A universal extender model without large cardinals in V

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Abstract

We construct, assuming that there is no inner model with a Woodin cardinal but without any large cardinal assumption, a model K^c which is iterable for set length iterations, which is universal with respect to all weasels with which it can be compared, and (assuming GCH) is universal with respect to set sized premice.

0 Introduction.

In [6] Steel constructs a core model K under the assumptions that there is no inner model with a Woodin cardinal and that there is a measurable cardinal Ω in V. This core model is an extender model of height Ω with the following three properties (among others):

1. ($\Omega+1$ iterablity) The model K is $\Omega+1$ iterable, i.e., it has an iteration strategy for iteration trees of height at most (and including) Ω.

2. (Universality) The model K wins the coiteration against any coiterable premouse M of height less than Ω in the sense that the final model in the iteration tree on M is an initial segment of the last model of the iteration tree on K.

3. (Weak covering) ${\kappa^+}^K = \kappa^+$ for all singular cardinals $\kappa < \Omega$.

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The assumption that there is no inner model with a Woodin cardinal is necessary; however the assumption that Ω is measurable has no precedent in earlier core models. Jensen has shown that this assumption can be weakened: he obtains iterability and universality assuming only an inaccessible cardinal, and additionally obtains the weak covering property with the stronger assumption of a subtle cardinal. This paper presents a partial answer to the question whether the large cardinal assumption can be completely eliminated:

Theorem 0.1. *Assume* $\mathsf{BGC}+$ *"there is no inner model with a Woodin cardinal." Then there is an extender model* K^c *with the following properties:*

1. K^c *is* OR*-iterable, i.e., it has an iteration strategy for iteration trees of height less than* OR.

2a. K^c *wins the iteration against any premouse against which it can be coiterated.*

2b. *(*GCH*) If* κ *is a cardinal of uncountable cofinality and* M *is a premouse of size* κ *then* K^c *wins the coiteration against* M *in fewer than* κ^+ *steps.*

The key new idea here is to weaken the concept of "countably certified" of [6, Definition 1.2], which is crucial for Steel's construction of K^c. The iterability proof of [6] still goes through with our weaker requirement on new extenders to be added to the K^c-sequence; in fact, our iterability proof is a clarified version of the one of [6]. Moreover, our argument for the universality of our K^c is much simpler than the traditional argument from [2] and [4].

It is still open whether this result can be extended to give a true core model K.[1] Steel gets K by constructing a model K^c with the properties listed on the first paragraph except that weak covering may hold only in a stationary class, and then obtains K as an elementary submodel of K^c. In order to do the same with the model K^c defined in this paper we would have to solve two remaining problems: we would have to prove full $\mathrm{OR}+1$-iterablity for K^c, or, alternatively show that the universality in theorem 0.1(2a,2b) is adequate in spite of the limited iterability; and we would have to prove that the weak covering property holds on a stationary class in K^c. We are inclined to conjecture that the model K^c defined here does have the required properties; however we would have more confidence in conjecturing that either K^c

[1] Cf. `wwwmath.uni-muenster.de/math/inst/logik/org/staff/rds/list.html`, problem #1. For the purpose of this introduction we may characterize the true core model K as a universal extender model which elementarily embeds into any other universal extender model.

has these properties, or else no model exists with these properties. It seems striking that the new ideas used in this paper give no hints as to how one would prove a weak covering lemma, and this failure may be a reason to take seriously the possibility that the existence of a core model does not follow from the nonexistence of a model with a Woodin cardinal.

The obvious further question whether the GCH is actually necessary for clause (2b) of theorem 0.1 is of less interest. The main goal, as described in the last paragraph, is to use the model K^c as an intermediate step towards the construction of the core model K. If the problems discussed in the last paragraph can be overcome then we could construct K^c in a generic extension $V[G]$ where the GCH does hold. This model K^c would depend on the forcing extension, but the core model K to which it gives rise probably would not; thus K would be an inner model in V, and would be the core model there.

This in turn would probably yield—among other things—a proof of the following conjecture: Suppose that there is an ω_2-saturated ideal on ω_1; then there is an inner model with a Woodin cardinal. We refer the reader to the discussion in the introduction to [3].

Historical note. The second author originally constructed a version of K^c which satisfied clauses 1 and 2a of theorem 0.1. After seeing notes presenting this work, the first author gave a different construction which satisfied clauses 1 and 2b. This paper is an amalgamation of these two constructions. John Steel came upon something very close to Definition 1.6 and Lemmas 2.5 and 2.5 in 1993 or 1994.

1 Preliminaries.

We build upon [3] and [6]; in particular, we use the concept of "extender" as in [3, Definition 1.0.1] and of "premouse" as in [3, Definition 3.5.1] (cf. also the discussion in [5]). We aim to define a model K^c which resembles the K^c of [6, §1] and which satisfies versions of [6, Corollary 3.6].

We first recall a familiar concept.

Definition 1.1. *Let F be an extender with critical point κ and length $\nu = \mathrm{lh}(\nu)$. Let U be a countable set. We say that F is* countably complete with respect to U *if and only if there is a map τ such that $\nu \cap U \subset \mathrm{dom}(\tau)$, $\tau \restriction \nu \cap U \colon \nu \cap U \to \kappa$ is order-preserving, $\tau \restriction (\kappa \cap U) = \mathrm{id}$, and for all $a \in [\nu \cap U]^{<\omega}$ and for every $X \in \mathcal{P}([\kappa]^{Card(a)}) \cap U$ which is measured by F_a we have that*

$$X \in F_a \;\Rightarrow\; \tau\text{"}a \in X.$$

The extender F is called countably complete *if and only if F is countably complete with respect to every countable U.*

The following lemma is part of the folklore.

Lemma 1.2. *Let $\mathcal{M}$ and $\mathcal{Q}$ be premice, and let $\bar{\mathcal{M}}$ and $\bar{\mathcal{Q}}$ be countable premice. Let $k \leq \omega$, and let $\bar{\sigma}\colon \bar{\mathcal{M}} \to \bar{\mathcal{Q}}$ be a weak k-embedding. Let $\sigma\colon \mathcal{M} \to_{r\Sigma_0} \mathcal{Q}$. Suppose that $E = E_\nu^{\mathcal{M}}$ is a total extender on $\mathcal{M}$ with critical point κ, $\mathcal{P}(\kappa) \cap \bar{\mathcal{M}} \subset \mathcal{M}$, and $\bar{\sigma} \upharpoonright \mathcal{P}(\kappa) \cap \bar{\mathcal{M}} = \sigma \upharpoonright \mathcal{P}(\kappa) \cap \bar{\mathcal{M}}$. Let $F = \sigma(E)$, or let F be the top extender of $\mathcal{Q}$ if E is the top extender of $\mathcal{M}$, and suppose further that F is countably complete with respect to $\operatorname{ran}(\sigma)$. Let τ be a witness. There is then some weak k-embedding $\varphi\colon Ult(\bar{\mathcal{M}}; E) \to \bar{\mathcal{Q}}$ defined by $[a, f]_E^{\bar{\mathcal{M}}} \mapsto \bar{\sigma}(f)(\tau"\sigma(a))$ for $a \in [\mathrm{lh}(E)]^{<\omega}$ and appropriate f (cf. [3, p. 34]).*

Countable completeness was exploited in [4] for building K^c below $0^\P$, the sharp for a class of strong cardinals. In general, it seems, we need a stronger criterion for new extenders to be added to the K^c-sequence. We have to introduce some prerequisites before being able to formulate it in Definition 1.6.

We shall let $\mathcal{L}_{\dot\in}$ denote the language of set theory, i.e., the language of ZFC. In what follows, though, we shall work in Bernays-Gödel's class theory BGC (cf. [1, p. 76f.]) which is conservative over ZFC (cf. [1, p. 197]). Let Γ denote Gödel's pairing function (cf. [1, p. 20f.]). Throughout the rest of this paper we shall fix a class $A \subset \mathrm{OR}$ such that for all infinite cardinals κ,

$$\mathcal{P}(\kappa) \subset \{\{\alpha < \kappa \;:\; \Gamma(\alpha, \xi) \in A\} \colon \xi < 2^\kappa\}. \tag{1}$$

In particular, $V = L[A]$. The existence of such a class A is provable in BGC.

The formal language of BGC, which we shall call $\mathcal{L}_{\dot\in,\dot A}$, has two sorts of variables, one for sets and one for classes. Moreover, as we construe it, $\mathcal{L}_{\dot\in,\dot A}$ has a predicate symbol $\dot A$ which is intended to denote A (rather than a function symbol denoting a global selector as in [1, p. 76f.]). Officially, BGC then consists of the axioms A through D of [1, p. 76] together with (1) above. We shall denote by BGC^- the theory $\mathsf{BGC} \setminus \{\mathsf{Power\ set}\}$.

If $\mathcal{X}$ is a model then we write $|\mathcal{X}|$ for the universe of $\mathcal{X}$. Any transitive model $\mathcal{X}$ of $\mathcal{L}_{\dot\in,\dot A}$ (in particular, any model $\mathcal{X}$ of BGC^-) is of the form $(|\mathcal{X}|; \in, \dot A^{\mathcal{X}})$.

Definition 1.3. *A Σ_{1+} formula of the language $\mathcal{L}_{\dot\in,\dot A}$ is a formula of the following form.*

$$\exists v_0 \; \exists v_1 \; \exists v_2 \; ({}^\omega v_0 \subset v_0 \wedge v_2 = \dot A \cap v_3 \wedge \varphi(v_0, v_1, v_2, v_3, v_4)), \tag{2}$$

where φ is a Σ_0 formula of $\mathcal{L}_{\dot\in}$ and v_0, v_1, v_2, v_3, and v_4 are set variables.

Let $\mathcal{X}$, $\mathcal{Y}$ be models of $\mathcal{L}_{\dot\in,\dot A}$. If $\vec{x} \in |\mathcal{X}|$ and $\vec{y} \in |\mathcal{Y}|$ then $(\mathcal{X};\vec{x}) \equiv_{\Sigma_{1+}} (\mathcal{Y};\vec{y})$ expresses that for all $\psi \in \Sigma_{1+}$, $\mathcal{X} \models \psi(\vec{x}) \Leftrightarrow \mathcal{Y} \models \psi(\vec{y})$. We write $\pi\colon \mathcal{X} \to_{\Sigma_{1+}} \mathcal{Y}$ to say that $\pi\colon |\mathcal{X}| \to |\mathcal{Y}|$ and for all $\vec{x} \in |\mathcal{X}|$, $(\mathcal{X};\vec{x}) \equiv_{\Sigma_{1+}} (\mathcal{Y};\pi(\vec{x}))$. We write $\mathcal{X} \prec_{\Sigma_{1+}} \mathcal{Y}$ for $\mathrm{id}\colon \mathcal{X} \to_{\Sigma_{1+}} \mathcal{Y}$.

