be weasels coiterating to a common weasel Q with iteration maps $i: W \to Q$ and $j: W' \to Q$ from the main branches. Let κ be a cardinal in both W and W' such that $\mathcal{J}^W_{\kappa^{+W}} = \mathcal{J}^{W'}_{\kappa^{+W'}}$. Suppose that $j \restriction \kappa = id$. Let F be an extender at κ, ν on W such that $Ult(W, F)$ is iterable.*

Then both $Ult(Q, F)$ and $Ult(W', F)$ are iterable, too, and in fact there is some $k: Ult(W', F) \to Ult(Q, F)$ with $k \restriction \nu = id$.

Proof. By the iterability of $Ult(W, F)$, we may copy the W-side iteration from the comparison of W, W' onto $Ult(W, F)$, with $i_F: W \to Ult(W, F)$ being the base step. We then get an iterate Q^* of $Ult(W, F)$ and a map $i^*: Q \to Q^*$. It is standard that i^* may be factorized as $i^* = k^* \circ j^*$ for some k^*, where $j^*: Q \to Ult(Q, F)$ is the ultrapower map. This already shows that $Ult(Q, F)$ is iterable.

But now there is an embedding $k: Ult(W', F) \to Ult(Q, F)$ defined by

$$k([a, f]_{Ult(W',F)}) = [a, j(f)]_{Ult(Q,F)} = j^*(j(f))(a)$$

for appropriate a, f. Note that for example

$$a \in F(\{u \in [\kappa]^{Card(a)} : W' \models \Phi(f(u))\}) \text{ iff}$$

$$a \in F(\{u \in [\kappa]^{Card(a)} : Q \models \Phi(j(f)(u))\})$$

by $j \restriction \kappa = id$. In particular, $Ult(W', F)$ is iterable and if $\xi < \nu$ then $k(\xi) = k([\{\xi\}, pr]_{Ult(W',F)}) = j^*(j(pr))(\{\xi\}) = \xi$. Hence $k \restriction \nu = id$.

$\square$ (1.2)

The following two Lemmata are now proved simultaneously by induction.

Lemma 1.3 *Let W be a universal weasel, and let κ be a cardinal in W such that W has the definability property at all $\bar\kappa$ with $\mathcal{J}_\kappa^W \models$ "$\bar\kappa$ is a strong cardinal." Then W has the hull property at κ, and in fact if*

$$\sigma: \bar W \to W$$

is the inverse of the transitive collapse of $H^W(\kappa \cup \Gamma)$ for some Γ being thick in W then $\mathcal{J}_\lambda^{\bar{W}} = \mathcal{J}_\lambda^W$ for $\lambda = \kappa^{+\bar{W}} = \kappa^{+W}$.

Lemma 1.4 *Let W, W' be universal weasels. Let κ be a cardinals in W as well as in W' with $\mathcal{J}_\kappa^W = \mathcal{J}_\kappa^{W'}$, and suppose that both W and W' have the definability property at all $\bar{\kappa}$ with $\mathcal{J}_\kappa^W \models$ "$\bar{\kappa}$ is a strong cardinal." Assume further that W' has the definability property at κ.*

Then the coiteration of W with W' is above $\kappa + 1$ along the main branch on the W-side.

Proof. The proof is by induction on κ. We first show:

Claim 1. W has the hull property at κ.

Proof. Let Γ be a class which is thick in W, and let

$$\sigma : \bar{W} \to W$$

be the uncollapse of $H^W(\kappa \cup \Gamma) \prec W$. W.l.o.g. we may assume that $\sigma^{-1}(\xi) = \xi$ for every $\xi \in \Gamma$, and that Γ is thick in $\bar{W}$, too. If $\bar{\Gamma} \subset \Gamma$ is thick in both $\bar{W}$ and W, and $\bar{\kappa}$ is a strong cardinal in $\mathcal{J}_\kappa^W$, then $\bar{\kappa} = \tau^W[\bar{\xi}]$ for some Skolem term τ and some $\bar{\xi} \in (\bar{\kappa} \cup \bar{\Gamma})^{<\omega}$, as W has the hull and definability property at $\bar{\kappa}$. (For the hull property we use the inductive hypothesis 1.3, applied to W and $\bar{\kappa}$.) Hence $\bar{\kappa} = \tau^{\bar{W}}[\bar{\xi}]$, so that $\bar{W}$ has the definability property at $\bar{\kappa}$, too.

But now the coiteration of $\bar{W}$, W is above κ along the main branches on both sides (by the inductive hypothesis 1.4, applied to $\bar{W}$, W, and all $\bar{\kappa} < \kappa$). So if Q' is the common coiterate, then

$$\mathcal{J}_{\kappa^{+\bar{W}}}^{\bar{W}} = \mathcal{J}_{\kappa^{+Q'}}^{Q'} = \mathcal{J}_{\kappa^{+W}}^{W},$$

showing that W has the hull property at κ.

$\square$ (Claim 1)

The very same argument also shows:

Claim 2. W' has the hull property at κ.

Now let $\mathcal{T} = (E_\alpha^{\mathcal{T}} \colon \alpha < \theta)$ and $\mathcal{U} = (E_\alpha^{\mathcal{U}} \colon \alpha < \theta)$ denote the iteration trees on W and W', respectively, arising from the comparison of W with W'. It is easy to see (by inductive hypothesis 1.4, applied to W, W' and all $\bar{\kappa} < \kappa$) that this coiteration is above κ along the main branches on both sides, i.e., if

$$\pi_{0\theta}^{\mathcal{T}}: W \to Q, \text{ and}$$

$$\pi_{0\theta}^{\mathcal{U}}: W' \to Q$$

denote the iteration maps from the main branches of the coiteration of W, W', where Q is the common coiterate, then $\pi_{0\theta}^{\mathcal{T}} \restriction \kappa = \pi_{0\theta}^{\mathcal{U}} \restriction \kappa = id$. We also have that

$$\mathcal{J}_{\kappa^{+W}}^{W} = \mathcal{J}_{\kappa^{+Q}}^{Q} = \mathcal{J}_{\kappa^{+W'}}^{W'}.$$

Let Γ be thick in W, W', and Q, such that $\pi_{0\theta}^{\mathcal{T}}(\xi) = \pi_{0\theta}^{\mathcal{U}}(\xi) = \xi$ for all $\xi \in \Gamma$.

Claim 3. $\pi_{0\theta}^{\mathcal{T}} \restriction \kappa + 1 = id$.

PROOF. Suppose not, i.e., $\kappa = c.p.(\pi_{0\theta}^{\mathcal{T}})$. Well, W' has the hull and definability property at κ by Claim 2 and by assumption. So if we let $x \in \mathcal{P}(\kappa) \cap W'$ then we may pick a term τ and $\bar{\xi} \in (\kappa \cup \Gamma)^{<\omega}$ such that

$$x = \tau^{W'}[\bar{\xi}].$$

Because $\bar{\eta} \in \tau^{W'}[\bar{\xi}]$ iff $\bar{\eta} \in \tau^{Q}[\bar{\xi}]$ iff $\bar{\eta} \in \tau^{W}[\bar{\xi}]$ for $\bar{\eta} < \kappa$, we have that

$$x = \tau^{W}[\bar{\xi}] \cap \kappa.$$

Now notice that $c.p.(\pi_{0\theta}^{\mathcal{U}}) = \kappa$, too: Otherwise $\pi_{0\theta}^{\mathcal{U}} \restriction \kappa + 1 = id$ and from the above reasoning by choosing $x = \kappa$ we get that

$$\kappa = \tau^{Q}[\bar{\xi}] \in ran(\pi_{0\theta}^{\mathcal{T}}),$$

contradicting $c.p.(\pi_{0\theta}^{\mathcal{T}}) = \kappa$.

We are now going to use the argument from the comparison lemma, cf. [7] §4 Lemma 2 (cf. [13] Lemma 7.2).

Let $\alpha + 1$ be least in $(0, \theta]_{\mathcal{T}}$, so that $F = E_{\alpha}^{\mathcal{T}}$ is the first extender used to form an ultrapower along the main branch on the W-side. Let $\beta + 1$ be least in $(0, \theta]_{\mathcal{U}}$, so that $F' = E_{\beta}^{\mathcal{U}}$ is the first extender used to form an ultrapower along the main branch on the W'-side. Suppose F to be at κ, ν, and F' to be at κ, ν'. Let $\lambda = min\{F(\kappa), F'(\kappa)\}$. We then have that for $\alpha < \lambda$ and $x \in \mathcal{P}(\kappa) \cap W = \mathcal{P}(\kappa) \cap W'$, if $x = \tau^{W'}[\bar{\xi}]$ for a term τ and $\bar{\xi} \in (\kappa \cup \Gamma)^{<\omega}$,

$$\alpha \in F'(x) = \tau^{Q}[\bar{\xi}] \text{ iff } \alpha \in \tau^{Q}[\bar{\xi}] \cap F(\kappa) = F(x).$$

So if $\nu = \nu'$ this means that $F = F'$, which is a plain contradiction.

Hence suppose w.l.o.g. that $\nu < \nu'$. Then $\mathcal{J}_\nu^{\mathcal{M}_\alpha^\mathcal{T}}$ with top extender F witnesses that

$$(\mathcal{J}_\nu^{\mathcal{M}_\beta^\mathcal{U}}, F' \upharpoonright\upharpoonright \nu)$$

is a pre-premouse (in fact a premouse, just because the two structures are equal). Hence $E_\nu^{\mathcal{M}_\beta^\mathcal{U}} \neq \emptyset$ by the initial segment condition, which implies that $E_\nu^Q \neq \emptyset$, as the tail of the comparison is beyond ν' on the W'-side.

But on the other hand $E_\nu^Q = \emptyset$, as F is applied on the W-side, and the tail of the comparison there is beyond ν. Contradiction!

$\Box$ (Claim 3)

But Claim 1 and Claim 3 taken together establish 1.3 and 1.4.

$\Box$ (1.3, 1.4)

The following is just the "symmetric version" of 1.4.

Corollary 1.5 *Let W, W' be universal weasels. Let κ be a cardinal in W as well as in W' with $\mathcal{J}_\kappa^W = \mathcal{J}_\kappa^{W'}$, and suppose that both W and W' have the definability property at all $\bar\kappa$ with $\mathcal{J}_\kappa^W \models$ "$\bar\kappa$ is a strong cardinal." Then the coiteration of W with W' is above κ along the main branches on both sides.*

The next lemma shows that iterates of weasels admitting the definability property at certain places have a "maximality property."

Lemma 1.6 *Let W and W' be universal weasels. Let κ be a cardinal in W and W' with $\mathcal{J}_\kappa^W = \mathcal{J}_\kappa^{W'}$, and suppose that both W and W' have the definability property at all $\bar\kappa$ with $\mathcal{J}_\kappa^W \models$ " $\bar\kappa$ is a strong cardinal." Assume further that there are only boundedly many strong cardinals in $\mathcal{J}_\kappa^W$, and that W has the definability property at κ.*

Let $\nu > \kappa^{+W'}$, and let F be an extender at κ, ν which can be applied to W'. Suppose that $Ult(W', F)$ is iterable, and set $\mathcal{N} = \mathcal{J}_\nu^{Ult(W',F)}$. Assume further that $(\mathcal{N}, F)$ is a premouse, and that $\mathcal{N}$ does not move in the comparison of $\mathcal{N}$ with W. Let $\bar{\mathcal{T}} = (E_\alpha^{\bar{\mathcal{T}}} \colon \alpha < \bar\theta)$ be the iteration tree arising on the W-side from the comparison of $\mathcal{N}$ with W. I.e., $\mathcal{N} \trianglelefteq \mathcal{M}_{\bar\theta}^{\bar{\mathcal{T}}}$.

In this situation, we have that in fact

$$F = E_\nu^{\mathcal{M}_{\bar\theta}^{\bar{\mathcal{T}}}}.$$

PROOF. Let $\mathcal{T} = (E_\alpha^\mathcal{T} : \alpha < \theta)$ and $\mathcal{U} = (E_\alpha^\mathcal{U} : \alpha < \theta)$ denote the iteration trees on W and $Ult(W', F)$, respectively, arising from the comparison of W with $Ult(W', F)$. Of course, $E_\alpha^\mathcal{T} = E_\alpha^{\bar{\mathcal{T}}}$ for all $\alpha < \bar{\theta}$. Set $Q = \mathcal{M}_\theta^\mathcal{T} = \mathcal{M}_\theta^\mathcal{U}$, i.e. the common coiterate of W, $Ult(W', F)$.

By 1.4, the coiteration is above κ along the main branches on both sides, and it is above $\kappa + 1$ along the main branch on the $Ult(W', F)$-side. Set $\lambda = F(\kappa)$. The coiteration is certainly beyond ν on the $Ult(W', F)$-side, as $\mathcal{N}$ doesn't move in the comparison with W, and hence 1.1 straightforwardly yields:

Claim 1. $\pi_{0\theta}^\mathcal{U} \restriction \lambda = id$, i.e., the iteration is in fact above λ along the main branch on the $Ult(W', F)$-side.

Now let $\sigma : W' \to Ult(W', F)$ be the ultrapower map, and pick a class Γ of fixed points under σ, $\pi_{0\theta}^\mathcal{T}$, and $\pi_{0\theta}^\mathcal{U}$, which is thick in W', $Ult(W', F)$, and W. Exactly as in the proof of 1.3 and 1.4 one can argue that for all $x \in \mathcal{P}(\kappa) \cap W = \mathcal{P}(\kappa) \cap W' = \mathcal{P}(\kappa) \cap Ult(W', F)$ there are a term τ and $\bar{\xi} \in (\kappa \cup \Gamma)^{<\omega}$ with

$$(*) \qquad x = \tau^W[\bar{\xi}] = \tau^{W'}[\bar{\xi}] \cap [\kappa]^{<\omega},$$

and that $c.p.(\pi_{0\theta}^\mathcal{T}) = \kappa$.

Let $\alpha + 1$ be minimal in $(0, \theta]_\mathcal{T}$, i.e., $G = E_\alpha^\mathcal{T}$ is the first extender used along the main branch on the W-side. Then G has critical point κ. Set $\tilde{\lambda} = G(\kappa)$, and let $\tilde{\nu}$ be such that $G = E_{\tilde{\nu}}^{\mathcal{M}_\alpha^\mathcal{T}}$. Again 1.1 gives:

Claim 2. $\pi_{\alpha+1\theta}^\mathcal{T} \restriction \tilde{\lambda} = id$.

Now using Claims 1 and 2 together with $(*)$, the argument from the comparison lemma as in the proof of 1.4 gives that $\lambda = \tilde{\lambda}$ and in fact:

Claim 3. $F = G$.

But with Claim 3 we now have that really $\alpha = \bar{\theta}$ and

$$F = G = E_{\bar{\theta}}^\mathcal{T} = E_\nu^{\mathcal{M}_{\bar{\theta}}^{\bar{\mathcal{T}}}},$$

as desired.

$\square$ (1.6)

So any reader of [18], say, will have noticed that 1.2 thru 1.6 are proved by fairly standard techniques, or even might be abstractly derivable from theorems in that book. They are nevertheless crucial for what is to come.

2 n-full mice, and $K^c_n(\mathcal{M})$.

In this section we shall isolate the necessary concepts that will enable us to projectively compute $K \cap HC$ later on. In turns out that what is required is a generalization of $K^c(\mathcal{M})$ (cf. [18] p.6 f. and p.59 f.). It will become clear that what makes $W = K^c(\mathcal{M})$ useless for our purposes is that $On \cap \mathcal{M}$ is required to be a cutpoint in W and thus in general there's no chance for W to have the definability property at certain $\kappa \in \mathcal{M}$ where we would like to have it.

We commence with a simple observation.

Lemma 2.1 *Let $\mathcal{M} = \mathcal{J}^K_\eta$ where η is a successor cardinal in K, and let W be a very soundness witness for $\mathcal{J}^K_{\eta+1}$. Let $\mathcal{T} = (E_\alpha : \alpha + 1 < \theta)$ be an iteration tree on W. Let ρ_α denote the (strict) sup of the generators of E_α for $\alpha + 1 < \theta$.*

Let $\alpha < \theta$ be such that $D^\mathcal{T} \cap (0, \alpha]_\mathcal{T} = \emptyset$, i.e., $\pi^\mathcal{T}_{0\alpha}: W = \mathcal{M}^\mathcal{T}_0 \to \mathcal{M}^\mathcal{T}_\alpha$ exists. Set $\rho^* = sup\{\rho_\beta : \beta + 1 \in (0, \alpha]_\mathcal{T}\}$. Let κ be such that $\rho^* \leq \kappa < On \cap \pi^\mathcal{T}_{0\alpha}(\mathcal{M})$. Then $\mathcal{M}^\mathcal{T}_\alpha$ has the definability property at κ.

Proof. Let Γ be a class of fixed points under $\pi^\mathcal{T}_{0\alpha}$ being thick in both W and $\mathcal{M}^\mathcal{T}_\alpha$. Set $\mathcal{N} = \pi^\mathcal{T}_{0\alpha}(\mathcal{M})$. We have that $\mathcal{N} = H^\mathcal{N}(\rho^* \cup \pi^\mathcal{T}_{0\alpha}{}''\mathcal{M})$. However, $\mathcal{M} \cup \{\mathcal{M}\} \subset H^W(\Gamma)$, and hence

$$\kappa \in H^\mathcal{N}(\rho^* \cup \pi^\mathcal{T}_{0\alpha}{}''\mathcal{M}) \subset H^{\mathcal{M}^\mathcal{T}_\alpha}(\kappa \cup \Gamma).$$

□ (2.1)

Definition 2.2 *Let $\mathcal{M}$ be a premouse. For $\alpha \in \mathcal{M}$, we let $d^\mathcal{M}(\alpha)$ denote the order type of the set of those $\beta < \alpha$ such that $J^\mathcal{M}_\alpha \models$ "β is a strong cardinal."*

Hence $d^\mathcal{M}(\alpha)$ counts the "degree of strength" of α in $\mathcal{M}$. We only have use for finite values of $d^\mathcal{M}(\alpha)$ in what follows.

Definition 2.3 *Let $\mathcal{M}$, $\mathcal{N}$ be premice with $\mathcal{M} \trianglelefteq \mathcal{N}$, and let $n < \omega$. Then $\mathcal{M}$ is called an n-cutpoint in $\mathcal{N}$ if the following hold.*

(a) $\mathcal{M}$ has a largest cardinal and is passive (i.e., has no top extender),

(b) every cardinal in $\mathcal{M}$ remains a cardinal in $\mathcal{N}$, and

(c) whenever $F = E^\mathcal{N}_\nu \neq \emptyset$ with $\kappa = c.p.(F) < On \cap \mathcal{M}$ and $\nu \geq On \cap \mathcal{M}$ then $d^\mathcal{M}(\kappa) < n$ or else F is partial, i.e. $\exists \xi \in [\nu, On \cap \mathcal{N}]$ $\rho_\omega(\mathcal{J}^\mathcal{N}_\xi) \leq \kappa$.

We can not forbid that there is some partial F as in (c) "overlapping" $On \cap \mathcal{M}$. However, if this happens to be the case then $d^\mathcal{M}(\kappa) = n$ (otherwise there would be some $\mu < \kappa$ with $d^{\mathcal{M}}(\mu) = n$ and some $E^{\mathcal{N}}_{\nu'}$ at μ, ν' with $\nu' \geq On \cap \mathcal{M}$), and the last clause of (c) explicitly requires F to be partialized.

It can now easily be checked that if $\mathcal{M}$ is an n-cutpoint in $\mathcal{N}$ and $\mathcal{N}^*$ is an iterate of $\mathcal{N}$ beyond $On \cap \mathcal{M}$ then $\mathcal{M}$ still is an n-cutpoint in $\mathcal{N}^*$. Also, we have that if λ is the largest cardinal of $\mathcal{M}$ then $\lambda < On \cap \mathcal{M} \leq \lambda^{+\mathcal{N}}$.

Definition 2.4 *Let $\mathcal{M}$ be a premouse with $\gamma = On \cap \mathcal{M}$ and largest cardinal λ, and let $n < \omega$. $\mathcal{M}$ is called n-full (with witness W) if W is a universal weasel with $\mathcal{M} = \mathcal{J}^W_\gamma$ and the following hold.*

(a) $\mathcal{M}$ is an n-cutpoint in W.

(b) Let $\mathcal{T} = (E^{\mathcal{T}}_\alpha : \alpha < \theta)$ be any iteration tree on W such that $D^{\mathcal{T}} \cap [0, \theta)_{\mathcal{T}} = \emptyset$ and

$$\alpha + 1 \in (0, \theta]_{\mathcal{T}} \Rightarrow \ d^{\pi^{\mathcal{T}}_{0\theta}(\mathcal{M})}(c.p.(E^{\mathcal{T}}_\alpha)) \geq n.$$

Set $\rho^* = sup\{\rho_\beta : \ \beta + 1 \in (0, \theta]_{\mathcal{T}}\}$ where ρ_β is the strict sup of the generators of E_β.

Then $\mathcal{M}^{\mathcal{T}}_\theta$ has the definability property at all $\mathcal{M}^{\mathcal{T}}_\theta$-cardinals κ such that $\rho^* \leq \kappa < \pi^{\mathcal{T}}_{0\theta}(\lambda)$ and $d^{\mathcal{M}^{\mathcal{T}}_\theta}(\kappa) < n$.

Notice that the fact that $\mathcal{M}$ is 0-full just "syntactically" strengthens the fact that there is a universal weasel extending $\mathcal{M}$. Such $\mathcal{M}$ with the latter property are called "strong" by Jensen in [6].

Definition 2.5 *Let $\mathcal{M}$ be a premouse with largest cardinal λ, and let $n < \omega$. $\mathcal{M}$ is called strongly n-full (with witness W) if $cf^{\mathcal{M}}(\lambda) = \omega$, W is a universal weasel witnessing that $\mathcal{M}$ is n-full, and $On \cap \mathcal{M} = \lambda^{+W}$.*

So if W witnesses that $\mathcal{M}$ is strongly n-full (with largest cardinal λ) and $\mathcal{T}$ is an iteration tree on W as in (b) of 2.4 then trivially $\mathcal{M}^{\mathcal{T}}_\theta$ also has the definability property at its largest cardinal, $\pi^{\mathcal{T}}_{0\theta}(\lambda)$.

If we had omitted (a) in 2.4 then every initial segment of K would have been n-full for every $n < \omega$ by 2.1. With (a), instead we have:

Lemma 2.6 *Let $n < \omega$, and let $\lambda = \kappa^{+K}$ for κ a cardinal in K such that $\mathcal{J}^K_\lambda \models$ "there are at most n strong cardinals." Then $\mathcal{J}^K_\eta$ is n-full for every $\eta \leq \lambda$ with $\mathcal{J}^K_\eta \prec \mathcal{J}^K_\lambda$. Moreover, if $cf^K(\kappa) = \omega$ then $\mathcal{J}^K_\lambda$ is strongly n-full.*

PROOF. Let W be a very soundness witness for $\mathcal{J}^K_{\lambda+1}$. If κ were measurable in K with $d^K(\kappa) > n$ then we would end up with more than n strong cardinals in $\mathcal{J}^K_\lambda$. Hence either κ is not measurable in K or else $d^K(\kappa) = n$.

Case 1. κ is not measurable in K.

Using 2.1, in this case it is easy to see that W also witnesses that $\mathcal{J}_\lambda^K$ is n-full, and even strongly n-full if $cf^K(\kappa) = \omega$.

Moreover, let $\eta < \lambda$ be such that $\mathcal{J}_\eta^K \prec \mathcal{J}_\lambda^K$. Suppose that there were some $F = E_\nu^W \neq \emptyset$ at $\bar{\kappa}$, ν with $\bar{\kappa} < \kappa$, $\nu \geq \eta$, and $d^{\mathcal{M}}(\bar{\kappa}) \geq n$. Then for any $\gamma < \eta$, $\mathcal{J}_\lambda^K$ knows that there is some such extender with $\nu > \gamma$. So, by $\mathcal{J}_\eta^K \prec \mathcal{J}_\lambda^K$, $\mathcal{J}_\eta^K$ knows that for any γ there is some such extender with $\nu > \gamma$. Applying $\mathcal{J}_\eta^K \prec \mathcal{J}_\eta^K$ once more we get that $\bar{\kappa}$ is one of the strong cardinals in $\mathcal{J}_\lambda^K$, and hence $d^K(\bar{\kappa}) < n$. Contradiction!

We have thus shown that $\mathcal{J}_\eta^K$ is an n-cutpoint in W, too, so that W also witnesses that $\mathcal{J}_\eta^K$ is n-full in this case.

Case 2. κ is measurable in K, and $d^K(\kappa) = n$.

Let $F = E_\nu^W$ where $\nu > \lambda$ is least such that E_ν^K has critical point κ. By 1.1 and repeating the arguments from Case 1 it is now easy to see that $\mathcal{J}_\eta^K$ is an n-cutpoint in $Ult(W, F)$ for all $\eta \leq \lambda$ with $\mathcal{J}_\eta^K \prec \mathcal{J}_\lambda^K$.

Moreover, $Ult(W, F)$ is still a very soundness witness for $\mathcal{J}_\kappa^K$. Let $\mathcal{T}$ be an iteration tree on $Ult(W, F)$ as in 2.4. We then have that for any $\mu < \pi_{0\theta}^{\mathcal{T}}(\kappa)$, $\mu = \pi_{0\theta}^{\mathcal{T}}(f)(a)$ for $f \in \mathcal{J}_\kappa^{Ult(W,F)}$ and $a \in [\rho^*]^{<\omega}$. (Notice that κ is not measurable in $Ult(W, F)$.) Hence if $\rho^* \leq \kappa$ then

$$f \in H^{Ult(W,F)}(\Gamma) \quad \Rightarrow \quad \mu = \pi_{0\theta}^{\mathcal{T}}(f)(a) \in H^{\mathcal{M}_\theta^{\mathcal{T}}}(\rho^* \cup \Gamma)$$

for any class Γ of fixed points under $\pi_{0\theta}^{\mathcal{T}}$ being thick in $Ult(W, F)$ as well as $\mathcal{M}_{0\theta}^{\mathcal{T}}$.

Thus $Ult(W, F)$ witnesses that $\mathcal{J}_\eta^K$ is n-full for all $\eta \leq \lambda$ with $\mathcal{J}_\eta^K \prec \mathcal{J}_\lambda^K$, and if $cf^K(\kappa) = \omega$ then $Ult(W, F)$ witnesses that $\mathcal{J}_\lambda^K$ is strongly n-full.

$\square$ (2.6)

3.7 below will prove a slight generalization of 2.6.

Definition 2.7 *Let $\mathcal{M} = (J_\alpha[E], F)$ be a premouse with top extender $F \neq \emptyset$, and let $n < \omega$. $\mathcal{M}$ is called an n-beaver provided $d^{\mathcal{M}}(\kappa) = n$ for $\kappa = c.p.(F)$, and there is a universal weasel W with the following properties.*

(i) $\mathcal{J}_\lambda^W = \mathcal{J}_\lambda^{\mathcal{M}}$ where $\lambda = \kappa^{+W} = \kappa^{+\mathcal{M}}$,

(ii) W has the definability property at all $\mu < \kappa$ such that $\mathcal{J}_\kappa^W \models$ "μ is strong," and

(iii) $Ult(W, F)$ is iterable.

We also say that a premouse $\mathcal{M}$ is a beaver if it is an n-beaver for some $n < \omega$.

Notice that we don't assume $J_\alpha[E] \lhd W$, but it clearly is the case that $J_\alpha[E] \lhd Ult(W,F)$.

0-beavers are what Jensen calls "beavers" in [6], his notes on the core model below $0^\P$. Of course there just are no n-beavers for $n > 0$ below $0^\P$.

The following is an immediate consequence of 1.4 and 1.2.

Lemma 2.8 *Let $\mathcal{M}$ be an n-beaver whose top extender has critical point κ. Suppose that W is a universal weasel with $J_\kappa^W = J_\kappa^{\mathcal{M}}$, κ is a cardinal in W, and W has the definability property at all $\bar\kappa$ which are strong cardinals in $\mathcal{J}_\kappa^{\mathcal{M}}$. Then $Ult(W,F)$ is iterable.*

Notice that 1.6 can now be thought of as providing information about when a given "beaver extender" is on the sequence of a certain weasel.

Following [5], we could call $(J_\alpha[E], F, G)$ a double beaver if both $(J_\alpha[E], F)$ and $(J_\alpha[E], G)$ are beavers. As in [5], double beavers resemble bicephali in that they always trivialize:

Lemma 2.9 *If $(J_\alpha[E], F)$ and $(J_\alpha[E], G)$ are beavers then $F = G$.*

Proof. Let W witness that $(J_\alpha[E], F)$ is a beaver. Let $\kappa = c.p.(F)$. By the argument for [18] Theorem 5.10 there is some $\xi \geq \kappa$ such that $\xi \in H^W(\kappa \cup \Gamma)$ for every Γ which is thick in W. Let $\tilde\kappa \geq \kappa$ be the minimal such ξ, and let Γ_0 be thick in W with $\tilde\kappa \cap H^W(\kappa \cup \Gamma_0) = \kappa$. Let $\sigma: \bar W \to W$ be the map obtained by collapsing $H^W(\kappa \cup \Gamma_0) \prec W$. Clearly, $\sigma(\kappa) = \tilde\kappa$.

Let $\Gamma \subset \Gamma_0$ be thick in $\bar W$, such that Γ consists of fixed points under σ. We may choose a term τ and $\bar\xi \in [\Gamma]^{<\omega}$ with $\tilde\kappa = \tau^W[\bar\xi]$. Hence using σ, $\kappa = \tau^{\bar W}[\bar\xi]$. We have shown that $\bar W$ has the definability property at κ. Moreover, it is easily seen that $\bar W$ still witnesses that $(J_\alpha[E], F)$ is a beaver. (Notice that $\kappa^{+\bar W} = \kappa^{+W}$ by 1.3 and σ induces $\hat\sigma: Ult(\bar W, F) \to Ult(W,F)$.)

For notational convenience, let us assume w.l.o.g. that $\bar W = W$. In the same way, we may now pick W' witnessing that $(J_\alpha[E], G)$ is a beaver and such that W' has the definability property at κ', where $\kappa' = c.p.(G)$.

Let $\pi: W \to_F Ult(W,F)$, and $\pi': W' \to_G Ult(W', G)$ be the ultrapower maps. By coiterating $Ult(W,F)$ with $Ult(W',G)$, we get a common coiterate Q and maps $i: Ult(W,F) \to Q$ and $j: Ult(W',G) \to Q$ obtained from the main branches. Let w.l.o.g. $\kappa = c.p.(F) \leq \kappa' = c.p.(G)$.

Let λ denote the largest cardinal in $J_\alpha[E]$. Note that $\lambda = F(\kappa) = G(\kappa')$.

Also notice that for $\bar\kappa < \kappa$, $\bar\kappa$ is strong in $\mathcal{J}_\kappa^W$ iff $\bar\kappa$ is strong in $\mathcal{J}_\lambda^Q$ iff $\bar\kappa$ is strong in $\mathcal{J}_{\kappa'}^{W'}$. Hence both $Ult(W,F)$ and $Ult(W',G)$ have the definability property at all $\bar\kappa < \kappa$ with

$$\mathcal{J}_\kappa^{Ult(W,F)} = \mathcal{J}_\kappa^{Ult(W',G)} \models \text{"}\bar\kappa \text{ is a strong cardinal,"}$$

and so the coiteration is above κ along the main branches on both sides.

Case 1. $\kappa < \kappa'$.

Considering the maps $i \circ \pi : W \to Q$ and $j \circ \pi' : W' \to Q$, by combining arguments from the proofs of 1.3 and 1.4, and 1.6, we get that $i \restriction \lambda = id$ and $c.p.(j) = \kappa$. As the coiteration of $Ult(W, F)$ with $Ult(W', G)$ is certainly beyond $\alpha + 1$, this means that if H is the first extender used along the main branch on the $Ult(W', G)$-side then H has an index $> \alpha$, and $c.p.(H) = \kappa$.

Moreover, by again considering $i \circ \pi$ and $j \circ \pi'$ and recalling that both W and W' have the hull- and definability property at κ, we get that $E^Q_\alpha \neq \emptyset$ as in the proof of Claim 3 in the proof of 1.3 and 1.4. But this is a plain contradiction!

Case 2. $\kappa = \kappa'$.

In this case we get that $i \restriction \lambda = j \restriction \lambda = id$ and hence $F = G$ using the hull- and definability property at κ of both W and W'. But this is what we were trying to show.

$\square$ (2.9)

Definition 2.10 *Let $\mathcal{M}$ be a premouse, and let $n < \omega$. $\mathcal{M}$ is called internally n-full if the following holds.*

Let $\mathcal{T} = (E^{\mathcal{T}}_\alpha : \alpha < \theta)$ be any iteration tree on W such that $D^{\mathcal{T}} \cap [0, \theta)_{\mathcal{T}} = \emptyset$ and

$$\alpha + 1 \in (0, \theta]_{\mathcal{T}} \Rightarrow \; d^{\pi^{\mathcal{T}}_{0\theta}(\mathcal{M})}(c.p.(E^{\mathcal{T}}_\alpha)) \geq n.$$

Set $\rho^* = sup\{\rho_\beta : \beta + 1 \in (0, \theta]_{\mathcal{T}}\}$ where ρ_β is the strict sup of the generators of E_β.

Then whenever there are $i < n$, ν, and F with critical point $\kappa \geq \rho^*$ such that $(\mathcal{J}^{\mathcal{M}^{\mathcal{T}}_\alpha}_\nu, F)$ is an i-beaver then $E^{\mathcal{M}^{\mathcal{T}}_\alpha}_\nu = F$.

We observe that any n-full premouse is also internally n-full, using 1.6.

Definition 2.11 *Let $\mathcal{M}$ be a premouse with largest cardinal κ. A premouse $\mathcal{N} \trianglerighteq \mathcal{M}$ is called an n-collapsing mouse for $\mathcal{M}$ if:*

(a) $\mathcal{N}$ is n-full,

(b) $\mathcal{M}$ is an n-cutpoint in $\mathcal{N}$, and

(c) $\rho_\omega(\mathcal{N}) \leq \kappa$.

Lemma 2.12 *Let $\mathcal{N}$, $\mathcal{N}'$ be n-collapsing mice for the premouse $\mathcal{M}$. Then $\mathcal{N} \trianglerighteq \mathcal{N}'$ or $\mathcal{N} \trianglelefteq \mathcal{N}'$.*

PROOF. Let W, W' witness that $\mathcal{N}$, $\mathcal{N}'$ are n-full, respectively. Let κ denote the largest cardinal of $\mathcal{M}$. By 1.5, the coiteration of W, W' is above κ along the main branches on both sides. Hence if Q is the common coiterate of W, W' then we have that

$$\mathcal{J}^{W}_{\kappa^{+W}} = \mathcal{J}^{Q}_{\kappa^{+Q}} = \mathcal{J}^{W'}_{\kappa^{+W'}}.$$

This implies that $\mathcal{N}$ and $\mathcal{N}'$ are lined up.

$\square$ (2.12)

We now present a variation of the K^c construction from §11 of [7] (cf. §1 of [18]). Let us fix a countable premouse $\mathcal{M}$ with a largest cardinal λ, and an integer $n < \omega$. Modulo breakdown, we shall define by recursion on $\xi < \Omega$ premice $\mathcal{N}_\xi$ and $\mathcal{M}_\xi$. (Recall that Ω is our fixed measurable from **A 2** .)

Set $\mathcal{N}_0 = \mathcal{M}$. Having defined $\mathcal{N}_\xi$ for some $\xi < \Omega$, we set $\mathcal{M}_\xi = core(\mathcal{N}_\xi)$, the core of $\mathcal{M}_\xi$ (cf. [7] §7 p. 9; notice that in Jensen the core is defined for all premice, regardless of whether they are iterable or not). We suppose that $\mathcal{N}_\xi$ (and so $\mathcal{M}_\xi$) is a mouse (i.e., an iterable premouse, cf. [7] §4 p. 28), and that $\mathcal{M}$ is an n-cutpoint in $\mathcal{N}_\xi$ as well as in $\mathcal{M}_\xi$. (However, we do not require that $\rho_\omega(\mathcal{N}_\xi) \geq On \cap \mathcal{M}$.) If one of these requirements is not fulfilled we let the construction break down.

If it does not break down, then in order to define $\mathcal{N}_{\xi+1}$ we consider three cases.

Case 0. $\mathcal{M}_\xi = (J_\gamma[E], \in, E \upharpoonright \gamma)$ is a passive premouse, and there is an extender F with critical point $\kappa < \lambda$ over $\mathcal{M}_\xi$ such that $(J_\gamma[E], \in, E \upharpoonright \gamma, F)$ is an i-beaver, where $i = d^{\mathcal{M}_\xi}(\kappa) < n$. In fact, F is unique by 2.9, and we then set

$$\mathcal{N}_{\xi+1} = (J_\gamma[E], \in, E \upharpoonright \gamma, F).$$

Case 1. $\mathcal{M}_\xi = (J_\gamma[E], \in, E \upharpoonright \gamma)$ is a passive premouse, Case 0 does not hold, but there is a certifiable extender F such that $(J_\gamma[E], \in, E \upharpoonright \gamma, F)$ is a premouse. We then choose some such F and set

$$\mathcal{N}_{\xi+1} = (J_\gamma[E], \in, E \upharpoonright \gamma, F).$$

("Certifiable" is defined in [7] §11 p. 2, and it corresponds to what is called "background certified" in [18] §1.)