Note that the predicate $\dot A$ is not allowed to appear in the formula ϕ of (2).

As usual, if κ is a regular cardinal then H_κ denotes the set of all sets which are hereditarily smaller than κ. If κ is a singular cardinal then we set $H_\kappa = \bigcup_{\lambda<\kappa} H_{\lambda^+}$. Now let α be an ordinal. We let $\bar H_\alpha$ ambiguously denote the set $L_\alpha[A\cap\alpha]$ as well as the model $(L_\alpha[A\cap\alpha];\in,A\cap\alpha)$ (which is a model of $\mathcal{L}_{\dot\in,\dot A}$). In particular, we intend that $\dot A^{L_\alpha[A\cap\alpha]} = A\cap\alpha$. We also confuse V with $(V;\in,A)$.

Let κ be an infinite cardinal. Trivially, $\bar H_\kappa \subset H_\kappa$. By (1) above, we'll have that if $2^{<\kappa} = \kappa$ (i.e., if $2^\alpha \leq \kappa$ for all $\alpha < \kappa$) then $H_\kappa \subset \bar H_\kappa$, and thus $H_\kappa = \bar H_\kappa$.

Definition 1.4. *Let α be an ordinal. We say that α is a* reflection point *if and only if $\bar H_\alpha \prec_{\Sigma_{1+}} V$.*

A cardinal κ is called *countably closed* if $\alpha^{\aleph_0} < \kappa$ for all $\alpha < \kappa$.

Lemma 1.5. *Let κ be an infinite cardinal such that $2^{<\kappa} = \kappa$ and κ is countably closed. Then κ is a reflection point.*

Proof. Let $\{x_3, x_4\} \subset \bar H_\kappa$. As κ is countably closed, we may pick some $\pi\colon \bar V \to_{\Sigma_2} V$ such that $\bar V$ is transitive, $\mathrm{Card}(\bar V) < \kappa$, $\mathrm{TC}(\{x_3, A\cap x_3, x_4\}) \in \mathrm{ran}(\pi)$, and ${}^\omega\bar V \subset \bar V$. Here, $\pi\colon \bar V \to_{\Sigma_2} V$ is intended to just mean Σ_2 elementarity with respect to $\mathcal{L}_{\dot\in}$. Suppose that $V \models \exists v_0\, \exists v_1\, \exists v_2\, ({}^\omega v_0 \subset v_0 \wedge v_2 = \dot A\cap x_3 \wedge \varphi(v_0,v_1,v_2,x_3,x_4))$. We shall then have that $\bar V \models \exists v_0\, \exists v_1\, ({}^\omega v_0 \subset v_0 \wedge \varphi(v_0,v_1,A\cap x_3,x_3,x_4))$. Let x_0, $x_1 \in \bar V$ be witnesses. Notice that ${}^\omega x_0 \subset x_0$ in V, as ${}^\omega\bar V \subset \bar V$. Moreover, $\bar V \subset \bar H_\kappa$ by $2^{<\kappa} = \kappa$. Therefore, x_0 and y_0 witness that $\bar H_\kappa \models \exists v_0\, \exists v_1\, \exists v_2\, ({}^\omega v_0 \subset v_0 \wedge \varphi(v_0,v_1,A\cap x_3,x_3,x_4))$. $\square$ (Lemma 1.5)

In particular, every strong limit cardinal κ of uncountable cofinality is a reflection point, as is every successor cardinal $\kappa = \mu^+$ where the cardinal μ satisfies $2^\mu = \mu^+$, $2^{<\mu} = \mu$, and $\mathrm{cf}(\mu) > \omega$. In particular, there is a proper class of cardinals which are reflection points, and under the GCH the only uncountable cardinals which are not reflection points are the successors of cardinals of cofinality ω Of course there are also many reflection points which are not cardinals, for example, if μ is as above then the set of reflection points is unbounded in μ^+.

We will frequently use reflection points where Steel uses cardinals, and we will use the sets $\bar{H}_\alpha$ in contexts where Steel uses the sets V_α, .

The following now gives the key definition of our paper.

Definition 1.6. *Let F be an extender with critical point κ and length $\nu = \mathrm{lh}(\nu)$. Let $\gamma(F)$ be the least cardinal $\gamma \geq (\mathrm{Card}(\nu)^{\aleph_0})^+$ such that $2^{<\gamma} = \gamma$ and γ is countably closed. We say that F is* certified *if and only if for all countable $U \prec_{\Sigma_{1+}} \bar{H}_{\gamma(F)} = H_{\gamma(F)}$ there is some $\tau \colon U \to_{\Sigma_{1+}} \bar{H}_\kappa$ witnessing that F is countably complete with respect to U.*

Thus if F is certified then F is countably closed. Certifiability requires that the witness for countable completeness respects a certain amount of elementary equivalence between two models (cf. below). We leave it as an exercise to the reader to figure out the relation between "countably certified" in the sense of [6, Definition 1.2] and "certified" in our sense (cf. the proof of Lemma 3.6 below).

Notice that the GCH implies that $\gamma(F) = \nu^+$ whenever $\mathrm{cf}(|\nu|) > \omega$.

The following is easy to verify, by Lemma 1.5.

Lemma 1.7. *Let F be a certified extender. Then* $\mathrm{crit}(F)$ *is a reflection point.* $\square$

In what follows we shall use the following convention: We will say that a map τ *witnesses that F is countably complete with respect to a set X* if and only if, setting $\gamma = \gamma(F)$, there is some countable $U \prec_{\Sigma_{1+}} \bar{H}_\gamma$ with $X \cap \bar{H}_\gamma \subset U$ and there is some $\tilde{\tau} \colon U \to_{\Sigma_{1+}} \bar{H}_\kappa$ such that $\tau \restriction (X \cap \nu) = \tilde{\tau} \restriction (X \cap \nu)$ and $\tilde{\tau}$ witnesses, in the sense of definition 1.1, that F is countably complete with respect to U.

We shall make use of "being certified" in the following equivalent form.

Lemma 1.8. *Let F be a certified extender with $\kappa = \mathrm{crit}(F)$ and $\nu = \mathrm{lh}(F)$. Let x_1, ..., $x_m \in H_{\gamma(F)}$, and let $(a_n, X_n \colon n < \omega)$ be such that for all $n < \omega$, $a_n \in [\nu]^{<\omega}$, $X_n \in \mathcal{P}([\kappa]^{\mathrm{Card}(a_n)})$, and $X_n \in F_{a_n}$. Then there are $\bar{x}_1$, ..., $\bar{x}_m \in \bar{H}_\kappa$ and there is an order preserving $\tau \colon \bigcup_{n<\omega} a_n \to \kappa$ such that*

$$(\bar{H}_\kappa; \bar{x}_1, ..., \bar{x}_m, \tau\text{"} \bigcup_{n<\omega} a_n) \equiv_{\Sigma_{1+}} (V; x_1, ..., x_m, \bigcup_{n<\omega} a_n)$$

and $\tau(a_n) \in X_n$ for each $n < \omega$.

Proof. The objects $\bar{x}_1$, ..., $\bar{x}_m$, and τ are obtained as follows. Let $\gamma = \gamma(F)$. Let $U \prec_{\Sigma_{1+}} \bar{H}_\gamma$ be such that U is countable and $\{x_1, ..., x_m, (a_n, X_n \colon n <$

$\omega)\} \subset U$. Let $\tilde{\tau}\colon U \to_{\Sigma_{1+}} \bar{H}_\kappa$ witness that F is countably complete with respect to U. We may then set $\bar{x}_1 = \tilde{\tau}(x_1)$, ..., $\bar{x}_n = \tilde{\tau}(x_n)$, and $\tau = \tilde{\tau} \upharpoonright \bigcup_{n<\omega} a_n$. □ (Lemma 1.8)

2 The existence of K^c.

In this section we define the model K^c and prove that it is iterable. The definition and the proof are both based on that in [6], and we will only describe aspects of the construction and proof that are different from that book. Hence we assume that the reader is familiar with and has immediate access to [6].

The models $\mathcal{N}_\xi$ and $\mathcal{M}_\xi$ are constructed exactly as in [6, p. 6f.], except that we don't require (2) and (3) at all in *Case 1* and that in (1) of *Case 1* we replace "countably certified" by "certified" in the sense of Definition 1.6 above.

Lemma 2.1. *Let $F = E_\nu^{\mathcal{N}_\xi}$ be a total extender on $\mathcal{N}_\xi$ with critical point κ. Then κ is a reflection point.*

Proof. The hypothesis gives some $\bar{\xi} \leq \xi$ such that $\mathcal{N}_{\bar{\xi}+1}$ has a top extender G with $\mathrm{crit}(G) = \kappa$ and G is certified. Lemma 1.7 then implies that κ is a reflection point. □ (Lemma 2.1)

We have to prove [6, Theorem 2.5], which asserts that if $\mathcal{N}_\theta$ exists then transitive collapses of countable submodels of $\mathfrak{C}_k(\mathcal{N}_\theta)$ are countably k-iterable for every $k < \omega$ (cf. [6] and Theorem 2.7 below for the exact statement). The proof will be presented as a modification of [6, §9]. As in [6] we will present the basic ideas in simplified cases in which the tree $\mathcal{T}$ has length ω or $\omega+1$ and has no drops. We will treat these cases by presenting full proofs of the new versions of what is shown in [6, §2, pp. 12ff.]. After this we will briefly indicate the necessary changes in the general iterability argument of [6, §9].

The case of $\mathrm{lh}(\mathcal{T}) = \omega$ is given below as lemma 2.5, and that of $\mathrm{lh}(\mathcal{T}) = \omega + 1$ is given as lemma 2.6.

Definition 2.2. *Let $\mathcal{P}$ be a countable premouse. We call the triple $(\sigma, \mathcal{Q}, \mathcal{R})$ a* realization *of $\mathcal{P}$ if and only if $\mathcal{R}$ is a transitive model of* BGC^- *with ${}^\omega\mathcal{R} \subset \mathcal{R}$, $\sigma\colon \mathcal{P} \to \mathcal{Q}$ is a weak k-embedding, and $\mathcal{Q} = \mathfrak{C}_k(\mathcal{N}_\xi^{\mathcal{R}})$ for some $\xi \in \mathcal{R} \cap \mathrm{OR}$ and $k < \omega$.*

Definition 2.3. *Let $\mathcal{T}$ be an iteration tree of length $\theta < \omega_1$ on a countable premouse. Set $\nu_i = \mathrm{lh}(E_i^{\mathcal{T}})$ for $i + 1 < \theta$. We call a sequence $(\sigma_i, \mathcal{Q}_i, \mathcal{R}_i \colon i < \theta)$ an* enlargement *of $\mathcal{T}$ if and only if for all $j \leq i < \theta$ we have that $(\sigma_i, \mathcal{Q}_i, \mathcal{R}_i)$ is a realization of $\mathcal{M}_i^{\mathcal{T}}$, $\dot{A}^{\mathcal{R}_i} \cap \sigma_j(\nu_j) = \dot{A}^{\mathcal{R}_j} \cap \sigma_j(\nu_j)$, and $\sigma_i \upharpoonright \nu_j = \sigma_j \upharpoonright \nu_j$.*

In [6], the definitions of a realization and an enlargement require that $\mathcal{P}(\sigma_j(\nu(E_j))) \cap \mathcal{R}_j \subset \mathcal{R}_i$ (in [6, §2] this requirement is added on page 21 for the case $\mathrm{lh}(\mathcal{T}) = \omega + 1$). We cannot use this requirement because of our use of the sets $\bar{H}_\alpha$ instead of the sets V_α.

Definition 2.4. *Let $\mathcal{R}$ be a transitive model of BGC^-, let $\mathcal{Q} \in \mathcal{R}$ be a premouse, and let $\theta \in \mathcal{R} \cap \mathrm{OR}$. We say that θ is a* cutoff point of $\mathcal{R}$ relative to $\mathcal{Q}$ *if and only if $\bar{H}^{\mathcal{R}}_\theta \models \mathsf{BGC}^-$, and $\mathcal{Q} = \mathcal{N}_\eta$ for some $\eta < \theta$ such that the sequences $(\mathcal{N}_\xi \colon \xi \leq \eta)^{\mathcal{R}}$ and $(\mathcal{N}_\xi \colon \xi \leq \eta)^{\bar{H}^{\mathcal{R}}_\theta}$ are both defined and are equal.*

Lemma 2.5. *Let $\sigma \colon \mathcal{P} \to \mathcal{N}_\eta$ be elementary with $\mathcal{N}_\eta \models \mathsf{ZFC}$, and let $\mathcal{T}$ be an iteration tree on $\mathcal{P}$ of length ω such that $\mathcal{D}^T = \emptyset$. Suppose that for all $i < \omega$,*[2]

$$\forall \kappa < \mathrm{lh}(E^{\mathcal{T}}_i)\ (\mathcal{M}^{\mathcal{T}}_i \models \kappa \text{ is a cardinal } \Leftrightarrow \mathcal{M}^{\mathcal{T}}_i || \mathrm{lh}(E^{\mathcal{T}}_i) \models \kappa \text{ is a cardinal }).$$

(This holds for instance if $\nu(E^{\mathcal{T}}_i)$ is a cardinal of $\mathcal{M}^{\mathcal{T}}_i$ for all $i < \omega$.) Then there are b and σ' such that b is a cofinal branch through T and $\sigma' \colon \mathcal{M}^{\mathcal{T}}_b \to \mathcal{N}_\eta$ with $\sigma' \circ \pi^{\mathcal{T}}_{0b} = \sigma$.

Proof. We closely follow [6, p. 12ff.]. For any $\tau \colon \mathcal{P} \to \mathcal{Q}$ we denote by

$$U(\tau, \mathcal{Q})$$

the tree of attempts to find b, τ' such that b is a cofinal branch through T and $\tau' \colon \mathcal{M}^{\mathcal{T}}_b \to \mathcal{Q}$ with $\tau' \circ \pi^{\mathcal{T}}_{0b} = \tau$. The nodes of $U = U(\tau, \mathcal{Q})$ are the functions $\phi \colon \mathcal{M}^{\mathcal{T}}_i \to \mathcal{Q}$ with $\phi \circ \pi^{\mathcal{T}}_{0i} = \tau$, and if $\phi \colon \mathcal{M}^{\mathcal{T}}_i \to \mathcal{Q}$ and $\phi' \colon \mathcal{M}^{\mathcal{T}}_k \to \mathcal{Q}$ then we put $\phi' <_U \phi$ if and only if iTk and $\phi' \circ \pi^{\mathcal{T}}_{ik} = \phi$.

Let us assume that $U(\sigma, \mathcal{N}_\eta)$ is well-founded. We shall define an enlargement $(\sigma_i, \mathcal{Q}_i, \mathcal{R}_i : i < \omega)$ of $\mathcal{T}$ such that for all $i < \omega$ the following requirements are met.

$(1)_i$ if $U = U(\sigma_i \circ \pi^{\mathcal{T}}_{0i}, \mathcal{Q}_i)$ then U is well-founded and $\mathcal{R}_i$ has (in order type) at least $|\sigma_i|_U$ many cutoff points relative to $\mathcal{Q}_i$, and

$(2)_i$ $i > 0 \Rightarrow \mathcal{R}_i \in \mathcal{R}_{i-1}$.

This will of course give a contradiction.

To commence, we let $\sigma_0 = \sigma$, $\mathcal{Q}_0 = \mathcal{N}_\eta$, and $\mathcal{R}_0 = V$.

Suppose now we are given an enlargement $(\sigma_l, \mathcal{Q}_l, \mathcal{R}_l : l \leq i)$ of $\mathcal{T} \restriction (i+1)$ such that $(1)_l$ and $(2)_l$ hold for all $l \leq i$. We want to construct σ_{i+1}, $\mathcal{Q}_{i+1}$,

[2] If $\mathcal{M}$ is a premouse and $\alpha \leq \mathcal{M} \cap \mathrm{OR}$ then in what follows we'll write $\mathcal{M}||\alpha$ for $\mathcal{M}$ being cut off at α; this is $\mathcal{J}^{\mathcal{M}}_\alpha$ in the notation of [3] and [6].

and $\mathcal{R}_{i+1}$ in such a way that $(\sigma_l, \mathcal{Q}_l, \mathcal{R}_l : l \leq i+1)$ is an enlargement of $\mathcal{T} \restriction (i+2)$ and $(1)_{i+1}$ and $(2)_{i+1}$ hold. Set $F = \sigma_i(E_i^{\mathcal{T}})$, $\kappa = \mathrm{crit}(F)$, and $\nu = \mathrm{lh}(E_i^{\mathcal{T}})$. We know that F is certified inside $\mathcal{R}_i$ (cf. [6, Lemma 2.6]).

Let $\Phi(\mathcal{V}, \tau)$ be the following Σ_{1+} statement:

1. $\mathcal{V}$ is an iteration tree of length ω on a countable premouse and $\tau \colon \mathrm{lh}(E_i^{\mathcal{V}})+1 \to \mathrm{OR}$,
2. there is a realization $(\sigma, \mathcal{Q}, \mathcal{R})$ of $\mathcal{M}_{i+1}^{\mathcal{V}}$ such that $\dot{A}^{\mathcal{R}} \cap \tau(\mathrm{lh}(E_i^{\mathcal{V}})) = \dot{A} \cap \tau(\mathrm{lh}(E_i^{\mathcal{V}}))$, and $\sigma \restriction \mathrm{lh}(E_i^{\mathcal{V}}) = \tau \restriction \mathrm{lh}(E_i^{\mathcal{V}})$, and
3. if $U = U(\sigma \circ \pi_{0i+1}^{\mathcal{V}}, \mathcal{Q})$ then U is well-founded and $\mathcal{R}$ has (in order type) at least $|\sigma|_U$ many cutoff points relative to $\mathcal{Q}$.

It is now straightforward to see that in order to construct σ_{i+1}, $\mathcal{Q}_{i+1}$, and $\mathcal{R}_{i+1}$ with the desired properties it suffices to prove that

$$\mathcal{R}_i \models \Phi(\mathcal{T}, \sigma_i \restriction (\nu+1)).$$

By Lemma 1.8, inside $\mathcal{R}_i$ we may pick some order preserving $\tau \colon \sigma_i{}''(\nu+1) \to \kappa$ such that

$$(\bar{H}_\kappa^{\mathcal{R}_i}; \mathcal{T}, \tau \circ \sigma_i \restriction (\nu+1)) \equiv_{\Sigma_{1+}} (\mathcal{R}_i; \mathcal{T}, \sigma_i \restriction (\nu+1))$$

and τ witnesses that F is countably complete with respect to $\mathrm{ran}(\sigma_i)$. We are therefore left with having to verify that

$$\bar{H}_\kappa^{\mathcal{R}_i} \models \Phi(\mathcal{T}, \tau \circ \sigma_i \restriction (\nu+1)).$$

Let $j = T\text{-pred}(i+1)$. By Lemma 1.2 we may define $\sigma' \colon \mathcal{M}_{i+1}^{\mathcal{T}} \to \mathcal{Q}_j$ by setting

$$\pi_{ji+1}(f)(a) \mapsto \sigma_j(f)(\tau(\sigma_i(a_i))).$$

We'll have that $\sigma' \circ \pi_{0i+1}^{\mathcal{T}} = \sigma_j \circ \pi_{0j}^{\mathcal{T}}$, and so $\sigma' \in U(\sigma_j \circ \pi_{0j}, Q_j)$. Moreover, clearly,

$$\epsilon = |\sigma'|_{U(\sigma_j \circ \pi_{0j}, Q_j)} < |\sigma_j|_{U(\sigma_j \circ \pi_{0j}, Q_j)},$$

and hence by $(1)_j$ we may let θ be the ϵ^{th} cutoff point of $\mathcal{R}_j$ relative to $\mathcal{Q}_j$.

Therefore,[3] $(\sigma', \mathcal{Q}_j, \bar{H}_\theta^{\mathcal{R}_j})$ witnesses that

$$\mathcal{R}_j \models \Phi(\mathcal{T}, \tau \circ \sigma_i \restriction (\nu+1)).$$

By Lemma 2.1, we therefore have that

$$\bar{H}_\kappa^{\mathcal{R}_j} \models \Phi(\mathcal{T}, \tau \circ \sigma_i \restriction (\nu+1)).$$

[3] Recall that ${}^\omega\mathcal{R}_j \subset \mathcal{R}_j$.