Case 2. If Cases 0 and 1 both fail, then we just "contruct one more step," as in [7] §10 p. 9 (cf. §1 of [18]).

This defines $\mathcal{N}_{\xi+1}$. (Notice that if $\rho_\omega(\mathcal{N}_\xi) < \lambda$ then $\mathcal{M}$ has no chance of being an n-cutpoint in $\mathcal{M}_{\xi+2}$, so the construction would break down.)

Having defined $\mathcal{N}_\xi$ for all $\xi < \eta$ where η is a limit ordinal, we define $\mathcal{N}_\eta$ exactly as in §10 p. 10 of [7] (cf. §1 of [18]).

This completes the inductive definition of the $\mathcal{N}_\xi$'s. If $\mathcal{N}_\xi$ is defined for all $\xi < \Omega$ then as in Lemma 1.3 of [18] one can verify that $lim\ inf_{\xi \to \Omega}\ \rho_\omega(\mathcal{M}_\xi) = \Omega$, and that therefore we can define a premouse, denoted by $K_n^c(\mathcal{M})$, of height Ω by setting

$$\mathcal{J}_\beta^{K_n^c(\mathcal{M})} = \text{ eventual value of } \mathcal{J}_\beta^{\mathcal{M}_\xi}\ ,\ \text{as } \xi \to \Omega$$

for all $\beta < \Omega$. We express this by saying that $K_n^c(\mathcal{M})$ exists.

Because $\mathcal{M}$ is countable, if $\mathcal{N}_\xi$ exists and $\mathcal{M}$ is an n-cutpoint in $\mathcal{N}_\xi$ then any "formerly certified extender" $F = E_\nu^{\mathcal{N}_\xi}$ with index $\nu \geq On \cap \mathcal{M}$ either has critical point $\geq On \cap \mathcal{M}$ or else has critical point λ and is *partial* (in the sense that there is $\tilde{\nu} \in [\nu, On \cap \mathcal{N}]$ with $\rho_\omega(\mathcal{N}_\xi) \leq \lambda$). Conversely, any "beaver extender" $F = E_\nu^{\mathcal{N}_\xi}$ with index $\geq On \cap \mathcal{M}$ is *total* (non-partial) and has critical point $< \lambda$. (Otherwise we would have let the construction break down earlier.)

Also, the reader will have noticed that "beaver extenders" have the right of way. We need a little lemma which says that this doesn's cause a problem if we want to show cheapo covering for $K_n^c(\mathcal{M})$.

Lemma 2.13 *Let $\mathcal{N}_{\xi+1} = (J_\alpha[E], \in, E \restriction \alpha, F)$ be a beaver, where $\mathcal{N}_{\xi+1}$ is as in the construction of $K_n^c(\mathcal{M})$ for some $\mathcal{M}$. Suppose that $K_n^c(\mathcal{M})$ exists.*

Let $\kappa = c.p.(F)$. Then there is no extender G with critical point $\kappa' > \kappa$ such that κ' is a cardinal in $K_n^c(\mathcal{M})$ and $(J_\alpha[E], \in, E \restriction \alpha, G)$ is a premouse.

Proof. Suppose otherwise. Let λ be the largest cardinal in $J_\alpha[E]$, so that $\lambda = F(\kappa) = G(\kappa')$. Set $W = K_n^c(\mathcal{M})$. Because κ' is a cardinal in W, we get that

$$\mathcal{J}_{\kappa'}^W \models \text{"}\kappa \text{ is a strong cardinal,"}$$

hence using the embedding G gives

$$\mathcal{J}_\lambda^W \models \text{"}\kappa \text{ is a strong cardinal."}$$

But from this it follows, using the extender F, that in $\mathcal{J}_\kappa^W$ there are arbitrarily large strong cardinals, contradicting the fact that $(J_\alpha[E], F)$ is a beaver.

□ (2.13)

The argument for [7] §11 Lemma 3 (cf. [18] Theorem 1.4 and Lemma 3.5) now shows the following. Note that by 2.13, putting certifiable extenders on the $K_n^c(\mathcal{M})$-sequence as in the proof of [7] §11 Lemma 3 ([18] Theorem 1.4) is not blocked due to the fact that at some point we instead would have to put a beaver extender on the sequence.

Lemma 2.14 *Let $\mathcal{M}$ be a premouse, and $n < \omega$. If $K^c_n(\mathcal{M})$ exists then it is iterable, satisfies cheapo covering, is universal, and On is thick in $K^c_n(\mathcal{M})$.*

We are now going to prove the main result of this section.

Lemma 2.15 *Let $\mathcal{M}$ be a countable premouse, and let $n < \omega$. The following are equivalent.*

(1) $\mathcal{M}$ is n-full,

(2) $\mathcal{M}$ is internally n-full, and $K^c_n(\mathcal{M})$ exists, and

(3) $\mathcal{M}$ is internally n-full with a largest cardinal λ, and the following holds true: IF $\mathcal{N}$ is any premouse such that
(a) $\mathcal{M} \trianglelefteq \mathcal{N}$,
(b) $\mathcal{M}$ is an n-cutpoint in $\mathcal{N}$,
(c) $\mathcal{N}$ is iterable using extenders with critical point $\geq \lambda$ and index $\geq On \cap \mathcal{M}$ (i.e., every iteration tree on $\mathcal{N}$ which is both above λ and beyond $On \cap \mathcal{M}$ is well-behaved), and
(d) for all maximal iterates $\mathcal{N}^*$ of $\mathcal{N}$ as in (c), if $F = E^{\mathcal{N}^*}_\nu$ with $\nu \geq On \cap \mathcal{M}$ has critical point $\kappa < \lambda$ then $(\mathcal{J}^{N^*}_\nu, F)$ is an i-beaver for $i = d^{\mathcal{M}}(\kappa) < n$,
THEN we have that
(a') $\mathcal{N}$ is a mouse, i.e., is iterable,
(b') $\mathcal{M} \trianglelefteq core(\mathcal{N})$ and $\mathcal{M}$ is an n-cutpoint in $core(\mathcal{N})$, and
(c') $\rho_\omega(\mathcal{N}) \geq \lambda$.

Proof. We prove (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Set $W = K^c_n(\mathcal{M})$, and let λ be the largest cardinal of $\mathcal{M}$. We first show:

Claim 1. W has the definability property at all $\kappa < \lambda$ such that $d^{\mathcal{M}}(\kappa) < n$.

Proof. Suppose not, and let $\kappa \in \mathcal{M}$ be the least counterexample. There is

$$\sigma : \bar{W} \to W$$

with critical point κ and $\bar{W}$ being universal, obtained as the inverse of the transitive collapse of $H^W(\kappa \cup \Gamma)$ for some Γ being thick in W such that $\kappa \notin H^W(\kappa \cup \Gamma)$. By 1.3 and the minimality of κ, W has the hull property at κ, and $\mathcal{J}^{\bar{W}}_\lambda = \mathcal{J}^W_\lambda$ where $\lambda = \kappa^{+\bar{W}} = \kappa^{+W}$.

Let $i = d^{\mathcal{M}}(\kappa) < n$. In particular, there is some $\eta < \kappa$ such that

$$\mathcal{J}_\kappa^{\bar{W}} = \mathcal{J}_\kappa^W \models \text{"all strong cardinals are } < \eta.\text{"}$$

Hence, using σ,

$$\mathcal{J}_{\sigma(\kappa)}^W \models \text{"all strong cardinals are } < \eta.\text{"}$$

This means that κ is not a strong cardinal in $\mathcal{J}_{\sigma(\kappa)}^W$.

On the other hand, we claim that in fact κ is strong in $\mathcal{J}_{\sigma(\kappa)}^W$ which of course gives a contradiction.

Let F be an extender at κ, ν derived from σ such that $(\mathcal{J}_\nu^W, F)$ is a premouse and $\mathcal{J}_\nu^W$ is a stage in the construction of W. Of course, $Ult(\bar{W}, F)$ is iterable, as there is $k\colon Ult(\bar{W}, F) \to W$ defined by $k([a, f]_{Ult(\bar{W},F)}) = \sigma(f)(a)$ for appropriate a, f. But this means that $(\mathcal{J}_\nu^W, F)$ is an i-beaver. Hence $F = E_\nu^W$ by 2.9 and Case 0 of the construction of $K_n^c(\mathcal{M})$, or by the internal n-fullness of $\mathcal{M}$, depending on whether $\nu \geq \mathcal{M} \cap On$ or $\nu < On \cap \mathcal{M}$.

But the proof of [7] §11 Lemma 3 (cf. Lemma 11.4 in [13]), replacing the bicephalus argument by 2.9, then shows that there are enough ν's such that $(\mathcal{J}_\nu^W, F)$ as above is a premouse to witness that in fact κ is strong in $\mathcal{J}_{\sigma(\kappa)}^W$. (To check premousehood, i.e. the initial segment condition, one uses Case 0 of the construction of $K_n^c(\mathcal{M})$ or the internal n-fullness of $\mathcal{M}$.)

$\square$ (Claim 1)

Claim 2. W witnesses that $\mathcal{M}$ is n-full.

Proof. Let us consider an iteration tree $\mathcal{T}$ on W as in 2.4, the definition of n-fullness, and let us suppose that $\mathcal{M}_\theta^\mathcal{T}$ does not have the definability property at some κ with $\rho^* \leq \kappa < \pi_{0\alpha}^\mathcal{T}(\lambda)$ and $d^{\mathcal{M}_\theta^\mathcal{T}}(\kappa) = i < n$. Moreover, let us assume that (θ, κ) is the lexicographically least pair with this property.

As above, we have an embedding

$$\sigma\colon \bar{W} \to \mathcal{M}_\theta^\mathcal{T}$$

with critical point κ and $\bar{W}$ being universal, obtained as the inverse of the transitive collapse of $H^{\mathcal{M}_\theta^\mathcal{T}}(\kappa \cup \Gamma)$ for some Γ being thick in $\mathcal{M}_\theta^\mathcal{T}$. Again, κ cannot be a strong cardinal in $\mathcal{J}_{\sigma(\kappa)}^{\mathcal{M}_\theta^\mathcal{T}}$.

Subclaim 1. $\mathcal{M}_\theta^\mathcal{T}$ has the definability property at all $\mu < \kappa$ with $\mathcal{J}_\kappa^{\mathcal{M}_\theta^\mathcal{T}} \models$ "μ is a strong cardinal."

Proof. Fix such a μ. Supposing that $\mathcal{M}_\theta^\mathcal{T}$ does not have the definability property at μ, there is $\alpha+1 \in (0,\theta]_\mathcal{T}$ such that $F = E_\alpha^\mathcal{T}$ has a generator in the half-open interval $[\mu,\kappa)$ (as $(\theta,\mu) <_{lex} (\theta,\kappa)$, we would otherwise have chosen (θ,μ) instead of (θ,κ).). I.e., μ does not generate F (in the obvious sense, i.e., $F \restriction \mu$ does not code the entire F, where $F \restriction \mu$ was defined during the proof of 1.1; cf. [7] §8 p. 7). This implies $F \restriction \mu \in \mathcal{M}_\theta^\mathcal{T}$ by [7] §8 Cor. 4.2.

In particular, $c.p.(F)$ is a strong cardinal in $\mathcal{J}_\mu^{\mathcal{M}_\theta^\mathcal{T}}$, and thus $d^{\mathcal{M}_\theta^\mathcal{T}}(c.p.(F)) < n$. This contradicts the choice of $\mathcal{T}$.

$\square$ (Subclaim 1)

We now split the remaining proof into two cases.

Case 1. $\sigma(\kappa) \leq \pi_{0\theta}^\mathcal{T}(On \cap \mathcal{M})$.

Then reasoning exactly as above, using the fact that $\mathcal{M}$ is internally n-full, we may deduce that κ is a strong cardinal in $\mathcal{J}_{\sigma(\kappa)}^{\mathcal{M}_\theta^\mathcal{T}}$. Contradiction!

Case 2. $\sigma(\kappa) > \pi_{0\theta}^\mathcal{T}(On \cap \mathcal{M})$.

Still we may deduce, as in Case 1, that $F = E_\nu^{\mathcal{M}_\theta^\mathcal{T}}$ whenever F is an extender derived from σ such that $(\mathcal{J}_\nu^{\mathcal{M}_\theta^\mathcal{T}}, F)$ is a premouse and $\nu < \pi_{0\theta}^\mathcal{T}(On \cap \mathcal{M})$.

Subclaim 2. κ is definable (without parameters) in $\pi_{0\theta}^\mathcal{T}(\mathcal{M})$.

Proof. Recall that λ is the largest cardinal of $\mathcal{M}$. Then $\tilde{\lambda} = \pi_{0\theta}^\mathcal{T}(\lambda)$ is the largest cardinal of $\pi_{0\theta}^\mathcal{T}(\mathcal{M})$. So if $\kappa^{+\pi_{0\theta}^\mathcal{T}(\mathcal{M})} = \tilde{\lambda}$ then the Claim holds trivially.

So let $\kappa^{+\pi_{0\theta}^\mathcal{T}(\mathcal{M})} < \tilde{\lambda}$. Because $\mathcal{M}$ is an n-cutpoint in W as well as in $\bar{W}$, λ is still a cardinal in W, and hence $\tilde{\lambda}$ is still a cardinal in $\mathcal{M}_\theta^\mathcal{T}$. But then the above reasoning shows that κ is the $(i+1)^{st}$ strong cardinal in $\mathcal{J}_{\tilde{\lambda}}^{\mathcal{M}_\theta^\mathcal{T}}$. Thus again κ is definable.

$\square$ (Subclaim 2)

This however means that $\kappa \in ran(\pi_{0\theta}^\mathcal{T})$. Let $\kappa = \pi_{0\theta}^\mathcal{T}(\bar{\kappa})$.

Let Γ be thick in both W and $\mathcal{M}_\theta^\mathcal{T}$ such that $\pi_{0\theta}^\mathcal{T}(\xi) = \xi$ for all $\xi \in \Gamma$. We then have

$$\bar{\kappa} \in H^W(\bar{\kappa} \cup \Gamma)$$

by Claim 1, and hence

$$\kappa \in H^{\mathcal{M}_\theta^\mathcal{T}}((\pi_{0\theta}^{\mathcal{T}}{}''\bar{\kappa}) \cup \Gamma) \subset H^{\mathcal{M}_\theta^\mathcal{T}}(\kappa \cup \Gamma),$$

which contradicts what we had assumed.

$\square$ (Claim 2)

This completes the proof of (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3). Let W witness that $\mathcal{M}$ is n-full, and let $\mathcal{N}$ be given with properties (a) thru (d) in (3). We try to coiterate W with $\mathcal{N}$. The only worry here is that on the $\mathcal{N}$-side we might have to use an extender with critical point $< \lambda$ (where $\lambda =$ the largest cardinal of $\mathcal{M}$). However, 1.6 is designed just for the purpose to remove this worry.

So let $\mathcal{U} = (E^{\mathcal{U}}_\alpha : \alpha < \theta)$ and $\mathcal{T} = (E^{\mathcal{T}}_\alpha : \alpha < \theta)$ be a coiteration of W, $\mathcal{N}$, i.e. iteration trees on W, $\mathcal{N}$, respectively, formed by "iterating the least disagreement," as in [7] §4 p. 15 (cf. §7 of [13]). Assume that $\mathcal{U}$ is as in (c) of (3).

Suppose that there is some ν with

$$\mathcal{J}_\nu^{\mathcal{M}^{\mathcal{U}}_\theta} \neq \mathcal{J}_\nu^{\mathcal{M}^{\mathcal{T}}_\theta},$$

and that for the least such ν we have that

$$F = E_\nu^{\mathcal{M}^{\mathcal{T}}_\theta} \neq \emptyset$$

is a total extender with critical point $\kappa < \lambda$.

Setting $i = d^{\mathcal{M}}(\kappa)$ we know that $i < n$, and also that $(J_\nu^{\mathcal{M}^{\mathcal{T}}_\theta}, F)$ is an i-beaver by assumption (d) on $\mathcal{N}$. Let W' witness that $(J_\nu^{\mathcal{M}^{\mathcal{T}}_\theta}, F)$ is an i-beaver. However then from 1.6 we know that $F = E_\nu^{\mathcal{M}^{\mathcal{U}}_\theta}$. Contradiction!

We thus cannot run into the above scenario, which means that W, $\mathcal{N}$ can be successfully coiterated by assumption (c) on $\mathcal{N}$. The universality of W implies that there can be no drop along the main branch on the $\mathcal{N}$-side of this comparison, and that the W-side cannot come out shorter. We thus have a Σ^*-elementary embedding

$$\sigma : \mathcal{N} \to \mathcal{N}^*$$

where $\mathcal{N}^*$ is an initial segment of an iterate of W, and is hence iterable. This means that $\mathcal{N}$ itself is iterable. This shows (a').

Notice that we also have shown that $\sigma : \mathcal{N} \to \mathcal{N}^*$ is such that $\sigma \restriction \lambda = id$. Hence if $\rho_\omega(\mathcal{N}) < \lambda$ then $\rho_\omega(\mathcal{N}^*) < \lambda$, too. But as the iteration is beyond $On \cap \mathcal{M} > \lambda$ on the W-side, this would mean that λ is no cardinal in W, contradicting the fact that $\mathcal{M}$ is an n-cutpoint in W. Hence $\rho_\omega(\mathcal{N}) \geq \lambda$, which shows (c').

From this in turn it follows that the core map

$$\pi : core(\mathcal{N}) \to \mathcal{N}$$

is such that $\pi \restriction \lambda = id$. Now regardless of whether $\rho_\omega(\mathcal{N}) = \lambda$ or $\rho_\omega(\mathcal{N}) \geq \lambda^{+\mathcal{N}}$ we get that that $\mathcal{M} \trianglelefteq core(\mathcal{N})$. Moreover, as π is Σ_1-elementary and $\mathcal{M}$ is an n-cutpoint in $\mathcal{N}$, $\mathcal{M}$ is an n-cutpoint in $core(\mathcal{N})$ as well. This shows (c').

This completes (1) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2). Essentially, we have to check that (3) supplies us with enough tools to deduce that the construction of $K^c_n(\mathcal{M})$ does not break down. This will be done inductively: assuming (3) and fixing $\bar{n} \leq n$, we shall show that if $K^c_i(\mathcal{M})$ exists for all $i < \bar{n}$ then $K^c_{\bar{n}}(\mathcal{M})$ exists as well.

For this in turn we shall verify that if $\xi < \Omega$ is such that $\mathcal{N} = \mathcal{N}_\xi$ exists and is as in the recursive definition of $K^c_{\bar{n}}(\mathcal{M})$ and such that $\mathcal{M}$ is an $\bar{n}$-cutpoint in $\mathcal{N}$, then the $K^c_{\bar{n}}(\mathcal{M})$-construction does not break down taking the next steps. I.e., we shall show that $\mathcal{N}$ is iterable (and so is $\mathcal{M}_\xi = core(\mathcal{N})$), and $\mathcal{M}$ is an n-cutpoint in $\mathcal{M}_\xi$ as well as in $\mathcal{N}_{\xi+1}$. (Notice that if η is a limit ordinal and $\mathcal{N}_\xi$ are defined for all $\xi < \eta$ then $\mathcal{N}_\eta$ is trivially defined and such that $\mathcal{M}$ is an n-cutpoint in $\mathcal{N}_\eta$.)