However, $\dot{A}^{\mathcal{R}_j} \cap \kappa = \dot{A}^{\mathcal{R}_i} \cap \kappa$, and thus

$$\bar{H}_{\kappa}^{\mathcal{R}_i} \models \Phi(\mathcal{T}, \tau \circ \sigma_i \restriction (\nu + 1)),$$

as desired. $\square$ (2.5)

We are now ready to present the case when $\mathrm{lh}(\mathcal{T}) = \omega + 1$, but first we give a brief comparison with Steel's argument in this case, which is based on the following proof (due to Jensen) of the wellfoundedness for a linear iteration of countably complete extenders: Suppose that $\sigma_0 \colon M_0 \to V$ where M_0 is a countable model, and $\langle M_k : k \leq \omega \rangle$ is a linear iterated ultrapower, using countably complete ultrafilters, of M_0 with $\pi_{k,k+1} \colon M_i \to M_{i+1} = \mathrm{ult}(M_i, E_i)$. To see that M_ω is wellfounded, we can use recursion on k to define $\sigma_{k+1} \colon M_{k+1} \to V$ satisfying $\sigma_k = \sigma_{k+1} \circ \pi_{k,k+1}$ by setting $\sigma_{k+1}([f]_{E_{k,a}}) = \sigma_k(f)(\tau(a))$ where τ witnesses that $\sigma_k(E_k)$ is countably complete with respect to $\mathrm{range}(\sigma_k)$. Then the direct limit σ_ω of the sequence of maps σ_n embeds M_ω into V, and it follows that M_ω is well founded.

Steel's proof for a tree $\mathcal{T}$ of height $\omega + 1$ uses a similar argument to define realizations $\sigma_{i+1} \colon M_{i+1}^{\mathcal{T}} \to \mathcal{R}_{i+1}$ for each $i + 1 \in (0, \omega)_{\mathcal{T}}$, where the sets $\mathcal{R}_{i+1}$ approximate V sufficiently closely that the discussion in the last paragraph makes sense. In this case, however, it may be that $i \notin [0, \omega)$, so that $M_{i+1} = \mathrm{ult}(M_j, E_i)$ where $j = \mathcal{T}\text{-}\mathrm{Pred}(i+1) < i$. In that case σ_i must be defined by another method to obtain $\sigma_i(E_i)$. The method used is an adaptation of the method used in lemma 2.5 for trees of height ω, but there is a further complication: The map τ must be a witness to the countable completeness of $\sigma_i(E_i)$ with respect to $\mathrm{range}(\sigma_j)$, not with respect to $\mathrm{range}(\sigma_i)$. In order to deal with this, the whole segment $\langle \mathcal{R}_j \dots \mathcal{R}_i \rangle$ of the enlargement is redefined by reflecting it, along with the generator σ_i"E_i of $\sigma_i(E_i)$, below the critical point of $\sigma_i(E_i)$. As a preliminary step it is necessary to first reflect this segment of the enlargement into $\mathcal{R}_{i+1}$. Steel can do all of this at stage $i + 1$ because his construction has $\mathcal{P}(\sigma_k(\mathrm{lh}(E_k))) \cap \mathcal{R}_k \subset \mathcal{R}_{i+1}$ for each $j \leq k \leq i$. Our proof requires more care at this point because the enlargements don't contain the whole power set; however we can construct them so as to contain any particular chosen subset, and furthermore it is known at stage k which set will be required. For this reason the preliminary step is performed at each of the stages $j \leq k \leq i$ instead of waiting to do it all at once at stage $i + 1$.

In the full proof of iterability for trees of arbitrary countable length, the construction at stage k may have to anticipate this situation for several future stages $i+1 > k$. However, as in Steel's proof, there are at most finitely many such stages $i + 1$; furthermore these stages can be determined by examining the tree $\mathcal{T}$ and hence are already known and can be provided for at stage k.

Lemma 2.6. *Let $\sigma\colon \mathcal{P} \to \mathcal{N}_\eta$ be elementary such that $\mathcal{N}_\eta \models \mathsf{ZFC}$ and $\mathcal{P}$ is countable and transitive, and let $\mathcal{T}$ be an iteration tree on $\mathcal{P}$ of length $\omega+1$ such that $\mathcal{D}^T = \emptyset$ and $\mathcal{T} \restriction \omega$ is simple. Suppose that for all $i < \omega+1$,*

$$\forall \kappa < \mathrm{lh}(E_i^{\mathcal{T}})\ (\mathcal{M}_i^{\mathcal{T}} \models \kappa \text{ is a cardinal } \Leftrightarrow \mathcal{M}_i^{\mathcal{T}}||\mathrm{lh}(E_i^{\mathcal{T}}) \models \kappa \text{ is a cardinal }).$$

(Again, this holds for instance if $\nu(E_i^{\mathcal{T}})$ is a cardinal of $\mathcal{M}_i^{\mathcal{T}}$ for all $i < \omega+1$.) $\mathcal{T}$ then admits an enlargement.

Proof. We closely follow [6, p. 19ff.]. This time for any $\tau\colon \mathcal{P} \to \mathcal{Q}$ we denote by

$$U(\tau, \mathcal{Q})$$

the tree of attempts to find b, τ' such that b is a cofinal branch through $T \restriction \omega$, $b \neq [0,\omega)_T$, and $\tau' \circ \pi_{0b}^{\mathcal{T}} = \tau$. The nodes of $U = U(\tau, \mathcal{Q})$ are the functions $\phi\colon \mathcal{M}_i^{\mathcal{T}} \to \mathcal{Q}$ with $\phi \circ \pi_{0i}^{\mathcal{T}} = \tau$, and if $\phi\colon \mathcal{M}_i^{\mathcal{T}} \to \mathcal{Q}$ and $\phi'\colon \mathcal{M}_k^{\mathcal{T}} \to \mathcal{Q}$ then we put $\phi' <_U \phi$ if and only if iTk, $\phi' \circ \pi_{ik}^{\mathcal{T}} = \phi$, *and* $k \notin [0,\omega)_T$ (i may be in $[0,\omega)_T$, though). As $\mathcal{T} \restriction \omega$ is simple, we know that $U(\tau, \mathcal{Q})$ is well-founded for all τ and $\mathcal{Q}$.

We shall first construct, by induction on $i < \omega$, enlargements $\mathcal{E}^i = (\sigma_l^i, \mathcal{Q}_l^i, \mathcal{R}_l^i \ :\ l \leq i)$ of $\mathcal{T} \restriction (i+1)$ such that for all $l \leq i$ the following requirements are met.

$(1)_l^i$ Let $U = U(\sigma_l^i \circ \pi_{0l}^{\mathcal{T}}, \mathcal{Q}_l^i)$. Then U is well-founded and $\mathcal{R}_l^i$ has (in order type) at least $2 \cdot |\sigma_l^i|_U$ many cutoff points relative to $\mathcal{Q}_l^i$. In addition, $\mathcal{R}_i^i$ has (in order type) at least $2 \cdot |\sigma_i^i|_U + 1$ many cutoff points relative to $\mathcal{Q}_i^i$.

$(2)^i$ Suppose that $i \notin [0,\omega)_T$, and let j_0 be the largest element of $[0,\omega)_T \cap i$. Then $\{\mathcal{R}_{j_0}^i, ..., \mathcal{R}_{i-1}^i\} \subset \mathcal{R}_i^i$.

$(3)^i$ For $l \leq j \leq i$ such that $[0,\omega)_T \cap (l,i] = \emptyset$, $\sigma_l^i \restriction \mathrm{lh}(E_l^{\mathcal{T}}) = \sigma_l^j \restriction \mathrm{lh}(E_l^{\mathcal{T}})$.

To commence, we let $\sigma_0^0 = \sigma$, $\mathcal{Q}_0^0 = \mathcal{N}_\eta$, and $\mathcal{R}_0^0 = V$. Let us now suppose that we are given $\mathcal{E}^i$. We aim to construct $\mathcal{E}^{i+1}$. Set $F = \sigma_i^i(E_i^{\mathcal{T}})$, $\kappa = \mathrm{crit}(F)$, and $\nu = \mathrm{lh}(E_i^{\mathcal{T}})$. We know that F is certified inside $\mathcal{R}_i^i$ (cf. [6, Lemma 2.6]). Let $j = T\text{-pred}(i+1)$.

Case 1. $i+1 \notin (0,\omega)_T$.

Let $j_0 = \max([0,\omega)_{\mathcal{T}} \cap (i+1))$. We set $(\sigma_l^{i+1}, \mathcal{Q}_l^{i+1}, \mathcal{R}_l^{i+1}) = (\sigma_l^i, \mathcal{Q}_l^i, \mathcal{R}_l^i)$ for all $l < j_0$.

Let $\Phi_*(\mathcal{V}, \tau)$ denote the following Σ_{1+} statement, which should be compared to the formula $\Phi(\mathcal{V}, \tau)$ in the proof of lemma 2.5:

1. $\mathcal{V}$ is an iteration tree of length $\omega+1$ on a countable premouse and $\tau\colon \mathrm{lh}(E_i^{\mathcal{V}})+1 \to \mathrm{OR}$,
2. there are realizations $(\sigma_l, \mathcal{Q}_l, \mathcal{R}_l)$ of $\mathcal{M}_l^{\mathcal{V}}$ for $j_0 \le l \le i$ such that for all $j_0 \le l \le i$, $\dot{A}^{\mathcal{R}_l} \cap \tau(\mathrm{lh}(E_l^{\mathcal{V}})) = \dot{A} \cap \tau(\mathrm{lh}(E_l^{\mathcal{V}}))$, $\sigma_l \restriction \mathrm{lh}(E_l^{\mathcal{V}}) = \tau \restriction \mathrm{lh}(E_l^{\mathcal{V}})$, and
3. if $U = U(\sigma_l \circ \pi_{0l}^{\mathcal{V}}, \mathcal{Q}_l)$ then U is well-founded and $\mathcal{R}_l$ has (in order type) at least $2 \cdot |\sigma_l|_U$ many cutoff points relative to $\mathcal{Q}_l$.

Let θ be the $(2 \cdot |\sigma_l^i|_{U(\sigma_i^i \circ \pi_{0i}^{\mathcal{T}}, \mathcal{Q}_i^i)})^{\text{th}}$ cutoff point of $\mathcal{R}_i^i$. Using $(2)^i$, the triples $(\sigma_l^i, \mathcal{Q}_l^i, \mathcal{R}_l^i)$ for $j_0 \le l < i$ and $(\sigma_i^i, \mathcal{Q}_i^i, \bar{H}_\theta^{\mathcal{R}_i^i})$ witness that

$$\mathcal{R}_i^i \models \Phi_*(\mathcal{T}, \sigma_i^i \restriction (\nu+1)).$$

Therefore by the definition of $\gamma(F)$ and Lemma 1.5,

$$(\bar{H}_{\gamma(F)})^{\mathcal{R}_i^i} \models \Phi_*(\mathcal{T}, \sigma_i^i \restriction (\nu+1)).$$

We let, for $j_0 \le l \le i$, $(\sigma_l^{i+1}, \mathcal{Q}_l^{i+1}, \mathcal{R}_l^{i+1})$ be witnesses to this fact. Notice that $\sigma_l^{i+1} \restriction \sigma_i^i(\nu+1) = \sigma_l^i \restriction \sigma_i^i(\nu+1)$ for all $l \le i$.

We now proceed similarly to the proof of Lemma 2.5 to define the realization $(\sigma_{i+1}^{i+1}, \mathcal{Q}_{i+1}^{i+1}, \mathcal{R}_{i+1}^{i+1})$. We have that

$$\vec{r} = (\mathcal{R}_l^{i+1} \,:\, j_0 \le l \le i) \in (\bar{H}_{\gamma(F)})^{\mathcal{R}_i^i}.$$

Let $\Phi(\mathcal{V}, \tau, \vec{v})$ be the following Σ_{1+} statement:

1. $\mathcal{V}$ is an iteration tree of length $\omega+1$ on a countable premouse and $\tau\colon \mathrm{lh}(E_i^{\mathcal{V}})+1 \to \mathrm{OR}$,
2. there is a realization $(\sigma, \mathcal{Q}, \mathcal{R})$ of $\mathcal{M}_{i+1}^{\mathcal{V}}$ such that $\vec{v} \in \mathcal{R}$, $\dot{A}^{\mathcal{R}} \cap \tau(\mathrm{lh}(E_i^{\mathcal{V}})) = \dot{A} \cap \tau(\mathrm{lh}(E_i^{\mathcal{V}}))$, $\sigma \restriction \mathrm{lh}(E_i^{\mathcal{V}}) = \tau \restriction \mathrm{lh}(E_i^{\mathcal{V}})$, and
3. if $U = U(\sigma \circ \pi_{0i+1}^{\mathcal{V}}, \mathcal{Q})$ then U is well-founded and $\mathcal{R}$ has (in order type) at least $2 \cdot |\sigma|_U + 1$ many cutoff points.

It is straightforward to see that in order to construct σ_{i+1}^{i+1}, $\mathcal{Q}_{i+1}^{i+1}$, and $\mathcal{R}_{i+1}^{i+1}$ with the desired properties it suffices to prove that

$$\mathcal{R}_i^i \models \Phi(\mathcal{T}, \sigma_i^i \restriction (\nu+1), \vec{r}).$$

By Lemma 1.8, inside $\mathcal{R}_i^i$ we may pick some order preserving $\tau\colon (\sigma_i^i)''(\nu+1) \to \kappa$ and some $\vec{s} \in \bar{H}_\kappa^{\mathcal{R}_i^i}$ such that

$$(\bar{H}_\kappa^{\mathcal{R}_i^i}; \mathcal{T}, \tau \circ \sigma_i^i \restriction (\nu+1), \vec{s}) \equiv_{\Sigma_{1+}} (V; \mathcal{T}, \sigma_i^i \restriction (\nu+1), \vec{r})$$

and τ witnesses that F is countably complete with respect to $\mathrm{ran}(\sigma_i^i)$. We are therefore left with having to verify that

$$\bar{H}_\kappa^{\mathcal{R}_i^i} \models \Phi(\mathcal{T}, \tau \circ \sigma_i \restriction (\nu+1), \vec{s}).$$

By Lemma 1.2 we may define $\sigma' \colon \mathcal{M}_{i+1}^{\mathcal{T}} \to \mathcal{Q}_j^i$ by

$$\pi_{ji+1}^{\mathcal{T}}(f)(a) \mapsto \sigma_j^i(f)(\tau(\sigma_i^i(a_i))).$$

Then $\sigma' \circ \pi_{0i+1}^{\mathcal{T}} = \sigma_j^i \circ \pi_{0j}^{\mathcal{T}}$, and so $\sigma' \in U(\sigma_j^i \circ \pi_{0j}^{\mathcal{T}}, Q_j^i)$. Moreover, clearly,

$$\epsilon = |\sigma'|_{U(\sigma_j^i \circ \pi_{0j}^{\mathcal{T}}, Q_j)} < |\sigma_j^i|_{U(\sigma_j^i \circ \pi_{0j}^{\mathcal{T}}, Q_j)},$$

and hence by $(1)_j^i$ we may let θ be the $(2 \cdot \epsilon + 1)^{\mathrm{st}}$ cutoff point of $\mathcal{R}_j^i$.

Therefore, σ', $\mathcal{Q}_j^i$, and $\bar{H}_\theta^{\mathcal{R}_j^i}$ witness that

$$\mathcal{R}_j^i \models \Phi(\mathcal{T}, \tau \circ \sigma_i^i \restriction (\nu+1), \vec{s}).$$

By Lemma 2.1, we therefore have that

$$\bar{H}_\kappa^{\mathcal{R}_j^i} \models \Phi(\mathcal{T}, \tau \circ \sigma_i^i \restriction (\nu+1), \vec{s}).$$

However, $\dot{A}^{\mathcal{R}_j^i} \cap \kappa = \dot{A}^{\mathcal{R}_i^i} \cap \kappa$, and thus

$$\bar{H}_\kappa^{\mathcal{R}_i^i} \models \Phi(\mathcal{T}, \tau \circ \sigma_i^i \restriction (\nu+1), \vec{s}),$$

as desired.

We have shown that

$$\mathcal{R}_i^i \models \Phi(\mathcal{T}, \sigma_i^i \restriction (\nu+1), \vec{r}),$$

and we may let $(\sigma_{i+1}^{i+1}, \mathcal{Q}_{i+1}^{i+1}, \mathcal{R}_{i+1}^{i+1})$ be a witness to this fact. Notice that $\sigma_{i+1}^{i+1} \restriction \sigma_i^i(\nu) = \sigma_i^i \restriction \sigma_i^i(\nu) = \sigma_i^{i+1} \restriction \sigma_i^i(\nu)$ which may be used for showing that $(3)^{i+1}$ holds.

Case 2. $i+1 \in (0,\omega)_T$.

Set $j_0 = \mathcal{T}\text{-}\mathrm{Pred}(i+1)$. We set $(\sigma_l^{i+1}, \mathcal{Q}_l^{i+1}, \mathcal{R}_l^{i+1}) = (\sigma_l^i, \mathcal{Q}_l^i, \mathcal{R}_l^i)$ for $l < j_0$, and we will have to define $(\sigma_l^{i+1}, \mathcal{Q}_l^{i+1}, \mathcal{R}_l^{i+1})$ for $j_0 \leq l \leq i+1$.

Let $\Phi_*(\mathcal{V}, \tau)$ be the following Σ_{1+} statement:

1. $\mathcal{V}$ is an iteration tree of length $\omega+1$ on a countable premouse,

2. there are realizations $(\sigma_l, \mathcal{Q}_l, \mathcal{R}_l)$ of $\mathcal{M}_l^{\mathcal{V}}$ for $j_0 \leq l \leq i$ such that for all $j_0 \leq l \leq i$, $\dot{A}^{\mathcal{R}_l} \cap \tau(\mathrm{lh}(E_l^{\mathcal{V}})) = \dot{A} \cap \tau(\mathrm{lh}(E_l^{\mathcal{V}}))$, $\sigma_l \restriction \sigma_l(\mathrm{lh}(E_l^{\mathcal{V}})) = \tau \restriction \sigma_l(\mathrm{lh}(E_l^{\mathcal{V}}))$, and
3. if $U = U(\sigma_l \circ \pi_{0l}^{\mathcal{V}}, \mathcal{Q}_l)$ then U is well-founded and $\mathcal{R}_l$ has (in order type) at least $2 \cdot |\sigma_l|_U$ many cutoff points relative to $\mathcal{Q}_l$.

Now $(2)^i$ implies that

$$\mathcal{R}_i^i \models \Phi_*(\mathcal{T}, \sigma_i^i \restriction \nu + 1).$$

By Lemma 1.8, we may pick some order preserving $\tau\colon (\sigma_i^i)"(\nu+1) \to \kappa$ such that

$$(\bar{H}_\kappa^{\mathcal{R}_i^i}; \mathcal{T}, \tau \circ \sigma_i^i \restriction (\nu+1)) \equiv_{\Sigma_{1+}} (\mathcal{R}_i^i; \mathcal{T}, \sigma_i^i \restriction (\nu+1))$$

and τ witnesses that F is countably complete with respect to $\mathrm{ran}(\sigma_i^i)$. We therefore have that

$$\bar{H}_\kappa^{\mathcal{R}_i^i} \models \Phi_*(\mathcal{T}, \tau \circ \sigma_i^i \restriction (\nu+1)).$$

We let $(\sigma_l^{i+1}, \mathcal{Q}_l^{i+1}, \mathcal{R}_l^{i+1} \colon j_0 \leq l \leq i)$ be a witness to this fact.

By $(3)^i$ and Lemma 1.2, we may define $\sigma'\colon \mathcal{M}_{i+1}^{\mathcal{T}} \to \mathcal{Q}_{j_0}^{j_0}$ by

$$\pi_{j_0 i+1}^{\mathcal{T}}(f)(a) \mapsto \sigma_{j_0}^{j_0}(f)(\tau(\sigma_i^i(a_i))).$$

We set $\sigma_{i+1}^{i+1} = \sigma'$, $\mathcal{Q}_{i+1}^{i+1} = \mathcal{Q}_{j_0}^{j_0}$, and $\mathcal{R}_{i+1}^{i+1} = V$.

It is important to notice that because $\tau \restriction (\kappa \cap \mathrm{ran}(\sigma_i^i)) = \mathrm{id}$ we shall in particular have that $\sigma_{i+1}^{i+1} \restriction \mathrm{crit}(E_i^{\mathcal{T}}) = \sigma_l^i \restriction \mathrm{crit}(E_i^{\mathcal{T}})$ for all $l < j_0$. This is needed for verifying that the defined object $\mathcal{E}^{i+1}$ is an enlargement.

Having now defined $\mathcal{E}^i$ for every $i < \omega$, we may then easily define our enlargement $(\sigma_i, \mathcal{Q}_i, \mathcal{R}_i \colon i \leq \omega)$ of $\mathcal{T} \restriction \omega + 1$.

Let $i < \omega$. Notice that $(\sigma_i^n, \mathcal{Q}_i^n, \mathcal{R}_i^n)$ is eventually constant as $n \to \omega$. We may therefore define $(\sigma_i, \mathcal{Q}_i, \mathcal{R}_i)$ to be the eventual value of $(\sigma_i^n, \mathcal{Q}_i^n, \mathcal{R}_i^n)$ as $n \to \omega$. Finally, we set $\mathcal{R}_\omega = V$ and $\mathcal{Q}_\omega = \mathcal{N}_\eta$. Notice that $\mathcal{N}_\eta = \mathcal{Q}_i^i$ for all $i \in [0, \omega)_T$. Moreover, the construction gives that

$$\sigma_i^i \circ \pi_{ji}^{\mathcal{T}} = \sigma_j^j$$

whenever jTi and $i \in [0, \omega)_T$. We may therefore define $\sigma_\omega \colon \mathcal{M}_\omega^{\mathcal{T}} \to \mathcal{Q}_\omega$ by setting

$$\sigma_\omega(x) = \sigma_i^i \circ (\pi_{i\omega}^{\mathcal{T}})^{-1}(x),$$

where $x \in \mathcal{M}_\omega^{\mathcal{T}}$, $x \in \mathrm{ran}(\pi_{i\omega}^{\mathcal{T}})$. It is straightforward to verify that we have in fact defined an enlargement $(\sigma_i, \mathcal{Q}_i, \mathcal{R}_i \colon i \leq \omega)$ of $\mathcal{T} \restriction \omega + 1$. $\square$ (Lemma 2.6)

Let us now turn towards a discussion of the general case. We shall point out the necessary changes to the argument in [6, §9]. It is to be emphasized that the basic idea and structure of this argument remains untouched.