We first notice that $\mathcal{N}$ is iterable using extenders with critical point $\geq \lambda$ and index $\geq On \cap \mathcal{M}$ (where $\lambda =$ the largest cardinal of $\mathcal{M}$) by the iterability proof of "Corrections to [7]" §3 ([13] §9). (Notice here that as $\mathcal{M}$ is countable, we may pick a potential witness $\pi: \bar{\mathcal{N}} \to \mathcal{N}$ for the failure of that much iterability of $\mathcal{N}$ in such a way that $\bar{\mathcal{N}}$ is countable, and $\pi \restriction \mathcal{M} = id$. But this makes sure that in the iteration of $\bar{\mathcal{N}}$ we only have to do with "certified" extenders [cf. the remark after the definition of $K^c_n(\mathcal{M})$; a beaver extender cannot become partial during the $K^c_n(\mathcal{M})$-construction], and hence we can run the argument of [13] §9.)

It hence suffices to check that (d) of (3) holds for $\mathcal{N}$. Because if this is the case then in fact (a) thru (d) of (3) hold for $\mathcal{N}$, and thus so do (a') thru (c'), as we assume (3). But then $\mathcal{N}$ and $core(\mathcal{N}) = \mathcal{M}_\xi$ are both iterable by (a'), and $\mathcal{M}$ is an n-cutpoint in $\mathcal{M}_\xi$ by (b'). Moreover by (c'), $\mathcal{M}$ will also be an n-cutpoint in $\mathcal{N}_{\xi+1}$.

So let $\mathcal{T} = (E_\alpha : \alpha < \theta)$ be an iteration tree on $\mathcal{N}$ witnessing (d) of (3) is false, i.e., setting $\mathcal{N}^* = \mathcal{M}^{\mathcal{T}}_\theta$, there is some total $F = E^{\mathcal{N}^*}_\nu$ with critical point $\kappa < \lambda$ and $d^{\mathcal{M}}(\kappa) = i < \bar{n}$ such that $(\mathcal{J}^{\mathcal{N}^*}_\nu, F)$ is not an i-beaver.

By our inductive assumption, $K^c_i(\mathcal{M})$ exists. Let us write $W = K^c_i(\mathcal{M})$. By the proof of (2) $\Rightarrow$ (1) above W witnesses that $\mathcal{M}$ is i-full. In particular, W is a universal weasel such that

$$\mathcal{J}^W_\lambda = \mathcal{J}^{\mathcal{N}^*}_\lambda, \text{ where } \lambda = \kappa^{+W} = \kappa^{+\mathcal{N}^*}$$

and W has the definability property at all $\mu < \kappa$ such that $\mathcal{J}^W_\kappa \models$ " μ is a strong cardinal." But as $(\mathcal{J}^{\mathcal{N}^*}_\nu, F)$ is assumed not to be an i-beaver this means that

$$Ult(W, F)$$

cannot be iterable.

By (the proof of) [18] Lemma 2.4 we may pick some ill-behaved putative iteration tree $\mathcal{U} = (F_\alpha : \alpha < \eta)$ of countable length (i.e., $\eta < \omega_1$) on $\mathcal{J}_\mu^{Ult(W,F)}$ for some large enough $\mu < \Omega$ (here, Ω is from **A 2**).

Using a variant of the proof of [18] Lemma 2.4, we may pick some

$$\pi : H \to V_\Omega$$

such that $\mathcal{T}, \mathcal{U} \in ran(\pi)$ as well as $\{F_\alpha : \alpha < \eta\} \subset ran(\pi)$. Because $\mathcal{M}$ is countable, we may suppose that $\pi \restriction \mathcal{M} = id$. Now $\bar{\mathcal{T}} = \pi^{-1}(\mathcal{T})$ is an iteration tree on $\bar{\mathcal{N}} = \pi^{-1}(\mathcal{N})$ such that its last model, $\bar{\mathcal{N}}^* = \pi^{-1}(\mathcal{N}^*)$, has $\bar{F} = \pi^{-1}(F)$ as an extender with critical point $\kappa < \lambda$.

Moreover, we claim that $Ult(W, \bar{F})$ is not iterable, as witnessed by a putative iteration tree $\mathcal{U}'$ with the same tree order as $\mathcal{U}$ on $Ult(W, F)$ which is obtained by "copying down" $\mathcal{U}$ as follows. (Cf. [13] p. 52 ff. on copying constructions.)

For any $\alpha < \eta$ we construct a map

$$\pi_\alpha : \mathcal{M}_\alpha^{\mathcal{U}'} \to \mathcal{M}_\alpha^{\mathcal{U}},$$

simultaneously verifying that $\mathcal{M}_\alpha^{\mathcal{U}} \cap ran(\pi) \subset ran(\pi_\alpha)$.

To commence, we set $\pi_0([a, f]_{Ult(W,\bar{F})}) = [\pi(a), f]_{Ult(W,F)}$ for appropriate a, f. Note that π_0 is well-defined as

$$\{u : W \models \Phi(f(u), ...)\} \in \bar{F}_a \text{ iff}$$

$$\{u : W \models \Phi(f(u), ...)\} \in F_{\pi(a)}$$

using π, noting that $\pi \restriction \mathcal{M} = id$. Also, $Ult(W, F) \cap ran(\pi) \subset ran(\pi_0)$ is straightforward.

We let the limit step of the construction as an exercise to the reader, as well as the task of checking the various agreements between the maps necessary to keep the construction going. We here give the successor step.

If π_β is defined for all $\beta \leq \alpha$ such that $\mathcal{M}_\beta^{\mathcal{U}} \cap ran(\pi) \subset ran(\pi_\beta)$, then we set $\hat{F} = \pi_\alpha^{-1}(F_\alpha)$, $P = dom(\pi_{\alpha^*\alpha+1}^{\mathcal{U}})$ for $\alpha^* = U\text{-}pred(\alpha + 1)$, and $\bar{P} = \pi_{\alpha^*}^{-1}(P)$, and define

$$\pi_{\alpha+1}([a, f]_{Ult(\bar{P},\hat{F})}) = [\pi_\alpha(a), \pi_{\alpha^*}(f)]_{Ult(P,F_\alpha)}.$$

$\pi_{\alpha+1}$ is well-defined because by the agreement $\pi_{\alpha^*} \restriction \mathcal{P}(c.p.(\hat{F}_\alpha)) = \pi_\alpha \restriction \mathcal{P}(c.p.(\hat{F}_\alpha))$ we get that

$$\{u : \mathcal{M}_{\alpha^*}^{\mathcal{U}'} \models \Phi(f(u), ...)\} \in \hat{F}_a \text{ iff}$$

$$\{u : \mathcal{M}_{\alpha^*}^{\mathcal{U}} \models \Phi(\pi_{\alpha^*}(f)(u), ...)\} \in F_{\alpha, \pi_\alpha(a)}.$$

Let $x \in \mathcal{M}^{\mathcal{U}}_{\alpha+1} \cap ran(\pi)$. Because $\mathcal{M}^{\mathcal{U}}_{\alpha+1} \in ran(\pi)$, by elementarity there are a, $f \in ran(\pi)$ such that $x = [a, f]_{Ult(P,F_\alpha)}$, and hence by the inductive hypothesis there are $\bar{a} \in \mathcal{M}^{\mathcal{U}'}_{\alpha}$ and $\bar{f} \in \bar{P}$ such that

$$x = [\pi_\alpha(\bar{a}), \pi_{\alpha^*}(\bar{f})]_{Ult(P,F_\alpha)}.$$

But this clearly implies $x \in ran(\pi_{\alpha+1})$.

But now it is straightforward to check that $\mathcal{U}'$ in fact witnesses that $Ult(W, \bar{F})$ is not iterable: If $\mathcal{U}'$ (and hence $\mathcal{U}$) has successor length then one can just use $\mathcal{M}^{\mathcal{U}}_{\eta-1} \cap ran(\pi) \subset ran(\pi_{\eta-1})$. If $\mathcal{U}'$ (and hence $\mathcal{U}$) has limit length one uses the fact that H is closed under sharps: if $\mathcal{U}'$ were well-behaved, then $\pi^{-1}(\mathcal{U})$ would have to be well-behaved, too, as we have the maps

$$\pi_\alpha^{-1} \circ \pi : \mathcal{M}^{\pi^{-1}(\mathcal{U})}_{\alpha} \to \mathcal{M}^{\mathcal{U}'}_{\alpha}$$

for any $\alpha < \eta$. But then then $H^{Col(\zeta,\omega)}$ would know that $\pi^{-1}(\mathcal{U})$ is well-behaved for any large enough ζ (by Σ^1_2-correctness), so, using π, $V_\Omega^{Col(\zeta,\omega)}$ would know that $\mathcal{U}$ is well-behaved for any large enough ζ, which implies that $\mathcal{U}$ were really well-behaved (as $\mathcal{U}$ is on W, so there cannot be two cofinal well-founded branches, and $Col(\zeta, \omega)$ is homogeneous).

What we have shown by now is the following. There is a countable iterate $\bar{\mathcal{N}}^*$ of $\bar{\mathcal{N}}$ as in (c) of (3) with a total extender $\bar{F} = E^{\bar{\mathcal{N}}^*}_{\bar{\nu}}$ with critical point $\kappa < \lambda$ and $i = d^{\mathcal{M}}(\kappa) < n$ such that $(\mathcal{J}^{\bar{\mathcal{N}}^*}_{\bar{\nu}}, \bar{F})$ is not an i-beaver. ¿From this we now derive a contradiction.

By (the proof of) "Addendum to [7] (cf. [18] Theorem 9.14), there are a $\bar{\xi} \leq \xi$ and an embedding

$$\sigma : \bar{\mathcal{N}}^* \to \mathcal{N}_{\bar{\xi}},$$

where $\sigma \restriction \mathcal{M} = id$. Let $\tilde{F} = \sigma(\bar{F})$. We now know from the definition of $\mathcal{N}_{\bar{\xi}}$ that

$$(\mathcal{J}^{\mathcal{N}_{\bar{\xi}}}_{\tilde{\nu}}, \tilde{F})$$

is an i-beaver (using 1.3), where $\tilde{\nu} = \sigma(\bar{\nu}) = lh(\tilde{F})$. In particular, $Ult(W, \tilde{F})$ is iterable by 1.2.

We may define

$$\pi : Ult(W, \bar{F}) \to Ult(W, \tilde{F})$$

by setting

$$\pi([a, f]_{Ult(W,\bar{F})}) = [\sigma(a), f]_{Ult(W,\tilde{F})}.$$

Note that

$$\{u : W \models \Phi(f(u), ...)\} \in \bar{F}_a \text{ iff}$$

$$\{u\colon W \models \Phi(f(u), ...)\} \in \sigma(\bar{F}_a) = \tilde{F}_{\sigma(a)},$$

and hence π is well-defined.

But this implies that in fact $Ult(W, \bar{F})$ is iterable, contradicting the fact that $(\mathcal{J}_{\bar{\nu}}^{\bar{\mathcal{N}}^*}, \bar{F})$ is not an i-beaver.

This gives (3) $\Rightarrow$ (2), and thus finishes the proof of the lemma.

$\square$ (2.15)

There are two variants of 2.15 for "strongly n-full" and "n-beaver." We state them without proof.

Lemma 2.16 *Let $\mathcal{M}$ be a countable premouse and let $n < \omega$. The following are equivalent.*

(1) $\mathcal{M}$ is strongly n-full, and

(2) $\mathcal{M}$ is internally n-full with largest cardinal λ, and the following holds true: IF $\mathcal{N}$ is any premouse such that

(a) $\mathcal{M} \trianglelefteq \mathcal{N}$,

(b) $\mathcal{M}$ is an n-cutpoint in $\mathcal{N}$,

(c) $\mathcal{N}$ is iterable using extenders with critical point $\geq On \cap \mathcal{M}$, and

(d) for all iterates $\mathcal{N}^*$ of $\mathcal{N}$ as in (c), if $F = E_\nu^{\mathcal{N}^*}$ with $\nu \geq On \cap \mathcal{M}$ has critical point $\kappa < \lambda$ then $(\mathcal{J}_\nu^{\mathcal{N}^*}, F)$ is an i-beaver for $i = d^{\mathcal{M}}(\kappa) < n$,

THEN we have that

(a') $\mathcal{N}$ is a mouse, i.e., is iterable, and

(b') $\rho_\omega(\mathcal{N}) \geq On \cap \mathcal{M}$.

Lemma 2.17 *Let $\mathcal{M}$ be a countable premouse with top extender $F \neq \emptyset$, let $\kappa =$ c.p.(F), and let $n < \omega$. The following are equivalent.*

(1) $\mathcal{M}$ is an n-beaver, and

(2) $\mathcal{J}_{\kappa^{+\mathcal{M}}}^{\mathcal{M}}$ is internally $(n-1)$-full, and the following holds true: IF $\mathcal{N}$ is any premouse such that

(a) $\mathcal{J}_{\kappa^{+\mathcal{M}}}^{\mathcal{M}} \trianglelefteq \mathcal{N}$,

(b) $\mathcal{J}_{\kappa^{+\mathcal{M}}}^{\mathcal{M}}$ is an $(n-1)$-cutpoint in $\mathcal{N}$,

(c) $\mathcal{N}$ is iterable using extenders with critical point $\geq \kappa^{+\mathcal{M}}$, and

(d) for all iterates $\mathcal{N}^*$ of $\mathcal{N}$ as in (c), if $G = E_\nu^{\mathcal{N}^*}$ with $\nu \geq On \cap \mathcal{M}$ has critical point $\mu < \kappa$ then $(\mathcal{J}_\nu^{\mathcal{N}^*}, G)$ is an i-beaver for $i = d^{\mathcal{M}}(\kappa) < n$,

THEN we have that
(a') $Ult(\mathcal{N}, F)$ is iterable, and
(b') $\rho_\omega(\mathcal{N}) \geq \kappa^{+\mathcal{M}}$.

We hence get:

Corollary 2.18 *Let $n < \omega$. The following sets of reals are Π^1_{n+3}.*
(a) $\{x : x$ codes an n-full premouse $\}$,
(b) $\{x : x$ codes a strongly n-full premouse $\}$,
(c) $\{x : x$ codes an n-collapsing mouse $\}$, and
(d) $\{x : x$ codes an n-beaver $\}$.

Proof. This is shown by simultaneous induction on $n < \omega$. Suppose it holds for all $i < n$. We then use 2.15 for computing the complexity of "x codes an n-full premouse." In fact it is easily checked that this statement can be written as

$$\forall y[\Phi_0(x, y) \ \wedge \ \forall y^* \forall f(\Phi_1(x, y, y^*, f) \to \Phi_2(y^*, f)) \to \Phi_3(x, y)].$$

Let us write c_r for the object coded by the real r (we assume some coding device being fixed). Then the displayed formula is supposed to express that (3) of 2.15 is satisfied with $\mathcal{M} = c_x$, $\mathcal{N} = c_y$, $\mathcal{N}^* = c_{y^*}$, and $F = c_f$. In particular, $\Phi_2(y^*, f)$ says that (y^*, f) codes an i-beaver for $i = d^{c_x}(c.p.(c_f)) < n$ and is Π^1_{n+2} by inductive assumption. Φ_0 and Φ_3 are Π^1_2 (stating iterability), and Φ_1 is Σ^1_1 (stating the shape of a tree giving c_{y^*} from c_y), and thus the entire statement is in fact uniformly $\Pi^1_{n+3}(x)$. This shows (a).

In the same way, (b), (c), and (d) are easily checked, using 2.16, 2.17, and 2.11.

$\square$ (2.18)

3 $K \cap HC$ is projective.

Using the tools of the previous section, we can now show that $K \cap HC$ is projective, given that it only has finitely many strong cardinals. We need just two more preliminary lemmas.

Lemma 3.1 *Let W witness that $\mathcal{M}$ is strongly n-full, let W' witness that $\mathcal{N}$ is strongly n-full, and let $\mathcal{U} = (E^{\mathcal{U}}_\alpha : \alpha < \theta)$ and $\mathcal{T} = (E^{\mathcal{T}}_\alpha : \alpha < \theta)$ denote the iteration trees arising from the comparison of W with W', respectively. Let $Q = \mathcal{M}^{\mathcal{U}}_\theta = \mathcal{M}^{\mathcal{T}}_\theta$ be the common coiterate.*

Let κ be a cardinal in Q such that

$$\kappa \in \pi_{0\theta}^{\mathcal{U}}(\mathcal{M}) \cap \pi_{0\theta}^{\mathcal{T}}(\mathcal{N}), \text{ and } d^{Q}(\kappa) < n.$$

Then if $\alpha + 1 \in (0, \theta]_{\mathcal{U}}$ is such that $F = E_{\alpha}^{\mathcal{U}} \neq \emptyset$, we have that $c.p.(F) \neq \kappa$. Symmetrically, if $\alpha + 1 \in (0, \theta]_{\mathcal{T}}$ is such that $F = E_{\alpha}^{\mathcal{T}} \neq \emptyset$, we have that $c.p.(F) \neq \kappa$.

Proof. The proof is by induction on κ. Supposing the lemma is wrong, we consider the least counterexample, and assume w.l.o.g. that the "bad" extender is used along the main branch of $\mathcal{U}$. I.e., κ is minimal such that

$$\kappa \in \pi_{0\theta}^{\mathcal{U}}(\mathcal{M}) \cap \pi_{0\theta}^{\mathcal{T}}(\mathcal{N}), \; d^{Q}(\kappa) < n,$$

κ is the critical point of an extender used along one of the branches $[0, \theta]_{\mathcal{U}}$ and $[0, \theta]_{\mathcal{T}}$, and as a matter of fact, $\kappa = c.p.(E_{\alpha}^{\mathcal{U}})$ for some $\alpha + 1 \in (0, \theta]_{\mathcal{U}}$.

Let $\gamma \in [0, \theta]_{\mathcal{T}}$ be minimal such that $\pi_{\gamma\theta}^{\mathcal{T}} \restriction \kappa = id$.

Claim 1. $\mathcal{M}_{\gamma}^{\mathcal{T}}$ has the definability property at all $\bar{\kappa} \leq \kappa$ such that either $\bar{\kappa} = \kappa$ or else $\bar{\kappa}$ is a strong cardinal in $\mathcal{J}_{\kappa}^{\mathcal{M}_{\gamma}^{\mathcal{T}}}$.

Proof. Let X_{β} denote the set of generators of $E_{\beta}^{\mathcal{T}}$, and let σ_{β} be the (strict) sup of X_{β}, for $\beta < \theta$.