To commence, $\mathbb{C}$ is defined as in the K^c-construction presented in [6, p. 6f.], varied as described in the first paragraph of this section (cf. [6, p. 89 middle]). Definition 9.2 of [6] is to be changed accordingly. A *coarse premouse* will be a transitive ω-closed model of BGC^- (cf. [6, p. 89 bottom]). For the record, if $\mathcal{M}$ is a coarse premouse then we may understand that $\delta^{\mathcal{M}} = \mathrm{OR} \cap \mathcal{M}$ (cf. [6, Definition 9.3]). In [6, Definition 9.11] the agreement requirement (2)(a) on coarse premice is changed as follows:

(2)(a) Let $\mu = \sigma^\beta \circ \pi_\beta(\lambda_\beta)$. Then $\dot{A}^{\mathcal{R}_\beta} \cap \mu = \dot{A}^{\mathcal{R}_\gamma} \cap \mu$, and whenever Φ is Σ_{1+} then $\mathcal{R}_\beta \models \Phi(\mathcal{T}, \pi_\beta \restriction (\nu_\beta + 1)) \Rightarrow \mathcal{R}_\gamma \models \Phi(\mathcal{T}, \pi_\beta \restriction (\nu_\beta + 1))$.

The following is identical to [6, Theorem 9.14], but of course is to be understood in the light of the definitions, including Definition 2.4, given this paper.

Theorem 2.7. *Let $\mathcal{B}$ be a hereditarily countable phalanx, and let $\mathcal{E}$ be a realization of $\mathcal{B}$ such that for all $\alpha < \mathrm{lh}(\mathcal{B})$, $\mathcal{R}^{\mathcal{E}}_\alpha$ has $\delta^{\mathcal{R}^{\mathcal{E}}_\alpha}$ many cutoff points relative to $\mathcal{M}^{\mathcal{B}}_\alpha$. Let $\mathcal{T}$ be a countable putative normal iteration tree on $\mathcal{B}$. Then either*

(1) $\mathcal{T}$ has a maximal, $\mathcal{E}$-realizable branch, or

(2) $\mathcal{T}$ has a last model, $\mathcal{M}^{\mathcal{T}}_\gamma$, and this model is $\mathcal{E}$-realizable.

The rest of this section will discuss how to vary [6, pp. 94-107] in order to arrive at a proof of Theorem 2.7. In this discussion we shall freely use the notation introduced in [6, pp. 94-107]. We leave it as a straightforward, although tedious, exercise to the reader to fill in the missing details.

Everything goes through unchanged until [6, p. 101 middle]. The statement of *Claim* 7 on [6, p. 101] becomes $\dot{A}^{\mathcal{R}_\beta} \cap \kappa = \dot{A}^{\mathcal{R}_\gamma} \cap \kappa$, which again easily follows from the fact that $\mathcal{H}$ is a realization.

In the case split of [6, pp. 94-107], *Case* 1 on [6, p. 97] (i.e., for some $\delta \leq \alpha$, δ survives at $\gamma+1$) will have to be dealt with as Case 2 was in the proof of Lemma 2.6 and *Case* 2 on [6, p. 97] (i.e., α is a break point at $\gamma+1$ and β does not survive at $\gamma+1$) will have to be dealt with as Case 1 was in the proof of Lemma 2.6. It is in dealing with these two cases where all the adaptation which remains to be performed takes place.

Let us first look at Case 1 (cf. [6, pp. 101-105]). In this case, we plan on setting

$$\mathcal{F} = \mathcal{H} \restriction \beta^\frown \bar{\mathcal{H}}^{*\frown}(\mathcal{R}_\beta, \mathcal{Q}^*_{\beta+1}, \varphi)$$

for certain objects $\bar{\mathcal{H}}^*$ and φ (cf. [6, p. 105]). In order to do this we proceed as follows. We first define a version of $\mathcal{H}^*$ (cf. [6, p. 101f.]), but not by collapsing hulls but by arguing as in the first part of the construction in Case 1 of the proof of Lemma 2.6. For this to work we need that at this point $((\mathcal{R}_\eta, \mathcal{Q}_\eta, \pi_\eta) \colon \beta \leq \eta < \gamma) \in \mathcal{R}_\gamma$ (cf. requirement $(2)^i$ in the proof of Lemma 2.6); happily, we may and shall assume this to hold. We also exploit at this point the second part of our rewritten clause (2)(a) of [6, Definition 9.11]. Having picked $\mathcal{H}^*$, we then obtain $\bar{\mathcal{H}}^*$ and φ in much the same way as we had obtained the corresponding objects in the construction of Case 2 in the proof of Lemma 2.6. We know that $\mathcal{R}_\gamma$ satisfies a certain Σ_{1+} statement which states that we get certain realizations of $\mathcal{M}_\eta^\mathcal{T}$, where $\beta \leq \eta \leq \gamma + 1$, as desired.

Let us finally look at Case 2 (cf. [6, p. 105f.]). In much the same way as in the second part of the construction in Case 1 of the proof of Lemma 2.6, or, for this matter, as in the proof of Lemma 2.5, in this case we verify that $\mathcal{R}_\gamma$ contains a certain realization of $\mathcal{M}_{\gamma+1}^\mathcal{T}$ by verifying that $\mathcal{R}_\gamma$ satisfies a certain Σ_{1+} statement. The realization thus obtained may be concatenated with $\mathcal{H}$ to get an $\mathcal{F}$ as desired.

3 K^c is universal.

In this section we shall define two different meanings of "universal," and we shall verify that K^c is universal according to both of them.

Definition 3.1. *Let W be a weasel. W is called* full *if and only if there is no cofinal $\pi \colon W||\tau \to_{\Sigma_0} W||\nu$ with $\alpha = \mathrm{crit}(\pi) \in \mathrm{Card}^W$, $\tau = \alpha^{+W}$, $\pi(\alpha) \in \mathrm{Card}^W$, and such that the extender derived from π is certified.*

The following is shown by sophisticated arguments in much the same way as [6, Claim, p. 8]. The proof, which we omit, uses a lot of the machinery developed in the later parts of [3].

Lemma 3.2. *K^c is full.* $\square$

We first aim to show that no set-sized premouse iterates past an initial segment of K^c in the sense of the following definition.

Definition 3.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be premice such that $\kappa = \mathcal{N} \cap \mathrm{OR}$ is a regular cardinal. We say that $\mathcal{M}$* iterates past *$\mathcal{N}$ provided the following holds true: $\mathcal{M} \cap \mathrm{OR} < \kappa$ and there are normal "padded" iteration trees $\mathcal{T}$ and $\mathcal{U}$ of length $\kappa + 1$ on $\mathcal{M}$ and $\mathcal{N}$ respectively arising from the comparison of $\mathcal{M}$ with $\mathcal{N}$.*

Lemma 3.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be premice, with $\kappa = \mathcal{N} \cap \mathrm{OR}$, and suppose that $\mathcal{T}$ on $\mathcal{M}$ and $\mathcal{U}$ on $\mathcal{N}$ witness that $\mathcal{M}$ iterates past $\mathcal{N}$. Then $\mathcal{M}^{\mathcal{U}}_{\kappa} \cap \mathrm{OR} = \kappa$, and $\mathcal{M}^{\mathcal{T}}_{\kappa} \rhd \mathcal{M}^{\mathcal{U}}_{\kappa}$.*[4]

Proof sketch. It is easy to verify that $\mathcal{M}^{\mathcal{U}}_{\kappa} \cap \mathrm{OR} > \kappa$ could only happen if $\mathcal{N}$ had a largest cardinal, λ, such that $cf^{\mathcal{N}}(\lambda)$ is measurable in $\mathcal{N}$. But then κ would be a successor cardinal in $\mathcal{M}^{\mathcal{U}}_{\kappa}$. However, $\mathrm{Card}(\mathcal{M}) < \kappa$ and $\mathcal{T}$ has length $\kappa + 1$, so κ is inaccessible in $\mathcal{M}^{\mathcal{T}}_{\kappa}$. On the other hand, $\mathcal{M}^{\mathcal{T}}_{\kappa}||\kappa = \mathcal{M}^{\mathcal{U}}_{\kappa}||\kappa$. □ (Lemma 3.4)

Lemma 3.5. *Let $\mathcal{M}$ and $\mathcal{N}$ be premice, with $\kappa = \mathcal{N} \cap \mathrm{OR}$, and suppose that $\mathcal{T}$ on $\mathcal{M}$ and $\mathcal{U}$ on $\mathcal{N}$ witness that $\mathcal{M}$ iterates past $\mathcal{N}$. Then there are club many $X \in [H_{\kappa^+}]^{<\kappa}$ such that $X \prec H_{\kappa^+}$, $X \cap \kappa \in \kappa$, and if $\pi \colon \bar{H} \cong X \prec H_{\kappa^+}$ is such that $\bar{H}$ is transitive then $\mathcal{P}(X \cap \kappa) \cap \mathcal{N} \subset \bar{H}$ and $X \cap \kappa$ is a cardinal in $\mathcal{N}$.*

Proof. Let $X \prec H_{\kappa^+}$ be such that $\mathrm{Card}(X) < \kappa$, $X \cap \kappa \in \kappa$, $\{\mathcal{M}, \mathcal{N}, \mathcal{T}, \mathcal{U}\} \subset X$, and $X \cap \kappa \in (0,\kappa)_T \cap (0,\kappa)_U$. Let us write α for $X \cap \kappa$. Let $\pi \colon \bar{H} \cong X \prec H_{\kappa^+}$ be such that $\bar{H}$ is transitive. Of course, $\alpha = \mathrm{crit}(\pi)$. We aim to show that $\mathcal{P}(\alpha) \cap \mathcal{N} \subset \bar{H}$ and that α is a cardinal in $\mathcal{N}$.