Let $\bar{\kappa}$ be as in the statement of the Claim. Suppose that for some $\beta + 1 \in (0, \theta]_{\mathcal{T}}$, we have that $\mu = c.p.(E_{\beta}^{\mathcal{T}}) < \bar{\kappa}$ and $\bar{\kappa} < \sigma_{\beta}$. Let $F = E_{\beta}^{\mathcal{T}} = E_{\nu}^{\mathcal{M}_{\beta}^{\mathcal{T}}}$, and consider

$$\mathcal{J}_{\nu}^{\mathcal{M}_{\beta}^{\mathcal{T}}} = (\mathcal{J}_{\nu}^{\mathcal{M}_{\beta}^{\mathcal{T}}}, F).$$

As $\bar{\kappa} < \sigma_{\beta}$, for $\xi < \bar{\kappa}$ we have that ξ does not generate F, i.e., $F \restriction \xi$ does not code the entire F. This implies $F \restriction \xi \in J_{\nu}^{\mathcal{M}_{\beta}^{\mathcal{T}}}$ by [7] §8 Cor. 4.2. By coherency, $J_{\nu}^{\mathcal{M}_{\beta}^{\mathcal{T}}} \subset Q$, and so $F \restriction \xi \in Q$ for all $\xi < \bar{\kappa}$.

Moreover,

$$\bar{\kappa} \subset \{ i_F(f)(a) : \; a \in [X_{\beta} \cap \kappa]^{<\omega}, \; f \colon [\mu]^{Card(a)} \to J_{\mu^{+Q}}^{Q}, \text{ and } f \in Q \},$$

and $\bar{\kappa} > \mu$ is inaccessible in Q, so that $X_{\beta} \cap \bar{\kappa}$ must be unbounded in $\bar{\kappa}$. But this means that the family $\{ F \restriction \xi : \; \xi < \bar{\kappa} \} \subset Q$ of extenders witnesses that

$$\mathcal{J}_{\bar{\kappa}}^{Q} \models \text{"}\mu \text{ is a strong cardinal,"}$$

and so $d^Q(\mu) < d^Q(\bar\kappa) \leq d^Q(\kappa) < n$. But as $\mu = c.p.(E^{\mathcal{T}}_\beta) < \bar\kappa \leq \kappa$, this contradicts the minimality of κ.

Hence for every $\beta + 1 \in (0, \theta]_{\mathcal{T}}$, whenever $c.p.(E^{\mathcal{T}}_\beta) < \bar\kappa$ then $\sigma_\beta \leq \bar\kappa$, too. Let $\bar\gamma \in [0, \gamma]_{\mathcal{T}}$ be minimal such that $\pi^{\mathcal{T}}_{\bar\gamma\gamma} \upharpoonright \bar\kappa = id$. Because W' witnesses that $\mathcal{N}$ is n-full, we have that $\mathcal{M}^{\mathcal{T}}_{\bar\gamma}$ has the definability property at $\bar\kappa$. But certainly $\pi^{\mathcal{T}}_{\bar\gamma\gamma}(\bar\kappa) = \bar\kappa$ by the minimality of κ. So if Γ is a class of fixed points under $\pi^{\mathcal{T}}_{\bar\gamma\gamma}$ thick in both $\mathcal{M}^{\mathcal{T}}_{\bar\gamma}$ and $\mathcal{M}^{\mathcal{T}}_{\gamma}$, then

$$\bar\kappa \in H^{\mathcal{M}^{\mathcal{T}}_{\bar\gamma}}(\bar\kappa \cup \Gamma) \;\Rightarrow\; \bar\kappa \in H^{\mathcal{M}^{\mathcal{T}}_{\gamma}}(\bar\kappa \cup \Gamma).$$

$\square$ (Claim 1)

Now let $\alpha^* = \mathcal{U}\text{-}pred(\alpha + 1)$. Notice that α^* is just minimal in $[0, \theta]_{\mathcal{U}}$ such that $\pi^{\mathcal{U}}_{\alpha^*\theta} \upharpoonright \kappa = id$.

In exactly the same way as we proved Claim 1 one can show:

Claim 2. $\mathcal{M}^{\mathcal{U}}_{\alpha^*}$ has the definability property at all $\bar\kappa \leq \kappa$ such that either $\bar\kappa = \kappa$ or else $\bar\kappa$ is a strong cardinal in $\mathcal{J}^{\mathcal{M}^{\mathcal{U}}_{\alpha^*}}_\kappa$.

Now we consider the phalanxes $\mathcal{P}$ and $\mathcal{P}'$ generated by $\mathcal{U} \upharpoonright \alpha + 1$ and $\mathcal{T} \upharpoonright \gamma + 1$, respectively. Their coiteration is given by the tails of the coiteration of W with W', so that the final model is Q and is above $\mathcal{M}^{\mathcal{U}}_{\alpha^*}$ on the $\mathcal{P}$-side, and above $\mathcal{M}^{\mathcal{T}}_{\gamma}$ on the $\mathcal{P}'$-side. Moreover, the embedding obtained from the main branch on the $\mathcal{P}$-side is in fact

$$\pi^{\mathcal{U}}_{\alpha^*\theta} : \mathcal{M}^{\mathcal{U}}_{\alpha^*} \to Q,$$

and so has critical point κ, because the first extender used along $[\alpha^*, \theta]_{\mathcal{U}}$ is $E^{\mathcal{U}}_\alpha$. However, this is a plain contradiction with (the proof of) 1.3 and 1.4.

$\square$ (3.1)

3.1 also holds for n-full $\mathcal{M}$, $\mathcal{N}$ whose largest cardinal is singular.

In [16] it is shown that for any cardinal $\kappa \geq \aleph_2$, $\mathcal{J}^K_\kappa$ is universal for coiterable premice $\mathcal{M}$ of height $\leq \kappa$. Whereas this is not true in general for $\kappa = \aleph_1$ by an observation of Jensen, the following lemma shows that at least $\mathcal{J}^K_{\omega_1}$ is universal for sufficiently full countable mice, which will suffice for our purposes.

Lemma 3.2 *Assume that $\omega_1 = \omega_1^V$ is inaccessible in K. Let $n < \omega$, and suppose that there are at most n strong cardinals in $\mathcal{J}^K_{\omega_1}$. Let $\mathcal{M}$ be countable and strongly n-full.*

Then there is a countable strongly n-full initial segment of K winning the coiteration against $\mathcal{M}$, i.e., there is a strongly n-full $\mathcal{N} = \mathcal{J}^K_\alpha \lhd \mathcal{J}^K_{\omega_1}$ such that if $\mathcal{M}^*$, $\mathcal{N}^*$ denote the comparable coiterates of $\mathcal{M}$, $\mathcal{N}$, respectively, then $\mathcal{M}^* \lhd \mathcal{N}^*$.

PROOF. Let $\mathcal{M}$ be countable, and let W witness that $\mathcal{M}$ is n-full. (If ω_1 is a successor in K where we assume that $\mathcal{M}$ is $(n+1)$-full, let W witness $(n+1)$-fullness.) Let W' be a very soundness witness for $\mathcal{J}^K_{\omega_1}$. Let $\mathcal{U} = (E^{\mathcal{U}}_\alpha : \alpha < \theta)$ and $\mathcal{T} = (E^{\mathcal{T}}_\alpha : \alpha < \theta)$ be the iteration trees arising from the comparison of W with W'. There is $\bar\theta$ such that $\bar{\mathcal{U}} = (E^{\mathcal{U}}_\alpha : \alpha < \bar\theta)$, $\bar{\mathcal{T}} = (E^{\mathcal{T}}_\alpha : \alpha < \bar\theta)$ are on $\mathcal{M}$, $\mathcal{J}^K_{\omega_1}$, resp., and may be considered as the successful comparison of $\mathcal{M}$ with $\mathcal{J}^K_{\omega_1}$.

Let η be a large enough regular cardinal, and pick

$$\sigma : \bar V \to V_\eta$$

such that $\{\mathcal{M}, \mathcal{J}^K_{\omega_1}, \bar{\mathcal{U}}, \bar{\mathcal{T}}\} \subset ran(\sigma)$, and $\bar V$ is countable and transitive. Letting $\alpha = \omega_1 \cap ran(\sigma)$, α is the critical point of σ, and $\sigma^{-1}(\bar{\mathcal{U}})$, $\sigma^{-1}(\bar{\mathcal{T}})$ is the successful comparison of $\mathcal{M}$ with $\mathcal{J}^K_\alpha$.

Set $\mathcal{N} = \mathcal{J}^K_\alpha$. By 2.6, $\mathcal{N}$ is n-full.

Claim. If $\mathcal{M}^\star$, $\mathcal{N}^\star$ denote the coiterates of $\mathcal{M}$, $\mathcal{N}$, resp., then $\mathcal{M}^\star \lhd \mathcal{N}^\star$.

PROOF. Let us assume towards contradiction that $\mathcal{N}^\star \trianglelefteq \mathcal{M}^\star$. By the elementarity of σ, this implies that $\mathcal{M}^{\bar{\mathcal{T}}}_{\bar\theta} \trianglelefteq \mathcal{M}^{\bar{\mathcal{U}}}_{\bar\theta}$.

Subclaim. $\bar\theta \in [0,\theta]_{\mathcal{T}}$, and $\pi^{\mathcal{T}}_{\bar\theta\theta} \upharpoonright (On \cap \pi^{\mathcal{T}}_{0\bar\theta}(\omega^V_1)) = id$.

PROOF. First notice that by $\mathcal{M}^{\bar{\mathcal{T}}}_{\bar\theta} \trianglelefteq \mathcal{M}^{\bar{\mathcal{U}}}_{\bar\theta}$ there can be no drop along $[0,\bar\theta]_{\bar{\mathcal{T}}}$, which implies that there is no drop along $[0,\bar\theta]_{\mathcal{T}}$ as well. Let $\lambda = On \cap \mathcal{M}^{\bar{\mathcal{T}}}_{\bar\theta}$, and notice that $\lambda = \pi^{\mathcal{T}}_{0\bar\theta}(\omega^V_1)$.

So if $\bar\theta \notin [0,\theta]_{\mathcal{T}}$, we may let $\beta \geq \bar\theta$ least such that $\beta+1 \in (0,\theta]_{\mathcal{T}}$. Then $\kappa = c.p.(E^{\mathcal{T}}_\beta) < \lambda$. As $lh(E^{\mathcal{T}}_\beta) > \lambda$ and λ is a cardinal in $\hat M = \mathcal{M}^{\mathcal{T}}_\beta$, we have that

$$\mathcal{J}^{\hat M}_\lambda \models \text{"}\kappa \text{ is a strong cardinal."}$$

But $\mathcal{J}^{\hat M}_\lambda = \mathcal{M}^{\bar{\mathcal{T}}}_{\bar\theta}$, so that

$$\mathcal{M}^{\bar{\mathcal{T}}}_{\bar\theta} \models \text{"}\kappa \text{ is a strong cardinal."}$$

Hence κ is in fact the image under $\pi^{\bar{\mathcal{T}}}_{0\omega_1}$ of one of the $\leq n$ many strong cardinals of $\mathcal{J}^K_{\omega_1}$, and we clearly get that

$$d^{\mathcal{M}^{\mathcal{T}}_\theta}(\kappa) = d^{\hat M}(\kappa) < n,$$

contradicting 3.1.

$\square$ (Subclaim)

Now using the assumption that $\mathcal{M}^{\bar{\mathcal{T}}}_{\bar{\theta}} \trianglelefteq \mathcal{M}^{\bar{\mathcal{U}}}_{\bar{\theta}}$ once more we get that

$$Card(\pi^{\mathcal{U}}_{0\theta}{}''W \cap \lambda) = \aleph_0,$$

whereas by the Subclaim we have that

$$Card(\pi^{\mathcal{T}}_{0\theta}{}''\omega_1 \cap \lambda) = Card(\pi^{\mathcal{T}}_{0\theta}{}''\omega_1) = \aleph_1.$$

However, this plainly contradicts [18] Theorem 6.5.

$\square$ (Claim)

Now to get an $\mathcal{N}$ as desired we can just replace $\mathcal{J}^K_\alpha$ by $\mathcal{J}^K_{\beta+K}$ for some $\beta \geq \alpha$, $\beta < \omega_1$, with $cf^K(\beta) = \omega$.

$\square$ (3.2)

Also, there is a version of 3.2 for an $(n+1)$-full $\mathcal{M}$ in the case that ω_1 is a successor cardinal in K and $\mathcal{J}^K_{\omega_1}$ has n strong cardinals.

We may now finally give our projective computations of $K \cap HC$. The case that ω_1^V is inaccessible in K is the more difficult one, and we deal with it first.

Theorem 3.3 *Assume that $\omega_1 = \omega_1^V$ is inaccessible in K. Let $m < \omega$, and suppose that $\mathcal{J}^K_{\omega_1} \models$ "there are exactly m strong cardinals." Let $m \leq n < \omega$.*

Then for all premice $\bar{\mathcal{M}}$ we have that

$$\bar{\mathcal{M}} \triangleleft \mathcal{J}^K_{\omega_1} \text{ iff}$$

there is a premouse $\mathcal{M} \triangleright \bar{\mathcal{M}}$ with the following properties:

$\mathcal{M}$ is countable and strongly n-full, and for all countable strongly n-full $\mathcal{N}$, IF $\mathcal{M}, \mathcal{N}$ coiterate to comparable premice $\mathcal{M}^*$, $\mathcal{N}^*$, respectively, such that $\mathcal{M}^* \trianglelefteq \mathcal{N}^*$, THEN $\mathcal{M}^*$ is an n-cutpoint in $\mathcal{N}^*$, there is no drop along the main branches on either side and if

$$i: \mathcal{M} \to \mathcal{M}^*, \text{ and}$$

$$j: \mathcal{N} \to \mathcal{N}^*$$

denote the maps coming from the main branches of the coiteration then

$$i''\mathcal{M} \subset j''\mathcal{N}.$$

By 2.18 (b) this readily implies:

Theorem 3.4 *If $\omega_1 = \omega_1^V$ is inaccessible in K and $\mathcal{J}^K_{\omega_1} \models$ "there are at most n strong cardinals" (where $n < \omega$), then $\mathcal{J}^K_{\omega_1}$ is (lightface) Δ^1_{n+5} in the codes.*

PROOF of 3.3. " $\Rightarrow$ ". Let $\bar{\mathcal{M}} \lhd \mathcal{J}^K_{\omega_1}$ be given. To commence, inside K there is $S \subset \omega_1$ stationary in ω_1 such that

$$\mathcal{J}^K_\kappa \prec \mathcal{J}^K_{\omega_1} \text{ and } cf^K(\kappa) = \omega$$

for all $\kappa \in S$. By 2.6, $\mathcal{J}^K_{\kappa^{+K}}$ is strongly n-full whenever $\kappa \in S$. So we may pick some $\kappa \in S$, $\kappa > On \cap \bar{\mathcal{M}}$, set

$$\mathcal{M} = \mathcal{J}^K_{\kappa^{+K}},$$

and let W witness that $\mathcal{M}$ is strongly n-full.

Now fix a countable strongly n-full $\mathcal{N}$ such that $\mathcal{M}^* \unlhd \mathcal{N}^*$ where $\mathcal{M}^*$, $\mathcal{N}^*$ are the comparable coiterates of $\mathcal{M}$, $\mathcal{N}$, respectively. Let $W' \rhd \mathcal{N}$ witness that $\mathcal{N}$ is strongly n-full. We may consider the comparison of $\mathcal{M}$, $\mathcal{N}$ as an "initial segment" of the comparison of W, W'. Let $\bar{\mathcal{U}} = (E^{\bar{\mathcal{U}}}_\alpha : \alpha < \bar{\theta})$, $\bar{\mathcal{T}} = (E^{\bar{\mathcal{T}}}_\alpha : \alpha < \bar{\theta})$ denote the coiteration of $\mathcal{M}$ with $\mathcal{N}$, and let $\mathcal{U} = (E^{\mathcal{U}}_\alpha : \alpha < \theta)$, $\mathcal{T} = (E^{\mathcal{T}}_\alpha : \alpha < \theta)$ denote the coiteration of W with W'. Hence $E^{\bar{\mathcal{U}}}_\alpha = E^{\mathcal{U}}_\alpha$ and $E^{\bar{\mathcal{T}}}_\alpha = E^{\mathcal{T}}_\alpha$ for $\alpha < \bar{\theta}$, and $\mathcal{M}^* = \mathcal{M}^{\bar{\mathcal{U}}}_{\bar{\theta}} \unlhd \mathcal{N}^* = \mathcal{M}^{\bar{\mathcal{T}}}_{\bar{\theta}}$. We let $Q = \mathcal{M}^{\mathcal{U}}_\theta = \mathcal{M}^{\mathcal{T}}_\theta$ be the common coiterate of W, W'.

It is clear that there cannot be a drop along $[0, \bar{\theta}]_{\bar{\mathcal{U}}}$, because otherwise $\mathcal{M}^*$ were not sound, so that $\mathcal{M}^* = \mathcal{N}^*$. But $\mathcal{N}$ is sound (as $W' \rhd \mathcal{N}$), so there would have been a drop along $[0, \bar{\theta}]_{\bar{\mathcal{T}}}$, too. Contradiction! - In fact:

Claim 1. $\bar{\theta} \in [0, \theta]_{\mathcal{U}}$.

PROOF. Suppose not. Let $\alpha \geq \bar{\theta}$ be minimal such that $\alpha + 1 \in [0, \theta]_{\mathcal{U}}$. It is easily verified that $\mathcal{M}^*$ is an n-cutpoint in $\mathcal{M}^{\mathcal{U}}_\alpha$. ($\mathcal{M}^*$ is an n-cutpoint in $\mathcal{M}^{\mathcal{U}}_{\bar{\theta}}$, and the tail of the comparison is beyond $On \cap \mathcal{M}^*$.) As $\mu = c.p.(E^{\mathcal{U}}_\alpha) < On \cap \mathcal{M}^*$ and $lh(E^{\mathcal{U}}_\alpha) \geq On \cap \mathcal{M}^*$, this implies that $d^Q(\mu) < n$. But this contradicts 3.1.

$\square$ (Claim 1)

This proof moreover shows:

Claim 2. $\pi^{\mathcal{U}}_{\bar{\theta}\theta} \restriction \mathcal{M}^* = id$, and $\pi^{\mathcal{U}}_{0\theta} \restriction \mathcal{M} = \pi^{\bar{\mathcal{U}}}_{0\bar{\theta}}$.

We also claim that there cannot be a drop along $[0, \bar{\theta}]_{\bar{\mathcal{T}}}$, and that in fact:

Claim 3. $\bar{\theta} \in [0, \theta]_{\mathcal{T}}$.