By elementarity, $((0,\kappa]_T \cap \mathcal{D}^{\mathcal{T}}) \cup ((0,\kappa]_U \cap \mathcal{D}^{\mathcal{U}}) \subset \alpha$. Also, $\pi^{\mathcal{T}}_{\alpha\kappa} \restriction \alpha = \pi^{\mathcal{U}}_{\alpha\kappa} \restriction \alpha = \mathrm{id}$. Therefore

$$\mathcal{M}^{\mathcal{T}}_{\alpha}||\alpha^{+\mathcal{M}^{\mathcal{T}}_{\alpha}} = \mathcal{M}^{\mathcal{T}}_{\kappa}||\alpha^{+\mathcal{M}^{\mathcal{T}}_{\kappa}} = \mathcal{M}^{\mathcal{U}}_{\kappa}||\alpha^{+\mathcal{M}^{\mathcal{U}}_{\kappa}} = \mathcal{M}^{\mathcal{U}}_{\alpha}||\alpha^{+\mathcal{M}^{\mathcal{U}}_{\alpha}}.$$

Let us write $\bar{\mathcal{T}}, \bar{\mathcal{U}} = \pi^{-1}(\mathcal{T}, \mathcal{U})$. It is easy to verify that $\mathcal{M}^{\mathcal{T}}_{\alpha} = \mathcal{M}^{\bar{\mathcal{T}}}_{\alpha}$. We thus know that

$$\mathcal{P}(\alpha) \cap \mathcal{M}^{\mathcal{U}}_{\alpha} \subset \mathcal{M}^{\mathcal{U}}_{\alpha}||\alpha^{+\mathcal{M}^{\mathcal{U}}_{\alpha}} = \mathcal{M}^{\bar{\mathcal{T}}}_{\alpha}||\alpha^{+\mathcal{M}^{\bar{\mathcal{T}}}_{\alpha}} \in \bar{H}.$$

We also have that $\pi^{\mathcal{U}}_{0\alpha}{}''\alpha \subset \alpha$ by Lemma 3.4 and elementarity, and

$$\pi^{\mathcal{U}}_{0\alpha} \restriction \alpha = \pi^{\bar{\mathcal{U}}}_{0\alpha} \restriction \alpha \in \bar{H}.$$

Now let $X \in \mathcal{P}(\alpha) \cap \mathcal{N}$. Set $Y = \pi^{\mathcal{U}}_{0\alpha}(X) \in \mathcal{P}(\alpha) \cap \mathcal{M}^{\mathcal{U}}_{\alpha}$. Then $Y \in \bar{H}$. Thus $Z = (\pi^{\mathcal{U}}_{0\alpha} \restriction \alpha)^{-1}{}''Y$ is then in $\bar{H}$ as well. But $Z = X$, and therefore $X \in \bar{H}$. We have shown that $\mathcal{P}(\alpha) \cap \mathcal{N} \subset \bar{H}$.

If α were not a cardinal in $\mathcal{N}$ then there would be some $X \in \mathcal{P}(\alpha) \cap \mathcal{N} \subset \bar{H}$ witnessing this; however, α is a cardinal in $\bar{H}$. □ (Lemma 3.5)

Lemma 3.6. *Let κ be an uncountable regular cardinal. Assume that $2^{<\kappa} = \kappa$ and κ is countably closed. Suppose that $\pi \colon \bar{H} \to H_{\kappa^+}$ is such that $\bar{H}$ is*

[4]If $\mathcal{M}, \mathcal{N}$ are premice then we write $\mathcal{M} \rhd \mathcal{N}$ for $\exists \alpha < \mathcal{M} \cap \mathrm{OR}\ \mathcal{N} = \mathcal{M}||\alpha$.

transitive, ${}^{\omega}\bar{H} \subset \bar{H}$, *and* $X \cap \kappa \in \kappa$. *Set* $\alpha = X \cap \kappa$. *Let* $\alpha < \nu < \kappa$, *and let* F *be the* (α, ν)*-extender derived from* π, *i.e.,*

$$X \in F_a \Leftrightarrow a \in \pi(X)$$

for $a \in [\nu]^{<\omega}$ *and* $X \in \mathcal{P}([\alpha]^{\mathrm{Card}(a)}) \cap \bar{H}$. *Then* F *is certified.*

Proof. Notice that $\gamma(F) \leq \kappa$, and that κ is a reflection point. Let $U \prec_{\Sigma_{1+}} H_\kappa$ be countable and let $\sigma\colon \bar{U} \cong U$ with $\bar{U}$ transitive. Let $(a_n, X_n\colon n < \omega)$ be a list of all pairs (a, X) such that $a \in [\nu \cap U]^{<\omega}$, $X \in \mathcal{P}([\alpha]^{\mathrm{Card}(a)}) \cap \bar{H} \cap U$, and $a \in \pi(X)$. Let $\bar{a}_n = \sigma^{-1}(a_n) = \sigma^{-1}{}''a_n$ for $n < \omega$. Notice that $(\bar{a}_n, X_n\colon n < \omega) \in \bar{H}$.

Now σ witnesses that in H_{κ^+} there is some $\varphi\colon \bar{U} \to_{\Sigma_{1+}} H_\kappa$ such that $\varphi''\bar{a}_n \in \pi(X_n)$ for all $n < \omega$. By elementarity of π, there is hence some $\varphi \in \bar{H}$, $\varphi\colon \bar{U} \to_{\Sigma_{1+}} \bar{H}_\alpha$, such that $\varphi''\bar{a}_n \in X_n$ for all $n < \omega$. Let $\varphi_0 \in \bar{H}$ be a witness, and set $\tau = \varphi_0 \circ \sigma$. Then $\tau\colon U \to_{\Sigma_{1+}} \bar{H}_\alpha$, and moreover $\tau(a_n) \in X_n$ for all $n < \omega$, i.e., τ witnesses that F is countably complete with respect to U. $\square$ (Lemma 3.6)

Definition 3.7. *Let* $\mathcal{N}$ *be a premouse. We say that* $\mathcal{N}$ *is* universal with respect to smaller premice *provided that there is an uncountable regular cardinal* κ *such that* $\mathcal{N} \cap \mathrm{OR} = \kappa$, *and there is no premouse* $\mathcal{M}$ *which iterates past* $\mathcal{N}$.

If W *is a weasel then we say that* W *is* universal with respect to set-sized premice *if and only if for arbitrary large regular cardinals* κ, $W||\kappa$ *is universal with respect to smaller premice.*

The following restatement of theorem 0.1(2b) is now an immediate consequence of Lemma 3.2, Lemma 3.5, and 3.6.

Theorem 3.8. *If* GCH *holds then* K^c *is universal with respect to set-sized premice.*[5]

We do not know how to prove a version of Lemma 3.8 if GCH fails from some point onward and there is no proper class of inaccessible cardinals.

We now aim to show that K^c is also universal in a different sense. Because we assume that there is no inner model with a Woodin cardinal, the results in §2 together with [6, Lemma 2.4 (b)] give that K^c is $<$ OR iterable. However, it may be the case that there is a definable tree on K^c of length OR with

[5]The observation right after [4, Corollary 4.6], on the other hand, shows that there can be a cardinal $\kappa > \omega_1$ and a premouse $\mathcal{M}$ of height κ such that $\mathcal{M}$ wins the coiteration against $\mathcal{J}_\kappa^{K^c}$.

no cofinal branch at all.[6] We should therefore define "universal" in such a way that the universality of a weasel W does not imply that W be $\mathrm{OR}+1$ iterable (cf. [6, Definition 3.4]).

The proofs of the following results will essentially be refinements of the proofs of Lemma 3.2, Lemma 3.5, and 3.6.[7]

Definition 3.9. *Let $\mathcal{M}$ be a premouse, and let W be a weasel. We say that $\mathcal{M}$* iterates past *W provided the following holds true. There are normal "padded" iteration trees $\mathcal{T}$ and $\mathcal{U}$ of length $\mathrm{OR}+1$ on $\mathcal{M}$ and W respectively arising from the comparison of $\mathcal{M}$ with W such that either $\mathcal{D}^{\mathcal{T}} \cap [0,\infty]_T \neq \emptyset$, or else $\pi^{\mathcal{T}}_{0\infty}$"$\mathrm{OR} \not\subset \mathrm{OR}$.*

The following is easy to verify.

Lemma 3.10. *Let $\mathcal{M}$ be a premouse, and let W be a weasel. Suppose that $\mathcal{T}$ on $\mathcal{M}$ and $\mathcal{U}$ on W witness that $\mathcal{M}$ iterates past W. Then $\mathcal{D}^{\mathcal{U}} \cap [0,\infty]_U = \emptyset$, $\pi^{\mathcal{U}}_{0\infty}$"$\mathrm{OR} \subset \mathrm{OR}$, and if $\beta \in [0,\infty)_T$ is such that $\mathcal{D}^{\mathcal{T}} \cap [0,\infty]_T \subset \beta$ then $\pi^{\mathcal{T}}_{\beta\infty}$"$\mathrm{OR} \not\subset \mathrm{OR}$.* $\square$

Let $\mathcal{M}$ be a premouse, let W be a weasel, and suppose that $\mathcal{T}$ on $\mathcal{M}$ and $\mathcal{U}$ on W witness that $\mathcal{M}$ iterates past W. Both $\mathcal{T}$ and $\mathcal{U}$ are then iteration trees of length $\mathrm{OR}+1$. To make do with the resources of our formal language (which only has variables for sets and classes) and to construe $\mathcal{T}$ and $\mathcal{U}$ as classes (rather than as hyper-classes) we have to identify these iteration trees with classes coding them. Cf. [3, Definition 5.0.6] for the official definition of an iteration tree. For instance, we may identify $\mathcal{T}$ with a class coding the 5-tuple

$$(T \restriction \mathrm{OR}, [0,\mathrm{OR})_T, \mathrm{deg}, D, (E_\alpha \colon \alpha \in \mathrm{OR})$$

(and similarly for $\mathcal{U}$). I.e., (the code for) $\mathcal{T}$ basically consists of (a code for) $\mathcal{T} \restriction \mathrm{OR}$ together with (a code for) the branch $[0,\mathrm{OR})_T$.

Construed in this way, we may then use $\mathcal{T}$ and $\mathcal{U}$ as predicates of the structure $(V;\mathcal{T},\mathcal{U})$, and it will be the case that if

$$(H_\kappa;\bar{\mathcal{T}},\bar{\mathcal{U}}) \prec_{\Sigma_n} (V;\mathcal{T},\mathcal{U})$$

for some sufficiently large $n<\omega$ then actually $\bar{\mathcal{T}}$ is (a code for) $\mathcal{T} \restriction \kappa+1$ and $\bar{\mathcal{U}}$ is (a code for) $\mathcal{U} \restriction \kappa+1$ (note the "+1" here which is important). In

[6]This result is due to Neeman, using a refinement of Woodin's genericity iteration. If δ is the Woodin cardinal of M_1, the minimum iterable model with one Woodin cardinal, then there is an iteration tree $\mathcal{T} \in M_1$ of height δ with no cofinal (well-founded) branch in M_1 at all; $\mathcal{T}$ may in fact be chosen to be definable over $V_\delta^{M_1}$.