PROOF. Suppose otherwise. Let $\alpha \geq \bar{\theta}$ be minimal such that $\alpha + 1 \in [0, \theta]_{\mathcal{T}}$. We then have that

$$\mu = c.p.(E_{\alpha}^{\mathcal{T}}) < On \cap \mathcal{M}^* \text{ and}$$
$$lh(E_{\alpha}^{\mathcal{T}}) \geq On \cap \mathcal{M}^*.$$

This implies that

$$\pi_{0\theta}^{\mathcal{T}}{}''W' \cap \mathcal{M}^* \subset \mu.$$

On the other hand, by Claim 2 we know that

$$\pi_{0\theta}^{\mathcal{U}} \restriction \mathcal{M} = \pi_{0\bar{\theta}}^{\bar{\mathcal{U}}},$$

and of course $\pi_{0\bar{\theta}}^{\bar{\mathcal{U}}}{}''(On \cap \mathcal{M})$ is unbounded in $On \cap \mathcal{M}^*$. However, as $\mathcal{M} \lhd K$, this contradicts

$$\pi_{0\theta}^{\mathcal{U}}{}''\mathcal{M} \subset \pi_{0\theta}^{\mathcal{T}}{}''W',$$

as given by [18] Theorem 5.5.

□ (Claim 3)

In turn this proof also established:

Claim 4. $(\pi_{0\theta}^{\mathcal{T}}{}''W') \cap \mathcal{M}^* = (\pi_{0\bar{\theta}}^{\bar{\mathcal{T}}}{}''\mathcal{N}) \cap \mathcal{M}^*$.

Moreover:

Claim 5. $\mathcal{M}^*$ is an n-cutpoint in $\mathcal{N}^*$.

PROOF. As $\mathcal{M}$ is an n-cutpoint in W, it is easily seen that $\mathcal{M}^*$ comes out as an n-cutpoint in Q. Let us assume that $\mathcal{M}^*$ is not an n-cutpoint in $\mathcal{N}^*$.

Then $\mathcal{M}^*$ is not an n-cutpoint in $\mathcal{M}_{\bar{\theta}}^{\mathcal{T}} \rhd \mathcal{N}^*$, too. Hence there must be some $\alpha \geq \bar{\theta}$ such that $\alpha + 1 \in [0, \theta]_{\mathcal{T}}$ and $c.p.(E_{\alpha}^{\mathcal{U}}) < On \cap \mathcal{M}^*$, as otherwise $\mathcal{M}^*$ were not an n-cutpoint in Q. This contradicts the proof of Claim 3.

□ (Claim 5)

But now if we denote

$$i = \pi_{0\bar{\theta}}^{\bar{\mathcal{U}}} : \mathcal{M} \to \mathcal{M}_{\bar{\theta}}^{\bar{\mathcal{U}}} = \mathcal{M}^* \text{ and}$$

$$j = \pi_{0\bar{\theta}}^{\bar{\mathcal{T}}} : \mathcal{N} \to \mathcal{M}_{\bar{\theta}}^{\bar{\mathcal{T}}} = \mathcal{N}^*,$$

by Claims 2 and 4 together with $\pi_{0\theta}^{\mathcal{U}}{}''\mathcal{M} \subset \pi_{0\theta}^{\mathcal{T}}{}''W'$ from [18] Theorem 5.6 we can deduce that

$$i''\mathcal{M} = (\pi_{0\theta}^{\mathcal{U}}{}''\mathcal{M}) \cap \mathcal{M}^* \subset (\pi_{0\theta}^{\mathcal{T}}{}''W') \cap \mathcal{M}^* = (j''\mathcal{N}) \cap \mathcal{M}^*.$$

This means that $i''\mathcal{M} \subset j''\mathcal{N}$, as desired.

" $\Leftarrow$ ". Now let $\bar{\mathcal{M}}$ and $\mathcal{M}$ be given with the properties as stated in the theorem. We know by 3.2 that there is a strongly n-full $\mathcal{N} \lhd \mathcal{J}_{\omega_1}^K$ winning the coiteration against $\mathcal{M}$, i.e. such that $\mathcal{M}^* \trianglelefteq \mathcal{N}^*$ where $\mathcal{M}^*$, $\mathcal{N}^*$ denote the comparable coiterates of $\mathcal{M}^*$, $\mathcal{N}^*$, respectively.

We hence know by the assumptions on $\mathcal{M}$ that $\mathcal{M}^*$ is an n-cutpoint in $\mathcal{N}^*$, there is no drop along the main branches of the coiteration of $\mathcal{M}$ and $\mathcal{N}$, and if

$$i\colon \mathcal{M} \to \mathcal{M}^*, \text{ and}$$

$$j\colon \mathcal{N} \to \mathcal{N}^*$$

denote the maps obtained from the comparison then

$$i''\mathcal{M} \subset j''\mathcal{N}.$$

Claim. $(j''\mathcal{N}) \cap \mathcal{M}^* \subset i''\mathcal{M}$.

Proof. This is shown using arguments as in " $\Rightarrow$ ". Let $\bar{\mathcal{U}} = (E_\alpha^{\bar{\mathcal{U}}}\colon \alpha < \bar{\theta})$, $\bar{\mathcal{T}} = (E_\alpha^{\bar{\mathcal{T}}}\colon \alpha < \bar{\theta})$ denote the coiteration of $\mathcal{M}$ with $\mathcal{N}$. Let W witness that $\mathcal{M}$ is n-full, let W' witness that $\mathcal{N}$ is n-full, and let $\mathcal{U} = (E_\alpha^{\mathcal{U}}\colon \alpha < \theta)$, $\mathcal{T} = (E_\alpha^{\mathcal{T}}\colon \alpha < \theta)$ denote the coiteration of W with W', which is an "extension" of $\bar{\mathcal{U}}$, $\bar{\mathcal{T}}$.

We have that

$$\bar{\theta} \in [0, \theta]_{\mathcal{U}} \text{ and } \pi_{0\theta}^{\mathcal{U}} \restriction \mathcal{M} = i$$

as in the proofs of Claims 1 and 2 of " $\Rightarrow$ ". Moreover, we know that $\mathcal{M}^*$ is an n-cutpoint in $\mathcal{N}^*$ and $\mathcal{N}^*$ is an n-cutpoint in $\mathcal{M}_{\bar{\theta}}^{\mathcal{T}}$, which implies that $\mathcal{M}^*$ is really an n-cutpoint in $\mathcal{M}_{\bar{\theta}}^{\mathcal{U}}$. But then again as in the proofs of Claims 1 and 2 we may use 3.1 to deduce that

$$\bar{\theta} \in [0, \theta]_{\mathcal{T}} \text{ and}$$

$$(\pi_{0\theta}^{\mathcal{T}}{}''W') \cap \mathcal{M}^* = (j''\mathcal{N}) \cap \mathcal{M}^*.$$

But now we get again from [18] Theorem 5.6 that

$$(j''\mathcal{N}) \cap \mathcal{M}^* = (\pi_{0\theta}^{\mathcal{T}}{}''W') \cap \mathcal{M}^* \subset (\pi_{0\theta}^{\mathcal{U}}{}''W) \cap \mathcal{M}^* = i''\mathcal{M},$$

as desired.

$\square$ (Claim)

On the other hand we have $i''\mathcal{M} \subset j''\mathcal{N}$ by the assumed properties of $\mathcal{M}$, and thus

$$i''\mathcal{M} = j''\mathcal{N} \cap \mathcal{M}^*,$$

where $\mathcal{M}^* \trianglelefteq \mathcal{N}^*$. But this clearly implies that $\bar{\mathcal{M}} \triangleleft \mathcal{N}$, and so $\bar{\mathcal{M}} \triangleleft K$ by $\mathcal{N} \triangleleft K$.

$\square$ (3.3)

We remark that in fact we did not use that ω_1 is inaccessible in K in the "$\Rightarrow$"-direction of the above proof.

We now turn to the case that ω_1^V is a successor in K.

Theorem 3.5 *Suppose that $\omega_1 = \omega_1^V$ is the successor of the cardinal δ in K. Let $m < \omega$, and assume that $\mathcal{J}^K_{\omega_1} \models$ "there are exactly m strong cardinals." Let $\delta < \gamma < \omega_1$ be such that $\mathcal{N} = \mathcal{J}^K_\gamma \prec \mathcal{J}^K_{\omega_1}$. Let $m \leq n < \omega$.*

Then for all countable premice $\mathcal{M}$ we have that

$$\mathcal{M} \triangleleft \mathcal{J}^K_{\omega_1} \text{ iff}$$

there is an n-collapsing mouse $\tilde{\mathcal{M}}$ for $\mathcal{N}$ with $\mathcal{M} \trianglelefteq \tilde{\mathcal{M}}$.

Using 2.18 (c) this readily gives:

Theorem 3.6 *If $\omega_1 = \omega_1^V$ is a successor cardinal in K, and $\mathcal{J}^K_{\omega_1} \models$ "there are at most n strong cardinals" (where $n < \omega$), then $\mathcal{J}^K_{\omega_1}$ is $\Delta^1_{n+4}(x)$ in the codes where x is a real coding an initial segment of $\mathcal{J}^K_{\omega_1}$.*

PROOF of 3.5. We first show that for any $\mathcal{M} \triangleright \mathcal{J}^K_{\omega_1}$ there is an n-collapsing mouse $\tilde{\mathcal{M}}$ with $\mathcal{M} \trianglerighteq \tilde{\mathcal{M}}$.

Let $\alpha = \mathcal{M} \cap On$, and let

$$\tilde{\mathcal{M}} = \mathcal{H}_1^{\mathcal{J}^K_{\omega_1}}(\delta \cup \{\alpha\}) \prec_{\Sigma_1} \mathcal{J}^K_{\omega_1}.$$

Then $\tilde{\mathcal{M}} = \mathcal{J}^K_\beta$ for some β such that $\alpha < \beta < \omega_1$. Moreover, $\rho_1(\tilde{\mathcal{M}}) = \delta < \alpha$, and any very soundness witness for $\mathcal{J}^K_{\omega_1}$ witnesses that $\tilde{\mathcal{M}}$ is n-full, too. Hence $\tilde{\mathcal{M}}$ is an n-collapsing mouse.

On the other hand, from the proof of 2.12 it follows that every n-collapsing mouse for $\mathcal{J}^K_\gamma$ is an initial segment of K.

$\square$ (3.5)

It can actually be shown that under the hypothesis of 3.6 $\mathcal{J}^K_{\omega_1}$ is $\mathbf{\Delta}^1_{n+3}$.

We now turn to the uniform version of the above computations, that is the projective computation of K up to its nth strong cardinal below true ω_1 without placing any bound on the total number of strong cardinals in $K \cap HC$. As in [4] we have to make additional assumptions about the shape of the universe we are working in. We shall assume that our universe is obtained as the Levy collapse of an inaccessible cardinal κ of the ground model to ω_1. (This is actually weakens the background assumption in the statement of Lemma 3.1 in [4].) We also assume K exists.

The first lemma yields a large stock of strongly full mice.

Lemma 3.7 *Let δ be a successor cardinal of K whose cardinal predecessor has cofinality ω in K. Then, for any $n \geq 0$, $\mathcal{J}^K_\delta$ is stronly n-full.*

Proof. Fix $n \geq 0$ and a very soundness witness W for $\mathcal{J}^K_\delta$ in which δ remains a cardinal. If $\mathcal{J}^K_\delta$ fails to be an n-cutpoint in W let E denote the extender of minimal length $> \delta$ from the sequence of W such that $c.p.(E) < \delta$ and $d^W(c.p.(E)) \geq n$. Set $R = Ult(W, E)$. We claim that R witnesses that $\mathcal{J}^K_\delta$ is strongly n-full. Since $\mathcal{J}^K_\delta$ is an n-cutpoint of R by the minimality of E, we are left to show the following

Claim. Suppose $(E^{\mathcal{T}}_\xi : \xi < \theta)$ defines an iteration tree $\mathcal{T}$ of length $\theta + 1$ on W such that $D^{\mathcal{T}} \cap [0, \theta]_{\mathcal{T}} = \emptyset$. Set $\rho^*_\xi = sup\{\rho(E^{\mathcal{T}}_\eta) : \eta < \xi\}$. Then $\mathcal{M}^{\mathcal{T}}_\theta$ has the definability property at its cardinals κ with $\rho^*_\theta \leq \kappa < i^{\mathcal{T}}_{0,\theta}(\delta)$ and $d^{\mathcal{M}^{\mathcal{T}}_\delta}(\kappa) < n$.

Proof of Claim. Let $\kappa_E = c.p.(E)$. We need to distinguish two cases.
Case 1. There is a $\xi + 1 \in (0, \theta]_{\mathcal{T}}$ such that $c.p.(E^{\mathcal{T}}_\xi) \geq i^{\mathcal{T}}_{0(\xi+1)^*}(\kappa_E)$.
Fix the minimal such $\xi+1$. By strong acceptability, $\mathcal{M}^{\mathcal{T}}_{(\xi+1)^*} \models$ "$i^{\mathcal{T}}_{0(\xi+1)^*}(\kappa_E)$ is strong up to μ" for any of its cardinals $\mu \in (i^{\mathcal{T}}_{0(\xi+1)^*}(\kappa_E), i^{\mathcal{T}}_{0(\xi+1)^*}(\delta))$. Hence $d^{\mathcal{M}^{\mathcal{T}}_{(\xi+1)^*}}(\mu) > n$ for any of these μ. It follows that $d^{\mathcal{M}^{\mathcal{T}}_{\xi+1}}(\mu) \geq n$ for cardinals μ in $\mathcal{M}^{\mathcal{T}}_{\xi+1}$ belonging to $(c.p.(E^{\mathcal{T}}_\xi), i^{\mathcal{T}}_{0\xi+1}(\delta))$. [For $\mu > i^{\mathcal{T}}_{(\xi+1)^*(\xi+1)}(c.p.(E^{\mathcal{T}}_\xi))$ this is immediate from the elementarity of $i^{\mathcal{T}}_{(\xi+1)*(\xi+1)}$. For $\mu \leq i^{\mathcal{T}}_{(\xi+1)^*(\xi+1)}(c.p.(E^{\mathcal{T}}_\xi))$ argue as follows: By case assumption there are $\geq n$ cardinals in $\mathcal{M}^{\mathcal{T}}_{(\xi+1)^*}$ which are each strong up to $c.p.(E^{\mathcal{T}}_\xi)$ and $< i^{\mathcal{T}}_{0(\xi+1)^*}(\kappa_E)$. They stay fixed under $i^{\mathcal{T}}_{(\xi+1)^*(\xi+1)}$ and are strong up to $i^{\mathcal{T}}_{(\xi+1)^*(\xi+1)}(c.p.(E^{\mathcal{T}}_\xi))$ in $\mathcal{M}^{\mathcal{T}}_{\xi+1}$ by elementarity.] But $c.p.(i^{\mathcal{T}}_{\xi+1\theta}) > c.p.(E^{\mathcal{T}}_\xi)$ which implies $d^{\mathcal{M}^{\mathcal{T}}_\theta}(\mu) \geq n$ for all cardinals of the final model $\mathcal{M}^{\mathcal{T}}_\theta$ belonging to $[\rho^*_\theta, i^{\mathcal{T}}_\theta(\delta))$.

Case 2. Otherwise.
By acceptability in $\mathcal{M}_\theta^\mathcal{T}$, $i_{0\theta}^\mathcal{T}(\kappa_E)$ is strong up to any cardinal $\mu < i_{0\theta}^\mathcal{T}(\delta)$. Since $d^{\mathcal{M}^\mathcal{T}}(i_{0\theta}^\mathcal{T}(\kappa_E)) \geq n$ this means that any cardinal $\mu < i_{0\theta}^\mathcal{T}(\delta)$ of $\mathcal{M}_\theta^\mathcal{T}$ with $d^{\mathcal{M}_\theta^\mathcal{T}}(\mu) < n$ must be $< i_{0\theta}^\mathcal{T}(\kappa_E)$. Thus it suffices to show that $\mathcal{M}_\theta^\mathcal{T}$ has the definability property at all ordinals in $[\rho_\theta^*, i_{0\theta}^\mathcal{T}(\kappa_E))$. This is proved by a routine induction on θ using the definability property of R below κ_E in the base step and the present case assumption in the successor step.

[If θ is a successor ordinal any $\mu < i_{\theta^*\theta}^\mathcal{T}(\kappa_E)$ can be written as $i_{\theta^*\theta}^\mathcal{T}(f)(a)$ with some function $f : [c.p.(E_{\theta-1}^\mathcal{T})]^{|a|} \rightarrow i_{0\theta*}^\mathcal{T}(\kappa_E)$ in $\mathcal{M}_{\theta^*}^\mathcal{T}$ and $a \in [\rho(E_{\theta-1}^\mathcal{T})]^{<\omega}]$. Since $c.p.(E_{\theta-1}^\mathcal{T}) < i_{0\theta^*}^\mathcal{T}(\kappa_E)$, $f \in \mathcal{J}_{i_{0\theta^*}^\mathcal{T}(\kappa_E)}^{\mathcal{M}_{\theta^*}^\mathcal{T}}$. Now $\mathcal{M}_{\theta^*}^\mathcal{T}$ has the definability property at all ordinals in $[\rho_{\theta*}^*, i_{0\theta*}^\mathcal{T}(\kappa_E))$ by inductive hypothesis. Using a thick class of fixed points of $i_{\theta^*\theta}^\mathcal{T}$ and $c.p.(E_{\theta-1}^\mathcal{T}) \geq \rho_{\theta^*}^*$ one sees that $\mathcal{M}_\theta^\mathcal{T}$ has the definability property at $\mu \geq \rho_\theta^*$.] $\square$ (claim and 3.7)

The following definition introduces some convenient notation.

Definition 3.8 *($n \geq 0$) A strongly n-full mouse $\mathcal{M}$ is n-special iff it has at most n strong cardinals and for any countable strongly n-full mouse $\mathcal{N}$ strictly outiterating $\mathcal{M}$ the following hold*

> *There are no drops along the main branches on either side of the comparison, and if $i : \mathcal{M} \rightarrow \mathcal{M}^*$ and $j : \mathcal{N} \rightarrow \mathcal{N}^*$ denote the maps along the main branches into the final models*
>
> *$\mathcal{M}^*$ is an n-cutpoint in $\mathcal{N}^*$ and $\mathcal{M}^* \cap On$ is a cardinal in $\mathcal{N}^*$*

and

> $i''\mathcal{M} \subset j''\mathcal{N}$.

For the record we state that n-speciality is a projective condition.

Lemma 3.9 *The property of being an n-special countable mouse is Π_{n+4}^1 in the codes.*

Proof. Follows from corollary 2.18 and the uniqueness of the comparison process of strongly full mice. $\square$ (3.9)

In terms of n-special mice one can characterize K up to its $n+1$st strong cardinal below true ω_1 uniformly (i.e., without imposing a bound on the total number of strong cardinals up to ω_1) in a projective way.

Theorem 3.10 *Suppose that the universe is of the form $V^{Coll(\omega,<\kappa)}$ for an inaccessible cardinal κ in the ground model. Let $\kappa_1 < \ldots < \kappa_{n+1}$ enumerate the first $n+1$ strong cardinals of K up to κ. Then the following are equivalent for a countable mouse $\bar{\mathcal{M}}$*

(1) $\bar{\mathcal{M}} \lhd \mathcal{J}^K_{\kappa_{n+1}}$

(2) $\exists \mathcal{M} \, (\mathcal{M} \rhd \bar{\mathcal{M}} \wedge \mathcal{M}$ *is n-special).*

Proof. (1) $\Rightarrow$ (2). In K, define a sequence of ordinals $\langle \gamma_n : n < \omega \rangle$ as follows: $\gamma_0 = min\{ht(\bar{\mathcal{M}}), \kappa_n + 1\}$ (where $\kappa_0 = 0$ in case $n = 0$). Let $\gamma_{n+1} =$ the least $\alpha < \kappa_{n+1}$ such that no ordinal $\in (\kappa_n, \gamma_n]$ is strong up to α. Let $\gamma_\omega = sup\{\gamma_n : n < \omega\}$ and set $\mathcal{M} = \mathcal{J}^K_{(\gamma_\omega^+)^K}$.

Fix a soundness witness for $\mathcal{J}^K_{\kappa_{n+1}}$, W witnessing also that $\mathcal{M}$ is strongly n-full. This is possible since $ht(\mathcal{M}) < \kappa_{n+1}$ (see 2.6). Any strongly n-full mouse $\mathcal{N}$ challenging the n-speciality of $\mathcal{M}$ can now be defeated with exactly the same argument as in the " $\Rightarrow$ " direction of the proof of theorem 3.3, i.e.: Pick a weasel W' witnessing that $\mathcal{N}$ is strongly n-full and compare W with W' via iteration trees $\mathcal{U}$ (on W) and $\mathcal{T}$ on W' ending in a common model $Q = \mathcal{M}^{\mathcal{U}}_\theta = \mathcal{M}^{\mathcal{T}}_\theta$. The comparison of the weasels "end-extends" the comparison of $\mathcal{M}$ with $\mathcal{N}$ in the sense that the main branches of the former extend the main branches of the latter. Moreover, all extenders used on either side of the "big" comparison along the main branches after $\mathcal{M}^*$ and $\mathcal{N}^*$ are lined up have critical points $\geq ht(\mathcal{M}^*)$. [Here we use $\mathcal{M}$ is an n-cutpoint of W together with 3.1 and the fact that $Def(Q)$ is cofinal in $i^{\mathcal{U}}$"$\mathcal{M}$ since $\mathcal{M} \subset Def(W)$.] It follows that neither side drops in the "small comparison", $\mathcal{M}^*$ is an n-cutpoint of $\mathcal{N}^*$ and $\mathcal{M}^* \cap Ord$ is a cardinal of $\mathcal{N}^*$. The final maps in the comparison of $\mathcal{M}$ with $\mathcal{N}$ behave as desired because $\mathcal{M} \subset Def(W)$. (We have not used any assumption about the shape of the universe we are in so far.)

(2) $\Rightarrow$ (1). By the inaccessibility of κ in the ground model we can find a successor cardinal of K, $\delta \in ((\kappa^+_{n+1})^K, \kappa)$ such that the cardinal predecessor of δ has cofinality ω in K and $\mathcal{J}^K_\delta$ strictly outiterates $\mathcal{M}$. (Reason: For any cardinal $\mu \geq \aleph_2$, $\mathcal{J}^K_\mu$ is universal for all mice of height $\leq \mu$ [16].) Set $\mathcal{N} = \mathcal{J}^K_\delta$. $\mathcal{N}$ is strongly n-full by lemma 3.7. Let W' be a soundness witness for $\mathcal{J}^K_\delta$ in which δ is still a cardinal and fix a weasel W witnessing that $\mathcal{M}$ is strongly n-full. As before denote by $\mathcal{U}$ and $\mathcal{T}$ the comparison trees on the weasels W and W' (resp.) ending in a common model $Q = \mathcal{M}^{\mathcal{U}}_\theta = \mathcal{M}^{\mathcal{T}}_\theta$. The comparison of the weasels "contains" the comparsion of $\mathcal{M}$ with $\mathcal{N}$ in the sense that the trees of the latter ($\bar{\mathcal{U}}$ and $\bar{\mathcal{T}}$) use the same extenders as the former and all of their models are initial models of the models appearing in the corresponding place in the "big comparison". Denote by $\bar{\theta}$ the length of the "small

comparison" and by $\mathcal{M}^*$ and $\mathcal{N}^*$ the final models of $\bar{\mathcal{U}}$ and $\bar{\mathcal{T}}$.

Claim. (1) $\bar{\theta} \in [0,\theta]_{\mathcal{U}}$ and $c.p.(i^{\mathcal{U}}_{\bar{\theta}\theta}) \geq ht(\mathcal{M}^*)$.
(2) $\bar{\theta} \in [\bar{\theta},\theta]_{\mathcal{T}}$ and $c.p.(i^{\mathcal{T}}_{\bar{\theta}\theta}) \geq ht(\mathcal{M}^*)$.

Proof of claim. An inspection of the proof of lemma 3.1. shows that the non-overlapping condition in part (a) of the definition of n-fullness (cf. 2.4) was never used. Thus with exactly the same argument as in the proof of claim 1 in 3.3, using that $\mathcal{M}$ is an n-cutpoint of W, we obtain (1).

If either part of (2) fails let $\xi+1$ be least in $[0,\theta]_{\mathcal{U}} \setminus \bar{\theta}+1$. Set $E = E^{\mathcal{T}}_{\xi}$ and $\kappa = crit(E)$. So $lh(E) > ht(\mathcal{M})^*$ while $\kappa < ht(\mathcal{M}^*)$. Because $\mathcal{M}$ is n-special and strictly outiterated by the strongly n-full mouse $\mathcal{N}$, there is no dropping along $[0,\bar{\theta}]_{\mathcal{T}}$ and $ht(\mathcal{M}^*)$ is a cardinal in $\mathcal{N}^*$ and therefore in $\mathcal{M}^{\mathcal{T}}_{\bar{\theta}}$. Note that κ^+ exists in $\mathcal{M}^*$ since the largest cardinal is of cofinality ω there. Therefore (by acceptability) the fragments of E witness that κ is a strong cardinal in $\mathcal{M}^*$. But $\mathcal{M}^*$ has at most n strong cardinals. This means that $d^{\mathcal{M}^*}(\kappa) < n$ - contradicting 3.1. (modulo the remark above). $\square$ (claim)

Denote by $i : \mathcal{M} \to \mathcal{M}^*$ and $j : \mathcal{N} \to \mathcal{N}^*$ the embeddings along the main branches in $\bar{\mathcal{U}}$ and $\bar{\mathcal{T}}$ respectively. Using $\mathcal{N} \subset Def(W')$, the claim yields as before $j''\mathcal{N} \cap \mathcal{M}^* \subset i''\mathcal{M}$. From the n-specialty of $\mathcal{M}$ we obtain $i''\mathcal{M} \subset j''\mathcal{N}$, i.e.,

$$i = j \restriction \mathcal{M}.$$

This implies $\bar{\mathcal{M}} \lhd \mathcal{N}$ (because it was a *strict* initial segment of $\mathcal{M}$), and therefore $\bar{\mathcal{M}} \lhd K$.

If $ht(\bar{\mathcal{M}}) \geq \kappa_{n+1}$ $\mathcal{M}^*$ has $n+1$ cardinals (namely $i(\kappa_0) = j(\kappa_0), \ldots i(\kappa_{n+1}) = j(\kappa_{n+1})$) which are each strong in $\mathcal{N}^*$. As $ht(\mathcal{M}^*)$ is a cardinal in $\mathcal{N}^*$ and $\mathcal{M}^* \models$ "There is a largest cardinal of cofinality ω", they are strong in $\mathcal{M}^*$ (by acceptability) -contradicting that $\mathcal{M}^*$ has at most n strongs. Thus $\bar{\mathcal{M}} \lhd \mathcal{J}^K_{\kappa_{n+1}}$. $\square$ (3.10)

An immediate corollary of the previous theorem and 3.9 is

Theorem 3.11 *For each integer $n \geq 1$ there is a Δ^1_{n+5} formula $\Phi^n(v_0)$ such that whenever the universe is of the form $V^{Coll(\omega,<\kappa)}$ with an inaccessible cardinal κ of the ground model and K exists and has at least $n+1$ cardinals strong up to κ, for all reals x,*

$$\Phi^n(x) \text{ iff } x \text{ codes an initial segment of } \mathcal{J}^K_{\kappa_{n+1}}$$

where κ_{n+1} denotes the $n+1$st ordinal strong in K up to κ. $\square$ (3.11)

The question whether there are at least $n+1$ countable ordinals which are strong in K up to true ω_1 is itself projective whenever the universe has the special form as in 3.10.

Theorem 3.12 *For each integer $n \geq 0$ there is a Σ^1_{n+6} sentence ψ^{n+1} such that in any universe of the form $V^{Coll(\omega,<\kappa)}$ with κ an inaccessible cardinal of the ground model where K exists,*

ψ^{n+1} *iff there are at least $n+1$ countable ordinals strong up to κ in K.*

Proof. By the similiar arguments as in the proof of 3.10 one shows that whenever the universe is of the prescribed form and K exists,

K has at most n strong cardinals up to true ω_1 iff
$\forall \alpha < \omega_1^V \, \exists \mathcal{M} \in HC \, \mathcal{M}$ is n-special and of height $> \alpha$.

Suitable initial segments of K which occur cofinal in true ω_1 provide the witnesses for the "if" direction as in the forward direction of 3.10. Conversely, if κ_{n+1} denotes the $n+1$st countable ordinal which is strong in K up to true ω_1 there cannot be any n-special countable mice of height $> \kappa_{n+1}$. [For any such $\mathcal{M}$ pick a strongly n-full countable initial segment of K strictly outiterating it (applying the aforementioned theorem from [16]) and derive a contradiction with the corresponding arguments from the proof of 3.10.] $\square$ (3.12)

A modification of its proof allows us to weaken the assumptions of the previous theorem at the cost of a slight increase in complexity.

Theorem 3.13 *For each integer $n \geq 0$ there is a Π^1_{n+7} sentence Ψ^n such that (provably) in the theory ZFC + K exists + ω_1^V is inaccessible in K,*

$\Psi^n \leftrightarrow$ *There are at most n countable ordinals strong in K up to true ω_1.*

Proof. Let Ψ^n express the following:

All countable $n+1$-special mice are lined up, and the union over all of them is a mouse of height ω_1 with at most n strong cardinals.

First assume that there are at most n countable ordinals which are strong in K up to true ω_1. If ω_1^V is inaccessible in K all countable $n+1$-special mice are initial segments of K and conversely such mice appear cofinally in $\mathcal{J}^K_{\omega_1^V}$ by 3.3. Therefore the union over all of them is a mouse of height ω_1 with at most n strong cardinals.

Conversely, assume toward a contradiction that Ψ^n holds yet K has at least $n+1$ cardinals strong up to ω_1^V. Let κ_{n+1} denote the $n+1$st such cardinal. Fix a successor cardinal of K, α in $(\kappa_{n+1}, \omega_1^V)$ such $\mathcal{J}_\alpha^K$ is an $n+1$-cutpoint of K with a largest cardinal of cofinality ω having at most $n+1$ strong cardinals (cf. 2.6). By the proof of 3.3 $\mathcal{J}_\alpha^K$ is $n+1$-special and therefore $\mathcal{J}_{\kappa_{n+1}}^K \trianglelefteq \mathcal{M}$ for any countable $n+1$-special mouse $\mathcal{M}$ of height $\geq \kappa_{n+1}$. Since the line-up of all such mice contains at most n strong cardinals by assumption there exists a countable $n+1$-special mouse $\mathcal{M}$ such that $\mathcal{M} \models$ "κ_{n+1}^+ exists and κ_{n+1} is not strong". Fix a universal weasel W witnessing that $\mathcal{M}$ is strongly $n+1$-full. Note that κ_{n+1} is not strong up to ω_1^V in W (by acceptability).

Let W' denote a very soundness witness for $\mathcal{J}_{\omega_1^V}^K$ which is obtained as the transitive collapse of the hull of a thick class taken in W and $\pi : W' \to W$ the inverse of that transitive collapse.

Claim $\pi \upharpoonright (\kappa_{n+1} + 1) = id$.

Proof. Compare W with W'. The comparison ends in a common model Q with embeddings $i : W \to Q$ and $j : W' \to Q$. Pick a thick (in W, W' and Q) class, Γ of fixed points for i, j and π. Now for any $x \in \mathcal{J}_{\omega_1^V}^K$, there is a term $t(\vec{v})$ and $\vec{\eta} \in \Gamma$ such that $x = t^{W'}[\vec{\eta}]$. Thus

$$\begin{aligned} i(x) &= t^Q[\vec{\eta}] \\ &= j(t^W[\vec{\eta}]) \\ &= j(\pi(t^{W'}[\vec{\eta}])) \\ &= j \circ \pi(x). \end{aligned}$$

This means

$$i''\omega_1 \subset range(j)$$

and

$$j^{-1} \circ i \upharpoonright \omega_1 = \pi \upharpoonright \omega_1.$$

From this the claim follows since both i and j are the identity below $\kappa_{n+1} + 1$ by lemma 1.4. □ (claim)

Together with the claim, the elementarity of π implies that κ_{n+1} is strong in W up to $\pi(\omega_1^V)$. Since $(\kappa_{n+1}^+)^W < \omega_1^V$, κ_{n+1} is also strong up to ω_1^V in W (by acceptability) - contradiction. □ (3.13)

There are two obvious ways in which one may try to generalize the results in this section. By initializing at the appropriate mouse condition, our computations routinely transfer to core models of higher order below PD. The second author has in the meantime generalized some of the above calculations beyond the projective hierarchy (cf. [17]).

We conclude this section by describing how the output of our projective calculations of $K \cap HC$ can be translated into the context of Steel's organization of K. Toward this end we denote by K_J and K_S the core model in Jensen's [7] and Steel's [18] stratification (resp.), and begin with a technical observation.

Fact 1. *Any (pre)mouse $\mathcal{M} \models$ ZFC - Pow in the sense of Jensen [7] can be re-organized as a (pre)mouse in the sense of Mitchell-Steel [13] having the same universe as $\mathcal{M}$ and vice versa.*

Proof. (sketch) The proof of the claim proceeds by induction along the active stages with the less complicated direction of going from a Steel indexing to a Jensen indexing being handled first. In this direction a given Mitchell-Steel extender gets re-indexed to appear on the Jensen sequence at the cardinal successor of the image of its critical point computed in the ultrapower. The Jensen sequence up to the new index is obtained by stretching the re-indexed sequence up to the cardinal successor of the critical point via the ultrapower of the extender. This will not conflict with stretches of the Jensen sequence obtained from earlier extenders on the Steel sequence. Moreover, the new model is fully sound.

In the other direction, one may potentially loose sections of the Jensen sequence when cutting down a Jensen extender to a Mitchell-Steel extender. However, this information is retrievable in a "sufficiently simple" way by taking an ultrapower via the shortened extender and applying the claim in the easier direction. This guarantees that the hulls of the new structure pick up everything, i.e., the new model is fully sound.

Further details are left to the reader. $\square$ (fact 1)

This allows us to conclude

Fact 2. *K_J and K_S have the same universe.*

Proof. Note that $K_S^{K_J}$ has the same universe as K_J. [This is seen as follows: By the previous fact, $\mathcal{N}$ has the same universe as K_J where $\mathcal{N}$ is the result of re-organizing K_J as a Mitchell-Steel model. Because of the Π^1_2 mouse condition $\mathcal{N}$ satisfies $\star$-iterability (cf. [2, section 2]). This implies $K_S^{\mathcal{N}} = \mathcal{N}$ (cf. [2, section 2]). Q.e.d.] By [18, 3.7(2)] $K_S^{K_J}$ is a universal Mitchell-Steel weasel, and therefore [18, 8.10] gives an elementary embedding $\sigma : K_S \to K_S^{K_J}$. Our initial remark implies that σ is also a fully elementary embedding between K^S and K^J with respect to the $\in$ language of ZFC.

Similiarly we obtain a fully elementary (in the $\in$ language of ZFC) embedding $\tau : K_J \to K_S$. But K_S can define its own extender sequence in the $\in$ language of

ZFC (this was originally proved by Steel (unpublished), cf. [2, section 2]), and thus the composition $\tau \circ \sigma$ is a fully elementary embedding of K_S into itself. This must be the identity by [18, 8.8] which proves the claim. $\square$ (fact 2)

The projective translations of the above computations of $K_J \cap HC$ into the setting of [18] can now be routinely obtained from fact 2 using local (i.e., inductive) definability of the core model together with strong acceptability. (Alternatively, one could apply the method of the proof of fact 1 above.)

4 Projective Uniformization Revisited

Recall the notion of Σ^1_k absoluteness [19]: A model N of set theory is Σ^1_k absolute iff for any N-generic G and any $N[G]$-generic H, $N[G] \prec_{\Sigma^1_k} N[G][H]$. Woodin [19] deduces from $PM + PB + PU$ for any integer k the existence of a transitive model of ZFC which is Σ^1_k absolute and Σ^1_k correct in V. Such a model is of the form $L_{\omega_1}[F]$ for some (partial) function $F : HC \to HC$. Roughly speaking, F is a Skolem function assigning to any forcing notion $\mathbb{P} \in HC$ and any $\mathbb{P}$-name τ for a real a canonical name for a real witnessing $\exists x \varphi_k(x, \tau)$ (provided it exists) where φ_k defines a universal Π^1_k set in $\mathbb{R} \times \mathbb{R}$. Since Π^1_k can be projectively uniformized and the forcing relation for posets in HC is projective in the codes, the function F can be chosen to be projective in the codes. By the regularity of ω_1, $L_{\omega_1}[F]$ satisfies all axioms of ZFC with the exception of the power set axiom which must hold because PM rules out an uncountable projective sequence of reals by a theorem of Shelah (cf. [15]). By construction, $L_{\omega_1}[F]$ is Σ^1_{k+1} correct in V. Similiarly, it satisfies Σ^1_{k+1} absoluteness because, under PB + PU, the projective theory of V is unchanged by Cohen forcing.

Until further notice fix an integer k and let N_k be a Σ^1_k absolute and correct model of the form just described. Suppose now that $k' \geq n + 10$ where the function F defining N_k is Π^1_n in some real parameter x. Let $N_{k',x}$ denote a $\Sigma^1_{k'}$ absolute and correct model of the above form (relativized to the real x). Working inside, $N_{k',x}$, the model N_k is definable by iterating the above construction of $L_{\omega_1}[F]$ into the transfinite. To define the class N_k in the interval (κ, κ^+), one has to work in $N_{k',x}^{Coll(\omega,\kappa)}$ and appeal to homogeneity and $\Sigma^1_{k'}$ absoluteness. Proceeding this way, N_k becomes an inner model of $N_{k',x}$ of the form $L[F']$ which is projective in the codes.