[7]We shall essentially replace the uncountable countably closed regular cardinal κ with $2^{<\kappa}=\kappa$ by OR. The reader who can easily see that BGC is strong enough to carry out the relevant arguments is invited to skip the rest of this paper.

this situation we shall in what follows write $\mathcal{T} \restriction \kappa+1$ for $\bar{\mathcal{T}}$ and $\mathcal{U} \restriction \kappa+1$ for $\bar{\mathcal{U}}$, so that with these conventions we shall have

$$(H_\kappa; \mathcal{T} \restriction \kappa+1, \mathcal{U} \restriction \kappa+1) \prec_{\Sigma_n} (V; \mathcal{T}, \mathcal{U}).$$

For the purposes of the following results we have to enrich our language $\mathcal{L}_{\dot{\in},\dot{A}}$ to a language $\mathcal{L}_{\dot{\in},\dot{A},\dot{\mathcal{T}},\dot{\mathcal{U}}}$ by introducing two more predicate symbols, $\dot{\mathcal{T}}$ and $\dot{\mathcal{U}}$. It is clear how we intend to interpret these predicate symbols. For example, $\dot{\mathcal{T}}$, as interpreted in $(H_\kappa; \mathcal{T} \restriction \kappa+1, \mathcal{U} \restriction \kappa+1)$, will be $\mathcal{T} \restriction \kappa+1$.

Let $m < \omega$, and let $\kappa \in \mathrm{OR}$. We say that $X \subset H_\kappa$ is Σ_m definable over $(H_\kappa; \mathcal{T} \restriction \kappa+1, \mathcal{U} \restriction \kappa+1)$ if and only if there is a Σ_m formula $\Phi(v, \vec{w})$ in the language $\mathcal{L}_{\dot{\in},\dot{A},\dot{\mathcal{T}},\dot{\mathcal{U}}}$ and there is a vector $\vec{z}$ of elements of H_κ such that for $x \in H_\kappa$,

$$x \in X \;\Leftrightarrow\; (H_\kappa; \mathcal{T} \restriction \kappa+1, \mathcal{U} \restriction \kappa+1) \models \Phi(x, \vec{z}).$$

Lemma 3.11. *Let $\mathcal{M}$ be a premouse, and let W be a weasel. Suppose that $\mathcal{T}$ on $\mathcal{M}$ and $\mathcal{U}$ on W witness that $\mathcal{M}$ iterates past W. There are then club many $\kappa \in \mathrm{OR}$ such that every element of $\mathcal{P}(\kappa) \cap W$ is Σ_2 definable over $(H_\kappa; \mathcal{T} \restriction \kappa+1, \mathcal{U} \restriction \kappa+1)$.*

Proof. Let $(H_\kappa; \mathcal{T} \restriction \kappa+1, \mathcal{U} \restriction \kappa+1) \prec_{\Sigma_n} (V; \mathcal{T}, \mathcal{U})$ for some sufficiently large $n < \omega$. In much the same way as in the proof of Lemma 3.5 we shall have that $\mathcal{P}(\kappa) \cap \mathcal{M}^{\mathcal{U}}_\kappa \subset \mathcal{M}^{\mathcal{T}}_\kappa$. Moreover, by Lemma 3.10, if $X \in \mathcal{P}(\kappa) \cap \mathcal{M}^{\mathcal{T}}_\kappa$ then there is some $\beta < \kappa$ such that $\beta \in [0, \kappa]_T$ and $\mathcal{D}^{\mathcal{T}} \cap [0, \infty]_T \subset \beta$, and there is some bounded subset $\bar{X}$ of κ, $\bar{X}$ in $\mathcal{M}^{\mathcal{T}}_\beta$, with $\pi^{\mathcal{T}}_{\beta\kappa}(\bar{X}) = X$. In particular, $\bar{X} \in H_\kappa$. For $\xi < \kappa$ we therefore have that $\xi \in X$ if and only if for all sufficiently large $\gamma \in [0, \kappa]_T \setminus \beta$, $\xi \in \pi^{\mathcal{T}}_{\beta\gamma}(\bar{X})$. This shows that X is Σ_2 over $(H_\kappa; \mathcal{T} \restriction \kappa+1, \mathcal{U} \restriction \kappa+1)$.

Now let $Y \in \mathcal{P}(\kappa) \cap W$, and set $Z = \pi^{\mathcal{U}}_{0\kappa}(Y)$. Of course, $\pi^{\mathcal{U}}_{0\kappa} \restriction \kappa$ is Σ_2 definable over $(H_\kappa; \mathcal{T} \restriction \kappa+1, \mathcal{U} \restriction \kappa+1)$, and for $\xi < \kappa$, $\xi \in Y$ if and only if $(\pi^{\mathcal{U}}_{0\kappa} \restriction \kappa)(\xi) \in Z$. But we have shown above that Z is Σ_2 definable over $(H_\kappa; \mathcal{T} \restriction \kappa+1, \mathcal{U} \restriction \kappa+1)$, so that Y is Σ_2 definable over $(H_\kappa; \mathcal{T} \restriction \kappa+1, \mathcal{U} \restriction \kappa+1)$ as well. □ (Lemma 3.11)

Lemma 3.12. *Let $\mathcal{M}$ be a premouse, and let W be a weasel. Suppose that $\mathcal{T}$ on $\mathcal{M}$ and $\mathcal{U}$ on W witness that $\mathcal{M}$ iterates past W. Let $\kappa < \lambda$ be cardinals such that $\mathrm{cf}(\kappa) > \omega$, $\mathrm{cf}(\lambda) > \omega$,*

$$(H_\kappa; \mathcal{T} \restriction \kappa, \mathcal{U} \restriction \kappa) \prec_{\Sigma_5} (H_\lambda; \mathcal{T} \restriction \lambda, \mathcal{U} \restriction \lambda) \prec_{\Sigma_5} (V; \mathcal{T}, \mathcal{U}),$$

and each $X \in \mathcal{P}(\kappa) \cap W$ is Σ_2 definable over $(H_\kappa; \mathcal{T} \restriction \kappa, \mathcal{T} \restriction \kappa)$. Let $\kappa < \nu < \lambda$, and let the (κ, ν)-extender F be defined by

$$\{u \in [\kappa]^{\mathrm{Card}(a)} \colon (H_\kappa; \mathcal{T} \restriction \kappa, \mathcal{U} \restriction \kappa) \models \Phi(u, \vec{x})\} \in F_a \;\Leftrightarrow\; (H_\lambda; \mathcal{T} \restriction \lambda, \mathcal{U} \restriction \lambda) \models \Phi(a, \vec{x}),$$

where $a \in [\nu]^{<\omega}$ *and* $\Phi \in \Sigma_2$ *defines an element of* $\mathcal{P}([\kappa]^{\mathrm{Card}(a)}) \cap W$ *over* $(H_\kappa; \mathcal{T} \restriction \kappa, \mathcal{U} \restriction \kappa)$ *using parameters* $\vec{x} \in H_\kappa$. *Then* F *is certified.*

Proof. Let us fix some $\vec{X} \subset H_\kappa \times [\kappa]^{<\omega}$ such that $\vec{X}$ is Σ_2 definable over $(H_\kappa; \mathcal{T} \restriction \kappa, \mathcal{U} \restriction \kappa)$ and if $X \subset [\kappa]^{<\omega}$ is Σ_2 definable over $(H_\kappa; \mathcal{T} \restriction \kappa, \mathcal{U} \restriction \kappa)$ then there is some $e \in H_\kappa$ with $X = \vec{X}_e = \{u\colon (e,u) \in \vec{X}\}$. Let $\vec{X}' \subset H_\lambda \times [\lambda]^{<\omega}$ be defined over $(H_\lambda; \mathcal{T} \restriction \lambda, \mathcal{U} \restriction \lambda)$ exactly as $\vec{X}$ is defined over $(H_\kappa; \mathcal{T} \restriction \kappa, \mathcal{U} \restriction \kappa)$. Let us write $\vec{X}'_e = \{u\colon (e,u) \in \vec{X}'\}$ for $e \in H_\lambda$. Notice that for all $e \in H_\kappa$ and $a \in [\nu]^{<\omega}$,

$$\vec{X}_e \in F_a \Leftrightarrow a \in \vec{X}'_e.$$

Now let U be countable, $U \prec_{\Sigma_{1+}} H_\lambda$. Let $\sigma\colon \bar{U} \cong U$ be such that $\bar{U}$ is countable and transitive. Let $(a_n, e_n\colon n < \omega)$ be a list of all pairs (a,e) such that $a \in [\nu \cap U]^{<\omega}$, $\vec{X}_e \in \mathcal{P}([\kappa]^{\mathrm{Card}(a)}) \cap U$, and $a \in \vec{X}'_e$. Let $\bar{a}_n = \sigma^{-1}(a_n) = \sigma^{-1}{}''a_n$ for $n < \omega$. Notice that $(\bar{a}_n, e_n\colon n < \omega) \in H_\kappa$.

Now σ witnesses that in H_λ there is some $\varphi\colon \bar{U} \to_{\Sigma_{1+}} H_\lambda$ such that $\varphi''\bar{a}_n \in \vec{X}_{e_n}$ for all $n < \omega$. As $(H_\kappa; \mathcal{T} \restriction \kappa, \mathcal{U} \restriction \kappa) \prec_{\Sigma_5} (H_\lambda; \mathcal{T} \restriction \lambda, \mathcal{U} \restriction \lambda)$, there is hence some $\varphi \in H_\kappa$, $\varphi\colon \bar{U} \to_{\Sigma_{1+}} H_\kappa$, such that $\varphi''\bar{a}_n \in \vec{X}_{e_n}$ for all $n < \omega$. Let $\varphi_0 \in H_\kappa$ be a witness, and set $\tau = \varphi_0 \circ \sigma^{-1}$. Then $\tau\colon U \to_{\Sigma_{1+}} H_\kappa$, and moreover $\tau(a_n) \in \vec{X}_{e_n}$ for all $n < \omega$, i.e., τ witnesses that F is countably complete with respect to U. □ (Lemma 3.12)

Definition 3.13. *A weasel* W *is* universal *if and only if there is no premouse which iterates past* W.

The last three lemmas give a proof of the following restatement of theorem 0.1(2a):

Theorem 3.14. K^c *is universal.*

This theorem can be also proved by a slight refinement of an argument of Jensen (cf. [2] and the proof of [4, Lemma 4.4]).

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