Lemma 4.1 *Any successor cardinal of $N_{k',x}$ is strongly inaccessible in N_k.*

Proof. First note that (in the notation of the previous paragraph) for any $\alpha < \omega_1$, $P(\alpha) \cap N_k \subset L_{(\alpha^+)^{N_{k',x}}}[F']$. To see this, consider a potential counterexample A. The

Σ^1_{n+5} statement expressing $A \in N_k$ becomes true in $N_{k',x}^{Coll(\omega,\gamma)}$ for large enough γ. But then it must already be true in $N^{Coll(\omega,\alpha)}$ by $\Sigma^1_{k'}$ absoluteness.

From this the lemma follows since otherwise one would obtain an uncountable Σ^1_{n+7} sequence of reals in $N_{k',x}^{Coll(\omega,\alpha)}$ contradicting that in this model every $\Sigma^1_{k'}$ set of reals is Lebesgue measurable by $\Sigma^1_{k'}$ absoluteness. □ (4.1))

We are now in a position where the machinery of the previous chapter can be applied to obtain a lower bound on the consistency strength of ZFC + PB + PM + PU.

Theorem 4.2 *(ZFC + $I\!R^\sharp$ exists.) If PB + PM + PU hold there is a transitive model M of ZFC containing all ordinals with infinitely many cardinals which are strong up to the cardinal successor of their supremum in M.*

Proof. PB + PM + PU are still true in $L(I\!R)$ which is also a model of DC (the axiom of dependent choices). Let G be an $L(I\!R)$ generic enumeration of the reals in ordertype ω_1 obtained from countable conditions. $L(I\!R)[G]$ has the same reals as $L(I\!R)$ (by DC) and is therefore a model of ZFC + PB + PM + PU. The lift of an $L(I\!R)$ measure on the first $I\!R$-indiscernible Ω coming from $I\!R^\sharp$ is weakly amenable to $L(I\!R)[G]$. This allows to build K up to Ω in $L(I\!R)[G]$ where the rest of the argument takes place.

Assume toward a contradiction that K has only n strongs up ω_1^V. Note that by PM and 3.6, ω_1^V is inaccessible in K. Thus the sentence Ψ^n from 3.13 holds.

Set $k = n + 10$ and work inside a fixed Σ^1_k absolute and correct model N_k as above. By the main theorem from [1], there are $n+1$ strong cardinals in K^{N_k}. (N_k is closed under the †-operation and without loss of generality does not contain an inner model with a Woodin which allows to build K^{N_k}.) Let μ be an inaccessible in N_k above the $n+1$st strong cardinal (cf. 4.1). From the Σ^1_k correctness and absoluteness of N_k it follows that $N_k^{Coll(\omega,<\mu)} \models \Psi^n$. This contradicts 3.13. □ (4.2)

It is also possible to give a proof of 4.2 using the methods from 3.11. The reason why we included $I\!R^\sharp$ in the assumptions of 4.2 was solely to be able to build K. While writing this paper we have learned from Jensen in personal communication that it is possible to construct the core model below an inaccessible limit of strong cardinals (and probably somewhat beyond this large cardinal assumption) in ZFC without any additional hypotheses. In fact, recent work of the second author done jointly with Steel confirms Jensen's remarks. Modulo these technicalities the previous theorem together with the theorem of Steel mentioned in the introduction determines the consistency strength of ZFC + PB + PM + PU in terms of large cardinals, namely as that of ZFC + "There are infinitely many cardinals each strong to the cardinal successor of their supremum." In addition, the previous theorem shows that PU and PS are of the same logical strength (in the presence of ZFC + PB + PM).

The proof of 4.2 does not require the full axiom of choice, in fact DC suffices.

We now drop choice altogether from the picture and work in the background theory ZF + PB + PM + PU. Our first observation is that in this framework the whole apparatus of the above described models N_k and their relativzaions can be obtained. The key point is that PU implies *projective DC* (the axiom of dependent choices restricted to projective binary relations). Projective DC allows to develop a theory of category and measure at the projective level and to derive the facts needed to implement the construction of the above described absolute and correct models.

Until further notice fix an integer $k > 3$ and a Σ^1_{k+10} absolute and correct model N of the above form. By the main theorem of [1], K^N has at least k strong cardinals. Let μ be an inaccessible cardinal in N greater than all of them (cf. 4.1), and g a real which is N generic for $Coll(\omega, < \mu)$. In $N[g]$ pick a real x, coding K up to its kth cardinal which is strong up to μ. Since this is a Σ^1_{k+6} property of x (cf. 3.11 and 3.12), it remains true in *any* Σ^1_{k+10} absolute and correct model of set theory of the above form of height ω_1 containing the real x. In this fashion we obtain a sequence of countable ordinals $\langle \kappa_n : n < \omega \rangle$ together with a sequence of mice $\langle \mathcal{M}_n : n < \omega \rangle$ where $ht(\mathcal{M}) = \kappa_n$. (No choice is being used here as the entire process is definable.) Absoluteness, correctness and theorem 3.11 imply that these mice end-extend one another. Set $\lambda = sup\{\kappa_n : n < \omega\}$ and let K^* be the premouse of height λ such that $\mathcal{J}^{K^*}_{\kappa_n} = \mathcal{M}_n$ for all $n < \omega$. Then κ_n is the $n + 1$st strong cardinal of K^*.

In order to obtain a ZF model we need to put at least L on top of K^*. However, this could introduce new sets which are no longer measured by the extenders witnessing the strength of the κ_n in K^*. The remainder of this section will solve this problem.

The following lemma tells us that in $L(I\!\!R)$, K exists and satisfies the inductive definition from [18, chapter 6] if we have $I\!\!R^\sharp$ and there is no inner class model with a Woodin cardinal in V.

Lemma 4.3 *Suppose A is a transitive set such that $A^\sharp$ exists and there is no inner model class model with a Woodin cardinal in $L(A)$. Then in $L(A)$, K exists and satisfies the inductive definition from [18, chapter 6].*

Proof. Let $G \subset Coll(\omega, A)$ be $L(A)$ generic. In $L(A)[G]$, which is a model of ZFC, K exists (cf. [1, chapter 2]) and satisfies the inductive definition. In particular K is OD in $L(A)[G]$ and therefore belongs to the ground model $L(A)$ by homogeneity.

Recall the inductive definition of α-strong from [18, chapter 6].

Claim. For any properly small set premouse or weasel $W \in L(A)$ and all cardinals

α of K,
$$L(A) \models \text{"}W \text{ is } \alpha\text{-strong" iff } L(A)[G] \models \text{"}W \text{ is } \alpha \text{ strong."}$$

PROOF OF CLAIM. This proceeds by induction on α.
" $\Leftarrow$ ". Let $\mathcal{M} \in L(A)$ be a properly small premouse or weasel which is $< \alpha$-strong in $L(A)$. So, in $L(A)[G]$, every pseudotree on $(\mathcal{M}, W, \alpha)$ is well behaved (by inductive hypothesis). Uniqueness of branches implies the same is true in $L(A)$.

" $\Rightarrow$ ". Let R be a soundness witness for $\mathcal{J}_\alpha^K$ which works in $L(A)[G]$ and is definable inside K, i.e., $R \in L(A)$. (Apply lemma 8.1 of [18] in $L(A)[G]$.) [Alternatively, one could take the common model at the end of the simultaneous coiteration of all possible evaluations of a term for a soundness witness in a large enough generic extension of $L(A)$ and then apply homogeneity to get such a witness in $L(A)$.] R is $< \alpha$ strong in $L(A)$ by inductive hypothesis. This means that we can compare R against the phalanx (R, W, α) in $L(A)$. The main branch in the pseudo tree on the phalanx must be above W and its associated embedding has critical point $\geq \alpha$ by construction. Moreover, all extenders on the R side have lengths $> \alpha$. This means that W is α strong in $L(A)[G]$ (cf. [18, chapter 6]. $\square$ (claim)

Suppose now that $\mathcal{M}$ is a set premouse which fails to be α-strong in $L(A)$. Fix any $< \alpha$ strong premouse $\mathcal{N}$ and an ill behaved pseudo tree $\mathcal{T}$ on $(\mathcal{N}, \mathcal{M}, \alpha)$. Working in $L(\mathcal{T})$ (which models ZFC), one can reflect to obtain a counterexample to the α-strength of $\mathcal{M}$ of the same cardinality as $\mathcal{M}$ in $L(A)$. $\square$ (4.3)

The rest of this section takes place in $L(\mathbb{R})$ under the assumption that in the surrounding universe $\mathbb{R}^\sharp$ exists and there is no inner class model with a Woodin cardinal. Thus we are still working in the background theory ZF + PB + PM + PU, and K exists and has the inductive definition from [18, chapter 6] by the previous lemma.

Our goal is to show that each of the κ_n defined above is a cardinal in K and that $K^* = \mathcal{J}_\lambda^K$. This will ultimately come down to the fact that the projective theory of V is the asymptotic limit of the projective theory of models of the form N_k.

From theorem 3.12 we obtain a Σ^1_{n+6} formula $\phi^n(v_0)$ such that in all models of "ZFC + K exists" which are of the form $V^{Coll(\omega,<\kappa)}$ for an inaccessible cardinal κ of the ground model

$$\phi^n(x) \leftrightarrow x \text{ codes } K \text{ up to its } n+1\text{st cardinal which is strong up to } \kappa.$$

Putting this together with [1, lemma 3.4], we obtain for each $n \geq 0$ a Π^1_{n+6} formula $\chi(v_0)$ such that for any countable premouse $\mathcal{M}$ (letting κ_n denote the n+1st cardinal of K strong up to ω_1)
$$\chi^n(\mathcal{M}) \text{ iff } \mathcal{M} \text{ is } \kappa_n\text{-strong}$$

in all models of the above form containing a real coding $\mathcal{M}$. In the following definition, all occurences of K^* and κ_n should be read as abbreviations for the corresponding expressions involving the formula ϕ^n.

Definition 4.4 *In the notation from above, for $n < \omega$ and α a cardinal in $\mathcal{J}^{K^*}_{\kappa_n}$, call a countable properly small premouse $\mathcal{M}$ α-strong* iff*

$\mathcal{J}^{\mathcal{M}}_\alpha = \mathcal{J}^{K^*}_\alpha$ *and* $\exists n < \omega\ \exists \bar{\mathcal{M}}, \mathcal{T}, \pi \in HC(\mathcal{J}^{\bar{\mathcal{M}}}_{\kappa_n} = \mathcal{J}^{K^*}_{\kappa_n} \wedge \chi^n(\bar{\mathcal{M}}) \wedge \mathcal{T}$ *is an iteration tree on $\bar{\mathcal{M}}$ all of whose extenders have lengths $> \alpha$, with last model $\bar{\mathcal{M}}^{\mathcal{T}}_\theta$ such that $\pi : \mathcal{M} \to \bar{\mathcal{M}}^{\mathcal{T}}_\theta$ with c.p.$(\pi) \geq \alpha$.)*

This definition is justified by the follwoing

Lemma 4.5 *In the current notation, for any premouse $\mathcal{M} \in HC$ and any $\alpha < \lambda$,*

$\mathcal{M}$ *is α-strong* iff* $N_k \models$ *"$\mathcal{M}$ is α-strong".*

whenever $k < \omega$ is chosen large enough and N_k is some Σ^1_k correct and absolute model of height ω_1 containing a real coding $\mathcal{M}$.

Proof. Fix an $n < \omega$ such that $\alpha < \kappa_n$ and any $k \geq n + 20$. Let N_k be any Σ^1_k absolute and correct model of height ω_1 as above containing a real coding $\mathcal{M}$. By correctness,

$$\mathcal{M} \text{ is } \alpha\text{-strong}^* \text{ iff } N_k \models \text{"}\mathcal{M} \text{ is } \alpha\text{-strong}^*\text{"}.$$

First suppose that $\mathcal{M}$ is α-strong*. Work in N_k and fix $\bar{\mathcal{M}}, \mathcal{T}$ and π witnessing this. There is a soundness witness for $\mathcal{J}^K_{\kappa_n}, W$ and a tree $\mathcal{U}$ on W such that $\bar{\mathcal{M}}$ can be embedded via some map σ with critical point $\geq \kappa_n$ into the last model of $\mathcal{U}$, $\mathcal{M}^{\mathcal{U}}_\gamma$ (cf. [18, chapter 6]). Moreover, all extenders used in $\mathcal{U}$ have lengths $> \alpha$. Copy the tree $\mathcal{T}$ via σ to a tree $\mathcal{T}^\sigma$ based on $\mathcal{M}^{\mathcal{U}}_\gamma$. Denote by σ^* the map at the end the copy process and note that $\sigma^* \restriction \alpha = id$ because all extenders used in $\mathcal{T}$ have lengths $\geq \alpha$ by assumption. Now $\mathcal{T}^\frown\mathcal{U}$ and $\sigma^* \circ \pi$ witness that $\mathcal{M}$ is α-strong.

Conversely, assume that $\mathcal{M}$ is α-strong in N_k. Working in that model, fix a soundness witness W for $\mathcal{J}^K_{\kappa_n}$. Compare W against the phalanx $(W, \mathcal{M}, \alpha)$ via trees $\mathcal{T}$ on W and $\mathcal{U}$ on $(W, \mathcal{M}, \alpha)$. The main branch in $\mathcal{U}$ is above $\mathcal{M}$ and yields an embedding of $\mathcal{M}$ into the final model of $\mathcal{T}$ with critical point $\geq \alpha$. By collapsing a suitable countable hull taken inside a sufficiently big rank, one obtains from W and $\mathcal{T}$ countable objects witnessing that $\mathcal{M}$ is α-strong*. $\square$ (4.5)

We are finally in a position to show that $\mathcal{J}^K_\lambda = K^*$. The only way this can fail is if there exist $n < \omega$ and $\alpha < \kappa_n$ such that:

(1) $\neg(\alpha$ is a cardinal in $K \Leftrightarrow \alpha$ is a cardinal in $K^*)$

or

(2) $\mathcal{J}_\alpha^K \neq \mathcal{J}_\alpha^{K^*}$

or

(3) $\exists \mathcal{M}(\mathcal{M}$ is a properly small countable premouse $\wedge\, \alpha$ is a cardinal in $K^* \wedge \mathcal{J}_\alpha^{\mathcal{M}} = \mathcal{J}_\alpha^K = \mathcal{J}_\alpha^{K^*} \wedge \neg(\mathcal{M}$ is α-strong $\Leftrightarrow \mathcal{M}$ is α-strong$^*))$.

Fix $n < \omega$ minimal so that such an $\alpha < \kappa_n$ exists, and let α be the first ordinal below κ_n where one of these three alternatives happen.

If either (1) or (2) happen, α must be a successor cardinal in both K and K^*. Set $\beta =$ the largest cardinal of K below $\alpha =$ the largest cardinal of K^* below α. Note that $\mathcal{J}_\beta^K = \mathcal{J}_\beta^{K^*}$. Recall that $\mathcal{J}_\alpha^K$ is computed as the union of β-strong mice projecting to β. The statement saying that for any δ, a cardinal in $\mathcal{J}_{\kappa_n}^{K^*}$, $\mathcal{J}_{\kappa_n}^{K^*}$ up to $(\delta^+)^{\mathcal{J}_{\kappa_n}^{K^*}}$ is computed as the union of δ-strong* mice projecting to δ, is (lightface) projective and true in N_k (for suitable N_k as above) using lemma 4.5. By correctness it remains true in V. Thus neither (1) or (2) can happen.

So it must be the case that (3) holds. However, with a similiar argument as the one just given, this plainly contradicts lemma 4.5. Therefore n and α from above cannot exist, i.e., $\mathcal{J}_\lambda^K = K^*$. This proves

Theorem 4.6 *(ZF +$\mathbb{R}^\sharp$ exists.) If PM + PB + PU hold then there is a transitive model of ZFC containing all ordinals with infinitely many cardinals that are each strong up to their supremum.* □ (4.6)

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References

[1] K. Hauser, *The consistency strength of projective absoluteness*, Ann. Pure Appl. Logic **74** (1995), pp. 245-295.

[2] K. Hauser, *A Minimal counterexample to universal Baireness*, to appear in J. Symb. Logic.

[3] K. Hauser, *Reflections on the last Delfino problem*, in *Proceedings of the European Summer Meeting of the Association of Symbolic Logic 1998*, Lecture Notes in Logic, Springer Verlag, 210-229 (1999).

[4] K. Hauser and G. Hjorth, *Strong cardinals in the core model*, Ann. Pure Appl. Logic **83** (1997), pp. 165-198.

[5] R. B. Jensen, *Measures of order zero*, handwritten notes.

[6] R. B. Jensen, *The core model for non-overlapping extender sequences*, handwritten notes.

[7] R. B. Jensen, *A new fine structure for higher core models*, handwritten notes.

[8] R. B. Jensen and M. Zeman, *The core model for one strong cardinal*, Lecture Notes in Logic, in preparation.

[9] A.S. Kechris, D.A. Martin and J.R. Steel (eds.), *Appendix: Victoria Delfino problems II,* in *Cabal Seminar 81-85*, Springer Lecture Notes in Math. **1333** (1988), pp.221-224.

[10] M. Kondo, *Sur l'uniformaization des complementaires analytiques et les ensembles projectifs de la second classe,* Jap. J. Math. **15** (1938) 197-230.

[11] N. Lusin, *Sur le problème de M. Hadamard d'uniformisation des ensembles,* Comptes Rendus Acad. Sci. Paris **190** (1930) 349-351.

[12] W.J. Mitchell, E. Schimmerling and J.R. Steel, *The covering lemma up to a Woodin cardinal,* Ann. Pure Appl. Logic **84** (1997), pp. 219 - 255.

[13] W. J. Mitchell and J. R. Steel, *Fine structure and iteration trees*, Lecture Notes in Logic **3**, Springer-Verlag (1994), 130 pages.

[14] Y. Moschovakis, *Descriptive set theory*, North-Holland (1980), xii + 637.

[15] J. Raisonnier, *A Mathematical proof of S. Shelah's theorem on the measure problem and related results,* Isr. J. Math. **48 (1)** (1984) 48-56.

[16] E. Schimmerling and J. R. Steel, *The maximality of the core model*, Trans. Amer. Math. Soc., to appear.

[17] R.-D. Schindler, *Strong cardinals and sets of seals in* $L_{\omega_1}(\mathbb{R})$, Math. Logic Quarterly, **45** (1999) 361 - 369.

[18] J. R. Steel, *The core model iterability problem*, Lecture Notes in Logic **8**, Springer Verlag (1996), 112 pages.

[19] W. H. Woodin, *On the consistency strength of projective uniformization*, in *Logic Colloquium '81*, J. Stern (ed.), North-Holland (1982), pp. 365-383.

[20] W.H. Woodin, A.R.D. Mathias and K. Hauser, *The axiom of determinacy*, monograph in preparation.